TIEA311 Tietokonegrafiikan perusteet

kevät 2018

("Principles of Computer Graphics" – Spring 2018)

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Frontpage of the local course version, held during Spring 2018 at the Faculty of Information technology, University of Jyväskylä:
`http://users.jyu.fi/~nieminen/tgp18/`

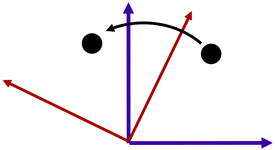
TIEA311 - Local plan for today

Pretty much the most important part of the theory coming up!
Focus! (Tell the lecturer to focus, too!)

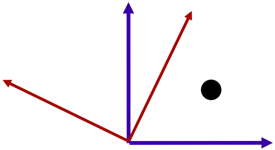
- Maybe some things I forgot to mention yesterday? evalCircle(), git diff?
- We recapitulate earlier definitions and notations about points and vectors.
- Then we think about frames.
- We think really, really hard about changing the coordinate representations of points and vectors between frames
- We think about matrices and affine transformations as tools for frame changes.

Matrices have two purposes

- (At least for geometry)
- Transform things
  - e.g. rotate the car from facing North to facing East



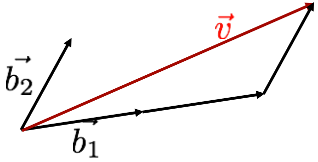
- Express coordinate system changes
  - e.g. given the driver's location in the coordinate system of the car, express it in the coordinate system of the world



Vectors (linear space)

- We can use a ***basis*** to produce all the vectors in the space:
  - Given n basis vectors $\vec{b_i}$ any vector $\vec{v}$ can be written as

$$\vec{v} = \sum_i c_i \vec{b_i}$$



here:
$$\vec{v} = 2\vec{b_1} + \vec{b_2}$$

Linear algebra notation

$$\vec{v} = c_1\vec{b_1} + c_2\vec{b_2} + c_3\vec{b_3}$$

- can be written as

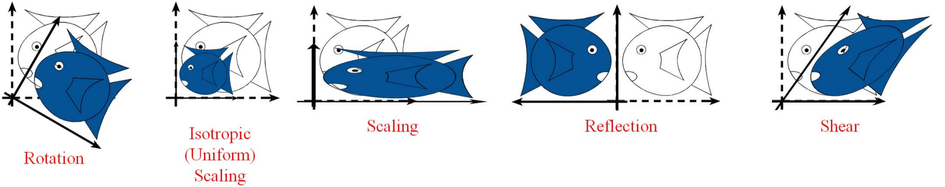
$$\begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

- Nice because it makes the basis (coordinate system) explicit
- Shorthand:

$$\vec{v} = \vec{\mathbf{b}}^t \mathbf{c}$$

- where bold means triplet, t is transpose

Linear transformation



- Transformation $\mathcal{L}$ of the vector space so that

$$\mathcal{L}(\vec{v} + \vec{u}) = \mathcal{L}(\vec{v}) + \mathcal{L}(\vec{u})$$

$$\mathcal{L}(\alpha \vec{v}) = \alpha \mathcal{L}(\vec{v})$$

- Note that it implies $\mathcal{L}(\vec{0}) = \vec{0}$
- Notation $\vec{v} \Rightarrow \mathcal{L}(\vec{v})$ for transformations

Recap, matrix notation

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

- Given the coordinates ***c*** in basis $\vec{\mathrm{b}}$
  the transformed vector has coordinates Mc in $\vec{\mathrm{b}}$

Affine space

- Points are elements of an affine space
- We denote them with a tilde $\tilde{p}$

- Affine spaces are an extension of vector spaces

Frames

- A frame is an origin $\tilde{o}$ plus a basis $\vec{\mathbf{b}}$
- We can obtain any point in the space by adding a vector to the origin

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b}_i$$

- using the coordinates ***c*** of the vector in $\vec{\mathbf{b}}$

Algebra notation

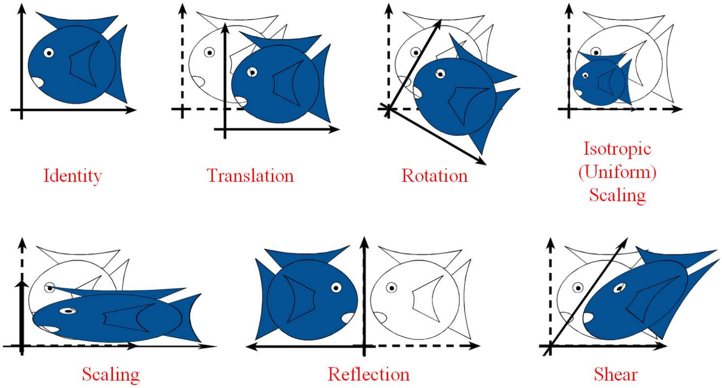
- We like matrix-vector expressions
- We want to keep track of the frame
- We're going to cheat a little for elegance and decide that 1 times a point is the point

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b}_i \quad = \quad \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} = \vec{\mathbf{f}}^t \mathbf{c}$$

- $\tilde{p}$ is represented in $\vec{\mathbf{f}}$ by 4 coordinate, where the extra dummy coordinate is always 1 (for now)

Affine transformations

- Include all linear transformations
  - Applied to the vector basis
- Plus translation



Full affine expression

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b_i} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

$$\Longrightarrow$$

$$\tilde{o} + \vec{t} + \sum_i c_i \mathcal{L}(\vec{b_i}) = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} & \tilde{o} \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

Which tells us both how to get a new frame ftM
or how to get the coordinates Mc after transformation

Note:

- Surfaces in Assignment 1 can be done by applying a suitable affine transform (4x4 matrix) to each vertex and normal of the profile curve.
- (Not the whole story! Especially about the normal vector, but quite enough for Assignment 1...)
- So you should go and complete it right about now.

Change of basis

- Critical in computer graphics
  - From world to car to arm to hand coordinate system
  - From Bezier splines to B splines and back

- problem with basis change:
  you never remember which is M or M^{-1}
  it's hard to keep track of where you are

Change of basis

- Assume we have two bases $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$
- And we have the coordinates of $\vec{\mathbf{a}}$ in $\vec{\mathbf{b}}$
- e.g.

$$\vec{a_1} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix} \begin{bmatrix} M_{11} \\ M_{21} \\ M_{31} \end{bmatrix}$$

- i.e.

$$\vec{\mathbf{a}}^t = \vec{\mathbf{b}}^t M$$

- which implies $\vec{\mathbf{a}}^t M^{-1} = \vec{\mathbf{b}}^t$

Change of basis

- We have $\vec{\mathbf{a}}^t = \vec{\mathbf{b}}^t M$ & $\vec{\mathbf{a}}^t M^{-1} = \vec{\mathbf{b}}^t$
- Given the coordinate of $\vec{v}$ in $\vec{\mathbf{b}}$: $\vec{v} = \vec{\mathbf{b}}^t \mathbf{c}$
- What are the coordinates in $\vec{\mathbf{a}}$?

Change of basis

- We have $\vec{\mathbf{a}}^t = \vec{\mathbf{b}}^t M$ & $\vec{\mathbf{a}}^t M^{-1} = \vec{\mathbf{b}}^t$
- Given the coordinate of $\vec{v}$ in $\vec{\mathrm{b}}$: $\vec{v} = \vec{\mathbf{b}}^t \mathbf{c}$
- Replace $\vec{\mathrm{b}}$ by its expression in $\vec{\mathbf{a}}$

$$\vec{v} = \vec{\mathbf{a}}^t M^{-1} \mathbf{c}$$

- $\vec{v}$ has coordinates $M^{-1}\mathbf{c}$ in $\vec{\mathbf{a}}$
- Note how we keep track of the coordinate system by having the basis on the left

Regarding the “left notation” and changing frames, the local lecturer of TIEA311 was dealing with something he hadn’t really used before.

Questions he had to **ask himself**: How is the math being used here, conceptually? How are the computations done? How does this relate to what he had learned before (on previous instantiations of the present local course and on Linear Algebra of the math dept.)?

Specifically, the order of A and A^{-1} and what it means to “move from a frame to another” were puzzling. How should we interpret “moving” from $\vec{\mathbf{f}}$ to $\vec{\mathbf{a}}$?

Were the slides correct (always possible to contain mistakes)? And assuming they were (which is more likely), what part of the concept did he not yet fully understand?

So…what did the lecturer have to do in order to understand?

(guesses, anyone?)

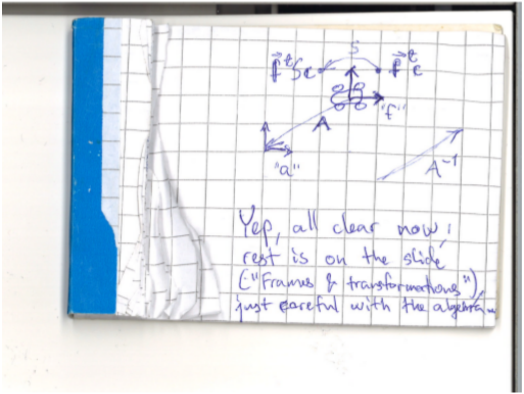
Take a **paper** and a **pen**, and use a **simple, concrete example** to **verify** that the equations **match the mental image**.

This time, it turns out that also the mental image needed to be adjusted (not much, but a little). This is called **learning**. It is painful, takes time, requires necessary tools (perhaps unique for everyone?), and then rewards.

Basic stuff. On the following slides, some scribbles from along the way.

Example: Back and forth between frames?

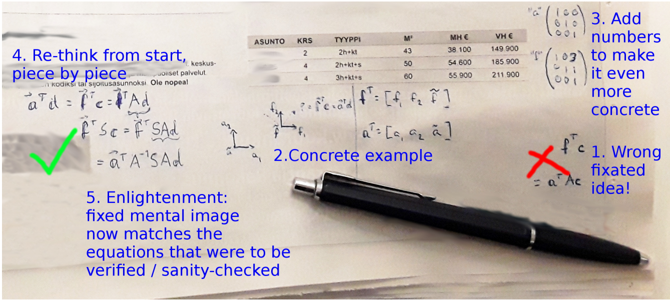
Pen and paper to help the brain (world + car + local origins and basis vectors + a point rotating in car frame):



Well, not yet enough.. provided only a momentary enlightenment that faded away overnight. . . followed an old fixation in thinking.

Example: Back and forth between frames!

More paper with whitespace, possibly same pen (very nice Ballograf one), re-start after thinking carefully about the slides covered on previous lectures:

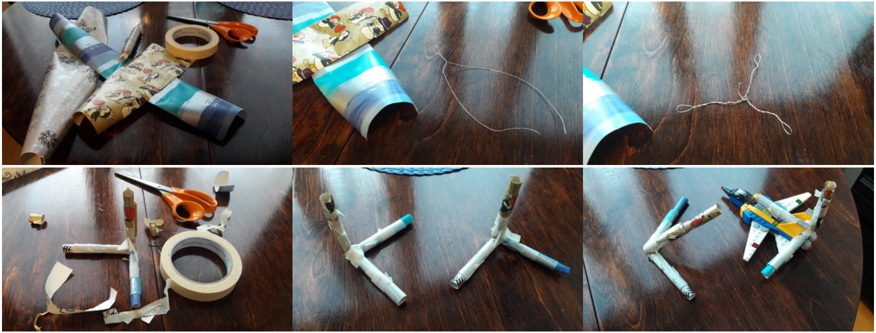


Finally, a corrected mental model of what is “a frame”, and “keeping track of the frame” as defined by the OCW slides. You **must** do this kind of stuff by **yourself** – in your **own way**!

(If you learn without, I think you have **superpowers** and should **go fight hostile aliens**, not waste those powers on IT studies)

TIEA311 Hobby Crafts Corner presents:

DIY Frame of Wire v0.1



The point being: Do whatever you need to do if you think it will help you understand things. . . Perhaps concrete and palpable artifacts (in ”true 3D”) work for you?

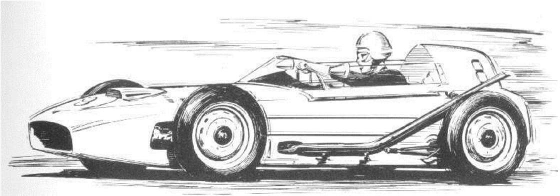
Recap

- Vectors can be expressed in a basis
  - Keep track of basis with left notation $\vec{v} = \vec{\mathbf{b}}^t \mathbf{c}$
  - Change basis $\vec{v} = \vec{\mathbf{a}}^t M^{-1} \mathbf{c}$
- Points can be expressed in a frame (origin+basis)
  - Keep track of frame with left notation
  - adds a dummy 4th coordinate always 1

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b}_i = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} = \vec{\mathbf{f}}^t \mathbf{c}$$

Frames & hierarchical modeling

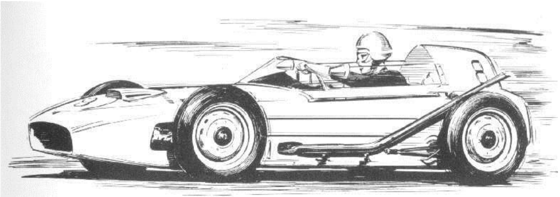
- Many coordinate systems (frames):
  - Camera
  - Static scene
  - car
  - driver
  - arm
  - hand
  - ...



- Need to understand nested transformations

Frames & hierarchical modeling

- Example: what if I rotate the wheel of the moving car:
- frame 1: world
- frame 2: car
- transformation: rotation



Frames & transformations

- Transformation S wrt car frame f

$$\tilde{p} = \vec{\mathbf{f}}^t \mathbf{c} \Rightarrow \vec{\mathbf{f}}^t S \mathbf{c}$$

- how is the world frame a affected by this?
- *we have* $\vec{a}^t = \vec{f}^t A \quad \vec{\mathbf{f}}^t = \vec{\mathbf{a}}^t A^{-1}$
- *which gives* $\vec{a}^t A^{-1} \Rightarrow \vec{a}^t A^{-1} S$

$$\vec{a}^t \Rightarrow \vec{a}^t A^{-1} S A$$

- *i.e. the transformation in a is A-1SA*
- *i.e., from right to left, A takes us from a to f, then we apply S, then we go back to a with A-1*

Ok, here is what **should** have been explained on the lecture:
Some transformation $\mathcal{S}$ w.r.t "car" frame $\vec{\mathbf{f}}$ is $\mathcal{S} : \vec{\mathbf{f}}^t\mathbf{c} \to \vec{\mathbf{f}}^t S\mathbf{c}$.
How is the world frame affected by this?
Breathe. Break it down:

- What do we have? We have a matrix S that can transform coordinates in $\vec{\mathbf{f}}$. Assume $\vec{\mathbf{a}}^t = \vec{\mathbf{f}}^t A$ and $\vec{\mathbf{f}}^t = \vec{\mathbf{a}}^t A^{-1}$.
- Mind the **types** and object identities – like in programming! One space, one transform, two frames. That means two matrices (one for each frame)...
- Rephrase "how is $\vec{\mathbf{a}}$ affected by $\mathcal{S}$": Give M such that $\vec{\mathbf{a}}^t\mathbf{d} \to \vec{\mathbf{a}}^t M\mathbf{d}$ is the same **transform** as $\vec{\mathbf{f}}^t\mathbf{c} \to \vec{\mathbf{f}}^t S\mathbf{c}$ for any **vector** $\vec{v} = \vec{\mathbf{f}}^t\mathbf{c} = \vec{\mathbf{a}}^t\mathbf{d}$. For identity, **coordinates** $\mathbf{c} = A\mathbf{d}$.
- Thus the transform can be written as $\vec{\mathbf{f}}^t(A\mathbf{d}) \to \vec{\mathbf{f}}^t S(A\mathbf{d})$.
- Just write $\vec{\mathbf{f}}$ in terms of $\vec{\mathbf{a}}$: $(\vec{\mathbf{a}}^t A^{-1})(A\mathbf{d}) \to (\vec{\mathbf{a}}^t A^{-1})S(A\mathbf{d})$
- Rearrange parentheses: $\vec{\mathbf{a}}^t(A^{-1}A)\mathbf{d} \to \vec{\mathbf{a}}^t(A^{-1}SA)\mathbf{d}$
- Observe $A^{-1}A = I$. We find that $M = A^{-1}SA$.