TIEA311
Tietokonegrafiikan perusteet
kevät 2018

("Principles of Computer Graphics" – Spring 2018)

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Adapted from: *Wojciech Matusik*, and *Frédo Durand*: 6.837 Computer Graphics. Fall 2012. Massachusetts Institute of Technology: MIT OpenCourseWare, https://ocw.mit.edu/.

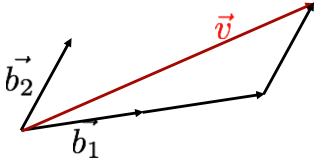
Frontpage of the local course version, held during Spring 2018 at the Faculty of Information technology, University of Jyväskylä:
`http://users.jyu.fi/~nieminen/tgp18/`

Vectors (linear space)

- Formally, a set of elements equipped with addition and scalar multiplication
  - plus other nice properties
- There is a special element, the zero vector
  - no displacement, no force

Vectors (linear space)

- We can use a ***basis*** to produce all the vectors in the space:
  - Given n basis vectors $\vec{b}_i$ any vector $\vec{v}$ can be written as $$\vec{v} = \sum_i c_i \vec{b}_i$$



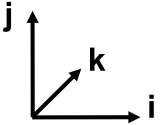
here:

$$\vec{v} = 2\vec{b}_1 + \vec{b}_2$$

Usual Vector Spaces

- In 3D, each vector has three components x, y, z
- But geometrically, each vector is actually the sum

$$v = x\,\vec{\boldsymbol{i}} + y\,\vec{\boldsymbol{j}} + z\,\vec{\boldsymbol{k}}$$



- **i**, **j**, **k** are basis vectors

- Vector addition: just add components
- Scalar multiplication: just multiply components

Polynomials as a Vector Space

- Polynomials $y(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$
- Can be added: just add the coefficients

$$(y+z)(t) = (a_0 + b_0) + (a_1 + b_1)t + \\ (a_2 + b_2)t^2 + \ldots + (a_n + b_n)t^n$$

- Can be multiplied by a scalar: multiply the coefficients

$$s \cdot y(t) = \\ (s \cdot a_0) + (s \cdot a_1)t + (s \cdot a_2)t^2 + \ldots + (s \cdot a_n)t^n$$

Polynomials as a Vector Space

- Polynomials $y(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$

- In the polynomial vector space, {1, *t*, ..., *t*n} are the basis vectors, *a*0, *a*1, ..., *a*n are the components

Subset of Polynomials: Cubic

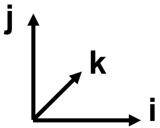
$$y(t) = a_0 + a_1\,t + a_2\,t^2 + a_3\,t^3$$

- Closed under addition & scalar multiplication
  - Means the result is still a cubic polynomial (verify!)
- Cubic polynomials also compose a vector space
  - A 4D **subspace** of the full space of polynomials
- The *x* and *y* coordinates of cubic Bézier curves belong to this subspace as functions of *t*.

Basis for Cubic Polynomials

$$\vec{v} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$

In 3D



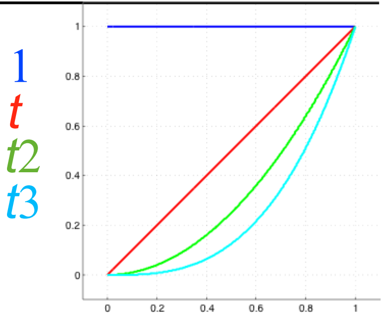
More precisely:

What's a basis?

- A set of "atomic" vectors
  - Called **basis vectors**
  - Linear combinations of basis vectors span the space
    - i.e. any cubic polynomial is a sum of those basis cubics
- Linearly independent
  - Means that no basis vector can be obtained from the others by linear combination
    - Example: **i**, **j**, **i**+**j** is not a basis (missing **k** direction!)

Canonical Basis for Cubics

$$\{1, t, t^2, t^3\}$$



- Any cubic polynomial is a linear combination of these:

$a0+a1t+a2t2+a3t3 = a0*1+a1*t+a2*t2+a3*t3$

- They are linearly independent
  - Means you cannot write any of the four monomials as a linear combination of the others. (You can try.)

Linear algebra notation

$$\vec{v} = c_1\vec{b_1} + c_2\vec{b_2} + c_3\vec{b_3}$$

- can be written as

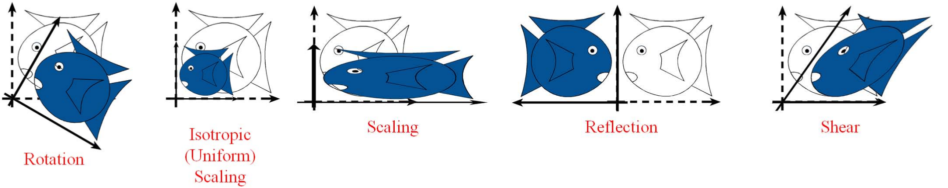
$$\begin{bmatrix} \vec{b_1} & \vec{b_2} & \vec{b_3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

- Nice because it makes the basis (coordinate system) explicit
- Shorthand:

$$\vec{v} = \vec{\mathbf{b}}^t \mathbf{c}$$

- where bold means triplet, t is transpose

Linear transformation



- Transformation $\mathcal{L}$ of the vector space so that

$$\mathcal{L}(\vec{v} + \vec{u}) = \mathcal{L}(\vec{v}) + \mathcal{L}(\vec{u})$$

$$\mathcal{L}(\alpha\vec{v}) = \alpha\mathcal{L}(\vec{v})$$

- Note that it implies $\mathcal{L}(\vec{0}) = \vec{0}$
- Notation $\vec{v} \Rightarrow \mathcal{L}(\vec{v})$ for transformations

Matrix notation

- Linearity implies

$$\mathcal{L}(\vec{v}) = \mathcal{L}\left(\sum_i c_i \vec{b_i}\right) = \sum_i c_i \mathcal{L}(\vec{b_i})$$

- i.e. we only need to know the basis transformation
- or in algebra notation

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{bmatrix} \mathcal{L}(\vec{b}_1) & \mathcal{L}(\vec{b}_2) & \mathcal{L}(\vec{b}_3) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Algebra notation

- The $\mathcal{L}(\vec{b_i})$ are also vectors of the space
- They can be expressed in the basis for example:

$$\mathcal{L}(\vec{b}_1) = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} M_{1,1} \\ M_{2,1} \\ M_{3,1} \end{bmatrix}$$

- which gives us

$$\begin{bmatrix} \mathcal{L}(\vec{b}_1) & \mathcal{L}(\vec{b}_2) & \mathcal{L}(\vec{b}_3) \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{bmatrix}$$

Recap, matrix notation

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

- Given the coordinates ***c*** in basis $\vec{\mathrm{b}}$
  the transformed vector has coordinates Mc in $\vec{\mathrm{b}}$

Example 1

Just one example of a linear transformation matrix useful in graphics – Counter-clockwise rotation of θ radians around the z-axis (pointing towards the viewer when right-handed coordinates are used):

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

How to remember / understand this? Take a pen, draw a unit circle on the xy-plane, and recall basic trigonometry from school times (or Wikipedia...)! The "canonical" basis vectors $[1, 0, 0]^t$ and $[0, 1, 0]^t$ must rotate along the unit circle to the expected new positions.

Really, do it, if you haven't already!

Example 2

Another example of a linear transformation matrix useful in graphics – Scaling of axes:

$$S(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}$$

How to remember / understand this? Take a pen, draw a unit box or some other simple shape, and see how different values of s_x,s_y, and s_z make isotropic and anisotropic scalings.

Really, do it, if you haven't already!

Further Examples

Look at the implementation of the Matrix3f class in our example codes, found in the files `Matrix3f.cpp` and `Matrix3f.h`

Make sure you **understand the implementation** of rotation and scaling matrices, and the overloaded **operators** for matrix-vector multiplication and matrix-matrix multiplication.

Really, do it, if you haven't already!

Also, see what else the class provides and how it all looks in C++. How is the code split in the header (.h) and implementation (.cpp). **Learn to use your IDE to navigate** the files easily!

Example: Inverse transforms

`Matrix3f.cpp` implements determining (by computing the "determinant") if an **inverse matrix** exists, and a formula for inverting an invertible matrix. It is important to understand the *concept* of the inverse transform ($M^{-1}M = MM^{-1} = I$). For most of our graphics transforms, we know the inverses explicitly (understand and verify):

$$R_z^{-1}(\theta) = R_z(-\theta) = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

BTW: The last identity means that the inverse **in this case** is the transpose of the original. How do we get that? (1) By symmetry properties of the trigonometric functions (draw it to believe) but also (2) they teach us in linear algebra courses that **this is true for any real-valued matrix that is "orthonormal", i.e., keeps orthogonality and distances the same**. This makes some inverses in graphics and other computation tasks trivial, maximally accurate, and blazingly fast!

Another inverse

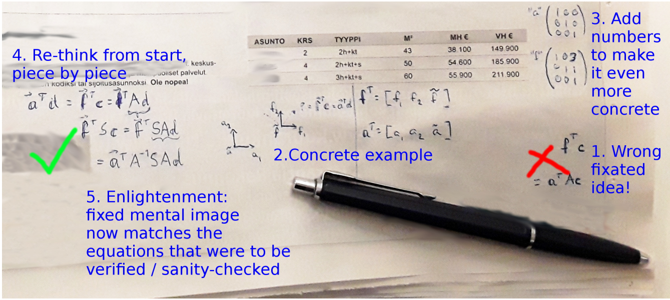
The inverse of scaling is easy to figure out:

$$S^{-1}(s_x, s_y, s_z) = S(1/s_x, 1/s_y, 1/s_z) = \begin{bmatrix} 1/s_x & 0 & 0 \\ 0 & 1/s_y & 0 \\ 0 & 0 & 1/s_z \end{bmatrix}$$

This one is **not** the same as S^T because scaling is not orthonormal (orthogonal yes, but not normal, i.e., it does not preserve lengths).

Remember the supertools

Your Super Tools: the Brain, the Pen and the Paper. Teacher's own example from last year:

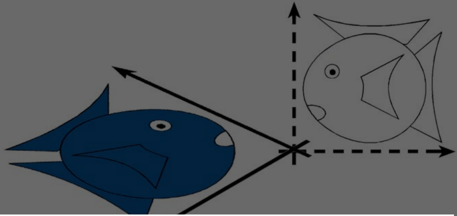


Why do we care

- We like linear algebra
- It's always good to get back to an abstraction that we know and for which smarter people have developed a lot of tools
- But we also need to keep track of what basis/coordinate system we use

Linear Transformations

- $L(p + q) = L(p) + L(q)$
- $L(ap) = a\,L(p)$



Translation is not linear:

$f(p) = p+t$

$f(ap) = ap+t \neq a(p+t) = a\,f(p)$

$f(p+q) = p+q+t \neq (p+t)+(q+t) = f(p) + f(q)$

Affine space

- Points are elements of an affine space
- We denote them with a tilde $\tilde{p}$
- Affine spaces are an extension of vector spaces

Point-vector operations

- Subtracting points gives a vector

$$\tilde{p} - \tilde{q} = \vec{v}$$

- Adding a vector to a point gives a point

$$\tilde{q} + \vec{v} = \tilde{p}$$

Frames

- A frame is an origin $\tilde{o}$ plus a basis $\vec{\mathrm{b}}$
- We can obtain any point in the space by adding a vector to the origin

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b}_i$$

- using the coordinates ***c*** of the vector in $\vec{\mathrm{b}}$

Algebra notation

- We like matrix-vector expressions
- We want to keep track of the frame
- We're going to cheat a little for elegance and decide that 1 times a point is the point

$$\tilde{p} = \tilde{o} + \sum_i c_i \vec{b}_i \quad = \quad \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} = \vec{\mathbf{f}}^t \mathbf{c}$$

- $\tilde{p}$ is represented in $\vec{\mathbf{f}}$ by 4 coordinate, where the extra dummy coordinate is always 1 (for now)

Further Examples (Affine transforms)

Look at the implementation of the Matrix4f class in our example codes, found in the files `Matrix4f.cpp` and `Matrix4f.h`

These are using the fourth coordinate to implement 3D frames and affine transforms of points. A straightforward way to do many things is to build a proper 4x4 matrix and then multiply. **Not much code**, actually!

In Assignment 1 you will **avoid a lot of tears** by figuring out how (and when and why) to use the provided constructors and the operator $\ast$ to transform points and frames suitably.