LECTURE NOTE FOR THE COURSE *PARTIAL DIFFERENTIAL EQUATIONS*, MATS230, 9 POINTS

MIKKO PARVIAINEN
UNIVERSITY OF JYVÄSKYLÄ

Contents

Date: December 29, 2021.

1. Introduction

This lecture note contains a sketch of the lectures. The illustrations and more examples are presented during the lectures. Lectures are mainly following Evans: PDEs.

A partial differential equation (PDE), is an equation of an unknown function of two or more variables, and its partial derivatives.

Example 1.1. *A simple PDE:*

$$u : \Omega \to \mathbb{R}, \quad \Omega \subset \mathbb{R}^n, \quad n \geq 2.$$

For example, $n = 2$, $(x, y) \in \mathbb{R}^2$

$$u_{x_1 x_1}(x_1, x_2) + u_{x_2 x_2}(x_1, x_2) = 0.$$

A boundary value problem: Quite often we are given boundary values g*, for example*

$$n = 2, (x, y) \in \Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2,$$

$$\begin{cases} u_{x_1 x_1} + u_{x_2 x_2} = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

and we should solve u*, or at least know some properties of* u*.*

Depending what we are modeling by a PDE, the unknown u may describe for example a physical quantity for the heat, electric potential. Partial differential equations have a great variety of applications to mechanics, electrostatics, quantum mechanics and many other fields of physics as well as to finance. In addition, PDEs have a rich mathematical theory.

Example 1.2. *We consider next an initial value problem*

$$\begin{cases} u_t + b \cdot Du = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where

$$\begin{aligned} &u : \mathbb{R}^n \times (0, \infty) \to \mathbb{R} \text{ (unknown, to be searched)} \\ &g : \mathbb{R}^n \to \mathbb{R} \text{ (given)}, \\ &b = (b_1, \ldots, b_n) \text{ (given)}, \\ &Du = (u_{x_1}, \ldots, u_{x_n}) \text{ the gradient.} \end{aligned}$$

This is called a transport equation. Roughly, the reason for the name of the equation is as follows. Consider a conveyor belt that is for simplicity modelled in 1D, and infinity long. Then denote the mass density (kg/m) at

x at time t by $u(x,t)$. The speed of the belt is b and thus the mass exiting at $x+h$ in a short time s is approximately

$$-sbu(x+h,t)$$

and similarly the mass entering $sbu(x,t)$. Then it holds that change of mass on $[x,x+h]$ over time $[t,t+s]$ $=$ mass entering at x over time $[t,t+s]-$mass exiting at $x+h$ over time $[t,t+s]$.

$$\int_x^{x+h} u(y,t+s)\,dy - \int_x^{x+h} u(y,t)\,dy \approx u(x,t)bs - u(x+h,t)bs,$$

$$\Rightarrow \frac{1}{h}\int_x^{x+h} \frac{1}{s}(u(y,t+s)-u(y,t))\,dy \approx \frac{u(x,t)-u(x+h,t)}{h}b.$$

Since $\frac{1}{h}\int_x^{x+h}$ is just the integral average by passing to a limit with s,h, we get

$$u_t(x,t) = -u_x(x,t)b.$$

What we naturally need to solve for mass density at given location x and time t is the initial mass density $g(x)$. We can guess that the solution is

$$u(x,t) = g(x-bt).$$

Even for $g \notin C^1$ the conveyor belt example makes sense, so already such a simple example suggests a need for the 'weak solutions' that are dealt in the later courses (PDE2 and Viscosity theory).

1.1. **Notations (review).** Basic notation

$$\mathbb{R}^n,\ n-\text{dimensional Euclidean space}$$

$$\mathbb{R} = \mathbb{R}^1$$

$$e_1 = (1,0,\dots,0),\dots,e_n = (0,\dots,0,1),\ \text{standard basis vectors}$$

$$\Omega, U \subset \mathbb{R}^n,\quad \text{open set, bounded unless otherwise stated}$$

$$|x| = \sqrt{x_1^2+\dots+x_n^2}\ \text{for}\ x \in \mathbb{R}^n,$$

$$\partial\Omega\ \text{boundary of a set}\ \Omega,$$

$$B(x,r),\ \text{a ball of radius}\ r\ \text{centered at}\ x$$

$$|B(x,r)| = \alpha_n r^n = \text{volume of a ball}$$

$$|\partial B(x,r)| = \omega_n r^{n-1} = \text{area of a sphere}$$

$$⨍_{B(0,\varepsilon)} \dots\,dy = \frac{1}{|B(0,\varepsilon)|}\int_{B(0,\varepsilon)} \dots\,dy\ \text{mean value integral over a ball}$$

$$⨍_{\partial B(0,\varepsilon)} \dots\,dy = \frac{1}{|\partial B(0,\varepsilon)|}\int_{\partial B(0,\varepsilon)} \dots\,dy\ \text{mean value integral over a sphere}$$

$$\Omega \Subset U,\quad \overline{\Omega} \subset U\ \text{and}\ \overline{\Omega}\ \text{is compact}$$

Functions and derivatives

$$f:\Omega\to\mathbb{R},\ \text{a function}$$
$$\operatorname{spt} f=\overline{\{x\in\Omega\,:\,f(x)\neq 0\}}=\text{the support of } f$$
$$\frac{\partial u}{\partial x_i}(x)=\lim_{h\to 0}\frac{u(x+he_j)-u(x)}{h},\ u\text{'s partial derivative to the direction } x_j$$
$$u_{x_j}=D_j u=\frac{\partial u}{\partial x_i},\ \text{shorthands for partial derivatives ,}$$
$$u_{x_ix_j}=D_{ij}u=\frac{\partial^2 u}{\partial x_i\partial x_j},\ \text{higher order derivatives}$$
$$Du=(\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}),\ \text{gradient}$$
$$\frac{\partial u}{\partial \nu}=Du\cdot\nu,\text{outward normal derivative, } \nu \text{ outward unit normal vector}$$

Multi-indexes and spaces

$$\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n\ \text{multi-index}$$
$$|\alpha|=\alpha_1+\ldots+\alpha_n$$
$$D^\alpha u=\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$
$$D^k u(x)=\{D^\alpha u(x)\,:\,|\alpha|=k\},\ \text{whenever } k\in\mathbb{N}$$
$$\begin{aligned}D^2u(x)&=\{D_{11}u(x),D_{12}u(x),\dots,D_{1n}u(x),D_{21}u(x),D_{22}u(x),\\&\dots,D_{2n}u(x),\dots D_{nn}u(x)\}\\&=\begin{pmatrix}D_{11}u(x)&\dots&D_{1n}u(x)\\ \vdots&\ddots&\vdots\\ D_{n1}u(x)&\dots&D_{nn}u(x)\end{pmatrix},\ \text{Hessian matrix}\end{aligned}$$
$$D^1u(x)=Du(x)$$
$$C(\Omega)=\{f\,:\,f\text{ continuous in }\Omega\}$$
$$C(\overline{\Omega})=\{f\,:\,f\text{ uniformly continuous on bounded subsets of }\Omega\}$$
$$C_0(\Omega)=\{f\in C(\Omega)\,:\,\operatorname{spt} f\text{ is compact subset of }\Omega\}$$
$$C^k(\Omega)=\{f\in C(\Omega)\,:\,f\text{ is } k \text{ times continuously differentiable}\}$$
$$C^k(\overline{\Omega})=\{u\in C^k(\Omega)\,:\,D^\alpha u\text{ is uniformly cont on all bdd subsets of }\Omega\text{ for all } |\alpha|\le k\}$$
$$C^k_0(\Omega)=C^k(\Omega)\cap C_0(\Omega)$$
$$C^\infty(\Omega)=\cap_{k=1}^\infty C^k(\Omega)=\text{smooth functions}$$

$$C^\infty_0(\Omega)=C^\infty(\Omega)\cap C_0(\Omega)=\text{compactly supported smooth functions}$$
$$||f||_{L^\infty(\Omega)}=\sup_\Omega|f|\ \text{ for } f\in C(\Omega).$$

Remark 1.3. *Recall that*

$$u \in C^k(\Omega) \iff D^\alpha u \in C(\Omega)$$

for multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ *and* $|\alpha| := \alpha_1 + \dots + \alpha_n \leq k$*, where*

$$D^\alpha u := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

Example 1.4. *(1)*

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

$$f \in C^1(\Omega) \setminus C^2(\Omega)$$

(2)

$$\varphi : \mathbb{R}^n \to \mathbb{R}, \quad \varphi(x) = \begin{cases} e^{1/(|x|^2-1)}, & |x| < 1 \\ 0, & |x| \geq 1. \end{cases}$$

$$\varphi \in C_0^\infty(\Omega), \operatorname{spt} \varphi \subset \overline{B}(0,1)$$

1.2. General form of a PDE and classifications.

Definition 1.5 (General form)**.** *Given a real valued function* F*, the expression of the form*

$$F(D^k u(x), D^{k-1}u(x), \dots, Du(x), u(x), x) = 0,$$

is k*th-order PDE i.e.* k *is the order of the highest order derivative. The unknown is a function*

$$u : \Omega \to \mathbb{R}.$$

Example 1.6. *Most of the examples on this course are of second order. Let* $n = 2$ *and consider*

$$\begin{aligned} &D^2 u(x) = \begin{pmatrix} D_{11}u(x) & D_{12}u(x) \\ D_{21}u(x) & D_{22}u(x) \end{pmatrix} \\ &F : \mathbb{R}^{2^2} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}, \\ &F(M, p, u, x) = m_{11} + m_{22}, \\ &M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}. \end{aligned}$$

Then

$$\begin{aligned} F(D^2u(x), D^1u(x), u(x), x) &= D_{11}u(x) + D_{22}u(x) \\ &= u_{x_1x_1}(x_1, x_2) + u_{x_2x_2}(x_1, x_2) = 0 \end{aligned}$$

i.e. the Laplace equation.

Definition 1.7 (Classifications)**.** *If PDE can be written in the forms below, then it is*

(1) linear, if

$$\sum_{|\alpha|\leq k} a_\alpha(x)D^\alpha u(x) = f(x).$$

(2) semilinear, if

$$\sum_{|a|=k} a_\alpha(x)D^\alpha u(x) + a_0(D^{k-1}u,\dots,Du,u,x) = 0$$

(3) Quasilinear, if

$$\sum_{|a|=k} a_\alpha(D^{k-1}u,\dots,Du,u,x)D^\alpha u(x) + a_0(D^{k-1}u,\dots,Du,u,x) = 0$$

(4) Fully nonlinear, if PDE depends nonlinearly on the highest order derivatives.

Remark 1.8. *In the second order case we get*

(1) linear, if

$$Lu(x) := -\sum_{i,j=1}^{n} a_{ij}(x)D_{ij}u(x) + \sum_{i=1}^{n} b_i(x)D_iu(x) + c(x)u(x) = f(x)$$

for given coefficients a_{ij}, b_i and c.

(2) Quasilinear, if

$$\sum_{i,j=1}^{n} a_{ij}(Du,u,x)D_{ij}u(x) + a_0(Du,u,x) = 0$$

Remark 1.9. *(1) In the linear case the LHS of PDE can be seen as a linear operator in the function space:*

$$L(au+bv) = aL(u) + bL(v)$$

where $a,b\in\mathbb{R}$ and u,v are functions (=linearity, L like linear), and PDE reads as

$$Lu = f.$$

Observe that the operator is linear, but naturally if there is a righ hand side i.e. $\Delta u = f, \Delta v = f$ then

$$\Delta(u+v) = \Delta u + \Delta v = 2f$$

so $u+v$ does not solve the same equation.

(2) Quasilinear equation is linear in highest order derivatives.

Example 1.10. *(1) Laplacian i.e. Δu is linear. Let $a_{ij} = 0$ if $i \neq j$ and $a_{ij} = 1$ if $i = j$. Then*

$$\Delta u = \sum_{i=1}^{n} D_{ii}u = \sum_{i,j=1}^{n} D_{ij}u$$

$$\Delta(au+bv)=\sum_i^n(au+bv)_{x_ix_i}=a\sum u_{x_ix_i}+b\sum v_{x_ix_i}=a\Delta u+b\Delta v.$$

(2) $\Delta u+|Du|^2=0$ is semilinear.

Remark 1.11. *There are further classifications. If the highest order term can be written in the form*

$$\operatorname{div}(\mathcal{A}(D^{k-1}u,\dots,u,x)),$$

then the equation is in divergence form. If not, then it is in non-divergence form

Example 1.12. *Observe that*

$$\begin{aligned}Lu=-\sum_{i,j=1}^n D_i(a_{ij}(x)D_ju(x))&=-\sum_{i=1}^n D_i(\sum_{j=1}^n a_{ij}(x)D_ju(x))\\&=-\operatorname{div}(\mathcal{A}(x)Du).\end{aligned}$$

where $\mathcal{A}$ is a matrix with the entries a_{ij} i.e. linear second order equation in divergence form reads as

$$Lu(x)=-\sum_{i,j=1}^n D_i(a_{ij}(x)D_ju(x))+\sum_{i=1}^n b_i(x)D_iu(x)+c(x)u(x)=f(x).$$

Remark 1.13. *There are further classifications. In particular:*

- *Elliptic="Laplace equation like"*
- *Parabolic="Heat equation like, time dependent"*
- *Hyperbolic="Wave equation like, time dependent"*

One could give more precise statements, but we do not pursue this direction.

Remark 1.14. *There are several boundary value problems. The most common on this course is: Dirichlet boundary value problem, the value of the solution is given at the boundary*

$$u=g \text{ on } \partial\Omega$$

Cf. the derivation of the minimal surface equation.

We also encounter the Neumann problem, where the outward normal derivative is given:

$$\frac{\partial u}{\partial\nu}=g \text{ on } \partial\Omega$$

where $\frac{\partial u}{\partial\nu}=Du\cdot\nu$, outward normal derivative, ν outward unit normal vector.

Remark 1.15.
- *There are several kind of solutions. On this course we consider classical solutions. It means that solution is smooth enough so that the derivatives in the equation make sense. For example, $u\in C^2(\Omega)$ such that $\Delta u=0$ is a classical solution to the Laplacian.*

- *Weak (distributional) solutions are considered in the course PDE2. Divergence form equations.*
- *Viscosity solution are considered in the course Viscosity theory (PDE 3). Control and game theory applications, probability and finance.*
- *Strong solutions...*

Remark 1.16 (A well-posed problem)**.** *A PDE problem is well-posed if it has*

(1) existence
(2) uniqueness
(3) stability: the solution depends continuously on data. In many cases in physics, data comes from measurements and it is crucial that small variations in the measurements only cause small change in the solution.

1.3. **Examples.**

Example 1.17. *(1) Laplace equation*

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = 0$$

(2) Poisson equation

$$-\Delta u(x) = f(x)$$

(3) Nonlinear Poisson equation (f not linear)

$$-\Delta u = f(u)$$

(4) Heat equation

$$u_t - \Delta u = 0$$

(5) Wave equation

$$u_{tt} - \Delta u = 0$$

(6) Linear transport equation

$$u_t + \sum_{i=1}^{n} b_i u_{x_i} = 0$$

(7) Eikonal equation

$$|Du|^2 = 1$$

(8) Eigenvalue equation or Helmholz equation

$$-\Delta u = \lambda u$$

(9) p-Laplace equation

$$\operatorname{div}(|Du|^{p-2} Du) = 0, \ p > 1$$

(10) Infinity Laplace equation

$$\Delta_\infty u = \sum_{i,j=1}^{n} u_{x_i x_j} u_{x_i} u_{x_j} = 0$$

(11) Monge-Ampère equation

$$\det(D^2 u) = f$$

(12) Hamilton-Jacobi equation

$$u_t + H(Du, x) = 0$$

(13) Parabolic p-Laplace/ p-parabolic equation

$$u_t = \operatorname{div}(|Du|^{p-2} Du)$$

(14) Porous medium equation

$$u_t = \Delta(u^m)$$

(15) Minimal surface equation

$$\operatorname{div}\left(\frac{Du}{(1+|Du|^2)^{\frac{1}{2}}}\right) = 0$$

(16) Navier-Stokes equation (system, n=3) (1 million \$ prize)

$$\begin{cases} (u_i)_t + \mathbf{u} \cdot Du_i - \nu \Delta u_i = -\frac{\partial p}{\partial x_i} \quad i = 1, 2, 3, \\ \operatorname{div} \mathbf{u} = 0, \mathbf{u} = (u_1, u_2, u_2) \end{cases}$$

Systems (= many equations) are often, like in this case, more involved.

Next we will derive an equation for the soap film (minimal surface equation). Recall the following counterparts of the integration by parts from the earlier courses. As a reminder, $\partial\Omega \in C^1$ roughly means that the boundary can be locally presented as a C^1 function. This suffices to guarantee that the normal vector below is well defined.

Theorem 1.18 (Gauss-Green theorem)**.** *Let $\partial\Omega \in C^1$ and $u \in C^1(\overline{\Omega})$. It holds that*

$$\int_\Omega u_{x_i}\, dx = \int_{\partial\Omega} u\nu_i \, dS, \quad i = 1, 2, \dots, n,$$

where $\nu = (\nu_1, \dots, \nu_n)$ is the unit normal vector.

In older versions, it is assumed above that $u \in C^1(\Omega) \cap C(\overline{\Omega})$ where integral on the right can be interpreted by taking approximations of the domain from inside.

Example 1.19. *In 1D the previous theorem is just the familiar fundamental theorem of calculus*

$$\int_a^b u'\,dx = u(b) - u(a).$$

From the previous theorem, we obtain (ex).

Theorem 1.20 (div-thm)**.** *Let $\varphi \in C_0^\infty(\Omega)$ and $F : \mathbb{R}^n \to \mathbb{R}^n$, $F_i \in C^1(\Omega)$. It holds that*

$$\int_\Omega F \cdot D\varphi\,dx = -\int_\Omega \operatorname{div} F\varphi\,dx, \quad i = 1, 2, \ldots, n,$$

Example 1.21. *In 1D this is just the familiar integration by parts with the zero boundary values*

$$\int_a^b F\varphi'\,dx = 0 - 0 - \int_a^b F'\varphi\,dx.$$

Example 1.22 (Minimal surface equation)**.** *Suppose you dip a wire frame into a soap solution, forming a soap film. The soap film tends to minimize the area i.e. it forms a minimal surface with boundary values fixed at the wire. $\Omega \subset \mathbb{R}^2$*

$$u : \Omega \to \mathbb{R}, \textit{unknown, the height of the soap film.}$$

Area of 3D-surface $z = u(x)$ is

$$A(u) = \int_\Omega \sqrt{1 + |Du|^2}\,dx$$

where

$$|Du|^2 = \sum_{i=1}^2 \Big(\frac{\partial u}{\partial x_i}\Big)^2.$$

Heuristically the idea is that since u is the minimizer, if we vary it a bit preserving the boundary values, and compute a suitable derivative (see (1.1) below), this derivative should be zero by the minimizing property. Let $\varphi \in C_0^\infty(\Omega)$. We need to compute the derivative in (1.1). Since

$$\begin{aligned}\frac{d}{d\varepsilon}\,|D(u+\varepsilon\varphi)|^2 &= \frac{d}{d\varepsilon}\sum_i^n\Big(\frac{\partial u}{\partial x_i} + \varepsilon\frac{\partial \varphi}{\partial x_i}\Big)^2 \\ &= \sum_{i=1}^n 2\Big(\frac{\partial u}{\partial x_i} + \varepsilon\frac{\partial \varphi}{\partial x_i}\Big)\frac{\partial \varphi}{\partial x_i} = 2(Du + \varepsilon D\varphi)\cdot D\varphi\end{aligned}$$

we get

$$\begin{aligned}\frac{d}{d\varepsilon}A(u+\varepsilon\varphi) &= \frac{d}{d\varepsilon}\int_\Omega \sqrt{1 + |D(u+\varepsilon\varphi)|^2}\,dx \\ &= \int_\Omega \frac{1}{2}(1 + |D(u+\varepsilon\varphi)|^2)^{-\frac{1}{2}}2(Du + \varepsilon D\varphi)\cdot D\varphi\,dx.\end{aligned}$$

As the soap film minimizes the area, the solution u should satisfy for some perturbation $\varphi \in C_0^\infty(\Omega)$

$$0 = \frac{d}{d\varepsilon} A(u+\varepsilon\varphi)\Big|_{\varepsilon=0} = \int_\Omega (1+|Du|^2)^{-\frac{1}{2}} Du \cdot D\varphi \, dx \qquad (1.1)$$

$$\overset{div\text{-}thm}{=} -\int_\Omega \operatorname{div}\Big(\frac{Du}{\sqrt{1+|Du|^2}}\Big)\varphi \, dx.$$

Since this holds for all $\varphi \in C_0^\infty(\Omega)$, we have

$$\operatorname{div}\Big(\frac{Du}{\sqrt{1+|Du|^2}}\Big) = 0.$$

12.9.2019

Example 1.23 (Hamilton-Jacobi equation and optimal control)**.** *Basic setup of an optimal control problem with (x,t) as a starting point We use the following terminology and notation*

$(x,t) \in \mathbb{R}^n \times [0,T]$, *starting point*

$\alpha : [t,T] \to A$, *control (we do not specify A in this sketch)*

$\mathcal{A}$, *the set of controls*

a trajectory $x(\cdot)$is given by

$$\begin{cases} x'(s) = f(x(s),\alpha(s)), & s \in [t,T] \\ x(t) = x \end{cases} \quad \text{dynamics given by } f$$

$f : \mathbb{R}^n \times A \to \mathbb{R}^n$,

$r : \mathbb{R}^n \times A \to \mathbb{R}$, *running payoff,*

$g : \mathbb{R}^n \to \mathbb{R}$, *terminal payoff.*

The value for this control problem is given by

$$u(x,t) = \sup_{\alpha\in\mathcal{A}} P_{x,t}(\alpha) := \sup_{\alpha\in\mathcal{A}} \int_t^T r(x(s),\alpha(s))\, ds + g(x(T)),$$

We could prove that for each $h > 0$ small enough so that $t + h \le T$, we have

$$u(x,t) = \sup_{\alpha\in\mathcal{A}} \{\int_t^{t+h} r(x(s),\alpha(s))\, ds + u(x(t+h), t+h)\},$$

where $x(\cdot)$ is the trajectory with the control α. This is called a dynamic programming principle. It is also easy to believe by heuristic consideration: Idea is that we can think that we play optimally from time $t+h$ and thus obtain $u(x(t+h), t+h)$. Getting there pays $\int_t^{t+h} r(x(s),\alpha(s))\, ds$.

Next we connect the value function to the PDE. The PDE then can provide (existence, uniqueness, solvers available...) us the way to access the value function and the optimal control.

The heuristics is that the PDE is an infinitesimal version of DPP. Formally supposing u is a smooth value we can start from the DPP

$$0 = \sup_{\alpha \in \mathcal{A}} \{ \int_t^{t+h} r(x(s), \alpha(s))\, ds + u(x(t+h), t+h) - u(x,t) \},$$

and assume that we are only using controls such that $\lim_{h \to 0+} \alpha(t+h) = a$. *Then dividing by h, taking limit, changing order of* $\lim$ *and* $\sup$, *we formally obtain*

$$\begin{aligned} 0 &= \sup_{a \in A} \{ r(x,a) + \frac{d}{dt}(u(x(t),t)) \} \\ &= \sup_{a \in A} \{ r(x,a) + Du(x(t),t) \cdot x'(t) + u_t(x(t),t)) \} \\ &= \sup_{a \in A} \{ r(x,a) + Du(x,t) \cdot f(x,a) \} + u_t(x,t)) \\ &=: H(x, Du(x,t)) + u_t(x,t). \end{aligned}$$

PDEs are useful with optimal control problems!

2. First order linear equations

We will solve some simple equations.

2.1. An equation with constant coefficients.

(1) ODE

$$\begin{cases} \frac{du}{dx} = 0 \\ u(0) = 1 \end{cases}$$
$$u : \mathbb{R} \to \mathbb{R}.$$

Then $u(x) = c$ and since $u(0) = 1$ the solution is $u(x) = 1$.

(2) PDE:

$$\frac{\partial u(x,y)}{\partial x} = 0$$
$$u : \mathbb{R}^2 \to \mathbb{R}.$$

Then the solution is constant along $y = c$ (characteristic curve) $u(x,y) = f(y)$. . Thus if we are given for example the initial condition $u(0,y) = y^2$ we get the whole solution

$$u(x,y) = y^2.$$

(3) Consider

$$\begin{aligned} &au_x + bu_y = 0 \\ &a, b \text{ are constant, not both } 0 \\ &u : \mathbb{R}^2 \to \mathbb{R}. \end{aligned}$$

a) Geometric method: $0 = au_x + bu_y = Du \cdot (a, b)$. This means that u is constant along the lines to the direction of (a, b). An equation for such a line is $bx - ay = c$ (characteristic line). Thus the solution only depends on $bx - ay$ i.e.

$$u(x, y) = f(bx - ay), \text{ differentiable } f : \mathbb{R} \to \mathbb{R}.$$

Let us check

$$\begin{aligned} &u_x(x, y) = bf'(bx - ay), \\ &u_y(x, y) = -af'(bx - ay), \\ \Rightarrow\ &au_x(x, y) + bu_y(x, y) = abf'(bx - ay) - baf'(bx - ay) = 0. \end{aligned}$$

Example:

$$\begin{cases} 4u_x - 3u_y = 0 \\ u(0, y) = y^3 \end{cases}$$

Then from the general solution and the initial condition

$$\begin{aligned} u(x, y) &= f(-3x - 4y) \\ y^3 &= u(0, y) = f(-4y) \overset{t=-4y}{\Rightarrow} f(t) = (-t/4)^3 = -t^3/64, \\ u(x, y) &= f(-3x - 4y) = -(-3x - 4y)^3/64 = (3x + 4y)^3/64. \end{aligned}$$

b) Method of characteristics: try to find a characteristic curve starting at some point (x_0, y_0)

$$\{(x(s), y(s)) : x(0) = x_0, y(0) = y_0\}$$

such that

$$z(s) := u(x(s), y(s))$$

is easy to solve along that curve of course using PDE:

$$\begin{aligned} \frac{d}{ds}z(s) &= \frac{d}{ds}u(x(s), y(s)) \\ &= u_x(x(s), y(s))\frac{dx(s)}{ds} + u_y(x(s), y(s))\frac{dy(s)}{ds} \\ &= au_x(x(s), y(s)) + bu_y(x(s), y(s)) = 0, \end{aligned}$$

i.e. $z(s) = u(x(s), y(s)) = c = u(x_0, y_0)$. It is enough to know the initial values along a curve.
Above we chose

$$\begin{cases} \frac{dx(s)}{ds} = a, \quad x(0) = x_0, \\ \frac{dy(s)}{ds} = b, \quad y(0) = y_0, \end{cases}$$

in order to use the PDE, i.e.

$$\begin{cases} x(s) = as + x_0 \\ y(s) = bs + y_0, \end{cases}$$

and further

$$bx_0 - ay_0 = bx - abs - ay + abs = bx - ay.$$

This gives us the equation of the characteristic curve on which u is a constant i.e. all the solutions are of the form

$$u(x, y) = f(bx - ay).$$

Example (same as above):

$$\begin{cases} 4u_x - 3u_y = 0 \\ u(0, y) = y^3 \end{cases}$$

Then

$$\begin{aligned} \frac{d}{ds}z(s) &= \frac{d}{ds}u(x(s), y(s)) \\ &= u_x(x(s), y(s))\frac{dx(s)}{ds} + u_y(x(s), y(s))\frac{dy(s)}{ds} \\ &= 4u_x(x(s), y(s)) - 3u_y(x(s), y(s)) = 0. \end{aligned}$$

Solving (observe $x_0 = 0$), we get

$$\begin{cases} x(s) = 4s \\ y(s) = -3s + y_0. \end{cases}$$

We want to solve the value of u at (x, y). Then we can solve

$$s = x/4, \quad y_0 = y + 3x/4,$$

and the constant value on the characteristic going through (x, y) is $u(0, y_0) = (y_0)^3 = (y + 3x/4)^3$ i.e.

$$u(x, y) = (y + 3x/4)^3.$$

2.2. **Nonconstant coefficients.** We consider the equations of the type, which read in 2D as

$$a(x, y)u_x + b(x, y)u_y = 0.$$

Example 2.1. *Consider*

$$yu_x - xu_y = 0$$

Again we use method of characteristics:

$$\begin{aligned} &\{(x(s), y(s)) \,:\, x(0) = x_0, y(0) = y_0\} \\ &z(s) := u(x(s), y(s)) \\ &\frac{d}{ds}z(s) = \frac{d}{ds}u(x(s), y(s)) = u_x(x(s), y(s))\frac{dx(s)}{ds} + u_y(x(s), y(s))\frac{dy(s)}{ds} \\ &= y(s)u_x(x(s), y(s)) - x(s)u_y(x(s), y(s)) = 0, \end{aligned}$$

i.e. $z(s) = u(x(s), y(s)) = c = u(x_0, y_0)$. *Above we chose*

$$\begin{cases} \frac{dx(s)}{ds} = y(s), & x(0) = x_0, \\ \frac{dy(s)}{ds} = -x(s), & y(0) = y_0, \end{cases} \tag{2.2}$$

As solved in the course 'differential equations', by elimination differenting the first equation we get

$$x'' = y' = -x$$

and the characteristic equation becomes

$$r^2 = -1, \quad r = \pm i$$

and $x(s) = c_1 \cos(s) + c_2 \sin(s)$ *and thus by the first equation* $y(s) = -c_1 \sin(s) + c_2 \cos(s)$. *In particular it holds that*

$$x^2(s) + y^2(s) = c_1^2(\sin^2(s) + \cos^2(s)) + c_2^2(\sin^2(s) + \cos^2(s)) = c_1^2 + c_2^2.$$

Thus the solution is of the form

$$u(x, y) = f(x^2 + y^2).$$

In the course 'differential equations' we have also learned to solve (2.2) directly in the phase plane

$$\frac{dy}{dx} = -x/y$$

so that $y\,dy = -x\,dx$ *i.e.* $x^2 + y^2 = c$.

We could have solved this also by the geometric method :

$$(y, -x) \cdot Du = 0$$

so that gradient is parallel to (x, y) *and the level sets are of the form* $x^2 + y^2 = c$.

3. Transport equation

3.1. Homogenous.

We consider

$$\begin{cases} u_t + b \cdot Du = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

where

$u : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ (unknown, to be searched)
$g : \mathbb{R}^n \to \mathbb{R}$ (given),
$b = (b_1, \ldots, b_n)$ (given),
$Du = (u_{x_1}, \ldots, u_{x_n})$ the gradient.

We already gave a rough derivation of this at the very beginning in 1D.

Let us try to solve this $z(s) = u(x(s), t(s))$ and

$$\begin{aligned}\frac{d}{ds}z(s) &= Du \cdot \frac{d}{ds}x(s) + u_t \frac{d}{ds}t(s)\\ &= Du \cdot b + u_t 1 = 0\end{aligned}$$

i.e.

$$\begin{cases}\frac{d}{ds}x(s) = b, & x(0) = x_0\\ \frac{d}{ds}t(s) = 1, & t(0) = 0,\end{cases}$$

giving

$$\begin{cases}x(s) = bs + x_0,\\ t(s) = s.\end{cases}$$

Thus u is constant along the lines $x = bt + x_0$ i.e. and since at $t = 0$ on this line u gets the value $u(x_0, 0) = g(x_0)$ we have

$$u(x,t) = g(x - bt).$$

Again after solving the characteristics it is enough to know the value at one point.

Remark 3.1. *In order, u to be a classical solution we require $g \in C^1$. However, even if the original mass distribution is rough, still $g(x-bt)$ seems to make sense as a solution. This suggests the need to have later a concept of a weak solution.*

3.2. **Inhomogenous.** We consider

$$\begin{cases}u_t + b \cdot Du = f(x,t) & \text{in } \mathbb{R}^n \times (0,\infty)\\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}.\end{cases}$$

where

$$f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R},$$

and the rest of the quantities are the same as above.

Example 3.2. *We continue the previous example but now we in addition drop material to the conveyor belt the amount $f(x,t)$ measured in kg/(ms). Thus*

$$\int_x^{x+h} u(y, t+s)\,dy - \int_x^{x+h} u(y,t)\,dy \approx u(x,t)bs - u(x+h,t)bs + s\int_x^{x+h} f(y,t)\,dy,$$

$$\Rightarrow \frac{1}{h}\int_x^{x+h} \frac{1}{s}(u(y,t+s) - u(y,t))\,dy \approx \frac{u(x,t) - u(x+h,t)}{h}b + \frac{1}{h}\int_x^{x+h} f(y,t)\,dy.$$

We get

$$u_t + bu_x = f$$

and f represents a source (or a sink depending on the sign).

Let us solve this $z(s) = u(x(s), t(s))$ and

$$\begin{aligned}\frac{d}{ds}z(s) &= Du \cdot \frac{d}{ds}x(s) + u_t \frac{d}{ds}t(s)\\ &= Du \cdot b + u_t 1 = f(x(s), t(s))\end{aligned}$$

i.e. similarly as above

$$\begin{cases} \frac{d}{ds}x(s) = b, & x(0) = x_0 \\ \frac{d}{ds}t(s) = 1, & t(0) = 0, \end{cases}$$

giving

$$\begin{cases} x(s) = bs + x_0, \\ t(s) = s. \end{cases}$$

Thus for given (x, t), we find the corresponding s and x_0 i.e. $s = t$, $x_0 = x - bt$, and observe

$$\begin{aligned}u(x(s), t(s)) - u(x(0), t(0)) = z(s) - z(0) &= \int_0^s \frac{d}{d\tilde{s}} z(\tilde{s})\, d\tilde{s}\\ &= \int_0^s f(x(\tilde{s}), t(\tilde{s}))\, d\tilde{s},\end{aligned}$$

i.e.

$$\begin{aligned}u(x, t) &= u(x_0, 0) + \int_0^s f(x(\tilde{s}), t(\tilde{s}))\, d\tilde{s}\\ &= g(x_0) + \int_0^t f(b\tilde{s} + x_0, \tilde{s})\, d\tilde{s}\\ &= g(x - bt) + \int_0^t f(b(\tilde{s} - t) + x, \tilde{s})\, d\tilde{s}.\end{aligned}$$

Example 3.3. *Suppose that there is a decay of mass for some weird reason comparable to the amount of mass and time by factor c, no source. Then*

$$\int_x^{x+h} u(y, t+s)\, dy - \int_x^{x+h} u(y, t)\, dy \approx u(x, t)bs - u(x+h, t)bs - sc \int_x^{x+h} u(y, t)\, dy,$$

$$\Rightarrow \frac{1}{h} \int_x^{x+h} \frac{1}{s}(u(y, t+s) - u(y, t))\, dy \approx \frac{u(x, t) - u(x+h, t)}{h} b - \frac{1}{h} \int_x^{x+h} cu(y, t)\, dy.$$

This gives an equation

$$u_t + bu_x + cu = 0.$$

4. Laplace equation

We consider the Laplace equation

$$\Delta u = \sum_{i=1}^n u_{x_i x_i} = 0.$$

and also the Poisson equation

$$-\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = f,$$

for $u \in C^2(\Omega)$. These are prime examples of a so called elliptic equation.

Definition 4.1. *Solutions $u \in C^2(\Omega)$ to the Laplace equation $\Delta u = 0$ are called harmonic.*

Usually we consider an open set $\Omega \subset \mathbb{R}^n$ and given boundary values $g : \partial\Omega \to \mathbb{R}$, $g \in C(\partial\Omega)$ and look for the solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to what is called a Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Example 4.2 (Equilibrium of diffusion)**.** *This models the equilibrium of diffusion. Consider $U \subset \Omega$ a smooth subset and consider the net flux through the boundary ∂U:*

$$0 \overset{\text{equilibrium}}{=} \int_{\partial U} F \cdot \nu \, dS \overset{\text{div-thm}}{=} \int_U \operatorname{div}(F)\, dx,$$

where ν is the exterior unit normal vector. If this holds for every $U \subset \Omega$, it is reasonable to assert that

$$\operatorname{div}(F) = 0.$$

In the case diffusion, think for example heat transfer, it is reasonable to assert that flux depends on the difference: heat flows from hot to cold, and faster the greater the difference. Thus we set

$$F = -aDu$$

and get

$$0 = \operatorname{div}(-aDu) = -a\Delta u.$$

Laplace equation can be used to describe for example temperature, chemical concentration or electrostatic potential. 19.9.2019

Next, suppose that there is a heat source/sink $f : \Omega \to \mathbb{R}$, then the net flux equals $\int_U f \, dx$

$$\int_U f \, dx = \int_{\partial U} F \cdot \nu \, dS \overset{\text{div-thm}}{=} \int_U \operatorname{div}(F)\, dx,$$

and with $F = -Du$ we get the Poisson equation

$$-\Delta u = f.$$

The boundary values $u = g$ in $\partial\Omega$ model a situation where temperatures, voltages or chemical concentrations are given/known at the boundary, and we try to find them inside.

Example 4.3. *Let* $\Omega = (0,1)$ *and*

$$\begin{cases} \Delta u(x) = u''(x) = 0 & \text{in } \Omega \\ u(0) = 0, u(1) = 2. \end{cases}$$

Then $u(x) = ax + b$ *and from*

$$u(0) = b = 0, \quad u(1) = a1 = 2,$$

so that the solution is $u(x) = 2x$.

Without the boundary values we, of course, couldn't have found the unique solution. This is natural also from the point of view of physical applications above. .

Poisson equation: $\Omega = (0,1)$ *and*

$$\begin{cases} -\Delta u = -u'' = 1 & \text{in } \Omega \\ u(0) = 0, u(1) = 0. \end{cases}$$

Then $u'(x) = -x + a$ *and* $u'' = -\frac{1}{2}x^2 + ax + b$, *and*

$$u(0) = b = 0, u(1) = -\frac{1}{2} + a = 0$$

i.e. $u(x) = -\frac{1}{2}x^2 + \frac{1}{2}x$.

4.1. **Fundamental solution.** We try to find a radial solution, so set

$$\begin{aligned} u(x) &= v(r) = v(r(x)), \\ r &= r(x) = (\sum_{j=1}^{n} x_j^2)^{1/2}, \\ r_{x_i}(x) &= \frac{1}{2}(\sum_{j=1}^{n} x_j^2)^{-1/2} 2x_i = \frac{x_i}{r}, \\ r_{x_i x_i}(x) &= \frac{1r - x_i r_{x_i}}{r^2} = \frac{r - x_i^2/r}{r^2} = \frac{1}{r} - \frac{x_i^2}{r^3}. \end{aligned}$$

Then by the chain rule

$$\begin{aligned} u_{x_i}(x) &= v'(r) r_{x_i} = v'(r)\frac{x_i}{r}, \\ u_{x_i x_i}(x) &= v''(r)\frac{x_i^2}{r^2} + v'(r) r_{x_i x_i} = v''(r)\frac{x_i^2}{r^2} + v'(r)\Big(\frac{1}{r} - \frac{x_i^2}{r^3}\Big). \end{aligned}$$

Thus

$$0 = \Delta u = \sum_{i=1}^{n} u_{x_i x_i} = \sum_{i=1}^{n} \Big\{ v''(r)\frac{x_i^2}{r^2} + v'(r)\Big(\frac{1}{r} - \frac{x_i^2}{r^3}\Big)\Big\}$$

$$\overset{\sum x_i^2 = r^2}{=} v''(r) + v'(r)\Big(\frac{n}{r} - \frac{r^2}{r^3}\Big)$$

$$= v''(r) + v'(r)\frac{n-1}{r}.$$

Setting $w(r) = v'(r)$, we have

$$w'(r) + w(r)\frac{n-1}{r} = 0.$$

Then solving we get

$$-\int_1^R \frac{w'}{w}\, dr = \int_1^R \frac{n-1}{r}\, dr,$$

$$\Rightarrow \log(|w(R)|) - \log(|w(1)|) = -(n-1)(\log(R) - \log(1)) = \log(R^{-(n-1)}),$$

$$\Rightarrow \big|v'(R)\big| = |w(R)| = e^{\log(R^{-(n-1)}) + \log(|w(1)|)} = aR^{1-n},$$

where log denotes the natural logarithm. Similar computation can be done replacing 1 in the integration bound by a positive constant, and this only affects the constant in the outcome. From this for $R > 0$

$$v(R) = \begin{cases} bR^{2-n} + c & \text{if } n \geq 3, \\ b\log(R) + c & \text{if } n = 2. \end{cases}$$

Definition 4.4 (Fundamental solution)**.**

$$\Phi(x) = \begin{cases} c_n \frac{1}{|x|^{n-2}} & \text{if } n \geq 3, \\ c_2 \log(|x|) & \text{if } n = 2, \end{cases}$$

where we fix the constant $c_n \geq 0 \geq c_2$ later in (4.4).

4.2. **Poisson equation.**

Remark 4.5. *Observe that when $x \neq 0$*

$$x \mapsto \Phi(x),$$
$$x \mapsto \Phi(x-y),$$
$$x \mapsto \Phi(x-y)f(y)$$
$$x \mapsto \Phi(x-y_1)f(y_1) + \Phi(x-y_2)f(y_2)$$

are harmonic, but still

$$x \mapsto \int_{\mathbb{R}^n} \Phi(x-y)f(y)\, dy$$

is not, even if one might be temped to calculate

$$\Delta_x \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy \overset{?}{=} \int_{\mathbb{R}^n} \Delta_x \Phi(x-y) f(y)\, dy = 0.$$

This is one of the cases when the change of the integral and the differential operator i.e. $\overset{?}{=}$ *is NOT ok, as we will soon see.*

Theorem 4.6. *Let* $f \in C_0^2(\mathbb{R}^n)$*. Let* u *be the convolution i.e.*

$$u(x) := (\Phi * f)(x) := \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy.$$

Then

(1) $u \in C^2(\mathbb{R}^n)$*,*
(2) $-\Delta u = f$ *in* $\mathbb{R}^n$*.*

Proof. (1) Observe

$$\int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy \overset{\text{chg vrbls}}{=} \int_{\mathbb{R}^n} \Phi(y) f(x-y)\, dy.$$

Let $e_i = (0, \dots, \underbrace{1}_{i \text{ th}}, 0, \dots, 0)$, $h > 0$. First we want to show that

$$\begin{aligned} \frac{\partial u}{\partial x_i}(x) &= \lim_{h \to 0} \frac{u(x+he_i) - u(x)}{h} \\ &= \lim_{h \to 0} \int_{\mathbb{R}^n} \Phi(y) \frac{f(x+he_i-y) - f(x-y)}{h}\, dy \\ &\overset{?}{=} \int_{\mathbb{R}^n} \Phi(y) \frac{\partial f(x-y)}{\partial x_i}\, dy. \end{aligned}$$

where the last step requires justification. Since $f \in C_0^2(\mathbb{R}^n)$ for $\varepsilon > 0$

$$\begin{aligned} &\left| \frac{f(x+he_i-y) - f(x-y)}{h} - \frac{\partial f(x-y)}{\partial x_i} \right| \\ &= \left| \frac{1}{h} \int_0^h \frac{d}{dr} f(x+re_i-y)\, dr - \frac{\partial f(x-y)}{\partial x_i} \right| \\ &\le \frac{1}{h} \int_0^h \left| \frac{\partial f(x+re_i-y)}{\partial x_i} - \frac{\partial f(x-y)}{\partial x_i} \right| dr \\ &\overset{f_{x_i} \text{ unif. cont.}}{\le} \frac{1}{h} \int_0^h \varepsilon\, dr = \varepsilon \end{aligned}$$

for small enough $h > 0$. Thus

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} \Phi(y) \Big(\frac{f(x+he_i-y) - f(x-y)}{h} - \frac{\partial f(x-y)}{\partial x_i} \Big)\, dy \right| \\ &\overset{\operatorname{spt} f(x-\cdot) \subset B(x,R)}{\le} \varepsilon \int_{B(x,R)} \Phi(y)\, dy = c\varepsilon, \end{aligned}$$

Above uniform continuity of f_{x_i} follows since f_{x_i} is continuous and compactly supported. We have shown $\stackrel{?}{=}$.

We can also show that $\frac{\partial u}{\partial x_i}(x) \in C(\mathbb{R}^n)$. Moreover, similarly we could show

$$\frac{\partial u}{\partial x_i \partial x_j}(x) = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial f(x-y)}{\partial x_i \partial x_j}\, dy$$

and $\frac{\partial u}{\partial x_i \partial x_j} \in C(\mathbb{R}^n)$.

(2) What we just proved gives us

$$\begin{aligned}\Delta u &= \int_{B(0,\varepsilon)} \Phi(y)\Delta f(x-y)\, dy + \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y)\Delta f(x-y)\, dy \\ &=: I_\varepsilon + J_\varepsilon.\end{aligned}$$

Then since when $n \geq 3$

$$\begin{aligned}\int_{B(0,\varepsilon)} \frac{1}{|y|^{n-2}}\, dy &= \int_0^\varepsilon \int_{\partial B(0,r)} r^{2-n}\, dS\, dr \\ &\stackrel{|\partial B(0,r)|=cr^{n-1}}{=} c\int_0^\varepsilon r^{n-1} r^{2-n}\, dr \\ &= c\int_0^\varepsilon r\, dr = c\frac{1}{2}\varepsilon^2\end{aligned}$$

and if $n = 2$

$$\begin{aligned}\int_{B(0,\varepsilon)} -\log(|y|)\, dy &= -\int_0^\varepsilon \int_{\partial B(0,r)} \log(r)\, dS\, dr \\ &\stackrel{|\partial B(0,r)|=cr=2\pi r}{=} -c\int_0^\varepsilon r\log(r)\, dr \\ &\stackrel{\text{int by parts}}{=} c(-\frac{1}{2}\varepsilon^2 \log(\varepsilon) + \int_0^\varepsilon \frac{1}{2} r^2 \frac{1}{r}\, dr) \\ &= c(-\frac{1}{2}\varepsilon^2\log(\varepsilon) + \frac{1}{4}\varepsilon^2) \leq c\varepsilon^2 |\log(\varepsilon)|.\end{aligned}$$

Thus

$$\begin{aligned}|I_\varepsilon| &= \left| \int_{B(0,\varepsilon)} \Phi(y)\Delta f(x-y)\, dy \right| \\ &\leq \max_{y\in\mathbb{R}^n} |\Delta f(x-y)| \int_{B(0,\varepsilon)} \Phi(y)\, dy \\ &\leq C \begin{cases} \varepsilon^2 |\log(\varepsilon)| & n = 2 \\ \varepsilon^2 & n \geq 3 \end{cases} \stackrel{\varepsilon \to 0}{\to} 0.\end{aligned}$$

For J_ε choosing $R>0$ large enough so that $\operatorname{spt} f \subset B(0,R)$, we can integrate by parts (or use Gauss-Green theorem/Green formula to be precise)

$$\begin{aligned}
J_\varepsilon &= \int_{B(0,R)\setminus B(0,\varepsilon)} \Phi(y)\Delta f(x-y)\,dy \\
&= -\int_{B(0,R)\setminus B(0,\varepsilon)} D\Phi(y)\cdot Df(x-y)\,dy + \int_{\partial B(0,\varepsilon)} \Phi(y)Df(x-y)\cdot\nu\,dS(y) \\
&= K_\varepsilon + L_\varepsilon,
\end{aligned}$$

where $\nu=\nu(y)=-y/|y|$ is the exterior unit normal.

Then

$$\begin{aligned}
|L_\varepsilon| &= \left|\int_{\partial B(0,\varepsilon)} \Phi(y)\cdot Df(x-y)\cdot\nu\,dS(y)\right| \\
&= C\varepsilon^{n-1} \max_{y\in\partial B(0,\varepsilon))} |\Phi(y)| \max_{y\in\partial B(0,\varepsilon))} |Df(x-y)| \\
&\le C\varepsilon^{n-1} \begin{cases} \varepsilon^{2-n} & n=3 \\ |\log(\varepsilon)| & n=2. \end{cases} \\
&\to 0,
\end{aligned}$$

as $\varepsilon\to 0$.

Then we integrate in K_ε by parts

$$\begin{aligned}
K_\varepsilon &= -\int_{B(0,R)\setminus B(0,\varepsilon)} D\Phi(y)\cdot Df(x-y)\,dy \\
&= \int_{B(0,R)\setminus B(0,\varepsilon)} \underbrace{\operatorname{div} D\Phi(y)}_{\Delta\Phi=0} f(x-y)\,dy - \int_{\partial B(0,\varepsilon)} D\Phi(y)\cdot\nu f(x-y)\,dS(y) \\
&= -\int_{\partial B(0,\varepsilon)} D\Phi(y)\cdot\nu f(x-y)\,dS(y) \\
&= -\int_{\partial B(0,\varepsilon)} \begin{cases} c_n(2-n)\,|y|^{1-n}\frac{y}{|y|} & n\ge 3 \\ c_2\frac{1}{|y|}\frac{y}{|y|} & n=2 \end{cases} \cdot (-\frac{y}{|y|}) f(x-y)\,dS(y) \\
&= \int_{\partial B(0,\varepsilon)} \begin{cases} c_n(2-n)\,|y|^{1-n} & n\ge 3 \\ c_2\frac{1}{|y|} & n=2 \end{cases} f(x-y)\,dS(y) \qquad (4.3) \\
&\overset{\text{choose } c_n}{=} -\frac{1}{|\partial B(0,\varepsilon)|}\int_{\partial B(0,\varepsilon)} f(x-y)\,dS(y) \overset{\text{aver.}}{\to} -f(x)
\end{aligned}$$

as $\varepsilon\to 0$. Observe that above $\nu=-\frac{y}{|y|}$ is the exterior normal to $B(0,R)\setminus B(0,\varepsilon)$ on $\partial B(0,\varepsilon)$. Above we had $|y|=\varepsilon$ and fixed c_n so that

$$|\partial B(0,\varepsilon)|^{-1} = -\begin{cases} c_n(2-n)\varepsilon^{-(n-1)}, & n\ge 3 \\ c_2\varepsilon^{-1}, & n=2. \end{cases}$$

and since $|\partial B(0,\varepsilon)| = \omega_n \varepsilon^{n-1} = n\alpha_n \varepsilon^{n-1}$, $|y| = 1$ above, we get

$$c_n = \frac{1}{(n-2)n\alpha_n},\ n \geq 3$$
$$c_2 = -\frac{1}{2\alpha_2} = -\frac{1}{2\pi},\ n = 2. \tag{4.4}$$

so that

.

What we showed was

$$\Delta u = I_\varepsilon + J_\varepsilon = I_\varepsilon + K_\varepsilon + L_\varepsilon \to 0 - f + 0,$$

as $\varepsilon \to 0$. □

Remark 4.7. *Formally,*

$$-\Delta u(x) = \int_{\mathbb{R}^n} -\Delta_x \Phi(x-y) f(y)\, dy = \int_{\mathbb{R}^n} \delta_x f(y)\, dy = f(x)$$

where δ_x is Dirac measure with unit mass at x (more about it in the course 'Measure and integration'). This motivates the notation

$$-\Delta \Phi = \delta_0 \text{ in } \mathbb{R}^n.$$

The above proof also motivates introduction of the following important tool.

4.3. **Convolution, mollifiers, and approximations.** Below we denote

$$\Omega_\varepsilon = \{x \in \Omega \,:\, \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$$

which is an open set by continuity of $\operatorname{dist}(x, \partial\Omega)$.

Definition 4.8 (Standard mollifier)**.** *Let*

$$\eta : \mathbb{R}^n \to \mathbb{R}, \quad \eta(x) = \begin{cases} Ce^{1/(|x|^2-1)} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

where C is chosen so that

$$\int_{\mathbb{R}^n} \eta\, dx = 1.$$

Then we set for $\varepsilon > 0$

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n} \eta(\frac{x}{\varepsilon})$$

which is called a standard mollifier.

Remark 4.9. *Observe that*

$$\eta_\varepsilon \in C_0^\infty(\mathbb{R}^n), \quad \operatorname{spt} \eta_\varepsilon \subset \overline{B}(0,\varepsilon)$$

and

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x)\,dx = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} \eta(\frac{x}{\varepsilon})\,dx \overset{y=x/\varepsilon,\varepsilon^n\,dy=\,dx}{=} \int_{\mathbb{R}^n} \eta(y)\,dy = 1.$$

Definition 4.10 (Standard mollification)**.** *Let*

$$f:\Omega \to [-\infty,\infty], \quad f \in C(\Omega).$$

Then we define the standard mollification for f by

$$f_\varepsilon : \Omega_\varepsilon \to \mathbb{R}, \quad f_\varepsilon := \eta_\varepsilon * f,$$

where $\eta_\varepsilon * f(x) = \int_\Omega \eta_\varepsilon(x-y)f(y)\,dy$ denotes the convolution for $x \in \Omega_\varepsilon$.

Definition 4.11.

$$u_j \to u \text{ locally uniformly in } \Omega$$

if

$$u_j \to u \text{ uniformly in } K \text{ for every } K \Subset \Omega\ .$$

Theorem 4.12. *The standard mollification has the following properties for $f \in C(\Omega)$*

(1)

$$D^\alpha f_\varepsilon = f * D^\alpha \eta_\varepsilon \quad \text{in } \Omega_\varepsilon$$

and

$$f_\varepsilon \in C^\infty(\Omega_\varepsilon).$$

(2) If $f \in C(\Omega)$, then

$$f_\varepsilon \to f, \qquad \text{locally uniformly } \Omega.$$

(3) If for $\Omega' \Subset \Omega'' \Subset \Omega$

$$\max_{\Omega'} |f_\varepsilon| \le \max_{\Omega''} |f|$$

for small enough $\varepsilon > 0$

The formula $D^\alpha f_\varepsilon = f * D^\alpha \eta_\varepsilon$ follows using similar techiques as in the previous proof. Then since $\eta_\varepsilon \in C_0^\infty$ by a direct calculation $f_\varepsilon \in C^\infty$ too.
26.9.2019 The detailed proof is given in the course 'PDE2'.

4.4. **Mean value property.** A Harmonic function has a remarkable property called mean value formula: it says that value at one point(!) determines the averages over a ball. It also heuristically connects harmonic functions to the stochastic process called Brownian motion and thus to stock prices, option pricing etc. It is also a key to many interesting mathematical properties.

Remember that $\fint_A = \frac{1}{|A|}\int_A$ denotes the integral average.

Theorem 4.13 (mean value property=mvp)**.** *Let $u \in C^2(\Omega)$. Then the following are equivalent*

(1) u is harmonic
(2)
$$u(x) = \fint_{\partial B(x,r)} u(y)\, dS(y) = \fint_{B(x,r)} u(y)\, dy,$$
as long as $B(x,r) \Subset \Omega$.

Proof. '(1)⇒(2)':

Idea: Set
$$\phi(r) = \fint_{\partial B(x,r)} u(y)\, dS(y) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)\, dS(y).$$
and show that $\phi'(r) = 0$.

To this end, let $z \in B(0,1)$ and perform the change of variables $y = rz+x$ so that $dS(y) = r^{n-1}\, dS(z)$ so that
$$\phi(r) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)\, dS(y) = \underbrace{\frac{|\partial B(0,1)|}{|\partial B(x,r)|} r^{n-1}}_{=1} \fint_{\partial B(0,1)} u(rz+x)\, dS(z)$$
$$= \fint_{\partial B(0,1)} u(rz+x)\, dS(z).$$
Since $u \in C^2 \subset C^1$ we have in the similar way as earlier , and changing variables back
$$\begin{aligned}\phi'(r) &= \fint_{\partial B(0,1)} D_y u(rz+x)\cdot z\, dS(z)\\ &\overset{\text{chg vrbls}}{=} \frac{|\partial B(x,r)|}{|\partial B(0,1)|}\fint_{\partial B(x,r)} D_y u(y)\cdot \frac{y-x}{r} r^{1-n}\, dS(y)\\ &= \fint_{\partial B(x,r)} D_y u(y)\cdot \frac{y-x}{r}\, dS(y)\\ &\overset{\text{div thm}}{=} \frac{1}{|\partial B(x,r)|}\int_{B(x,r)} \underbrace{\operatorname{div} Du(y)}_{\Delta u = 0}\, dy = 0,\end{aligned}$$
since $(y-x)/r$ is the exterior unit normal.

Since $\phi'(r) = 0$, $\phi(r)$ has a constant value, and the value has to be

$$\lim_{r\to 0} \fint_{\partial B(x,r)} u(y)\, dS(y) = u(x).$$

Moreover,

$$\begin{aligned}
\fint_{B(x,r)} u(y)\, dy &= \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\, dy \\
&= \frac{1}{|B(x,r)|} \int_0^r \int_{\partial B(x,s)} u\, dS\, ds \\
&= \frac{1}{|B(x,r)|} \int_0^r |\partial B(x,s)| \fint_{\partial B(x,s)} u\, dS\, ds \\
&= \frac{1}{|B(x,r)|} \int_0^r |\partial B(x,s)|\, u(x)\, ds \\
&= \frac{|B(x,r)|}{|B(x,r)|} u(x) = u(x).
\end{aligned}$$

'(1)$\Leftarrow$(2)':

Assume thriving for a contradiction that u is not harmonic even if the mean value theorem holds i.e. that there is x so that $\Delta u(x) > 0$ and by continuity even in a small ball around x. Then using the above calculation

$$\phi'(r) = \frac{1}{|\partial B(x,r)|} \int_{B(x,r)} \Delta u(y)\, dy > 0$$

so that the mean value cannot be constant, a contradiction. □

Example 4.14. *Let $\Omega = (0,1)$ and $\Delta u = u'' = 0$. Then $u(y) = ay + b$ and*

$$\begin{aligned}
\frac{1}{2r} \int_{x-r}^{x+r} u(y)\, dy = \frac{1}{2r} \int_{x-r}^{x+r} (ay+b)\, dy &= \frac{1}{2r} \frac{a}{2}((x+r)^2 - (x-r)^2) + b \\
&= \frac{a}{2r}(2xr) + b = ax + b.
\end{aligned}$$

4.5. **Properties of harmonic functions.**

Example 4.15. *Let $\Omega = (0,1)$ and $\Delta u(x) = u''(x) = 0$. Then $u(x) = ax+b$. In particular, u obtains its largest (and smallest) values at the boundary. This also holds in higher dimensions as seen in the next theorem.*

Theorem 4.16 (Max principles)**.** *Let Ω be a bounded open set and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ a harmonic in Ω. Then*

(1) (weak max principle) $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$.
(2) (strong max principle) if Ω is connected and there is $x_0 \in \Omega$

$$u(x_0) = \max_{\overline{\Omega}} u$$

it follows that

$$u \text{ is constant in } \Omega.$$

Proof. (2): Suppose that the assumption in (2) hold. Then for $r > 0$ such that $B(x_0, r) \Subset \Omega$ we have

$$M := u(x_0) \overset{\text{mvp}}{=} \fint_{B(x_0,r)} u \, dy$$

by the mean value property. Since M is max, the average on the right can only be equal if

$$u \equiv M \text{ in } B(x_0, r).$$

Thus

$$u \equiv M \text{ in } \Omega.$$

(1): Suppose for contradiction that $\max_{\overline{\Omega}} u > \max_{\partial\Omega} u$. Then there is a max point inside the domain and by the strong max principle this is a contradiction.

Second proof: For the later use we also give a proof that does not use the strong max principle. Assume without loss of generality that $\Omega = B(0, 1)$ and $\max_{\overline{\Omega}} u > \max_{\partial\Omega} u + 2\varepsilon$ for some $\varepsilon > 0$. Then $v(x) = u(x) + \varepsilon |x|^2 / 2$ also attains max at some $z_0 \in \Omega$. At the max point z_0 it holds that

$$\Delta v(z_0) \leq 0$$

but on the other hand $\Delta v = \Delta u + \varepsilon n = 0 + \varepsilon n > 0$, a contradiction. □

Remark 4.17. *Also $-u$ is harmonic, and thus we obtain a minimum principle.*

Remark 4.18. *Obviously the mean value principle or maximum principle does not hold for the Poisson equation. Recall that for*

$$\begin{cases} -\Delta u = -u''(x) = 1 & \text{in } \Omega = (0,1) \\ u(0) = 0, u(1) = 0. \end{cases}$$

we have $u(x) = -\frac{1}{2}x^2 + \frac{1}{2}x$.

Theorem 4.19 (Uniqueness to Dirichlet problem)**.** *Let Ω be a bounded open set. Let $g \in C(\partial\Omega)$ and $f \in C(\Omega)$. Then the problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

has at most one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

Proof. Let u and v be two solutions. Then $w = u - v$ solves

$$\Delta w = \Delta(u - v) = f - f = 0$$

with the boundary values $w = 0$. By the weak max principle

$$w = u - v \le 0.$$

By setting, $w = v - u$ we also get

$$v - u \le 0.$$

□

Mean value property has also other perhaps surprising consequences.

Theorem 4.20 (Smoothness)**.** *If $u \in C(\Omega)$ and satisfies the mean value property*

$$u(x) = \fint_{\partial B(x,r)} u \, dS$$

for every $B(x_0, r) \Subset \Omega$. Then

$$u \in C^\infty(\Omega).$$

In particular, harmonic functions are smooth.

Proof. Fix $\varepsilon > 0$. Let $u_\varepsilon = \eta_\varepsilon * u$ mollification by convolution as in Theorem 4.12 and $\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$. Theorem 4.12 says that $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$ so that idea is to show that $u = u_\varepsilon$ when mvp holds. To this end, let $x \in \Omega_\varepsilon$

$$\begin{aligned}
u_\varepsilon(x) &= \eta_\varepsilon * u \\
&= \int_{\mathbb{R}^n} \eta_\varepsilon(x - y) u(y) \, dy \\
&= \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \eta(\frac{x - y}{\varepsilon}) u(y) \, dy \\
&\overset{\eta \text{ radial}}{=} \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta(\frac{re_1}{\varepsilon}) \int_{\partial B(x,r)} u(y) \, dS(y) \, dr \\
&\overset{\text{mvp}}{=} \frac{u(x)}{\varepsilon^n} \int_0^\varepsilon \eta(\frac{re_1}{\varepsilon}) \, |\partial B(x,r)| \, dr \\
&= \frac{u(x)}{\varepsilon^n} \int_0^\varepsilon \eta(\frac{re_1}{\varepsilon}) \omega_n r^{n-1} \, dr \\
&\overset{\text{chg vrbls}}{=} \frac{u(x)}{\varepsilon^n} \int_{B(0,\varepsilon)} \eta(\frac{y}{\varepsilon}) \, dy \\
&= u(x) \int_{B(0,\varepsilon)} \eta_\varepsilon \, dy = u(x).
\end{aligned}$$

□

By mvp to harmonic function it holds

$$|u(x)| \le \fint_B |u| \; dy.$$

Also for the derivatives we have the integral estimates.

Theorem 4.21 (Derivative estimates)**.** *Let u be harmonic in Ω. Then*

$$|u_{x_i}(x_0)| \le \frac{c_1}{r^{n+1}} \int_{B(x_0,r)} |u| \; dy$$

$$\left|u_{x_i x_j}(x_0)\right| \le \frac{c_2}{r^{n+2}} \int_{B(x_0,r)} |u| \; dy$$

where $c_i = c(n,i)$ for $B(x_0,r) \Subset \Omega$.

Proof. Idea is to differentiate under the integral. If we can justify * below, we get

$$\begin{aligned}
|u_{x_i}(x_0)| &\overset{*}{=} \left| \fint_{B(x_0,r/2)} u_{x_i}(y)\, dy \right| \\
&\overset{\text{Gauss-Green}}{=} \frac{|\partial B(x_0,r/2)|}{|B(x_0,r/2)|} \left| \fint_{\partial B(x_0,r/2)} u\nu_i \, dS \right| \\
&\le \frac{c}{r} \max_{\partial B(x_0,r/2)} |u| \, .
\end{aligned}$$

Let $x \in \partial B(x_0, r/2)$ and observe that $B(x,r/2) \subset B(x_0,r)$ so that

$$\begin{aligned}
|u(x)| &= \left| \frac{1}{|B(x,r/2)|} \int_{B(x,r/2)} u \, dy \right| \\
&\le \left| \frac{|B(x_0,r)|}{|B(x,r/2)|} \fint_{B(x_0,r)} u \, dy \right| \\
&\le c \fint_{B(x_0,r)} |u| \; dy.
\end{aligned}$$

Combining the last two estimates yields the result if we can justify *: To show * observe

$$u_{x_i}(x_0) = \frac{\partial}{\partial (x_0)_i} \fint_{B(0,r/2)} u(x_0+y)\, dy = \fint_{B(0,r/2)} \partial_i u(x_0+y)\, dy$$

where the last equality follows by observing that for any $\varepsilon > 0$ there is $h > 0$ such that

$$\left| \int_{B(x_0,r/2)} \partial_i u(x_0+y) - \frac{u(x_0+y+he_i) - u(x_0+y)}{h} \, dy \right| \le \varepsilon,$$

similarly as in the proof of Theorem 4.6. This implies *. Another alternative is simply to observe that $\partial_i \Delta u = \Delta \partial_i u = 0$ so that also the partial derivatives need to satisfy the mean value theorem in *.

Then for the second derivatives we get similarly as above by slightly adjusting the radii:

$$
\begin{aligned}
|u_{x_i x_j}| &= \left| \fint_{B(x_0,r/2)} u_{x_i x_j}\, dx \right| \\
&\overset{\text{Gauss-Green}}{=} \frac{|\partial B(x_0,r/2)|}{|B(x_0,r/2)|} \left| \fint_{\partial B(x_0,r/2)} u_{x_i} \nu_j \, dS \right| \\
&\overset{\text{first step}}{=} \frac{c}{r} \max_{\partial B(x_0,r/2)} |u_{x_i}| \le \frac{cc_1}{r^{n+2}} \int_{B(x_0,r)} |u| \; dy.
\end{aligned}
$$

□

Example 4.22. *Let $\Omega = (0,1)$, and u harmonic i.e. $u'' = 0$, then $u(x) = ax + b$, and let for simplicity $a, b \ge 0$, and observe*

$$
\begin{aligned}
|u_{x_i}| &= a, \\
\frac{1}{r^2} \int_{x-r}^{x+r} |u| \; dy &= \frac{1}{r^2} (\frac{1}{2} a((x+r)^2 - (x-r)^2) + b2r) \\
&= \frac{4axr}{2r^2} + \frac{2b}{r} \ge \frac{2ax}{r} \overset{r<x}{\ge} 2a
\end{aligned}
$$

assuming $(x-r, x+r) \subset \Omega$.

Observe that the result is independent of the domain. Also observe that by $u(x) = ax + b$ immediately tells that it is natural to have dependence of the size of u on the right hand side.

Next we utilize the observation that the estimate is independent of the domain.

Corollary 4.23 (Liouville theorem)**.** *If u is bounded and harmonic in $\mathbb{R}^n$. Then u is constant.*

Proof. Since there is a constant $M \ge 0$ such that $|u| \le M$, by the previous theorem

$$
|u_{x_i}(x)| \le \frac{c}{r^{n+1}} \int_{B(x,r)} |u| \; dy \le \frac{c}{r} \fint_{B(x,r)} M \, dy = \frac{c_1 M}{r} \to 0,
$$

as $r \to \infty$, we see that $u_{x_i}(x) = 0$ at every point for any $i = 1, \dots, n$, so that

$$
u(x+sy) - u(x) = \int_0^s \frac{d}{dt} u(x+yt) \, dt = \int_0^s Du(x+yt) \cdot y \, dt = 0.
$$

□

Corollary 4.24 (Uniqueness in $\mathbb{R}^n$)**.** *Let $f \in C_0^2(\mathbb{R}^n)$ and $n \ge 3$. Then every bounded solution $u \in C^2(\mathbb{R}^n)$ to*

$$
-\Delta u = f,
$$

is of the form

$$u(x) = (\Phi * f)(x) + c = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy + c,$$

where c is a constant.

Proof. Let $v(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy$. We have shown that $v \in C^2(\mathbb{R}^n)$, $-\Delta v = f$. Let $\operatorname{spt} f \subset B(0,r)$. There is M such that $|v| \le M$ in $B(0,2r)$. Let $x \notin B(0,2r)$. Then

$$|v(x)| \le \left| \int_{\mathbb{R}^n} \Phi(\underbrace{x-y}_{|\cdot|>r, \text{ if } y\in \operatorname{spt} f}) f(y)\, dy \right| \le cr^{2-n} \int_{\mathbb{R}^n} |f|\; dy \le cr^{2-n},$$

i.e. v is a bounded solution. Let u be another bounded solution. Then

$$\Delta(u-v) = 0$$

and Liouville's theorem implies that $u - v = c$. □

Remark 4.25. *Previous thm false without boundedness.*

3.10.2019

Theorem 4.26. *Let u be harmonic in Ω. Then u is real analytic in Ω.*

Sketch of a proof. Aim: We have shown that $u \in C^\infty(\Omega)$, and we want to show that u can even be presented by a convergent power series around a point.

Set

$$R_N(x) := u(x) - \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{D^\alpha u(x_0)(x-x_0)^\alpha}{\alpha!}$$

where $(x-x_0)^\alpha = (x-x_0)_1^{\alpha_1} \dots (x-x_0)_n^{\alpha_n}$ and $\alpha! = \alpha_1! \dots \alpha_n!$. By the Taylor theorem that

$$R_N(x) = \sum_{|\alpha|=N} \frac{D^\alpha u(x_0 + t(x-x_0))(x-x_0)^\alpha}{\alpha!}$$

for some $t \in [0,1]$. One could establish higher derivative estimates and with sharp coefficient similarly as in Theorem 4.21, and plugging in such an estimate, we see that

$$|R_N(x)| \to 0$$

at the vicinity of x_0. □

Theorem 4.27 (Harnack's inequality)**.** *Let $u \ge 0$ be harmonic in Ω and $B(x_0, 4r) \subset \Omega$. Then for $c = 3^n$ it holds that*

$$\sup_{B(x_0,r)} u \le c \inf_{B(x_0,r)} u.$$

Proof. Let $x, y \in B(x_0, r)$, then

$$u(y) \overset{\text{mvp}}{=} \fint_{B(y,3r)} u\,dz \overset{B(x,r)\subset B(y,3r)}{\geq} \frac{|B(x,r)|}{|B(y,3r)|} \fint_{B(x,r)} u\,dz \overset{\text{mvp}}{=} \frac{1}{3^n} u(x).$$

Fix $\eta > 0$ and choose

$$u(y) < \inf_{B(x_0,r)} u + \eta, \quad u(x) + \eta > \sup_{B(x_0,r)} u.$$

□

Corollary 4.28 (Harnack's inequality, general form)**.** *Let $u \geq 0$ be harmonic in Ω and $V \Subset \Omega$ be a connected open set. Then there is $c = c(n, V) > 0$ s.t.*

$$\sup_V u \leq c \inf_V u.$$

Proof. Idea: covering argument. Let $r = \operatorname{dist}(\overline{V}, \partial\Omega)/4$,

$$\overline{V} \subset \{B(x_\gamma, r)\}_\gamma.$$

By compactness, there is a subcover

$$\overline{V} \subset \{B(x_i, r)\}_{i=1}^N.$$

Then for $x, y \in V$, use Harnack N times

$$u(y) \geq (1/3^n)^N u(x).$$

□

Remark 4.29. *$u \geq 0$ essential: let $\Omega = (-1, 1)$ and $u(x) = x$.*

Harnack's inequality implies strong maximum principle. We have already proved this starting from the mean value property but for many equations Harnack ie holds but mvp not.

Corollary 4.30 (Strong max principle)**.** *Let Ω be a bounded open set and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ harmonic in Ω. Then if Ω is connected and there is $x_0 \in \Omega$ such that*

$$u(x_0) = \max_{\overline{\Omega}} u$$

it follows that

$$u \text{ is constant in } \Omega.$$

Proof. Let $M = u(x_0) = \max_{\overline{\Omega}} u$. Then

$$v = M - u \geq 0 \text{ is harmonic}, \quad v(x_0) = 0.$$

Choose connected $V \ni x_0$ s.t. $V \Subset \Omega$

$$0 \leq \sup_V v \leq C \inf_V v \leq C v(x_0) = 0.$$

□

4.6. **Green's functions.** We are going to look for a so called Green function that helps to represent the solution to the boundary value Poisson problem.

Theorem 4.31. *Let $\partial\Omega \in C^1$, $u \in C^2(\overline{\Omega})$. If u solves*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

then

$$u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu}(x,y)\, dS(y) + \int_{\Omega} f(y)G(x,y)\, dy,$$

where G is the Green function.

Remark 4.32. *Observe that this resemples Φ since $u = \Phi * f$ solved*

$$-\Delta u = f$$

in $\mathbb{R}^n$ under suitable assumptions. Now in addition we have boundary conditions.

The theorem says: if there is such u and we can find G, then we have solved the Poisson problem. However, finding G can be difficult, and usually we can derive explicit formulas in the simple domains (like ball, later) only.

First recall that from Gauss-Green $\int_U u_{x_i}\, dx = \int_{\partial U} u\nu_i\, dx$ it follows integration by parts formula

$$\int_U u_{x_i} v\, dx = \int_{\partial U} uv\nu_i\, dS(x) - \int_U uv_{x_i}\, dx \tag{4.5}$$

and using this twice for $u \in C^2(\overline{\Omega})$ Green's formula

$$\begin{aligned} \int_U v\Delta u\, dx &= \int_{\partial U} \frac{\partial u}{\partial \nu} v\, dS(x) - \int_U \sum u_{x_i} v_{x_i}\, dx \\ &= \int_{\partial U} \frac{\partial u}{\partial \nu} v\, dS(x) - \int_{\partial U} u\frac{\partial v}{\partial \nu}\, dS + \int_U u\Delta v\, dx. \end{aligned} \tag{4.6}$$

Let Ω be bounded, $\partial\Omega \in C^1$, $u \in C^2(\overline{\Omega})$. Let $B(x,\varepsilon) \Subset \Omega$. Then

$$\begin{aligned} &\int_{\Omega\setminus B(x,\varepsilon)} u(y)\underbrace{\Delta\Phi(y-x)}_{=0} - \Phi(y-x)\Delta u(y)\, dy \\ &\overset{\text{Green}}{=} \int_{\partial(\Omega\setminus B(x,\varepsilon))} u(y)\frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\, dS(y) \end{aligned} \tag{4.7}$$

where ν is the exterior unit normal vector to $\Omega \setminus B(x,\varepsilon)$. Further,

$$\left| \int_{\partial B(x,\varepsilon)} \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\, dS(y) \right| \overset{u \in C^1}{\leq} c\varepsilon^{n-1} \max_{\partial B(x,\varepsilon)} |\Phi(y-x)|$$

$$\leq \varepsilon^{n-1} \begin{cases} \varepsilon^{2-n} & n \geq 3 \\ |\log \varepsilon| & n = 2 \end{cases} \to 0, \quad \varepsilon \to 0. \tag{4.8}$$

Similarly

$$\int_{\partial B(x,\varepsilon)} u(y) \frac{\partial \Phi}{\partial \nu}(y-x)\, dS(y) \overset{\text{cf. (4.3)}}{=} \int_{\partial B(x,\varepsilon)} u(y) \omega_n^{-1} \varepsilon^{1-n}\, dS(y)$$

$$= \fint_{\partial B(x,\varepsilon)} u(y)\, dy \to u(x), \quad \varepsilon \to 0. \tag{4.9}$$

Recall that above ν is exterior unit normal to $\Omega \setminus B(x,\varepsilon)$ on $\partial B(x,\varepsilon)$ so that it points towards x. Now

$$\begin{aligned}
u(x) &\overset{(4.9)}{=} \lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} u(y) \frac{\partial \Phi}{\partial \nu}(y-x)\, dS(y) \\
&\overset{(4.8)}{=} \lim_{\varepsilon \to 0} \Big\{ \int_{\partial B(x,\varepsilon)} u(y) \frac{\partial \Phi}{\partial \nu}(y-x)\, dS(y) - \int_{\partial B(x,\varepsilon)} \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\, dS(y) \Big\} \\
&\overset{(4.7)}{=} \lim_{\varepsilon \to 0} \Big\{ - \int_{\partial \Omega} u(y) \frac{\partial \Phi}{\partial \nu}(y-x)\, dS(y) + \int_{\partial \Omega} \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\, dS(y) \\
&\qquad - \int_{\Omega \setminus B(x,\varepsilon)} \Phi(y-x) \Delta u(y)\, dy \Big\} \\
&= - \int_{\partial \Omega} u(y) \frac{\partial \Phi}{\partial \nu}(y-x)\, dS(y) + \int_{\partial \Omega} \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\, dS(y) \\
&\qquad - \int_{\Omega} \Phi(y-x) \Delta u(y)\, dy.
\end{aligned}$$

Collecting

$$u(x) = - \int_{\partial \Omega} u(y) \frac{\partial \Phi}{\partial \nu}(y-x)\, dS(y) + \int_{\partial \Omega} \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\, dS(y) - \int_{\Omega} \Phi(y-x) \Delta u(y)\, dy \tag{4.10}$$

for any $x \in \Omega$ and any $u \in C^2(\overline{\Omega})$.

Above there is an extra term that we need to get rid of. Let us find a suitable corrector φ^x:

$$0 = \int_{\partial \Omega} u(y) \frac{\partial \varphi^x}{\partial \nu}(y) - \varphi^x(y) \frac{\partial u}{\partial \nu}(y)\, dS(y) + \int_{\Omega} \varphi^x(y) \Delta u(y)\, dy - \int_{\Omega} \Delta_y \varphi^x(y) u(y)\, dy.$$

To cancel the extra term, we require

$$\begin{cases} \Delta_y \varphi^x(y) = 0, & y \in \Omega \\ \varphi^x(y) = \Phi(y-x), & y \in \partial\Omega. \end{cases}$$

Thus

$$0 = \int_{\partial\Omega} u(y) \frac{\partial \varphi^x}{\partial \nu} \, dS(y) - \Phi(y-x) \frac{\partial u}{\partial \nu}(y) \, dS(y) + \int_{\Omega} \varphi^x(y) \Delta u(y) \, dy + 0 \tag{4.11}$$

Summing (4.10) and (4.11)

$$\begin{aligned} u(x) = &- \int_{\partial\Omega} u(y) \frac{\partial \Phi}{\partial \nu}(y-x) \, dS(y) + \int_{\partial\Omega} u(y) \frac{\partial \varphi^x}{\partial \nu}(y) \, dS(y) \\ &+ \int_{\partial\Omega} \Phi(y-x) \frac{\partial u}{\partial \nu}(y) \, dS(y) - \int_{\partial\Omega} \Phi(y-x) \frac{\partial u}{\partial \nu}(y) \, dS(y) \\ &+ \int_{\Omega} \varphi^x(y) \Delta u(y) \, dy - \int_{\Omega} \Phi(y-x) \Delta u(y) \, dy \end{aligned}$$

The second line cancels and thus

$$\begin{aligned} u(x) &= - \int_{\partial\Omega} u(y) \left(\frac{\partial \Phi}{\partial \nu}(y-x) - \frac{\partial \varphi^x}{\partial \nu}(y)\right) dS(y) \\ &\quad - \int_{\Omega} (\Phi(y-x) - \varphi^x(y)) \Delta u(y) \, dy \\ &= - \int_{\partial\Omega} g(y) \left(\frac{\partial \Phi}{\partial \nu}(y-x) - \frac{\partial \varphi^x}{\partial \nu}(y)\right) dS(y) \\ &\quad + \int_{\Omega} (\Phi(y-x) - \varphi^x(y)) f(y) \, dy, \end{aligned}$$

where the last line holds if $-\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$. This motivates:

Definition 4.33. *Green function for the region Ω is*

$$G(x,y) = \Phi(y-x) - \varphi^x(y), \quad x, y \in \Omega, x \neq y.$$

Remark 4.34.

- *Formally $-\Delta_y G(x,y) = -\Delta_y(\Phi(y-x) - \varphi^x(y)) = \delta_x - 0 = \delta_x$ in Ω, and $G(x,y) = \Phi(y-x) - \Phi(y-x) = 0$ in $y \in \partial\Omega$ i.e.*

$$\begin{cases} -\Delta_y G(x,y) = \delta_x, & y \in \Omega, \\ G(x,y) = 0, & y \in \partial\Omega. \end{cases}$$

- *If*

$$\begin{cases} \Delta u = 0 & \Omega \\ u = g & \partial\Omega, \end{cases}$$

we get the Poisson formula

$$u(x) = \int_{\partial\Omega} K(x,y) g(y) \, dS(y)$$

where

$$K(x,y) = -\frac{\partial G(x,y)}{\partial \nu}$$

is called the Poisson kernel.

4.7. **Green function on the half space.** Denote

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \,:\, x_n > 0\}.$$

Reflection of $x = (x_1, \dots, x_n)$ is

$$x^* = (x_1, \dots, -x_n).$$

Let

$$\varphi^x(y) = \Phi(y - x^*) = \Phi(y_1 - x_1, \dots, y_{n-1} - x_{n-1}, y_n + x_n).$$

where Φ is the fundamental solution. For $y \in \partial\mathbb{R}^n_+$ and

$$\varphi^x(y) = \Phi(y - x^*) \overset{\text{radial}}{=} \Phi(y - x),$$

so that

$$\begin{cases} \Delta\varphi^x(y) = 0, & \mathbb{R}^n_+ \\ \varphi^x(y) = \Phi(y - x), & \partial\mathbb{R}^n_+, \end{cases}$$

i.e. $G(x,y) = \Phi(y-x) - \varphi^x(y) = \Phi(y-x) - \Phi(y-x^*)$ is the Green function. If $y \in \partial\mathbb{R}^n_+$, then $|y-x| = |y-x^*|$, $\nu = (0, \dots, 0, -1)$ and

$$\begin{aligned} \frac{\partial G(x,y)}{\partial \nu} &= DG(x,y)\cdot\nu \\ &= -\frac{\partial G(x,y)}{\partial y_n} \\ &= -\frac{\partial \Phi(y-x)}{\partial y_n} + \frac{\partial \Phi(y-x^*)}{\partial y_n} \\ &\overset{n\geq 3}{=} -c_n(2-n)\{|y-x|^{-n+1}\frac{y_n - x_n}{|y-x|} - |y-x^*|^{-n+1}\frac{y_n + x_n}{|y-x^*|}\} \\ &= \frac{1}{n\alpha(n)}\{\frac{y_n - x_n}{|y-x|^n} - \frac{y_n + x_n}{|y-x|^n}\} \\ &= -\frac{1}{n\alpha(n)}\frac{2x_n}{|y-x|^n}. \end{aligned}$$

Thus the solution to

$$\begin{cases} \Delta u = 0, & \mathbb{R}^n_+ \\ u = g, & \partial\mathbb{R}^n_+. \end{cases}$$

is

$$u(x) = -\int_{\partial\mathbb{R}^n_+} \frac{\partial G(x,y)}{\partial \nu} g(y)\, dy = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|y-x|^n}\, dy =: \int_{\partial\mathbb{R}^n_+} K(x,y) g(y)\, dy.$$

This also holds when $n = 2$. Next we verify that this indeed gives a solution.

Theorem 4.35. *If $g \in C(\partial\mathbb{R}^n_+)$ is bounded and u as above, then*

(1) $u \in C^\infty(\mathbb{R}^n_+)$, u is bounded, $\Delta u = 0$ in $\mathbb{R}^n_+$
(2) $\lim_{\mathbb{R}^n_+ \ni x \to x_0} u(x) = g(x_0)$ for all $x_0 \in \partial\mathbb{R}^n_+$.

Proof. (1): Sketch (smoothness and details for below through difference quotients as before):

$$\begin{aligned}\Delta u(x) &= \Delta_x \int_{\partial\mathbb{R}^n_+} K(x,y) g(y)\, dy \\ &= \int_{\partial\mathbb{R}^n_+} \underbrace{\Delta_x K(x,y)}_{=0, ex.} g(y)\, dy = 0,\end{aligned}$$

observing that $|y-x| \geq |x_n| > 0$ so that $K(x,y)$ is smooth in the integration domain.

(2): $x_0 \in \partial\mathbb{R}^n_+, \varepsilon > 0, x \in \mathbb{R}^n_+$. Then

$$\begin{aligned}&|u(x) - g(x_0)| = \left| \int_{\partial\mathbb{R}^n_+} K(x,y) g(y)\, dy - g(x_0) \underbrace{\int_{\partial\mathbb{R}^n_+} K(x,y)\, dy}_{=1} \right| \\ &\leq \int_{\partial\mathbb{R}^n_+} K(x,y) \left| g(y) - g(x_0) \right| \, dy \\ &= \int_{\partial\mathbb{R}^n_+ \cap B(x_0,\delta)} K(x,y) \left| g(y) - g(x_0) \right| \, dy + \int_{\partial\mathbb{R}^n_+ \setminus B(x_0,\delta)} K(x,y) \left| g(y) - g(x_0) \right| \, dy \\ &= I + J,\end{aligned}$$

where the computation for $\int_{\partial\mathbb{R}^n_+} K(x,y)\, dy = 1$ is omitted. Then by continuity of g

$$\begin{aligned}I &\leq \int_{\partial\mathbb{R}^n_+ \cap B(x_0,\delta)} K(x,y) \left| g(y) - g(x_0) \right| \, dy \\ &\leq \int_{\partial\mathbb{R}^n_+ \cap B(x_0,\delta)} K(x,y) \varepsilon \, dy \leq \varepsilon.\end{aligned}$$

Further if

$$|x - x_0| < \delta/2, |y - x_0| \geq \delta$$

then

$$\begin{aligned}|y-x_0| &\le |y-x| + |x-x_0| \\ &\le |y-x| + \frac{\delta}{2} \\ &\le |y-x| + \frac{1}{2}|y-x_0|,\end{aligned}$$

so that

$$\frac{1}{2}|y-x_0| \le |y-x|.$$

Using this

$$\begin{aligned}J &= \int_{\partial\mathbb{R}^n_+\setminus B(x_0,\delta)} K(x,y)\,|g(y)-g(x_0)|\,dy \\ &\le \max_{y\in\mathbb{R}^n_+} 2\,|g(y)| \int_{\partial\mathbb{R}^n_+\setminus B(x_0,\delta)} K(x,y)\,dy \\ &= \max_{y\in\mathbb{R}^n_+} 2\,|g(y)|\,\frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+\setminus B(x_0,\delta)} |y-x|^{-n}\,dy \\ &= cx_n \int_{\partial\mathbb{R}^n_+\setminus B(x_0,\delta)} \underbrace{|y-x|^{-n}}_{\le(\frac{1}{2}|y-x_0|)^{-n}}\,dy \\ &\le cx_n \int_\delta^\infty \int_{\partial\mathbb{R}^n_+\cap\partial B(x_0,r)} |y-x_0|^{-n}\,dS(y)\,dr \\ &= cx_n \int_\delta^\infty cr^{n-2}r^{-n}\,dr = cx_n\delta^{-1} \to 0\end{aligned}$$

as $x_n \to 0$. Thus we have shown that $|u(x)-u(x_0)| \le 2\varepsilon$ when $|x-x_0|$ is small enough. □

4.8. **Green function on the ball:** $B(0,1)$**.** We know $\Phi(y-x)$ but need to solve the corrector:

$$\begin{cases}\Delta\varphi^x(y) = 0 & y\in B(0,1) \\ \varphi^x(y) = \Phi(y-x) & y\in\partial B(0,1).\end{cases}$$

to find $G(x,y) = \Phi(y-x) - \varphi^x(y)$.

We define an inversion through $\partial B(0,1)$ for $x \ne 0$

$$x^* = \frac{x}{|x|}\frac{1}{|x|} = \frac{x}{|x|^2}$$

If $y\in\partial B(0,1)$, $x\ne 0$, then

$$\begin{aligned}|x|^2\,|y-x^*|^2 &= |x|^2\,(|y|^2 - 2x^*\cdot y + |x^*|^2) = |x|^2\,(|y|^2 - 2\frac{x}{|x|^2}\cdot y + \frac{1}{|x|^2}) \\ &= |x|^2 - 2x\cdot y + 1 = |x-y|^2. \qquad (4.12)\end{aligned}$$

Then for $y \in \partial B(0,1)$ for $x \neq 0$

$$\Phi(|x|\,(y - x^*)) = c_n \,||x|\,(y - x^*)|^{2-n} = c_n \,|x - y|^{2-n} = \Phi(y - x)$$

and

$$\Delta_y \Phi(|x|\,(y - x^*)) = |x|^2 \,\Delta\Phi = 0$$

so that

$$\varphi^x(y) = \Phi(|x|\,(y - x^*)).$$

Thus

$$\begin{aligned} G(x,y) &= \Phi(y - x) - \varphi^x(y) \\ &= \Phi(y - x) - \Phi(|x|\,(y - x^*)) \end{aligned}$$

Also holds when $n = 2$.

Example 4.36. *Consider*

$$\begin{cases} \Delta u = 0 & B(0,1) \\ u = g & \partial B(0,1). \end{cases}$$

Then

$$u(x) = -\int_{\partial B(0,1)} g(y) \frac{\partial G(x,y)}{\partial \nu} \, dS(y),$$

with

$$\begin{aligned} D_y G(x,y) &= D_y \Phi(y - x) - D_y \Phi(|x|\,(y - x^*)) \\ &= c_n (2 - n)(|y - x|^{1-n} \,\frac{y - x}{|y - x|} - ||x|\,(y - x^*)|^{1-n} \,\frac{|x|\,(y - x^*)\,|x|}{|x|\,|y - x^*|} \\ &\overset{(4.12)}{=} c_n (2 - n)(\frac{y - x}{|y - x|^n} - \frac{|x|^2 \, y - x}{|y - x|^n}) \\ &= c_n (2 - n) \frac{y(1 - |x|^2)}{|y - x|^n}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial G(x,y)}{\partial \nu} &= D_y G(y,x) \cdot \nu = D_y G(y,x) \cdot \frac{y}{|y|} \\ &= c_n (2 - n) \frac{y(1 - |x|^2)}{|y - x|^n} \cdot \frac{y}{|y|} \\ &\overset{|y|=1}{=} c_n (2 - n) \frac{1 - |x|^2}{|y - x|^n}. \end{aligned}$$

Recalling (4.4) i.e. $c_n = 1/(n(n-2)\alpha(n))$, we have arrived at the Poisson's 10.10.2019 *representation formula*

$$u(x) = \frac{1-|x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(y)}{|y-x|^n}\, dS(y).$$

Origin should be checked separately, but we omit this.

Next we consider

$$\begin{cases} \Delta v = 0 & B(0,r) \\ v = g & \partial B(0,r). \end{cases}$$

Then $u(x) := v(xr)$ *solves*

$$\begin{cases} \Delta_x u(x) = r^2 \Delta v(xr) = 0 & x \in B(0,1) \\ u(x) = v(xr) = g(xr) & x \in \partial B(0,1). \end{cases}$$

Thus by the previous formula

$$u(x) = \frac{1-|x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(yr)}{|y-x|^n}\, dS(y)$$

so that setting $z = xr$

$$\begin{aligned} v(z) = u(z/r) &= \frac{1-|z|^2/r^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(yr)}{|y-z/r|^n}\, dS(y) \\ &\overset{y'=yr,\, dS(y')=r^{n-1}\, dS(y)}{=} \frac{1-|z|^2/r^2}{n\alpha(n)} \int_{\partial B(0,r)} \frac{g(y')}{|y'-z|^n} \frac{r^n}{r^{n-1}}\, dS(y') \\ &= \frac{r^2-|z|^2}{rn\alpha(n)} \int_{\partial B(0,r)} \frac{g(y')}{|y'-z|^n}\, dS(y'). \end{aligned}$$

4.9. **Variational method.** Some heuristics: Let

$$\begin{cases} -\Delta u = f & \Omega \\ u = 0 & \partial\Omega. \end{cases}$$

Then

$$\begin{aligned} \int_\Omega fu\, dx = \int_\Omega -\Delta u u\, dx &= \int_\Omega -\operatorname{div}(Du) u\, dx \\ &\overset{\text{int by parts}}{=} \int_\Omega Du \cdot Du\, dx \\ &= \int_\Omega |Du|^2\, dx \geq \frac{1}{2} \int_\Omega |Du|^2\, dx. \end{aligned}$$

We define

Definition 4.37 (Energy/variational integral)**.**

$$I(w) = \int_\Omega \frac{1}{2} |Dw|^2 - fw\, dx, \tag{4.13}$$

where $w \in \mathcal{A} = \{w \in C^2(\overline{\Omega}) : w = g \text{ on } \partial\Omega\}$.

We show variational principle sometimes also called Dirichlet principle.

Theorem 4.38 (Variational principle)**.** *A function $u \in C^2(\overline{\Omega})$ solves the Dirichlet problem*

$$\begin{cases} -\Delta u = f & \Omega \\ u = g & \partial\Omega, \end{cases}$$

if and only if

$$I(u) = \min_{w \in \mathcal{A}} I(w).$$

Proof. " $\Rightarrow$ " : $w \in \mathcal{A}$

$$0 = \int_\Omega (-\Delta u - f)(u - w)\,dx \overset{\text{int by parts}}{=} \int_\Omega Du \cdot D(u - w) - f(u - w)\,dx,$$

where no boundary term in int by parts since $u - w = g - g = 0$. Thus

$$\begin{aligned} \int_\Omega |Du|^2 - fu\,dx &= \int_\Omega Du \cdot Dw - fw\,dx \\ &\overset{\text{Cauchy}}{\leq} \frac{1}{2}\int_\Omega |Du|^2 + \frac{1}{2}\int_\Omega |Dw|^2 - \int_\Omega fw\,dx, \end{aligned}$$

since by Cauchy's inequality $|Du \cdot Dw| \leq \frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2$. Thus

$$I(u) \leq I(w).$$

" $\Leftarrow$ " : $v \in C_0^\infty(\Omega)$. Then we vary u by εv so that

$$u + \varepsilon v = g \text{ on } \partial\Omega \Rightarrow u + \varepsilon v \in \mathcal{A},$$

and further

$$I(u) \leq I(u + \varepsilon v).$$

Setting

$$\Psi(\varepsilon) := I(u + \varepsilon v)$$

it follows that $\Psi'(0) = 0$. Observe

$$\begin{aligned} \Psi(\varepsilon) &= \int_\Omega \frac{1}{2}|D(u + \varepsilon v)|^2 - f(u + \varepsilon v)\,dx \\ &= \int_\Omega \frac{1}{2}|D(u + \varepsilon v)|^2 - f(u + \varepsilon v)\,dx \\ &= \int_\Omega \frac{1}{2}(|Du|^2 + 2\varepsilon Du \cdot Dv + \varepsilon^2 |Dv|^2) - f(u + \varepsilon v)\,dx. \end{aligned}$$

Thus

$$\begin{aligned} 0 = \Psi'(0) &= \int_\Omega Du \cdot Dv - fv\,dx \\ &\overset{\text{int by parts}}{=} \int_\Omega (-\Delta u - f)v\,dx. \end{aligned}$$

for every $v \in C_0^\infty(\Omega)$. As we have shown in the exercises this implies $-\Delta u - f = 0$. $\square$

Remark 4.39. *The existence of the Dirichlet problem could be proven using variational integral. This is done in 'PDE2'.*

4.10. **Eigenvalue problem/ Helmholz equation.** Consider $u \in C^2(\overline{\Omega}), \partial\Omega \in C^1$.

Definition 4.40. *If*

$$\begin{cases} -\Delta u = \lambda u & \Omega \\ u = 0 & \partial\Omega. \end{cases}$$

with $\lambda > 0$ has a nontrivial solution i.e. $u \not\equiv 0$, then λ is eigenvalue of Δ in Ω. The corresponding u is an eigenfunction.

Remark 4.41. *(1) If u is an eigenfunction, so is cu.*
(2) If $\lambda \leq 0$, then there is no nontrivial solution. Let

$$D = \{x \in \Omega \,:\, u > 0\}.$$

Then

$$\begin{cases} -\Delta u = \lambda u < 0 & D \\ u = 0 & \partial D. \end{cases}$$

By a maximum principle for subharmonic functions (defined as $-\Delta u < 0$) (see Demo 4), we have

$$u \leq 0 \text{ in } D \Rightarrow D = \emptyset.$$

Similarly, $\{x \in \Omega \,:\, u < 0\} = \emptyset$. Thus $u \equiv 0$.

Let $w \in \mathcal{A} = \{w \in C^2(\overline{\Omega}) \,:\, w = 0 \text{ on } \partial\Omega,\ w \not\equiv 0\}$ and

Definition 4.42 (Rayleigh quotient (Rayleighin osamäärä))**.**

$$Q[w] = \frac{\int_\Omega |Dw|^2 \, dx}{\int_\Omega w^2 \, dx}$$

Set

$$m := \inf_{w \in \mathcal{A}} Q[w].$$

From Sobolev-Poincaré inequality it follows that (omitted till course 'PDE2')

$$m \geq \frac{n}{4 \operatorname{diam}(\Omega)} > 0.$$

Lemma 4.43. *If λ is an eigenvalue of Δ, then $\lambda \geq m$.*

Proof. Since

$$-\Delta u = \lambda u,$$

we have

$$\int_\Omega \lambda u u \, dx = \int_\Omega -\Delta u u \, dx \overset{\text{int by parts}}{=} \int_\Omega |Du|^2 \, dx.$$

Thus

$$m \le \frac{\int_\Omega |Du|^2 \, dx}{\int_\Omega u^2 \, dx} = \lambda. \quad \square$$

Theorem 4.44 (Rayleigh's principle)**.** *If there is $u \in \mathcal{A}$ s.t.*

$$Q[u] = m,$$

then m is the smallest eigenvalue of Δ.

Proof. Let $v \in C_0^\infty(\Omega)$, $\varepsilon \in \mathbb{R}$

$$\Psi(\varepsilon) = \frac{\int_\Omega |D(u+\varepsilon v)|^2 \, dx}{\int_\Omega (u+\varepsilon v)^2 \, dx}.$$

By the assumption

$$\Psi(\varepsilon) = Q[u+\varepsilon v] \ge Q[u] = \Psi(0).$$

Thus

$$0 \overset{\text{min at } \varepsilon = 0}{=} \Psi'(0) = \frac{\int_\Omega 2Du \cdot Dv \, dx \int_\Omega u^2 \, dx - \int_\Omega |Du|^2 \, dx \int_\Omega 2uv \, dx}{(\int_\Omega u^2 \, dx)^2}.$$

It follows

$$\begin{aligned} \int_\Omega Du \cdot Dv \, dx &= \frac{\int_\Omega |Du|^2 \, dx}{\int_\Omega u^2 \, dx} \int_\Omega uv \, dx \\ &= m \int_\Omega uv \, dx. \end{aligned}$$

$$\overset{\text{int by parts}}{\Rightarrow} -\int_\Omega \Delta uv \, dx = m \int_\Omega uv \, dx$$

for every $v \in C_0^\infty(\Omega)$. From this it follows (as shown in Ex) that

$$-\Delta u = mu.$$

$\square$

Showing that such u really exists is beyond our scope, see for example Jost: Partial differential equations. Taking the existence for granted, this then gives the smallest eigenvalue/ the principal eigenvalue/first eigenvalue which is often denoted by λ_1.

Theorem 4.45. *Let u be an eigenfunction corresponding to λ_1. Then either $u > 0$ or $u < 0$ in Ω.*

We omit the proof.

Theorem 4.46. *The first eigenspace is one dimensional i.e. the first eigenvalue is simple.*

Proof. Let u, v be two such eigenfunctions and set

$$k = u(x_0)/v(x_0).$$

Then $w = u - kv$ satisfies the Helmholz equation and by the above $w > 0$, $w < 0$ or $w \equiv 0$, and since $w(x_0) = 0$ the third case must apply. □

Lemma 4.47. *Let λ, μ be the eigenvalues corresponding to the eigenfunctions $u, v \in C^2(\overline{\Omega})$. Then*

$$\text{either } \lambda = \mu \text{ or } \int_\Omega uv\,dx = 0.$$

Proof.

$$\begin{aligned}
\lambda \int_\Omega uv\,dx &= -\int_\Omega \Delta u v\,dx \\
&\overset{\text{int by parts}}{=} \int_\Omega Du \cdot Dv\,dx \\
&\overset{\text{int by parts}}{=} -\int_\Omega u \Delta v\,dx \\
&= \mu \int_\Omega uv\,dx.
\end{aligned}$$

□

Remark 4.48.

- *We state without a proof that*

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots, \quad \lambda_i \to \infty.$$

5. Heat equation

We consider the heat equation

$$u_t = \Delta u$$

and

$$u_t = \Delta u + f,$$

where $u = u(x,t)$ depends on space and time, u_t is time derivative, and Δ is taken only respect to the space variable x:

$$\Delta u(x,t) = \sum_{i=1}^n \frac{\partial^2 u(x,t)}{\partial x_i^2}$$

Dirichlet problem

$$\begin{cases} u_t = \Delta u & \text{in } \Omega_T := \Omega \times (0,T) \\ u = g & \text{on } \partial_p \Omega_T, \end{cases}$$

where $\partial_p \Omega_T = \Omega \times \{0\} \cup (\partial\Omega \times [0,T])$.

Cauchy problem

$$\begin{cases} u_t = \Delta u & \text{in } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n. \end{cases}$$

Example 5.1. *Harmonic function u, $\Delta u = 0$ is a solution $u(x,t) := u(x)$ (constant in time) $u_t = 0 = \Delta u$.*

Example 5.2 (Time evolution of diffusion)**.** *Change is caused by the diffusion:*

$$\int_U \partial_t u \, dx = \underbrace{\partial_t \int_U u \, dx}_{\text{change of amount of heat}} = \underbrace{-\int_{\partial U} F \cdot \nu \, dS}_{\text{net flux}} \overset{\text{div-thm}}{=} -\int_U \operatorname{div}(F) \, dx,$$

where ν is the exterior unit normal vector. Thus

$$\partial_t u = -\operatorname{div}(F).$$

If again the flux density is proportional to the gradient (heat flow from hot to cold, proportional to difference)

$$F = -aDu$$

and setting for simplicity $a = 1$ we get

$$u_t = -\operatorname{div}(-Du) = \Delta u.$$

If in addition, there is a heat source, change is flux plus the added heat:

$$\int_U \partial_t u \, dx = \underbrace{\partial_t \int_U u \, dx}_{\text{change of amount of heat}} = \underbrace{-\int_{\partial U} F \cdot \nu \, dS}_{\text{net flux}} + \int_U f \, dx$$
$$\overset{\text{div-thm}}{=} \int_U -\operatorname{div}(F) \, dx + \int_U f \, dx,$$

i.e.

$$u_t = \Delta u + f.$$

More concretely take 1D steel rod on $\Omega = (0,1)$ that is insulated except at the ends

$$\begin{aligned} u_t &= u_{xx} \\ u(x,0) &= g(x,0) \text{ initial temperature distribution} \\ u(0,t) &= g(0,t) \text{ known outside temperature at } x = 0 \\ u(1,t) &= g(1,t) \text{ known outside temperature at } x = 1. \end{aligned}$$

Then solution $u(x,t)$ tells the temperature at x at later time t in the rod.

5.1. **Fundamental solution.** It is known that for parabolic equations it is useful to search solutions in self-similar form: Assume

$$u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$$

and set $\lambda = t^{-1}$. Then

$$u(x,t) = t^{-\alpha} u(t^{-\beta} x, 1) =: t^{-\alpha} v(t^{-\beta} x)$$

so that we look for the solution in the form

$$u(x,t) = t^{-\alpha} v(t^{-\beta} x), \ |x| \neq 0. \tag{5.14}$$

There are other ways to get the fundamental solution without such a guess but this is quick.

Then

$$\begin{aligned} u_t(x,t) &= -\alpha t^{-\alpha-1} v(t^{-\beta} x) - t^{-\alpha} \beta x t^{-\beta-1} Dv(t^{-\beta} x) \\ &\overset{y=t^{-\beta}x}{=} -\alpha t^{-\alpha-1} v(y) - t^{-\alpha-1} \beta y Dv(y), \end{aligned}$$

$$\Delta u(x,t) = t^{-\alpha-2\beta} \Delta v(t^{-\beta} x) = t^{-\alpha-2\beta} \Delta v(y)$$

i.e. plugging these into heat eq

$$0 = \alpha t^{-\alpha-1} v(y) + t^{-\alpha-1} \beta y Dv(y) + t^{-\alpha-2\beta} \Delta v(y).$$

We seek to simplify and select $\beta = \frac{1}{2}$, and thus

$$\begin{aligned} &0 = \alpha t^{-\alpha-1} v(y) + t^{-\alpha-1} \frac{1}{2} y Dv(y) + t^{-\alpha-1} \Delta v(y) \\ &\Rightarrow \ 0 = \alpha v(y) + \frac{1}{2} y Dv(y) + \Delta v(y). \end{aligned}$$

To further simplify, let us look for radial solution w s.t. $v(y) = w(|y|)$ (as for Laplace), so that in particular $y \cdot Dv(y) = y \cdot w'(|y|) \frac{y}{|y|}$, and recall radial Laplacian

$$0 = \alpha w(r) + \frac{1}{2} r w'(r) + w''(r) + \frac{n-1}{r} w'(r)$$

where $r = |y|$. Now

$$\begin{aligned} 0 &= \alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' \\ &\overset{(\alpha := n/2)}{=} (\frac{1}{2}(r^n w) + r^{n-1} w')' r^{1-n}. \end{aligned}$$

Thus

$$\frac{1}{2}(r^n w) + r^{n-1} w' = a.$$

Assume to again simplify that $a = 0$ and thus

$$w' = -\frac{1}{2} r w$$

which has a solution

$$w(r) = ce^{-r^2/4}.$$

Recalling all the selections 17.10.2019

$$u(x,t) = t^{-\alpha} v(t^{-\beta} x) = t^{-n/2} w(t^{-\frac{1}{2}} |x|) = \frac{c}{t^{n/2}} e^{-\frac{|x|^2}{4t}}.$$

Definition 5.3 (Fundamental solution to heat equation)**.**

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & (x,t) \in \mathbb{R}^n \times (0,T) \\ 0, & t \le 0. \end{cases}$$

The selection of the constant:

Lemma 5.4.

$$\int_{\mathbb{R}^n} \Phi(x,t)\, dx = 1, \quad t > 0.$$

Proof.

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(x,t)\, dx &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}\, dx \\ &\overset{y=x/(4t)^{1/2},\, dy = dx/(4t)^{n/2}}{=} \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|y|^2}\, dy \\ &\overset{|y|^2 = \sum y_i^2}{=} \frac{1}{\pi^{n/2}} \Pi_{i=1}^n \int_{-\infty}^{\infty} e^{-y_i^2}\, dy_i = \frac{\pi^{n/2}}{\pi^{n/2}} = 1 \end{aligned}$$

where

$$\begin{aligned} (\int e^{-y_1^2}\, dy_1)^2 &= \int e^{-y_1^2}\, dy_1 \int e^{-y_2^2}\, dy_2 \\ &= \int \int e^{-y_1^2 - y_2^2}\, dy_1\, dy_2 \\ &= \int_0^\infty \int_{\partial B(0,r)} e^{-r^2}\, dS\, dr \\ &\overset{\text{2D}}{=} \int_0^\infty 2\pi r e^{-r^2}\, dr \\ &= |_0^\infty - \pi e^{-r^2} = \pi. \end{aligned}$$

□

5.2. **Cauchy problem.**

Theorem 5.5 (Cauchy problem for heat eq)**.** *Let* $g \in C(\mathbb{R}^n)$ *be a bounded function and*

$$u(x,t) = (\Phi * g)(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)\,dy = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy,$$

$t > 0$*. Then*

(1) $u \in C^2(\mathbb{R}^n \times (0,\infty))$
(2) $u_t - \Delta u = 0$ *in* $\mathbb{R}^n \times (0,\infty)$
(3) $\lim_{(\mathbb{R}^n \times (0,\infty)) \ni (x,t) \to (x_0,0)} u(x,t) = g(x_0)$.

Proof. (1): (Sketch) Similarly as before chg the order of int and diff and recall $t > 0$:

$$u_{x_i x_j}(x,t) \overset{\text{cf. (4.6)}}{=} (\Phi_{x_i x_j} * g)(x,t) \in C(\mathbb{R}^n \times (0,\infty))$$
$$u_t(x,t) \overset{\text{cf. (4.6)}}{=} (\Phi_t * g)(x,t) \in C(\mathbb{R}^n \times (0,\infty)).$$

Thus $u \in C^2(\mathbb{R}^n \times (0,\infty))$.
(2):

$$\begin{aligned} &u_t(x,t) - \Delta u(x,t) \\ &\overset{\text{chg ord int, der}}{=} \int_{\mathbb{R}^n} \underbrace{(\Phi_t(x-y,t) - \Delta_x \Phi(x-y,t))}_{=0} g(y)\,dy = 0. \end{aligned}$$

(3): $x_0 \in \mathbb{R}^n, \varepsilon > 0$, then there is $\delta > 0$ s.t.

$$|g(y) - g(x_0)| < \varepsilon, \text{ when } |y - x_0| < \delta.$$

$$\begin{aligned} |u(x,t) - g(x_0)| &\overset{L5.4}{=} \left| \int_{\mathbb{R}^n} \Phi(x-y,t)(g(y)-g(x_0))\,dy \right| \\ &\le \int_{\mathbb{R}^n \cap B(x_0,\delta)} \Phi(x-y,t)\,|g(y)-g(x_0)|\,dy \\ &\quad + \int_{\mathbb{R}^n \setminus B(x_0,\delta)} \Phi(x-y,t)\,|g(y)-g(x_0)|\,dy = I + J. \end{aligned}$$

$$I = \varepsilon \int_{\mathbb{R}^n \cap B(x_0,\delta)} \Phi(x-y,t)\,dx \le \varepsilon.$$

$$|x - x_0| < \delta/2, |y - x_0| \ge \delta \overset{\text{cf. Gr in half space}}{\Rightarrow} |y - x| \ge \frac{1}{2}\,|y - x_0|\,.$$

$$
\begin{aligned}
J &\le 2\max_{\mathbb{R}^n}|g|\int_{\mathbb{R}^n\setminus B(x_0,\delta)}\Phi(x-y,t)\,dy\\
&\le \frac{c}{t^{n/2}}\int_{\mathbb{R}^n\setminus B(x_0,\delta)} e^{-\frac{|x-y|^2}{4t}}\,dy\\
&\le \frac{c}{t^{n/2}}\int_{\mathbb{R}^n\setminus B(x_0,\delta)} e^{-\frac{(\frac12|y-x_0|)^2}{4t}}\,dy\\
&= \frac{c}{t^{n/2}}\int_{\mathbb{R}^n\setminus B(x_0,\delta)} e^{-\frac{|y-x_0|^2}{16t}}\,dy\\
&= c\int_{\mathbb{R}^n\setminus B(x_0,\delta/\sqrt{t})} e^{-\frac{|z|^2}{16}}\,dy \to 0,\ \text{ as } t\to 0,
\end{aligned}
$$

where we did the change of variables $z=(y-x_0)/\sqrt{t},\ \ dz=t^{-n/2}\,dy$. Thus first choosing δ small enough and then $t>0$ small enough we get

$$
|u(x,t)-g(x_0)|\le I+J\le \varepsilon+J\le 2\varepsilon.
$$

□

Remark 5.6. *(1) It is often denoted that*

$$
\begin{cases}\Phi_t-\Delta\Phi=0 & \text{in } \mathbb{R}^n\times(0,\infty)\\ \Phi=\delta_0 & \text{on } \mathbb{R}^n\times\{t=0\},\end{cases}
$$

where δ_0 is Dirac's delta at the origin.

(2) Observe that if $g>0$ and $t>0$, then

$$
u(x,t)=\frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}g(y)\,dy>0
$$

for any $x\in\mathbb{R}^n$. This means that the heat equation has an infinite speed of propagation.

5.3. **Inhomogenous Cauchy problem.** Consider

$$
\begin{cases}u_t-\Delta u=f & \text{in } \mathbb{R}^n\times(0,\infty)\\ u=0 & \text{on } \mathbb{R}^n\times\{t=0\}.\end{cases}
$$

We use so called Duhamel's principle and define

$$
\begin{aligned}
u(x,t)&=\int_0^t\int_{\mathbb{R}^n}\Phi(x-y,t-s)f(y,s)\,dy\,ds\\
&=\int_0^t\frac{1}{(4\pi(t-s))^{n/2}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}}f(y,s)\,dy\,ds.
\end{aligned}
$$

Heuristic explanation (similar to that of wave equation later in Section 6.6) was given in the lectures.

Theorem 5.7. *Let f has a compact support, f, D_xf, D_x^2f, f_t continuous. Then for the above u it holds that*

(1) u, D_xu, D_x^2u, u_t are continuous in $\mathbb{R}^n\times(0,\infty)$

(2) $u_t - \Delta u = f$ *in* $\mathbb{R}^n \times (0,\infty)$
(3) $\lim_{(\mathbb{R}^n\times(0,\infty))\ni(x,t)\to(x_0,0)} u(x,t) = 0.$

Proof. (1): (Sketch) We want to avoid the singularity and change variables

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y,t-s)\,dy\,ds$$

so that we may change the order of int and diff

$$u_t(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_t(x-y,t-s)\,dy\,ds + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\,dy$$

$$u_{x_ix_j}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f_{x_ix_j}(x-y,t-s)\,dy\,ds,$$

and they are continuous using similar techniques as before. Other derivatives follow similarly.

(2): We divide the integral to the cases close and far away from the singularity:

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\left(\frac{\partial}{\partial t} - \Delta_x\right) f(x-y,t-s)\,dy\,ds \\ &\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\,dy \\ &= \int_\varepsilon^t \int_{\mathbb{R}^n} + \int_0^\varepsilon \int_{\mathbb{R}^n} + \int_{\mathbb{R}^n} \\ &= I + J + K. \end{aligned}$$

$$|J| \le \max(|f_t| + |\Delta f|) \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s)\,dy\,ds \overset{\text{L5.4}}{\le} c\varepsilon.$$

$$\begin{aligned} I &= \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)\left(\frac{\partial}{\partial t} - \Delta_x\right) f(x-y,t-s)\,dy\,ds \\ &= \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)\left(-\frac{\partial}{\partial s} - \Delta_y\right) f(x-y,t-s)\,dy\,ds \\ &\overset{\text{int by parts, } f \text{ cmp supp}}{=} \int_\varepsilon^t \int_{\mathbb{R}^n} \underbrace{\left(\frac{\partial}{\partial s} - \Delta_y\right)\Phi(y,s)}_{=0} f(x-y,t-s)\,dy\,ds \\ &\quad - \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\,dy + \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\,dy \\ &= -K + \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\,dy. \end{aligned}$$

Thus

$$I + K = \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\,dy$$

and

$$u_t(x,t) - \Delta u(x,t) = I + J + K = \lim_{\varepsilon\to 0}(I + c\varepsilon + K)$$
$$= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y, t-\varepsilon)\, dy \overset{\text{cf. Thm 5.5, ex}}{=} f(x,t).$$

(3):

$$|u(x,t)| \le \max|f| \left|\int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\, dy\, ds\right|$$
$$\le ct \to 0, \text{ as } t \to 0.$$

□

5.4. **Max principle.** Let Ω be a bounded domain, recall $\Omega_T = \Omega \times (0,T)$, $\partial_p\Omega_T = \Omega \times \{0\} \cup (\partial\Omega \times [0,T])$ and consider

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega_T \\ u = g & \text{on } \partial_p\Omega_T. \end{cases}$$

Theorem 5.8 (Weak min/max principle, bdd set)**.** *Let Ω be a bounded set $u \in C^2(\Omega_T) \cap C(\overline{\Omega}_T)$. If*

$$u_t - \Delta u \ge 0 (supersolution) \tag{5.15}$$

then u attains its min on $\partial_p\Omega_T$, and if

$$u_t - \Delta u \le 0 (subsolution). \tag{5.16}$$

then u attains its max on $\partial_p\Omega_T$.

Proof. Assume first that $u_t - \Delta u < 0$ consider Ω_τ, $\tau \in (0,T)$. If max is attained at

$$(x_0, t_0) \in \Omega \times \{t = \tau\}$$

then

$$u_t(x_0,t_0) \ge 0, \Delta u(x_0,t_0) \le 0$$

i.e.

$$u_t(x_0,t_0) - \Delta u(x_0,t_0) \ge 0$$

a contradiction. Thus u cannot attain its max at any interior point of Ω_τ, and by continuity of u

$$\max_{\overline{\Omega}_T} u = \lim_{\tau\to T} \max_{\overline{\Omega}_\tau} u = \lim_{\tau\to T} \max_{\partial_p\Omega_\tau} u = \max_{\partial_p\Omega_T} u.$$

Consider then the general case $u_t - \Delta u \le 0$, and consider instead

$$v = u - \varepsilon t,$$

$$v_t - \Delta v = u_t - \varepsilon - \Delta u \le -\varepsilon < 0.$$

Thus by the above,

$$\begin{aligned}\max_{\overline{\Omega}_T} u &\le \max_{\overline{\Omega}_T}(v+\varepsilon t)\\ &\le \max_{\overline{\Omega}_T}(v+\varepsilon T)\\ &\overset{\text{above}}{=} \max_{\partial_p\Omega_T}(v+\varepsilon T)\\ &\overset{v\,\le\,u}{\le} \max_{\partial_p\Omega_T}(u+\varepsilon T).\end{aligned}$$

By $\varepsilon \to 0$

$$\max_{\overline{\Omega}_T} u \le \max_{\partial_p\Omega_T} u.$$

Since $\max_{\partial_p\Omega_T} u \le \max_{\overline{\Omega}_T} u$

$$\max_{\overline{\Omega}_T} u = \max_{\partial_p\Omega_T} u.$$

The proof of the minimum principle for the supersolutions is similar. □

Heat equation also has a mean value property when interpreted correctly.

Definition 5.9 (Heat ball)**.**

$$E(x,t,r) = \{(y,s) \in \mathbb{R}^{n+1} \,:\, s<t, \Phi(x-y,t-s) > \frac{1}{r^n}\}.$$

Remark 5.10. *Observe that this does not look ball in $\mathbb{R}^{n+1}$ in the usual Euclidean metric.*

Theorem 5.11 (Mean value property for the heat equation)**.** *If u is a solution to the heat equation in Ω_T, then*

$$u(x,t) = \frac{1}{4r^n}\int\int_{E(x,t,r)} u(y,s)\frac{|x-y|^2}{(t-s)^2}\,dy\,ds,$$

for every $E(x,t,r) \Subset \Omega_T$

We omit the proof.

This implies the strong max principle:

Theorem 5.12 (Strong max principle, bdd set)**.** *Let $u \in C^2(\Omega_T) \cap C(\overline{\Omega}_T)$ be a solution to the heat equation in Ω_T, and Ω bounded, connected, and $(x_0,t_0) \in \Omega_T$ such that*

$$u(x_0,t_0) = \max_{\overline{\Omega}_T} u,$$

then

$$u \equiv c \text{ in } \overline{\Omega}_{t_0}.$$

Remark 5.13.

- *It suffices to assume subsolution above. Strong minimum principle for supersolutions.*

- *In a connected domain, $u \geq 0$ is positive somewhere, then positive everywhere from there on: Infinite speed of propagation.*

24.10.2019

Theorem 5.14 (Uniqueness in a bounded set)**.** *Let $g \in C(\partial_p \Omega_T)$ and $f \in C(\Omega_T)$. Then the problem*

$$\begin{cases} u_t = \Delta u + f & \Omega_T \\ u = g & \partial_p \Omega_T \end{cases}$$

has at most one solution in $C^2(\Omega_T) \cap C(\overline{\Omega}_T)$.

Proof. As before: Let u, v be two solutions. Then $u - v$ and $v - u$ have zero boundary values and thus the max for both is 0. □

Theorem 5.15 (Max principle for the Cauchy problem)**.** *Let $u \in C^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solves*

$$\begin{cases} u_t = \Delta u & \mathbb{R}^n \times (0,T) \\ u = g & \mathbb{R}^n \times \{t = 0\} \end{cases}$$

and

$$u(x,t) \leq A e^{a|x|^2}, (x,t) \in \mathbb{R}^n \times [0,T],$$

for some $a, A > 0$. Then

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g.$$

Proof. Fix $y \in \mathbb{R}^n$, $\mu > 0$ and define

$$v(x,t) = u(x,t) - \frac{\mu}{(T + \varepsilon - t)^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}}.$$

It holds (ex)

$$v_t - \Delta v = 0 \text{ in } \mathbb{R}^n \times (0,T).$$

Let $r > 0$, $\Omega = B(y,r)$, $\Omega_T = B(y,r) \times (0,T)$. By the max principle in a bdd set

$$\max_{\overline{\Omega}_T} v = \max_{\partial_p \Omega_T} v$$

We estimate that on $\partial_p \Omega_T$ it holds $v \leq \sup_{\mathbb{R}^n} g$:

$$v(x,0) = u(x,0) - \frac{\mu}{(T+\varepsilon)^{n/2}} e^{\frac{|x-y|^2}{4(T+\varepsilon)}} \leq u(x,0) = g(x),$$

for all $x \in \mathbb{R}^n$. Next we assume $4aT < 1$ (this will have to be guaranteed at the end), and that $\varepsilon > 0$ is s.t.

$$4a(T + \varepsilon) < 1.$$

If $|x-y|=r$, $0\le t\le T$, then

$$\begin{aligned} v(x,t) &= u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon-t)}} \\ &\le Ae^{a|x|^2} - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon-t)}} \\ &\overset{|x|\le|x-y|+|y|=r+|y|}{\le} Ae^{a(|y|+r)^2} - \frac{\mu}{(T+\varepsilon)^{n/2}} e^{\frac{r^2}{4(T+\varepsilon)}} \\ &\overset{\frac{1}{4(T+\varepsilon)}>a\Rightarrow\frac{1}{4(T+\varepsilon)}=a+\gamma,\gamma>0}{=} Ae^{a(|y|+r)^2} - \frac{\mu}{(T+\varepsilon)^{n/2}} e^{(a+\gamma)r^2} \\ &\overset{\text{choose large } r, a+\gamma>a}{\le} \sup_{\mathbb{R}^n} g. \end{aligned}$$

Thus

$$v(y,t) \le \sup_{\overline{\Omega}_T} v \le \sup_{\partial_p\Omega_T} v \le \sup_{\mathbb{R}^n} g$$

if $4aT<1$, and further

$$u(y,t) = \lim_{\mu\to 0} v(y,t) \le \sup_{\mathbb{R}^n} g.$$

If $4aT\ge 1$, then iterate

$$[0,T'],[T',2T'],\dots$$

where $T'=1/(8a)$. □

Theorem 5.16 (Uniqueness to the Cauchy problem)**.** *Let $g\in C(\mathbb{R}^n)$ and $f\in C(\mathbb{R}^n\times[0,T])$. Then the problem*

$$\begin{cases} u_t = \Delta u + f & \mathbb{R}^n\times(0,T) \\ u = g & \mathbb{R}^n\times\{t=0\}, \end{cases}$$

has at most one solution $C^2(\mathbb{R}^n\times(0,T))\cap C(\mathbb{R}^n\times[0,T])$ satisfying the growth condition

$$|u(x,t)| \le Ae^{a|x|^2}, (x,t)\in\mathbb{R}^n\times[0,T].$$

Proof. Let u and v be solutions. Then $u-v$ satisfies

$$\begin{cases} (u-v)_t = u_t - v_t = \Delta u - \Delta v + f - f = \Delta(u-v) & \mathbb{R}^n\times(0,T) \\ u-v = g-g = 0 & \mathbb{R}^n\times\{t=0\} \end{cases}$$

and

$$|u(x,t)-v(x,t)| \le |u(x,t)| + |v(x,t)| \le 2Ae^{a|x|^2}.$$

Thus by the max principle for the Cauchy problem

$$u-v \le \max_{\mathbb{R}^n} g = 0,$$

and by the similar argument $v-u\le 0$. Thus $u=v$. □

Remark 5.17. *The growth condition is essential. The problem*

$$\begin{cases} u_t = \Delta u & \mathbb{R}^n \times (0,T) \\ u = 0 & \mathbb{R}^n \times \{t=0\} \end{cases}$$

has infinitely many solutions without the growth condition that all except $u \equiv 0$ *grow fast as* $|x| \to \infty$*. For a counterexample of Tychonov, see for example DiBenedetto: PDEs, p146.*

5.5. **Energy methods and backwards in time uniqueness.** Let Ω be bounded and smooth, and consider

$$\begin{cases} u_t = \Delta u + f & \Omega_T \\ u = g & \partial_p \Omega_T. \end{cases}$$

We have shown that this has only one solution, but here is another way. If u and v are solutions $w = u - v$ solves

$$\begin{cases} (w)_t = \Delta w + f - f = \Delta w & \Omega_T \\ w = u - v = g - g = 0 & \partial_p \Omega_T. \end{cases}$$

Let

$$I(t) = \int_\Omega w(x,t)^2 \, dx.$$

Then

$$\begin{aligned} I'(t) &= \frac{d}{dt}\Big(\int_\Omega w(x,t)^2 \, dx\Big) \\ &\overset{\text{chg order int diff}}{=} \int_\Omega \frac{\partial}{\partial t} w(x,t)^2 \, dx \\ &= 2\int_\Omega w(x,t) \frac{\partial}{\partial t} w(x,t) \, dx \\ &= 2\int_\Omega w(x,t) \Delta w(x,t) \, dx \\ &\overset{\text{int by parts}}{=} -2\int_\Omega |Dw(x,t)|^2 \, dx \le 0. \end{aligned}$$

Thus

$$I(t) \le I(0) = 0$$

so that $w = u - v = 0$ for all $0 \le t \le T$.

If we know that the lateral boundary values are the same and the solutions are the same at some time instant $t = T$, then the solutions have been same in the past. In particular, below we do not assume that $u = v$ on $\Omega \times \{t = 0\}$.

The next theorem skipped in 2019 lectures.

Theorem 5.18 (Backwards in time uniqueness)**.** *Let* $u, v \in C^2(\overline{\Omega}_T)$

$$\begin{cases} u_t = \Delta u, v_t = \Delta v & \Omega_T \\ u = v = g & \partial\Omega \times [0,T] \quad \textit{(only lateral boundary)}, \end{cases}$$

and

$$u(x,T) = v(x,T).$$

Then

$$u = v \ \textit{in} \ \Omega_T.$$

Proof. Let $w = u - v$ and

$$I(t) = \int_\Omega w(x,t)^2 \, dx.$$

As above

$$I'(t) = -2 \int_\Omega |Dw|^2 \, dx.$$

Then

$$\begin{aligned} I''(t) &= -2\frac{d}{dt}\int_\Omega |Dw|^2 \, dx. \\ &\overset{\text{chg order int, diff}}{=} -2\int_\Omega \frac{\partial}{\partial t} |Dw|^2 \, dx \\ &\overset{\text{chain rule}}{=} -2\int_\Omega 2\,|Dw| \frac{Dw}{|Dw|} \cdot \frac{\partial}{\partial t} Dw \, dx \\ &= -4\int_\Omega Dw \cdot D\frac{\partial}{\partial t} w \, dx \\ &\overset{\text{int by parts, } w_t = 0 \text{ on bdr}}{=} 4\int_\Omega \Delta w \frac{\partial w}{\partial t} \, dx \\ &\overset{w_t = \Delta w}{=} 4\int_\Omega (\Delta w)^2 \, dx. \end{aligned}$$

$$\begin{aligned} \int_\Omega |Dw|^2 \, dx &\overset{\text{int by parts}}{=} -\int_\Omega w \Delta w \, dx \\ &\leq \int_\Omega |w|\,|\Delta w| \, dx \\ &\overset{\text{Cauchy-Schwarz ie}}{\leq} (\int_\Omega |w|^2 \, dx)^{1/2} (\int_\Omega |\Delta w|^2 \, dx)^{1/2}. \end{aligned}$$

Thus

$$\begin{aligned}(I'(t))^2 &= 4(\int_\Omega |Dw|^2\ dx)^2 \\ &\overset{\text{above}}{\leq} \underbrace{(\int_\Omega |w|^2\ dx)}_{=I(t)}\underbrace{(4\int_\Omega |\Delta w|^2\ dx)}_{=I''(t)} \qquad (5.17) \\ &\leq I(t)I''(t).\end{aligned}$$

If $I(t) = 0,\ 0 \leq t \leq T$, then

$$w = 0 \text{ in } \Omega_T,$$

and the claim follows. Otherwise, there exists $[t_1, t_2] \subset [0, T]$ such that

$$\begin{cases} I(t) > 0, & t_1 \leq t < t_2 \text{ and} \\ I(t_2) = 0 & \text{since } w(x,T) = u(x,T) - v(x,T). \end{cases}$$

Define

$$\Psi(t) := \log(I(t)),\ t_1 \leq t \leq t_2.$$

Then

$$\Psi'(t) = \frac{I'(t)}{I(t)},$$

and

$$\begin{aligned}\Psi''(t) &= \frac{I''(t)I(t) - I'(t)I'(t)}{I(t)^2} \\ &= \frac{I''(t)}{I(t)} - \frac{(I'(t))^2}{I(t)^2} \\ &\overset{(5.17)}{\geq} \frac{(I'(t))^2}{I(t)}\frac{1}{I(t)} - \frac{(I'(t))^2}{I(t)^2} = 0.\end{aligned}$$

Thus

$$\Psi \text{ convex over } (t_1, t_2)$$

i.e.

$$\Psi((1-\lambda)t_1 + \lambda t) \leq (1-\lambda)\Psi(t_1) + \lambda\Psi(t), \quad t_1 < t \leq t_2,\ 0 < \lambda < 1.$$

In other notation

$$\begin{aligned}\log I((1-\lambda)t_1 + \lambda t) &\leq (1-\lambda)\log I(t_1) + \lambda \log I(t) \\ &= \log I(t_1)^{1-\lambda} I(t)^\lambda.\end{aligned}$$

From this

$$0 \leq I((1-\lambda)t_1 + \lambda t_2) \leq I(t_1)^{1-\lambda}\underbrace{I(t_2)^\lambda}_{=0} = 0,$$

i.e.

$$I((1-\lambda)t_1+\lambda t_2)=0,\quad 0<\lambda<1,\qquad \Rightarrow I(t)=0,\quad t_1\le t\le t_2,$$

a contradiction. □

5.6. **Regularity results.**

Theorem 5.19. *Let $u \in C^2(\Omega_T)$ be a solution to the heat equation in Ω_T. Then*

$$u \in C^\infty(\Omega_T).$$

Moreover, solutions to the heat equation also have derivative estimates (cf. Laplace case). However, solutions to the heat equation are not necessarily real analytic in t .

5.6.1. *Integral regularity.* This section skipped in the 2019 lectures. Let us investigate

$$\begin{cases} u_t-\Delta u=f & \mathbb{R}^n\times(0,T)\\ u=g & \mathbb{R}^n\times\{t=0\}\end{cases}$$

a smooth solution. Then

$$\begin{aligned}\int_{\mathbb{R}^n} f^2\,dx &= \int_{\mathbb{R}^n}(u_t-\Delta u)^2\,dx\\ &= \int_{\mathbb{R}^n}(u_t^2-2u_t\Delta u+(\Delta u)^2)\,dx \\ &\overset{\text{int by parts}}{=} \int_{\mathbb{R}^n}(u_t^2+2Du_t\cdot Du+(\Delta u)^2)\,dx.\end{aligned} \tag{5.18}$$

Since

$$2Du\cdot Du_t = 2Du\cdot\frac{\partial}{\partial t}Du=\frac{\partial}{\partial t}|Du|^2$$

we have

$$\begin{aligned}\int_0^s\int_{\mathbb{R}^n}2Du\cdot Du_t\,dx\,dt &= \int_0^s\int_{\mathbb{R}^n}\frac{\partial}{\partial t}|Du|^2\,dx\,dt\\ &= |_0^s\int_{\mathbb{R}^n}|Du|^2\,dx \\ &\overset{Du(x,0)=Dg(x)}{=} \int_{\mathbb{R}^n}|Du(x,s)|^2\,dx-\int_{\mathbb{R}^n}|Dg(x)|^2\,dx\end{aligned} \tag{5.19}$$

On the other hand, assuming that the boundary terms vanish

$$\begin{aligned}
\int_{\mathbb{R}^n} (\Delta u)^2\,dx &= \int_{\mathbb{R}^n} \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}\,dx \\
&= \sum_{i,j=1}^n \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial x_i^2} \frac{\partial^2 u}{\partial x_j^2}\,dx \\
&\overset{\text{int by parts}}{=} -\sum_{i,j=1}^n \int_{\mathbb{R}^n} \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \frac{\partial u}{\partial x_j}\,dx \\
&\overset{\text{int by parts}}{=} \sum_{i,j=1}^n \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}\,dx \\
&= \int_{\mathbb{R}^n} \left|D^2 u\right|^2\,dx,
\end{aligned} \tag{5.20}$$

where we denoted

$$D^2 u = \begin{bmatrix} \frac{\partial^2 u}{\partial x_1^2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\ \frac{\partial^2 u}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 u}{\partial x_2 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 u}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 u}{\partial x_n^2} \end{bmatrix}$$

and $\left|D^2 u\right|^2 = \sum_{i,j=1}^n \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2$. Choose $t_0 \in [0,T]$ in (5.19) such that

$$\int_{\mathbb{R}^n} |Du(x,t_0)|^2\,dx = \sup_{0\le t\le T} \int_{\mathbb{R}^n} |Du(x,t)|^2\,dx.$$

Combining the estimates

$$\begin{aligned}
\sup_{0\le t\le T} \int_{\mathbb{R}^n} |Du(x,t)|^2\,dx &= \int_{\mathbb{R}^n} |Du(x,t_0)|^2\,dx \\
&\overset{(5.19)}{=} \int_0^{t_0} \int_{\mathbb{R}^n} 2Du \cdot Du_t\,dx\,dt + \int_{\mathbb{R}^n} |Dg|^2\,dx \\
&\le \int_0^{t_0} \int_{\mathbb{R}^n} (\underbrace{u_t^2}_{\ge 0} + 2Du \cdot Du_t + \underbrace{(\Delta u)^2}_{\ge 0})\,dx\,dt + \int_{\mathbb{R}^n} |Dg|^2\,dx \\
&\overset{(5.18)}{=} \int_0^{t_0} \int_{\mathbb{R}^n} f^2\,dx\,dt + \int_{\mathbb{R}^n} |Dg|^2\,dx \\
&\le \int_0^{T} \int_{\mathbb{R}^n} f^2\,dx\,dt + \int_{\mathbb{R}^n} |Dg|^2\,dx.
\end{aligned}$$

Moreover,

$$\int_0^T \int_{\mathbb{R}^n} ((u_t)^2 + \left|D^2 u\right|^2)\,dx\,dt \overset{(5.20)}{=} \int_0^T \int_{\mathbb{R}^n} ((u_t)^2 + |\Delta u|^2)\,dx\,dt$$
$$\overset{(5.18)}{=} \int_0^T \int_{\mathbb{R}^n} (f^2 - 2Du \cdot D(u_t))\,dx\,dt$$
$$\overset{(5.19)}{=} \int_0^T \int_{\mathbb{R}^n} f^2\,dx\,dt + \int_{\mathbb{R}^n} |Dg|^2\,dx - \int_{\mathbb{R}^n} |Du(x,T)|^2\,dx$$
$$\le \int_0^T \int_{\mathbb{R}^n} f^2\,dx\,dt + \int_{\mathbb{R}^n} |Dg|^2\,dx.$$

Collecting

$$\sup_{0\le t\le T} \int_{\mathbb{R}^n} |Du(x,t)|^2\,dx + \int_0^T \int_{\mathbb{R}^n} ((u_t)^2 + \left|D^2 u\right|^2)\,dx\,dt \le c\Big(\int_0^T \int_{\mathbb{R}^n} f^2\,dx\,dt + \int_{\mathbb{R}^n} |Dg|^2\,dx\Big).$$

Remark 5.20. *This was formal since we assumed smoothness but this can be done without this assumption. This is an example of the principle that "u has two more derivatives in space than f", in a suitable integral sense.*

5.6.2. *Harnack.* We denote

$$\begin{aligned}\tilde{Q} &= B(0,R) \times (-3R^2, 3R^2),\\ Q^+ &= B(0,R/2) \times (2R^2 - (R/2)^2, 2R^2 + (R/2)^2),\\ Q^- &= B(0,R/2) \times (-2R^2 - (R/2)^2, -2R^2 + (R/2)^2).\end{aligned} \tag{5.21}$$

Theorem 5.21 (Harnack)**.** *Let $u \ge 0$ be a solution to the heat equation in $\tilde{Q}$. Then*

$$\sup_{Q^-} u \le c \inf_{Q^+} u,$$

where $c = c(n)$.

Discussion about the proof is postponed to PDE2.

Example 5.22. *"Elliptic" Harnack's ie., where we have same cylinder on both sides, does not hold in the parabolic case: the equation $\frac{\partial u}{\partial t} - u_{xx} = 0$ has a nonnegative solution in $(-R,R) \times (-R^2, R^2)$ (translated fundamental solution)*

$$u(x,t) = \frac{1}{\sqrt{t + 2R^2}} e^{-\frac{(x+\xi)^2}{4(t+2R^2)}}$$

where ξ is a constant. Let $x \in (-R/2, R/2)$, $x \ne 0$ and $t \in (-R^2, R^2)$. Then

$$\frac{u(0,t)}{u(x,t)} = e^{-\frac{\xi^2 - (x+\xi)^2}{4(t+2R^2)}} = e^{-\frac{-x^2 - 2x\xi}{4(t+2R^2)}} = e^{\frac{x^2 + 2x\xi}{4(t+2R^2)}} \to 0$$

as $\xi \operatorname{sign} x \to -\infty$.

6. Wave equation (aaltoyhtälö)

Let $\Omega \subset \mathbb{R}^n$ We study the wave equation

$$u_{tt}(x,t) = \Delta u(x,t)$$

and its the solution

$$u : \Omega \times (0,T) \to \mathbb{R}.$$

Remark 6.1. *The behaviour is essentially different from the heat equation: finite speed of propagation, usually nonsmooth solutions.*

Example 6.2 (Physical interpretations)**.**

$$\begin{aligned} n &= 1, \quad \textit{vibrating string} \\ n &= 2, \quad \textit{vibrating membrane} \\ n &= 3, \quad \textit{vibrating elastic body.} \end{aligned}$$

Let $U \subset \Omega$ *smooth set. Then the net accelaration within* U *is*

$$\partial_{tt}\Big(\int_U u(x,t)\,dx\Big) \overset{\text{chg order int, diff}}{=} \int_U u_{tt}(x,t)\,dx$$

and net contact force is

$$-\int_{\partial U} F \cdot \nu \, dS$$

where $F = (F_1, \ldots, F_n)$ *is the force caused by the oscillation. According to Newtons law "mass* $\times$ *acceleration=total force at the boundary" (as we assume no other forces are present, and assume mass density to be unity) i.e.*

$$\int_U u_{tt}\,dx = -\int_{\partial U} F \cdot \nu \, dS \overset{\text{div thm}}{=} -\int_U \operatorname{div} F \, dx.$$

For the elastic bodies, F *is a function of the displacement gradient* Du*, and often for small* Du*, the linearization* $\approx -aDu$*.) . We get*

$$u_{tt} = a\Delta u$$

and for simplicity we set $a = 1$ *to get the wave equation.*

Think about the string: it seems credible that we need

$$\begin{aligned} u(x,0) &= g(x) \ \textit{the initial displacement} \\ u_t(x,0) &= h(x) \ \textit{the initial velocity.} \end{aligned}$$

to solve the problem.

6.1. $n = 1$**, d'Alembert formula.** We study

$$\begin{cases} u_{tt} - u_{xx} = 0, & \text{in } \mathbb{R} \times (0, \infty) \\ u(x,0) = g(x), & \text{on } \mathbb{R} \times \{t = 0\} \\ u_t(x,0) = h(x), & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

and look for an explicit solution u assuming it is smooth. Observe

$$\begin{aligned}(\frac{\partial}{\partial t} + \frac{\partial}{\partial x})(\frac{\partial}{\partial t} - \frac{\partial}{\partial x})u &= (\frac{\partial}{\partial t} + \frac{\partial}{\partial x})\frac{\partial}{\partial t}u - (\frac{\partial}{\partial t} + \frac{\partial}{\partial x})\frac{\partial}{\partial x}u \\ &= \frac{\partial}{\partial t}\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}\frac{\partial}{\partial t}u - (\frac{\partial}{\partial t}\frac{\partial}{\partial x}u + \frac{\partial}{\partial x}\frac{\partial}{\partial x}u) \\ &= u_{tt} - u_{xx}.\end{aligned}$$

Denote

$$v(x,t) := (\frac{\partial}{\partial t} - \frac{\partial}{\partial x})u$$

so that

$$\frac{\partial}{\partial t}v + \frac{\partial}{\partial x}v = v_t + v_x = 0.$$

This is first order equation, whose solution as we remember (Section 2) with

$$v(x,0) = a(x)$$

is

$$v(x,t) = a(x-t).$$

Thus

$$\begin{cases} u_t(x,t) - u_x(x,t) = a(x-t), & \mathbb{R} \times (0,T) \\ u(x,0) = g(x), & \mathbb{R}. \end{cases}$$

This is inhomogenous transport equation (Section 3) whose solution is as we remember $(f(x,t) := a(x-t), g = g, b = -1)$

$$\begin{aligned} u(x,t) &= g(x - bt) + \int_0^t f(b(s-t) + x, s)\, ds \\ &= g(x+t) + \int_0^t a(-(s-t) + x - s)\, ds \\ &= g(x+t) + \int_0^t a(x + t - 2s)\, ds \\ &\overset{y = x+t-2s}{=} g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} a(y)\, dy. \end{aligned}$$

Since

$$a(x) = v(x,0) = u_t(x,0) - u_x(x,0) = h(x) - g'(x)$$

we get

$$\begin{aligned}
u(x,t) &= g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} (h(y) - g'(y))\,dy \\
&= g(x+t) - \frac{1}{2}g(x+t) + \frac{1}{2}g(x-t) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy \\
&= \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.
\end{aligned}$$

This is d'Alembert's formula.

Remark 6.3. *By d'Alembert's formula:*

- *If $g \in C^k$ and $h \in C^{k-1}$ then*
$$u \in C^k.$$
No instant smoothening in contrast to the heat equation.
- *The solution at (x,t) is determined by the values of g and h on $[x-t, x+t]$. Huygens principle. On the other hand every y on the intial boundary only affects on conical area: Finite speed of propagation.*
- *Suppose that u and v are solutions. Then $u - v$ solves the problem with zero initial values. By d'Alembert's formula this is $\equiv 0$. Uniqueness!*
- *Stability: let u have inititial values g_1, h_1 and v have initial values g_2, h_2 :*

$$\begin{aligned}
|u(x,t) - v(x,t)| &\le \frac{1}{2}\left|g_1(x+t) - g_2(x+t)\right| + \frac{1}{2}\left|g_1(x-t) - g_2(x-t)\right| \\
&\quad + \frac{1}{2}\int_{x-t}^{x+t} \left|h_1(y) - h_2(y)\right|\,dy \\
&\le \sup_{y\in\mathbb{R}} |g_1 - g_2| + t \sup_{y\in\mathbb{R}} |h_1 - h_2| \\
&\le \varepsilon + t\varepsilon = (1+t)\varepsilon.
\end{aligned}$$

Example 6.4 (String)**.**

$$\begin{cases} u_{tt} - u_{xx} = 0, & \textit{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = g(x), & \textit{on } \mathbb{R} \times \{t = 0\} \\ u_t(x,0) = 0, & \textit{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

where

$$g(x) = \begin{cases} 1 - |x|, & -1 \le x \le 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Not regular, but lets not worry about it.

Then by d'Alembert's formula

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy = \frac{1}{2}(g(x+t) + g(x-t)).$$

Draw the following pictures

$$\begin{aligned} t = \frac{1}{2}: &\quad u(x, \frac{1}{2}) = \frac{1}{2}(g(x+\frac{1}{2}) + g(x-\frac{1}{2})), \ , \\ t = 1: &\quad u(x,1) = \frac{1}{2}(g(x+1) + g(x-1)), \\ t = 2: &\quad u(x,2) = \frac{1}{2}(g(x+2) + g(x-2)), \end{aligned}$$

31.10.2019

6.2. **Reflection method.**

$$\begin{cases} u_{tt} - u_{xx} = 0, & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u(x,0) = g(x), & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u_t(x,0) = h(x), & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u(0,t) = 0, & \text{on } \{x = 0\} \times (0,\infty) \end{cases}$$

Let us continue the functions in the whole of $\mathbb{R}$ by odd reflection:

$$\tilde{u}(x,t) = \begin{cases} u(x,t), & x \geq 0,\ t \geq 0 \\ -u(-x,t), & x < 0,\ t \geq 0, \end{cases}$$

$$\tilde{g}(x) = \begin{cases} g(x), & x \geq 0, \\ -g(-x), & x < 0, \end{cases}$$

$$\tilde{h}(x) = \begin{cases} h(x), & x \geq 0, \\ -h(-x), & x < 0. \end{cases}$$

Also assume that g, h are such that their reflections are C^2 and C^1 respectively, in particular $g(0) = 0 = h(0)$, $g''(0) = 0$. Then we get the solution to

$$\begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} = 0, & \text{in } \mathbb{R} \times (0,\infty) \\ \tilde{u}(x,0) = \tilde{g}(x), & \text{on } \mathbb{R} \times \{t = 0\} \\ \tilde{u}_t(x,0) = \tilde{h}(x), & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

By d'Alembert's formula:

$$\begin{aligned} \tilde{u}(x,t) &= \frac{1}{2}(\tilde{g}(x+t) + \tilde{g}(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)\,dy \\ &= \begin{cases} \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy, & x \geq t \geq 0 \\ \frac{1}{2}(g(x+t) - g(t-x)) + \frac{1}{2}\int_{t-x}^{x+t} h(y)\,dy & 0 \leq x \leq t, \end{cases} \end{aligned} \tag{6.22}$$

since in the second case

$$\int_{x-t}^{x+t} \tilde{h}(y)\,dy = -\int_{x-t}^{0} h(-y)\,dy + \int_{0}^{x+t} h(y)\,dy = \int_{t-x}^{x+t} h(y)\,dy.$$

Example 6.5. *If $h = 0$, then*

$$\tilde{u}(x,t) = \begin{cases} \frac{1}{2}(g(x+t) + g(x-t)), & x \geq t \geq 0 \\ \frac{1}{2}(g(x+t) - g(t-x)), & 0 \leq x \leq t. \end{cases}$$

Draw the following pictures

$$g(x) = \begin{cases} 1/2 - |x - 1.5| & 1 \leq x \leq 2 \\ 0 & \textit{otherwise}. \end{cases}$$

$$t = 0 :$$
$$t = 1 :$$
$$t = 2 : \ .$$

6.3. **Spherical means.** First recall Laplacian for a radially symmetric function $u(t,x) = v(t,|x|)$

$$u_{tt} - \Delta u = v_{tt} - v_{rr} - \frac{n-1}{r} v_r = 0.$$

This is called Euler-Poisson-Darboux equation.

Let $n \geq 2$ and $u \in C^2(\mathbb{R}^n \times (0,\infty))$ solves

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x), & \text{on } \mathbb{R}^n \times \{t = 0\} \\ u_t(x,0) = h(x), & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Denote

$$U(x,r,t) = \fint_{\partial B(x,r)} u(y,t)\,dS(y)$$
$$G(x,r) = \fint_{\partial B(x,r)} g(y)\,dS(y)$$
$$H(x,r) = \fint_{\partial B(x,r)} h(y)\,dS(y).$$

Fix x and regard U as a function of r, t.

Lemma 6.6. *It holds that $U \in C^2(\overline{\mathbb{R}}_+ \times [0,\infty))$ and*

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0, & \textit{in } \mathbb{R}_+ \times (0,\infty) \\ U = G, U_t = H, & \textit{on } \mathbb{R}_+ \times \{t = 0\}. \end{cases}$$

Proof. Initial conditions hold:

$$U(x,r,0) = \fint_{\partial B(x,r)} u(y,0)\,dS(y) = \fint_{\partial B(x,r)} g(y)\,dS(y) = G(x,r)$$
$$U_t(x,r,0) = \fint_{\partial B(x,r)} u_t(y,0)\,dS(y) = \fint_{\partial B(x,r)} h(y)\,dS(y) = H(x,r).$$

When proving mean value property for Laplacian, we obtained the formula

$$U_r(x,r,t) = \frac{r}{n}\underbrace{\fint_{B(x,r)} \Delta u(y,t)\,dy}_{\text{bounded}} = \frac{1}{\omega_n r^{n-1}}\int_{B(x,r)} \Delta u(y,t)\,dy,$$

so that

$$\lim_{r\to 0+} U_r(x,r,t) = 0.$$

After similar computations (omitted) for U_{rr}, we see that $U \in C^2(\overline{\mathbb{R}}_+ \times [0,\infty))$.

Then

$$\begin{aligned}
\partial_r(\omega_n r^{n-1} U_r) &= \partial_r(\int_{B(x,r)} \Delta u(y,t)\,dy) \\
&= \partial_r(\int_0^r \int_{\partial B(x,\rho)} \Delta u(y,t)\,dS(y)\,d\rho) \\
&= \int_{\partial B(x,r)} \Delta u(y,t)\,dS(y) \\
&= \int_{\partial B(x,r)} u_{tt}(y,t)\,dS(y) \\
&= \omega_n r^{n-1} \fint_{\partial B(x,r)} u_{tt}(y,t)\,dS(y) \\
&= \omega_n r^{n-1} U_{tt}.
\end{aligned}$$

Since

$$(r^{n-1}U_r)_r = r^{n-1}(\frac{n-1}{r}U_r + U_{rr})$$

this implies the claim. □

6.4. **Solution when** $n = 3$**.** Letting $r \to 0$ gives solution to the original equation:

$$\begin{aligned}
u(x,t) &= \lim_{r\to 0} U(x,r,t), \\
h(x) &= \lim_{r\to 0} H(x,r), \\
g(x) &= \lim_{r\to 0} G(x,r).
\end{aligned}$$

Denote

$$\tilde{U} = rU, \quad \tilde{G} = rG, \quad \tilde{H} = rH.$$

Then

Lemma 6.7.

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 & \mathbb{R}_+ \times (0, \infty) \\ \tilde{U} = \tilde{G}, \tilde{U}_t = \tilde{H}, & \mathbb{R}_+ \times \{t = 0\} \\ \tilde{U} = 0, & \{r = 0\} \times (0, \infty). \end{cases}$$

Proof. Equation:

$$\begin{aligned} \tilde{U}_{tt} &= rU_{tt} \overset{\text{prev. lemma, n-1=2}}{=} r(U_{rr} + \frac{2}{r}U_r) \\ &= rU_{rr} + 2U_r \\ &= (\underbrace{U + rU_r}_{\tilde{U}_r})_r \\ &= \tilde{U}_{rr}. \end{aligned}$$

Initial conditions hold by previous lemma:

$$\begin{aligned} \tilde{U}(x, r, 0) &= rU(x, r, 0) = rG(x, r) = \tilde{G}(x, r), \\ \tilde{U}_t(x, r, 0) &= rU_t(x, r, 0) = rH(x, r) = \tilde{H}(x, r), \\ \tilde{U}(x, 0, t) &= \lim_{r \to 0} r \fint_{\partial B(x,r)} u(y, t)\, dS(y) = u(x, t) \lim_{r \to 0} r = 0. \end{aligned}$$ □

This is one dimensional wave equation so that we may use the d'Alembert formula with reflection (6.22) (a brief computation shows that $\tilde{G}, \tilde{H}$ satisfy the assumptions):

$$\tilde{U}(x, r, t) = \frac{1}{2}(\tilde{G}(r + t) - \tilde{G}(t - r)) + \frac{1}{2}\int_{t-r}^{t+r} \tilde{H}(y)\, dy, \ 0 \le r \le t.$$

Thus

$$\begin{aligned}
u(x,t) &= \lim_{r\to 0} \fint_{\partial B(x,r)} u(y,t)\,dS(y)\\
&= \lim_{r\to 0} U(x,r,t) = \lim_{r\to 0} \frac{\tilde U(x,r,t)}{r}\\
&= \lim_{r\to 0} \Big[\frac{1}{2r}(\tilde G(r+t)-\tilde G(t-r)) + \frac{1}{2r}\int_{t-r}^{r+t} \tilde H(y)\,dy\Big]\\
&= \tilde G'(t) + \tilde H(t)\\
&= (t\fint_{\partial B(x,t)} g(y)\,dS(y))_t + t\fint_{\partial B(x,t)} h(y)\,dS(y)\\
&= \fint_{\partial B(x,t)} g(y)\,dS(y) + t(\fint_{\partial B(x,t)} g(y)\,dS(y))_t + t\fint_{\partial B(x,t)} h(y)\,dS(y)
\end{aligned}$$

where

$$\begin{aligned}
&(\fint_{\partial B(x,t)} g(y)\,dS(y))_t\\
&\overset{y=x+tz}{=} (\fint_{\partial B(0,1)} g(x+tz)t^{n-1}t^{-(n-1)}\,dS(z))_t\\
&= \fint_{\partial B(0,1)} Dg(x+tz)\cdot z\,dS(z)\\
&= \fint_{\partial B(x,t)} Dg(y)\cdot \frac{y-x}{t}\,dS(y).
\end{aligned}$$

Thus

$$\begin{aligned}
u(x,t) &= (t\fint_{\partial B(x,t)} g(y)\,dS(y))_t + t\fint_{\partial B(x,t)} h(y)\,dS(y)\\
&= \fint_{\partial B(x,t)} g(y)\,dS(y) + t(\fint_{\partial B(x,t)} g(y)\,dS(y))_t + t\fint_{\partial B(x,t)} h(y)\,dS(y)\\
&= \fint_{\partial B(x,t)} (g(y) + Dg(y)\cdot(y-x) + th(y))\,dS(y). \qquad (6.23)
\end{aligned}$$

This is called the Kirchhoff formula in three dimensions.

Remark 6.8. *(1) Implies uniqueness, stability cf. remark after d'Alemberts formula.*

(2) The value at (x,t) is determined by the values of g and h on $\partial B(x,t)$: Huygens principle. On the other hand, every point $y\in\mathbb{R}^2$, $t=0$ affect the values on:

$$\{(x,t)\in\mathbb{R}^3\times(0,\infty)\ :\ |x-y|=t\}.$$

Finite speed of propagation.

6.5. **Solution when $n = 2$.** Assume $u \in C^2(\mathbb{R}^2 \times (0,\infty))$ and

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{in } \mathbb{R}^2 \times (0,\infty) \\ u(x,0) = g(x), & \text{on } \mathbb{R}^2 \times \{t = 0\} \\ u_t(x,0) = h(x), & \text{on } \mathbb{R}^2 \times \{t = 0\}. \end{cases}$$

Define $\tilde{u} : \mathbb{R}^3 \times (0,\infty) \to \mathbb{R}$ setting

$$\tilde{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t),$$

and similarly

$$\tilde{g}(x_1, x_2, x_3) = g(x_1, x_2)$$
$$\tilde{h}(x_1, x_2, x_3) = h(x_1, x_2),$$

i.e. trivial extension. Then

$$\begin{cases} \tilde{u}_{tt} - \Delta \tilde{u} = 0, & \text{in } \mathbb{R}^3 \times (0,\infty) \\ \tilde{u}(x,0) = \tilde{g}(x), & \text{on } \mathbb{R}^3 \times \{t = 0\} \\ \tilde{u}_t(x,0) = \tilde{h}(x), & \text{on } \mathbb{R}^3 \times \{t = 0\}. \end{cases}$$

Denote

$$x = (x_1, x_2) \in \mathbb{R}^2 \quad \tilde{x} = (x_1, x_2, 0) \in \mathbb{R}^3.$$

By Kirchhoff's formula

$$u(x,t) = \tilde{u}(\tilde{x},t) = \fint_{\partial B^3(\tilde{x},t)} (\tilde{g}(y) + D\tilde{g}(y) \cdot (y - \tilde{x}) + t\tilde{h}(y))\, dS(y)$$

where we integrate over the boundaries of 3D balls. Here

$$\begin{aligned} \fint_{\partial B^3(\tilde{x},t)} \tilde{g}(y)\, dS(y) &= \frac{1}{4\pi t^2} \int_{\partial B^3(\tilde{x},t)} \tilde{g}(y)\, dS(y) \\ &\overset{\tilde{g}(y_1,y_2,y_3) = g(y_1,y_2)}{=} \frac{2}{4\pi t^2} \int_{B^2(x,t)} g(y) \sqrt{1 + |D\gamma(y)|^2}\, dy, \end{aligned}$$

where B^3 is 3D ball and B^2 is 2D ball , and $\gamma : B^2(x,t) \to \mathbb{R}$,

$$\gamma(y) = \sqrt{t^2 - |y - x|^2},$$

is parametric presentation of the one half of the sphere The factor 2 comes from the fact that sphere has two parts, upper and lower. Now

$$\begin{aligned} D\gamma(y) &= \frac{1}{2}(t^2 - |y - x|^2)^{-\frac{1}{2}} D_y(-|y - x|^2) \\ &= \frac{1}{2}(t^2 - |y - x|^2)^{-\frac{1}{2}} (-2) |y - x| \frac{y - x}{|y - x|} \\ &= -(t^2 - |y - x|^2)^{-\frac{1}{2}} (y - x). \end{aligned}$$

Thus

$$|D\gamma(y)| = (t^2 - |y - x|^2)^{-\frac{1}{2}} |y - x|.$$

Further,

$$
\begin{aligned}
(1+|D\gamma(y)|^2)^{\frac{1}{2}} &= \left(1+\frac{|y-x|^2}{t^2-|y-x|^2}\right)^{\frac{1}{2}} \\
&= \left(\frac{t^2-|y-x|^2+|y-x|^2}{t^2-|y-x|^2}\right)^{\frac{1}{2}} \\
&\overset{t\geq 0}{=} (t^2-|y-x|^2)^{-\frac{1}{2}}t.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\fint_{\partial B^3(\tilde{x},t)} \tilde{g}\,dS &= \frac{1}{2\pi t}\int_{B^2(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}\,dy \\
&\overset{|B^2(y,t)|=\pi t^2}{=} \frac{t}{2}\fint_{B^2(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}\,dy.
\end{aligned}
$$

Similarly

$$
t\fint_{\partial B^3(\tilde{x},t)} \tilde{h}\,dS = \frac{t^2}{2}\fint_{B^2(x,t)} \frac{h(y)}{\sqrt{t^2-|y-x|^2}}\,dy
$$

and

$$
\fint_{\partial B^3(\tilde{x},t)} D\tilde{g}(y)\cdot(y-x)\,dS(y) = \frac{t}{2}\fint_{B^2(x,t)} \frac{Dg(y)\cdot(y-x)}{\sqrt{t^2-|y-x|^2}}\,dy.
$$

This gives us the formula

$$
u(x,t) = \frac{1}{2}\fint_{B^2(x,t)} \frac{tg(y)+t^2h(y)+tDg(y)\cdot(y-x)}{\sqrt{t^2-|y-x|^2}}\,dy
$$

for the 2D problem. First solving 3D and then dropping to 2D is called method of descent.

Remark 6.9.

- *The value at (x,t) is determined by the values on $B(x,t)$ (different from 3D case). On the other hand each point $y \in \mathbb{R}^2$, $t=0$ affects the values in the cone*

$$
\{(x,t)\in\mathbb{R}^2\times(0,\infty)\,:\,|x-y|\leq t\}.
$$

- *Also Dg present. Irregularities may focus, i.e. solution may be more irregular than the initial data.*
- *The above approach can be generalzed to higher dimensions: solve odd n problem and then use method of descend to get to $n-1$.*

6.6. **Inhomogenous problem.** Assume $u \in C^2(\mathbb{R}^n \times (0,\infty))$ and

$$\begin{cases} u_{tt} - \Delta u = f, & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = 0, & \text{on } \mathbb{R}^n \times \{t=0\} \\ u_t(x,0) = 0, & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

Duhamel's principle: Solve

$$\begin{cases} u_{tt}(x,t,s) - \Delta u(x,t,s) = 0, & (x,t) \in \mathbb{R}^n \times (s,\infty) \\ u(x,s,s) = 0, & (x,t) \in \mathbb{R}^n \times \{t=s\} \\ u_t(x,s,s) = f(x,s), & \text{in } (x,t) \in \mathbb{R}^n \times \{t=s\}. \end{cases}$$

and set

$$u(x,t) = \int_0^t u(x,t,s)\,ds.$$

This solves the inhomogeneous problems since

$$u_t(x,t) = \underbrace{u(x,t,t)}_{=0} + \int_0^t u_t(x,t,s)\,ds = \int_0^t u_t(x,t,s)\,ds.$$

$$\begin{aligned} u_{tt}(x,t) &= u_t(x,t,t) + \int_0^t u_{tt}(x,t,s)\,ds \\ &= f(x,t) + \int_0^t u_{tt}(x,t,s)\,ds. \end{aligned}$$

$$\begin{aligned} \Delta u(x,t) &= \Delta \int_0^t u(x,t,s)\,ds \\ &= \int_0^t \Delta u(x,t,s)\,ds \\ &= \int_0^t u_{tt}(x,t,s)\,ds \end{aligned}$$

Thus

$$\begin{aligned} u_{tt}(x,t) &= f(x,t) + \int_0^t u_{tt}(x,t,s)\,ds \\ &= f(x,t) + \Delta u(x,t). \end{aligned}$$

Moreover,

$$u(x,0) = u_t(x,0) = (\int_0^0 \dots\, ds) = 0.$$

Solution to general inhomogenous problem is then solved by

$$v + u$$

where v is the solution to

$$\begin{cases} v_{tt} - \Delta v = 0, & \text{in } \mathbb{R}^n \times (0,\infty) \\ v(x,0) = g(x), & \text{on } \mathbb{R}^n \times \{t=0\} \\ v_t(x,0) = h(x), & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

obtained by the Euler-Poisson-Darboux equation and spherical means, and u by the Duhamel's principle as above.

Example 6.10. *Inhomogenous problem:*

- $n = 1:$ *d'Alembert*

$$u(x,t,s) = \frac{1}{2}\int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dy.$$

Duhamel:

$$\begin{aligned} u(x,t) &= \int_0^t u(x,t,s)\,ds \\ &= \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dy\,ds \\ &\overset{s=t-r}{=} \frac{1}{2}\int_0^t \int_{x-r}^{x+r} f(y,t-r)\,dy\,dr \\ &= \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,t-s)\,dy\,ds. \end{aligned}$$

- $n = 3$: *Kirchhoff formula:*

$$u(x,t,s) = (t-s) \fint_{\partial B(x,t-s)} f(y,s)\,dS(y).$$

Duhamel:

$$\begin{aligned} u(x,t) &= \int_0^t u(x,t,s)\,ds \\ &= \int_0^t (t-s) \fint_{\partial B(x,t-s)} f(y,s)\,dS(y)\,ds \\ &\overset{|\partial B(x,t-s)|=4\pi(t-s)^2}{=} \frac{1}{4\pi}\int_0^t \int_{\partial B(x,t-s)} \frac{f(y,s)}{t-s}\,dS(y)\,ds \\ &\overset{r=t-s}{=} \frac{1}{4\pi}\int_0^t \int_{\partial B(x,r)} \frac{f(y,t-r)}{r}\,dS(y)\,dr \\ &= \frac{1}{4\pi}\int_{B(x,t)} \frac{f(y,t-|y-x|)}{|y-x|}\,dy. \end{aligned}$$

6.7. **Energy method.** Let $\Omega \subset \mathbb{R}^n$ be a smooth open set, $u = 0$ on $\partial\Omega$.

Definition 6.11.

$$I(t) = \frac{1}{2}\int_\Omega ((u_t(x,t))^2 + |Du(x,t)|^2)\,dx.$$

Now

$$\begin{aligned}
I'(t) &\overset{\text{chg int diff}}{=} \frac{1}{2}\int_\Omega \partial_t((u_t(x,t))^2 + |Du(x,t)|^2)\,dx \\
&= \frac{1}{2}\int_\Omega 2u_t(x,t)u_{tt}(x,t) + 2\,|Du(x,t)|\,\frac{Du(x,t)}{|Du(x,t)|}Du_t(x,t)\,dx \\
&= \int_\Omega u_t(x,t)u_{tt}(x,t) + Du(x,t)\cdot Du_t(x,t)\,dx \\
&\overset{\text{int by parts, } u=0 \text{ on } \partial\Omega}{=} \int_\Omega u_t(x,t)u_{tt}(x,t) - \Delta u(x,t)u_t(x,t)\,dx \\
&= \int_\Omega u_t(x,t)(\underbrace{u_{tt}(x,t) - \Delta u(x,t)}_{=0})\,dx = 0.
\end{aligned}$$

We have proven

Theorem 6.12 (Conservation of energy)**.** *If u is a solution to $u_{tt} - \Delta u = 0$ and $u = 0$ on $\partial\Omega$, then the energy is conserved i.e.*

$$I(t) \equiv C.$$

7.11.2019

Recall $\partial_p\Omega_T = (\Omega \times \{0\}) \cup (\partial\Omega \times [0,T])$.

Theorem 6.13 (Uniqueness by energy method)**.** *Assume $\Omega \subset \mathbb{R}^n$ bounded. The problem*

$$\begin{cases} u_{tt} - \Delta u = f, & \text{in } \Omega_T \\ u(x,0) = g(x), & \text{on } \partial_p\Omega_T \\ u_t(x,0) = h(x), & \text{on } \Omega \times \{t=0\} \end{cases}$$

has at most one solution.

Proof. Let u, v be solutions and set $w = u - v$ is a solution to

$$\begin{cases} w_{tt} - \Delta w = 0, & \text{in } \Omega_T \\ w(x,0) = 0, & \text{on } \partial_p\Omega_T \\ w_t(x,0) = 0, & \text{on } \Omega \times \{t=0\}. \end{cases}$$

Then by the energy conservation above,

$$\begin{aligned}
I(t) &= \frac{1}{2}\int_\Omega ((w_t(x,t))^2 + |Dw(x,t)|^2)\,dx \\
&= I(0) = 0, \quad 0 \le t \le T.
\end{aligned}$$

Thus

$$(w_t(x,t))^2 + |Dw(x,t)|^2 = 0$$

thus $w_t(x,t) = 0$ and $Dw(x,t) = 0$ in Ω_T so that

$$w = u - v \equiv C.$$

Since

$$w = 0 \text{ on } \Omega \times \{t = 0\}$$

it holds

$$w = 0 \text{ on } \Omega_T.$$

□

Finite speed of propagation also follows from the energy method. We denote

$$C = \{(x,t) : 0 \le t \le t_0, |x - x_0| \le t_0 - t\}$$

with fixed $x_0 \in \mathbb{R}^n$ and $t_0 > 0$. We are going to show that the external disturbances outside C do not affect the value at (x_0, t_0). This follows from the Kirchhoff's formula, but energy method gives easier and more flexible proof.

Skipped at the lectures:

Theorem 6.14 (Finite speed of propagation)**.** *Let*

$$u_{tt} - \Delta u = 0, \text{ in } \mathbb{R}^n \times (0, \infty).$$

If

$$u(x,0) = 0 \text{ and } u_t(x,0) = 0$$

for all $x \in B(x_0, t_0)$*, then*

$$u(x,t) = 0 \text{ for all } (x,t) \in C.$$

Proof.

$$I(t) = \frac{1}{2} \int_{B(x_0, t_0 - t)} ((u_t(x,t))^2 + |Du(x,t)|^2)\, dx, \ 0 \le t \le t_0.$$

$$
\begin{aligned}
I'(t) &= \partial_t \frac{1}{2}\int_0^{t_0-t}\int_{\partial B(x_0,s)} ((u_t(x,t))^2 + |Du(x,t)|^2)\, dS\, ds \\
&= -\frac{1}{2}\int_{\partial B(x_0,t_0-t)} ((u_t(x,t))^2 + |Du(x,t)|^2)\, dS \\
&\quad + \frac{1}{2}\int_0^{t_0-t}\int_{\partial B(x_0,s)} \partial_t((u_t(x,t))^2 + |Du(x,t)|^2)\, dS\, ds \\
&= -\frac{1}{2}\int_{\partial B(x_0,t_0-t)} ((u_t(x,t))^2 + |Du(x,t)|^2)\, dS \\
&\quad + \int_0^{t_0-t}\int_{\partial B(x_0,s)} (u_t(x,t)u_{tt}(x,t) + Du(x,t)\cdot Du_t(x,t))\, dS\, ds \\
&= -\frac{1}{2}\int_{\partial B(x_0,t_0-t)} ((u_t(x,t))^2 + |Du(x,t)|^2)\, dS \\
&\quad + \int_{B(x_0,t_0-t)} (u_t(x,t)(\underbrace{u_{tt}(x,t) - \Delta u(x,t)}_{=0})\, dx \\
&\quad + \int_{\partial B(x_0,t_0-t)} \frac{\partial u}{\partial \nu} u_t\, dS \\
&= \int_{\partial B(x_0,t_0-t)} (\frac{\partial u}{\partial \nu}u_t - \frac{1}{2}u_t^2 - \frac{1}{2}|Du|^2)\, dS \\
&\overset{\text{below}}{\leq} 0,
\end{aligned}
$$

where the last inequality follows as

$$
\begin{aligned}
\left|\frac{\partial u}{\partial \nu}u_t\right| &\overset{\text{Cauchy-Schwarz}}{\leq} |Du|\underbrace{|\nu|}_{=1}|u_t| \\
&\overset{\text{Cauchy/Young's ie}}{\leq} \frac{1}{2}|Du|^2 + \frac{1}{2}|u_t|^2.
\end{aligned}
$$

Thus

$$
0 \leq I(t) \leq I(0) = 0,\ 0 \leq t \leq t_0,
$$

where the last equality holds because

$$
\begin{aligned}
u = 0 \Rightarrow Du &= 0 \text{ in } B(x_0,t_0), \\
u_t &= 0 \text{ in } B(x_0,t_0).
\end{aligned}
$$

We have proven

$$
\begin{gathered}
I(t) = 0,\ 0 \leq t \leq t_0, \\
u_t = 0, Du = 0 \text{ in } B(x_0, t_0 - t) \text{ for all } 0 \leq t \leq t_0.
\end{gathered}
$$

Thus

$$
u = 0 \text{ in } C.
$$

□

7. Other ways of representation of solutions

Examples of methods of finding explicit solutions.

7.1. **Fourier series.** In this section the underlying space is $\mathbb{C}$.

Definition 7.1. *The space $L^2([-\pi,\pi])$ consists of the functions $f:[-\pi,\pi]\to\mathbb{C}$ such that*

$$\int_{-\pi}^{\pi} |f(t)|^2 \, dt < \infty,$$

where $|f(t)|$ denotes the modulus or length of the corresponding complex number.

Example 7.2. $f:[-\pi,\pi]\to\mathbb{R}$,

$$f(t) = \begin{cases} \frac{1}{\sqrt{|t|}}, & t \neq 0 \\ 0 & t = 0. \end{cases}$$

$$\int_{-\pi}^{\pi} |f(t)|^2 \, dt = \int_{-\pi}^{0} 1/\,|t| \, dt + \int_{0}^{\pi} 1/t \, dt = \infty,$$

i.e. $f \notin L^2([-\pi,\pi])$.

Definition 7.3. *Inner product in L^2 is defined as*

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g}(t)\, dt.$$

The norm

$$||f||_2 = \sqrt{\langle f, f\rangle} = \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{f}(t)\, dt\Big)^{1/2}$$

$$\overset{z\overline{z}=(x+iy)(x-iy)=x^2-(iy)^2=|z|^2}{=} \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 \, dt\Big)^{1/2}.$$

Remark 7.4. *For inner product we need*

$$\langle f, f\rangle = 0 \ \Rightarrow \ f = 0$$

by agreeing that $f = g$ in L^2 sense if

$$\int_{-\pi}^{\pi} |f-g|^2 \, dt = 0.$$

Recall the following inequalities:
Cauchy-Schwarz:

$$|\langle f, g\rangle| \le ||f||_2\, ||g||_2 = (\int_{-\pi}^{\pi} |f|^2 \, dt)^{1/2} (\int_{-\pi}^{\pi} |g|^2 \, dt)^{1/2}.$$

Triangle inequality

$$||f+g||_2 \le ||f||_2 + ||g||_2\,.$$

Next we denote

$$e_j : [-\pi, \pi] \to \mathbb{C}, \quad e_j(t) = e^{ijt}, \quad j \in \mathbb{Z}.$$

Recall Euler's formula

$$e_j(t) = e^{ijt} = \cos(jt) + i\sin(jt).$$

Now

$$\begin{aligned}
\langle e_j, e_k \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ijt} \overline{e^{ikt}} \, dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ijt} e^{-ikt} \, dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(j-k)t} \, dt \\
&= \begin{cases} \frac{1}{2\pi} \frac{1}{i(j-k)} \big|_{-\pi}^{\pi} e^{it(j-k)} = 0, & j \neq k \\ 1, & j = k. \end{cases}
\end{aligned}$$

Thus $\{e_j : j \in \mathbb{Z}\}$ is an orthonormal set in $L^2([-\pi, \pi])$.

Definition 7.5. *The jth Fourier coefficient of f is*

$$\hat{f}(j) = \langle f, e_j \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ijt} \, dt, \quad j \in \mathbb{Z}.$$

The partial sum of Fourier series is

$$S_k(t) = S_k f(t) = \sum_{j=-k}^{k} \langle f, e_j \rangle e_j = \sum_{j=-k}^{k} \hat{f}(j) e^{ijt}, \quad k = 0, 1, 2, \dots$$

The Fourier series is the limit of the partial sum (if exists)

$$f(t) \overset{later}{=} \lim_{k \to \infty} S_k(t) = \lim_{k \to \infty} S_k f(t) = \lim_{k \to \infty} \sum_{j=-k}^{k} \hat{f}(j) e^{ijt} = \sum_{j=-\infty}^{\infty} \hat{f}(j) e^{ijt}.$$

For the next proofs that were covered in the lectures, see for example Juha Kinnunen: Partial differential equations https://koppa.jyu.fi/en/courses/216483/kinnunenpde.pdf.

Theorem 7.6 (Best approximation)**.** *If $f \in L^2([-\pi, \pi])$, then*

$$||f - S_k||_2 \leq \left|\left| f - \sum_{j=-k}^{k} \alpha_k e_j \right|\right|_2 ,$$

and $S_k(t)$ is the orthogonal projection on $\operatorname{Span}\{e_j : j = -k, \dots, k\}$.

We call $\sum_{j=-k}^{k} \alpha_k e_j$ kth order trigonometric polynomial. This says that the partial sum of Fourier series gives the best approximation in L^2 in the class of kth order trigonometric polynomials.

Theorem 7.7 (L^2-convegence)**.** *If $f \in L^2([-\pi,\pi])$, then*

$$||f - S_k||_2 \to 0$$

as $k \to \infty$.

This justifies writing

$$f(t) = \sum_{j=-\infty}^{\infty} \hat{f}(j) e^{ijt} \text{ in } L^2\text{-sense.}$$

The following theorem and remark were not proved in the lectures.

Theorem 7.8. *Let $f \in L^2([-\pi,\pi])$ and f differentiable at t_1. Then*

$$f(t_1) = \lim_{k\to\infty} S_k(t_1)$$

in the pointwise sense.

Remark 7.9.

- *The pointwise converge does not necessarily hold if f only continuous.*
- *If $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic, and f satisfies the Lipschitz condition*

  $$|f(t) - f(s)| \leq C\,|t - s| \quad \textit{for all } s, t \in \mathbb{R},$$

  then

  $$\max_{t\in[-\pi,\pi]} |S_k(t) - f(t)| \to 0, \; \textit{when } k \to \infty,$$

  i.e. $S_k \to f$ uniformly.

Theorem 7.10 (Parseval)**.** *If $f \in L^2([-\pi,\pi])$, then*

$$||f||_2 = \sum_{j=-\infty}^{\infty} \left|\hat{f}(j)\right|^2.$$

Theorem 7.11 (Uniqueness)**.** *Let $f, g \in L^2([-\pi,\pi])$ and $\hat{f}(j) = \hat{g}(j)$, then*

$$f = g \textit{ in } L^2.$$

Naturally also: if $f = g$, then $\hat{f}(j) = \hat{g}(j)$.

Definition 7.12. *A function $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic if*

$$f(t + 2\pi) = f(t).$$

Example 7.13.

- $$f : \mathbb{R} \to \mathbb{C}$$
  $$f(t) = e^{ijt}$$

  is 2π-periodic, since

  $$f(t + 2\pi) = e^{ij(t+2\pi)} = e^{ijt} \underbrace{e^{ij2\pi}}_{=1} = f(t).$$

- *Thus, the partial sum of Fourier series*

$$S_k(t) = \sum_{j=-k}^{k} \hat{f}(j) e^{ijt}$$

is 2π-periodic.
- *From the previous, the limit $\lim_{k\to\infty} S_k$ is 2π-periodic (if exists). Thus we can only directly approximate 2π-periodic functions by Fourier series, or $f : [-\pi, \pi] \to \mathbb{R}$.*

7.1.1. *Fourier series in real form (vs. complex form).*

$$\begin{aligned}
S_k &= \sum_{j=-k}^{k} \hat{f}(j) e^{ijt} \\
&= \sum_{j=-k}^{-1} \hat{f}(j) e^{ijt} + \hat{f}(0) + \sum_{j=1}^{k} \hat{f}(j) e^{ijt} \\
&= \hat{f}(0) + \sum_{j=1}^{k} (\hat{f}(j) e^{ijt} + \hat{f}(-j) e^{-ijt}) \\
&= \hat{f}(0) + \sum_{j=1}^{k} (\hat{f}(j)(\cos(jt) + i\sin(jt)) + \hat{f}(-j)(\cos(jt) - i\sin(jt))) \\
&= \hat{f}(0) + \sum_{j=1}^{k} (\hat{f}(j) + \hat{f}(-j)) \cos(jt) + \sum_{j=1}^{k} i(\hat{f}(j) - \hat{f}(-j)) \sin(jt),
\end{aligned}$$

where

$$\hat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt,$$

$$\begin{aligned}
\hat{f}(j) + \hat{f}(-j) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)(e^{-ijt} + e^{ijt})\, dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) 2\cos(jt)\, dt \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(jt)\, dt,
\end{aligned}$$

and

$$\begin{aligned}
i(\hat{f}(j) - \hat{f}(-j)) &= \frac{i}{2\pi}\int_{-\pi}^{\pi} f(t)(e^{-ijt} - e^{ijt})\,dt \\
&= \frac{i}{2\pi}\int_{-\pi}^{\pi} f(t)(-2i)\sin(jt)\,dt \\
&= \frac{-i^2}{\pi}\int_{-\pi}^{\pi} f(t)\sin(jt)\,dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(jt)\,dt.
\end{aligned}$$

Thus we have obtained the following real form Fourier series

$$\begin{aligned}
S_k &= \sum_{j=-k}^{k} \hat{f}(j)e^{ijt} \\
&= \frac{a_0}{2} + \sum_{j=1}^{k}(a_j\cos(jt) + b_j\sin(jt)) \qquad (7.24)
\end{aligned}$$

where

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt, \\
a_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(jt)\,dt, \quad j = 0, 1, \dots \\
b_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(jt)\,dt, \quad j = 1, 2, \dots
\end{aligned}$$

Remark 7.14.

- *If $f : \mathbb{R} \to \mathbb{R}$, then there are only real numbers visible in the above form.*
- *If Fourier series is given in the real form, we can transform it back to the complex form by recalling*

$$\begin{aligned}
\cos(jt) &= \frac{1}{2}(e^{ijt} + e^{-ijt}), \\
\sin(jt) &= \frac{1}{2i}(e^{ijt} - e^{-ijt}).
\end{aligned}$$

7.1.2. *Fourier series on the general interval, odd and even functions.* If $f : [a, b] \to \mathbb{C}$, then

$$S_k(t) = \sum_{j=-k}^{k} \hat{f}(j)e^{\frac{2\pi ijt}{b-a}}.$$

where

$$\hat{f}(j) = \frac{1}{b-a}\int_a^b f(t)e^{-\frac{2\pi ijt}{b-a}}\,dt, \quad j \in \mathbb{Z}.$$

This is due to the change of variables. In particular for $f : [-L, L] \to \mathbb{C}$ in the real form

$$S_k = \frac{a_0}{2} + \sum_{j=1}^{k} (a_j \cos(\frac{\pi j t}{L}) + b_j \sin(\frac{\pi j t}{L}))$$

where

$$a_j = \frac{1}{L} \int_{-L}^{L} f(t) \cos(\frac{\pi j t}{L})\, dt, \quad j = 0, 1, \ldots$$
$$b_j = \frac{1}{L} \int_{-L}^{L} f(t) \sin(\frac{\pi j t}{L})\, dt, \quad j = 1, 2, \ldots$$

Recall

Definition 7.15. *The function $f : \mathbb{R} \to \mathbb{C}$ is odd if*

$$f(-t) = -f(t)$$

and even if

$$f(-t) = f(t).$$

In the Fourier series of odd function cos-terms vanish i.e.

$$a_j = \frac{1}{L} \int_{-L}^{L} f(t) \cos(\frac{\pi j t}{L})\, dt = 0,$$
$$\Rightarrow f(t) = \sum_{j=1}^{\infty} b_j \sin(\frac{\pi j t}{L})$$

and of even function

$$b_j = \frac{1}{L} \int_{-L}^{L} f(t) \sin(\frac{\pi j t}{L})\, dt = 0$$
$$\Rightarrow f(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos(\frac{\pi j t}{L}).$$

These are called sin- and cos-series.

Example 7.16.

$$f : [-\pi, \pi] \to \mathbb{R}, \quad f(t) = t.$$

Then we can calculate that

$$\sum_{j=-k}^{k} \hat{f}(j) e^{ijt} = \sum_{j=-k, j\neq 0}^{k} \frac{i}{j} (-1)^j e^{ijt},$$

and in the real form

$$-\sum_{j=1}^{n} \frac{2}{j} (-1)^j \sin(jt).$$

Observe that the cos terms vanish.

Example 7.17.

$$f : [-\pi, \pi] \to \mathbb{R}, \quad f(t) = \begin{cases} -1, & -\pi \le t < 0 \\ 1, & 0 \le t \le \pi. \end{cases}.$$

Then we can calculate that the Fourier series in real form is

$$\operatorname{sgn}(t) = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\sin((2j+1)t)}{2j+1}, \;\; -\pi < t < \pi.$$

Observe that the cos terms vanish.

Remark 7.18. *Observe that Fourier series can be formed for discontinuous functions unlike Taylor's expansion.*

Remark 7.19. *Applications to*
- *PDEs (below)*
- *image compression*
- *signal processing*

7.2. **Separation of variables.** Strategy

(1) Separation of variables: In rectangular, cylindrical etc domain separate the variables i.e. try finding the solution in the form $v(x)w(y)$. Using this we obtain two simpler separated equations.
(2) Solving the separated equations.
(3) Solving the full problem: Look for general solution as series. Boundary values determine the coefficients in the series.

Example 7.20. *Consider*

$$u : [0, a] \times [0, b] \to \mathbb{R}, \quad \Omega = (0, a) \times (0, b).$$

and

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0, & \Omega \\ u(x, 0) = 0, & 0 < x < a \\ u(x, b) = 0, & 0 < x < a \\ u(0, y) = 0, & 0 \le y \le b \\ u(a, y) = g(y), & 0 \le y \le b, \end{cases}$$

where $g \in C^1$.

Step 1 (Separation of variables): *Set $u(x, y) = v(x)w(y)$*

$$0 = \Delta u(x, y) = v''(x)w(y) + v(x)w''(y)$$

i.e.

$$\frac{v''(x)}{v(x)} = \lambda = -\frac{w''(y)}{w(y)}, \quad \lambda \in \mathbb{R},$$

and further

$$\begin{cases} v''(x) = \lambda v(x), & v(0) = 0 \\ -w''(y) = \lambda w(y), & w(0) = 0 = w(b). \end{cases}$$

Step 2 (Solving the separated equations): Case 1: $\lambda < 0$: $\lambda = -\mu^2,\ \mu > 0$

$$w''(y) = \mu^2 w(y),\ r^2 - \mu^2 = 0,\ r = \pm\mu,$$
$$w(y) = c_1 \sinh(\mu y) + c_2 \cosh(\mu y)$$

where $\sinh(x) = \frac{1}{2}(e^x - e^{-x}),\ \sinh(0) = 0,\ \cosh(x) = \frac{1}{2}(e^x + e^{-x})$ *and*

$$\begin{aligned} 0 &= w(0) = c_2 \\ 0 &= w(b) = c_1 \sinh(\mu b) \\ &\Rightarrow \quad c_1 = c_2 = 0. \end{aligned}$$

Case 2: $\lambda = 0$:

$$w''(y) = 0, \quad w(y) = c_1 y + c_2, \quad w(0) = 0 = w(b) \qquad \Rightarrow \quad w = 0$$

Case 3: $\lambda > 0$: $\lambda = \mu^2,\ \mu > 0$

$$\begin{cases} v''(x) = \mu^2 v(x) \\ -w''(y) = \mu^2 w(y) \end{cases} \quad \text{and thus} \quad \begin{cases} r_1^2 = \mu^2 \\ -r_2^2 = \mu^2 \end{cases} \quad \text{i.e.} \begin{cases} r_1 = \pm\mu \\ r_2 = \pm i\mu. \end{cases}$$

which gives

$$\begin{cases} v(x) = c_1 \sinh(\mu x) + c_2 \cosh(\mu x) \\ w(y) = d_1 \sin(\mu y) + d_2 \cos(\mu y). \end{cases}$$

From boundary conditions

$$v(0) = 0 \Rightarrow c_2 = 0$$
$$w(0) = 0 \Rightarrow d_2 = 0,$$

and from $w(b) = 0$ *we get that one of the two holds*

$$d_1 = 0 \Rightarrow w(y) = 0 \text{ discarded, or}$$
$$\sin(\mu b) = 0 \Rightarrow \mu = \frac{j\pi}{b}, \quad j = 1, 2, \ldots.$$

Thus

$$\begin{cases} v(x) = c_1 \sinh(\frac{j\pi x}{b}) \\ w(y) = d_1 \sin(\frac{j\pi y}{b}). \end{cases}$$

Thus

$$u_j(x, y) = v(x)w(x) = a_j \sinh(\frac{j\pi x}{b}) \sin(\frac{j\pi y}{b})$$

are nontrivial special solutions.

Step 3 (Solving the full problem): *The full solution is looked for as a series*

$$u(x,y) = \sum_{j=1}^{\infty} a_j \sinh(\frac{j\pi x}{b}) \sin(\frac{j\pi y}{b}) \tag{7.25}$$

The last boundary condition

$$g(y) = u(a,y) = \sum_{j=1}^{\infty} a_j \sinh(\frac{j\pi a}{b}) \sin(\frac{j\pi y}{b}).$$

Extend g as an odd function to the whole of $\mathbb{R}$. Then its Fourier series is a sin*-series*

$$g = \sum_{j=1}^{\infty} b_j \sin(\frac{j\pi y}{b}),$$

$$b_j = \frac{1}{b}\int_{-b}^{b} g(y) \sin\frac{j\pi y}{b}\,dy = \frac{2}{b}\int_0^b g(y)\sin\frac{j\pi y}{b}\,dy.$$

Comparing the coefficients, we get

$$a_j \sinh(\frac{j\pi a}{b}) = \frac{2}{b}\int_0^b g(y)\sin\frac{j\pi y}{b}\,dy \text{ i.e.}$$

$$a_j = \frac{2}{b\sinh(\frac{j\pi a}{b})}\int_0^b g(y)\sin\frac{j\pi y}{b}\,dy$$

and inserting this into (7.25) gives a representation formula for the solution. This is a formal solution at this point, as we didn't consider the convergence and the regularity of the limit.

Example 7.21. *Ω open and bounded, $\Omega_T = \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^n$.*

$$\begin{cases} u_t - \Delta u = 0, & \text{in } \Omega_T \\ u = 0, & \text{on } \partial\Omega \\ u(x,0) = g(x), & \text{on } \Omega. \end{cases}$$

Step 1 (Separation of variables):

$$\begin{aligned} u(x,t) &= v(t)w(x), \\ u_t(x,t) &= v'(t)w(x), \\ \Delta u(x,t) &= v(t)\Delta w(x), \end{aligned}$$

so that

$$\begin{aligned} 0 &= u_t(x,t) - \Delta u(x,t) \\ &= v'(t)w(x) - v(t)\Delta w(x). \end{aligned}$$

In other words

$$\frac{v'(t)}{v(t)} = -\lambda = \frac{\Delta w(x)}{w(x)}.$$

i.e.

$$\begin{cases} v'(t) = -\lambda v(t) \\ \Delta w(x) = -\lambda w(x). \end{cases}$$

Step 2 (Solving the separated equations): *Then*

$$0 = v'(t)e^{\lambda t} + \lambda v(t)e^{\lambda t} = (v(t)e^{\lambda t})'$$

so that

$$v(t)e^{\lambda t} = c \Rightarrow \quad v(t) = ce^{-\lambda t}, \quad c \in \mathbb{R}.$$

Recall

Definition 7.22. *If*

$$\begin{cases} -\Delta w = \lambda w & \Omega \\ w = 0 & \partial\Omega. \end{cases}$$

with $\lambda > 0$ has a nontrivial solution i.e. $w \not\equiv 0$, then λ is eigenvalue of Δ in Ω. The corresponding w is an eigenfunction.

Also remember that we showed in connection to the eigenvalue problem that for $\lambda \leq 0$ the above problem has no nontrivial solutions.

We have

$$u(x,t) = ce^{-\lambda t}w(x).$$

solves

$$\begin{cases} u_t - \Delta u = 0, & \Omega_T \\ u = 0, & \partial\Omega. \end{cases}$$

Step 3 (Solving the full problem): *Let*

$$\lambda_j,\ j = 1, 2, 3, \dots$$
$$w_j,\ j = 1, 2, 3, \dots$$

be eigenvalues and eigenfunctions. Then we try the linear combination

$$u(x,t) = \sum_{j=1}^{\infty} c_j e^{-\lambda_j t} w_j(x)$$

requiring that

$$u(x,0) = \sum_{j=1}^{\infty} c_j \underbrace{e^0}_{=1} w_j(x) = \sum_{j=1}^{\infty} c_j w_j(x) = g(x).$$

As partly stated before there is an infinite number of eigenvalues and eigenfunctions can be chosen to form an orthonormal basis in $L^2(\Omega)$. Similarly as in the theory Fourier series, if $g \in L^2(\Omega)$, and

$$c_j = \langle g, w_j \rangle := \int_\Omega g(y) w_j(y)\, dy,\ j = 1, 2, \dots$$

where $\langle .,. \rangle$ is the inner product in L^2, then the series

$$g = \sum_{j=1}^{\infty} c_j w_j = \sum_{j=1}^{\infty} \langle g, w_j \rangle w_j$$

converges in L^2. Thus we have the representation formula

$$\begin{aligned} u(x,t) &= \sum_{j=1}^{\infty} \langle g, w_j \rangle e^{-\lambda_j t} w_j(x) \\ &= \sum_{j=1}^{\infty} \Big(\int_{\Omega} g(y) w_j(y)\, dy \Big) e^{-\lambda_j t} w_j(x) \\ &= \int_{\Omega} \Big(\sum_{j=1}^{\infty} e^{-\lambda_j t} w_j(x) w_j(y) \Big) g(y)\, dy \\ &= \int_{\Omega} K(x,y,t) g(y)\, dy, \end{aligned}$$

where

$$K(x,y,t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} w_j(x) w_j(y),$$

is the heat kernel in Ω.

Remark 7.23. *Existence of a solution can be proved using separation of variables and eigenfunction expansions in bounded domains. Requires: finding enough eigenfunctions and eigenvalues and proving convergence and regularity.*

Example 7.24. *Separation of variables also sometimes works for nonlinear PDEs. Porous medium equation in $\mathbb{R}^n$, $n \geq 2$, $x \neq 0$*

$$u_t = \Delta(u^m), \quad m > 1.$$

Try $u(x,t) = w(x)v(t)$ so that

$$v'(t)w(x) = v^m(t)\Delta w^m(x).$$

Thus

$$\frac{v'(t)}{v^m(t)} = \lambda = \frac{\Delta w^m(x)}{w(x)}.$$

A solution for $v' = \lambda v^m$ is

$$v(t) = ((1-m)\lambda t + a)^{\frac{1}{1-m}},$$

for $a \in \mathbb{R}$ that we take to be positive, since

$$\begin{aligned} v'(t) &= \frac{1}{1-m}((1-m)\lambda t + a)^{\frac{m}{1-m}}(1-m)\lambda, \\ \lambda v^m(t) &= \lambda((1-m)\lambda t + a)^{\frac{m}{1-m}}. \end{aligned}$$

Then we solve

$$\Delta w^m(x) = \lambda w(x).$$

We try $w(x) = |x|^{\alpha}$, $f(x) = |x|^{\alpha m}$ *and first observe*

$$\begin{aligned} f(x) &= |x|^{\alpha m}, \\ Df(x) &= \alpha m\, |x|^{\alpha m-2}\, x \\ D^2 f(x) &= \alpha m(\alpha m - 2)\, |x|^{\alpha m-2}\, \frac{x}{|x|} \otimes \frac{x}{|x|} + \alpha m\, |x|^{\alpha m-2}\, I, \\ \Delta f(x) &= \alpha m\, |x|^{\alpha m-2}\, (\alpha m - 2 + n) \end{aligned}$$

where we recall the shorthand $\overline{a} \otimes \overline{b}$ *is the matrix with the entries* $a_i b_j$. *Thus*

$$\begin{aligned} 0 &= \Delta w^m(x) - \lambda w(x) \\ &= \alpha m(\alpha m - 2 + n)\, |x|^{\alpha m-2} - \lambda\, |x|^{\alpha} \\ &\overset{choose\ \alpha m-2=\alpha}{=} (\alpha m(\alpha m - 2 + n) - \lambda)\, |x|^{\alpha} \\ &\overset{\alpha m(\alpha m-2+n)-\lambda=0}{=} 0 \end{aligned}$$

i.e. $\alpha = 2/(m-1)$, $\lambda = \alpha m(\alpha m - 2 + n) > 0$. *Thus for every* $a > 0$

$$u(x,t) = w(x)v(t) = |x|^{\alpha}\, ((1-m)\lambda t + a)^{\frac{1}{1-m}}$$

is a solution. This represents a solution that blows up when $(1-m)\lambda t + a \to 0+$.

Example 7.25. *Separation of variables also sometimes works in terms of addition instead of multiplication. This is also nonlinear example.*

Hamilton-Jacobi:

$$\begin{cases} u_t + H(Du) = 0, & \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x), & x \in \mathbb{R}^n, \end{cases}$$

where $H : \mathbb{R}^n \to \mathbb{R}$ *is given. Try*

$$u(x,t) = w(x) + v(t).$$

Then

$$\begin{aligned} 0 &= u_t(x,t) + H(Du(x,t)) \\ &= v'(t) + H(Dw(x)). \end{aligned}$$

Now we get

$$H(Dw(x)) = \mu = -v'(t).$$

Thus

$$\begin{aligned} v(t) &= -\mu t + b \\ u(x,t) &= w(x) - \mu t + b. \end{aligned}$$

In particular, if we select $w(x) = x \cdot a$ *for which* $H(a) = \mu$*, then*

$$u(x,t) = a \cdot x - H(a)t + b$$

is a solution for the initial condition $g(x) = a \cdot x + b$*.*

Observe that in general the Hamilton-Jacobi equation is nonlinear and we cannot sum up the solutions.

7.3. **Fourier transform.** Fourier series associates a function and its Fourier coefficients. Similarly Fourier transform associates the functions to its Fourier transform. Now the functions do not need to be periodic. The use of Fourier transform in solving the PDEs is based on the fact that it changes derivarives to a multiplication.

This also works on $\mathbb{R}^n$, but we restrict ourselves first on $\mathbb{R}$.

We define a Fourier transform of $f \in L^1(\mathbb{R}),\ f : \mathbb{R} \to \mathbb{C}$ i.e. $\int_{\mathbb{R}} |f|\ dx < \infty$ as

$$F(f) = \hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi}\, dx. \tag{7.26}$$

Remark 7.26. *(i)* $e^{-2\pi i x \xi} = \cos(2\pi x \xi) - i \sin(2\pi x \xi)$*, (even part in real, and odd in imaginary).*

(ii) Theory generalizes to $\mathbb{R}^n$ *(then* $\mathbf{x} \cdot \xi = \sum_{i=1}^n x_i \xi_i$ *and* $e^{-2\pi i \mathbf{x} \cdot \xi}$*) as we will see later.*

It is common to start consideration of Fourier transform of smooth and rapidly decreasing functions, the so called Schwarz class. In particular, for $f \in L^1$ it maybe that $\hat{f} \notin L^1$ (we will see example later). Then writing down inverse Fourier transform later would require more care. For Schwarz class these problems do not occur. Many properties hold for L^1 nonetheless so we state properties for both the classes.

Definition 7.27. *A function* f *is in the Schwartz class* $S(\mathbb{R})$ *if*

(i) $f \in C^\infty(\mathbb{R})$

(ii)

$$\sup_{x \in \mathbb{R}} |x|^k \left| \frac{d^l f(x)}{dx^l} \right| < \infty, \quad \textit{for every} \quad k, l \in \mathbb{N}.$$

In other words, every derivative decays at least as fast as any power of $|x|$*.*

14.11.2019

Many properties also hold for L^1 functions instead of S, so we state properties whichever assumption is the most appropriate.

Lemma 7.28. *(i)* $\alpha, \beta \in \mathbb{C},\ f \in L^1 \Rightarrow \widehat{(\alpha f + \beta g)} = \alpha \hat{f} + \beta \hat{g},$.

(ii) $f \in S(\mathbb{R}^n) \Rightarrow \widehat{\left(\frac{d^k f}{dx^k}\right)}(\xi) = (2\pi i \xi)^k \hat{f}(\xi), k \in \mathbb{N}.$

(iii) $f \in L^1 \Rightarrow f\widehat{(x+h)} = \hat{f}(\xi) e^{2\pi i h \xi},$

Proof. (i) Integral is linear.

(ii) We check $k = 1$, other cases are similar:

$$\widehat{\left(\frac{df}{dx}\right)}(\xi) = \int_{\mathbb{R}} \left(\frac{df}{dx}\right) e^{-2\pi i x\xi}\, dx \overset{\text{integrate by parts}}{=} -\int_{\mathbb{R}} f(x)\frac{d}{dx}e^{-2\pi i x\xi}\, dx = 2\pi i\xi \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}\, dx = 2\pi i\xi \hat{f}(\xi).$$

(iii)

$$\widehat{f(x+h)} = \int_{\mathbb{R}} f(x+h)e^{-2\pi i x\xi}\, dx \overset{y=x+h,\, dy=dx}{=} \int_{\mathbb{R}} f(y)e^{-2\pi i(y-h)\xi}\, dy = \hat{f}(\xi)e^{2\pi i h\xi}.$$

□

Theorem 7.29. *If $f \in S(\mathbb{R})$, then*

(i) $\hat{f} \in S(\mathbb{R})$ (similar result does not hold in L^1),

(ii)

$$F^{-1}(f) := \int_{\mathbb{R}} f(\xi)e^{2\pi i x\xi}\, d\xi \in S(\mathbb{R})$$

whenever $f \in S(\mathbb{R})$, where F^{-1} is called the inverse Fourier transform.

Lemma 7.30. *If $f, g \in L^1(\mathbb{R})$, then*

$$\widehat{f * g} = \hat{f}\hat{g}$$

Proof. We take for granted that conditions to Fubini's theorem are ok. Then

$$\widehat{f*g} = \int_{\mathbb{R}}\int_{\mathbb{R}} f(y)g(x-y)\, dy\; e^{-2\pi i x\xi}\, dx \overset{\text{Fubini}}{=} \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} g(x-y)e^{-2\pi i x\xi}\, dx\, dy \overset{x-y=z,\, dx=dz}{=} \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} g(z)e^{-2\pi i(z+y)\xi}\, dz\, dy = \int_{\mathbb{R}} f(y)e^{-2\pi i y\xi}\, dy \int_{\mathbb{R}} g(z)e^{-2\pi i z\xi}\, dz = \hat{f}\hat{g}.$$

□

Theorem 7.31 (Fourier inversion)**.** *If $f \in S(\mathbb{R})$, then*

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i x\xi}\, d\xi,$$

or with the other notation $f(x) = F^{-1}(F(f)) = F^{-1}(\hat{f})$.

We can state the following L^1 version as well.

Theorem 7.32 (Fourier inversion for L^1)**.** *If $f \in L^1(\mathbb{R})$, $\hat{f} \in L^1$, then*

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi,$$

or with the other notation $f(x) = F^{-1}(F(f)) = F^{-1}(\hat{f})$.

Remark 7.33. *Above a function can be recovered from its Fourier transform. This corresponds to the Fourier series representation of a periodic function.*

Example 7.34 (Bessel potential)**.**

$$-u'' + u = f \quad \mathbb{R},$$

where $f \in S(\mathbb{R})$. Then

$$\widehat{u''}(\xi) = (2\pi i \xi)^2 \hat{u}(\xi) = -4\pi^2 \xi^2 \hat{u}(\xi).$$

Thus by the equation

$$(4\pi^2 \xi^2 + 1)\hat{u}(\xi) = \hat{f}(\xi)$$

and thus

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{4\pi^2 \xi^2 + 1}$$

i.e. by Lemma 7.30 $u = g * f$ where

$$\hat{g}(\xi) = \frac{1}{4\pi^2 \xi^2 + 1}.$$

We need to find the g i.e. inverse Fourier transform for $\hat{g}$. It holds that

$$\int_0^\infty e^{-ta} \, dt = \frac{1}{a}$$

so that

$$\int_0^\infty e^{-t(4\pi^2\xi^2+1)} \, dt = \frac{1}{4\pi^2 \xi^2 + 1}.$$

Thus

$$\begin{aligned} g(x) &= \int_{\mathbb{R}} \hat{g}(\xi) e^{2\pi i x \xi} \, d\xi \\ &= \int_{\mathbb{R}} \frac{1}{4\pi^2 \xi^2 + 1} e^{2\pi i x \xi} \, d\xi \\ &= \int_{\mathbb{R}} \int_0^\infty e^{-t(4\pi^2\xi^2+1)} e^{2\pi i x \xi} \, dt \, d\xi \\ &= \int_0^\infty e^{-t} \int_{\mathbb{R}} e^{2\pi i x \xi - t4\pi^2 \xi^2} \, d\xi \, dt. \end{aligned} \tag{7.27}$$

Then we perform a change of variables $z = \sqrt{b}\xi - \frac{a}{2\sqrt{b}}i$ *so that*

$$\int_{-\infty}^{\infty} e^{ia\xi - b\xi^2}\, d\xi = \frac{e^{-\frac{a^2}{4b}}}{\sqrt{b}} \int_{\Gamma} e^{-z^2}\, dz,$$

where $\Gamma = \{Im(z) = -\frac{a}{2\sqrt{b}}\}$.

Further from the complex analysis it follows

$$\int_{\Gamma} e^{-z^2}\, dz = \int_{-\infty}^{\infty} e^{-\xi^2}\, d\xi \overset{Lemma\ 5.4}{=} \sqrt{\pi}.$$

Thus

$$\int_{-\infty}^{\infty} e^{ia\xi - b\xi^2}\, d\xi = e^{-\frac{a^2}{4b}} \sqrt{\frac{\pi}{b}} \tag{7.28}$$

It follows that $(a = 2\pi x,\ \ b = t4\pi^2)$

$$\int_{\mathbb{R}} e^{2\pi i x\xi - t4\pi^2\xi^2}\, d\xi = e^{-\frac{(2\pi x)^2}{4(t4\pi^2)}} \sqrt{\frac{\pi}{t4\pi^2}} = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.$$

Recalling (7.27) we get

$$\begin{aligned} g(x) &= \int_0^{\infty} e^{-t} \int_{\mathbb{R}} e^{2\pi i x\xi - t4\pi^2\xi^2}\, d\xi\, dt \\ &= \int_0^{\infty} e^{-t} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}\, dt. \end{aligned}$$

This is sometimes called the Bessel potential. Then

$$u(x) = (g * f)(x) = \frac{1}{\sqrt{4\pi}} \int_0^{\infty} \int_{\mathbb{R}} \frac{1}{\sqrt{t}} e^{-t - \frac{|x-y|^2}{4t}} f(y)\, dy\, dt.$$

is a solution to the equation.

Example 7.35.

$$\begin{cases} u_{tt} - u_{xx} = 0 & \mathbb{R} \times (0, \infty) \\ u = g,\ \ u_t = h & \mathbb{R} \times \{t = 0\}. \end{cases}$$

Fourier transform wrt x *gives*

$$\begin{cases} \hat{u}_{tt}(\xi, t) + 4\pi^2\xi^2 \hat{u}(\xi, t) = 0, \\ \hat{u} = \hat{g},\ \ \hat{u}_t = \hat{h}. \end{cases}$$

Solving the ODE we first get the characteristic equation

$$r^2 = -4\pi^2\xi^2 \Rightarrow r = \pm i2\pi\xi,$$

and we get

$$\hat{u}(\xi, t) = A(\xi)\cos(2\pi\xi t) + B(\xi)\sin(2\pi\xi t).$$

By the initial conditions

$$\hat{u}(\xi,0) = A(\xi) = \hat{g}(\xi),$$
$$\hat{u}_t(\xi,0) = 0 + 2\pi\xi\cos(0)B(\xi) = \hat{h}(\xi).$$

Thus for $\xi \neq 0$

$$\hat{u}(\xi,t) = \hat{g}(\xi)\cos(2\pi\xi t) + \frac{\hat{h}(\xi)}{2\pi\xi}\sin(2\pi\xi t).$$

Then let $H(x) = \int_{-\infty}^{x} h(y)\,dy$ *(assume that exists) so that* $H' = h$ *and further by Lemma 7.28*

$$2\pi i\xi\hat{H}(\xi) = \hat{h}(\xi).$$

From this we get

$$\begin{aligned}\hat{u}(\xi,t) &= \hat{g}(\xi)\cos(2\pi\xi t) + \frac{\hat{h}(\xi)}{2\pi\xi}\sin(2\pi\xi t)\\ &= \hat{g}(\xi)\frac{1}{2}(e^{2\pi i\xi t} + e^{-2\pi i\xi t}) + \underbrace{\frac{\hat{h}(\xi)}{2\pi\xi}\frac{1}{2i}}_{=\frac{1}{2}\hat{H}}(e^{2\pi i\xi t} - e^{-2\pi i\xi t})\\ &\overset{L7.28}{=} \frac{1}{2}(\widehat{g(x+t)}(\xi) + \widehat{g(x-t)}(\xi) + \frac{1}{2}\Big(\widehat{H(x+t)}(\xi) - \widehat{H(x-t)}(\xi)\Big).\end{aligned}$$

From this it follows

$$\begin{aligned}u(x,t) &= \frac{1}{2}(g(x+t) + g(x-t) + \frac{1}{2}\Big(H(x+t) - H(x-t)\Big)\\ &= \frac{1}{2}(g(x+t) + g(x-t) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy,\end{aligned}$$

i.e. D'Alembert formula. Naturally we need suitable assumptions so that the above computations are valid.

7.3.1. *Fourier transform in* $\mathbb{R}^n$. The Fourier transform extends to $\mathbb{R}^n$ in a straightforward manner: The definitions of $L^1(\mathbb{R}^n)$ and $S(\mathbb{R}^n)$ are analogous as well as

$$F(f) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx. \tag{7.29}$$

and inverse

$$F^{-1}(f)(\xi) := \int_{\mathbb{R}^n} f(\xi)e^{2\pi i x\cdot\xi}\,d\xi.$$

Lemma 7.36. *Suppose that* $f, g \in S(\mathbb{R}^n)$. *Then*

(i) $\widehat{\left(\frac{df}{dx_j}\right)}(\xi) = 2\pi i\xi_j\hat{f}(\xi)$.

(ii) $\widehat{f * g} = \hat{g}\hat{f}$

Proofs are similar to the one dimensional case.

Example 7.37.

$$\begin{cases} u_t - \Delta u = 0 & \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x) & \mathbb{R}^n \times \{t=0\}. \end{cases}$$

We assume that g is suitable for the following computation. Now we Fourier transform only with respect to x:

$$\begin{cases} \hat{u}_t(\xi,t) + \sum_{j=1}^n 4\pi^2 \xi_j^2 \hat{u}(\xi,t) = \hat{u}_t(\xi,t) + 4\pi^2 |\xi|^2 \hat{u}(\xi,t) = 0 \\ \hat{u}(\xi,0) = \hat{g}(\xi) \end{cases}.$$

We solve the ordinary differential equation $\hat{u}_t + 4\pi^2 |\xi|^2 \hat{u} = 0$ as usual to get

$$\hat{u}(\xi,t) = Ce^{-4\pi^2 t|\xi|^2}.$$

By

$$\hat{u}(\xi,0) = C = \hat{g}(\xi).$$

Thus

$$\hat{u}(\xi,t) = \hat{g}(\xi)e^{-4\pi^2 t|\xi|^2}$$

and

$$\begin{aligned} u &= F^{-1}(\hat{g}(\xi)e^{-4\pi^2 t|\xi|^2}) \\ &= F^{-1}(\widehat{g * \Phi}) \\ &= g * \Phi, \end{aligned}$$

where Φ is such that $\hat{\Phi}(\xi) = e^{-4\pi^2 t|\xi|^2}$. Now

$$\begin{aligned} \Phi(x) &= F^{-1}(e^{-4\pi^2 t|\xi|^2})(x) \\ &= \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi - 4\pi^2 t|\xi|^2}\, d\xi \\ &= \int_{\mathbb{R}^n} e^{\sum_{j=1}^n 2\pi i x_j \xi_j - 4\pi^2 t \xi_j^2}\, d\xi \\ &= \Pi_{j=1}^n \int_{-\infty}^{\infty} e^{2\pi i x_j \xi_j - 4\pi^2 t \xi_j^2}\, d\xi_j. \end{aligned}$$

Selecting $(a = 2\pi x_j,\ b = 4\pi^2 t)$ in (7.28), we get

$$\begin{aligned} \int_{-\infty}^{\infty} e^{ia\xi_j - b\xi_j^2}\, d\xi_j &= e^{-\frac{a^2}{4b}} \sqrt{\frac{\pi}{b}} \\ &= e^{-\frac{(2\pi x_j)^2}{4(4\pi^2 t)}} \sqrt{\frac{\pi}{4\pi^2 t}} \\ &= e^{-\frac{x_j^2}{4t}} \frac{1}{\sqrt{4\pi t}}. \end{aligned}$$

Combining

$$\begin{aligned}\Phi(x) &= \Pi_{j=1}^{n} \int_{-\infty}^{\infty} e^{2\pi i x_j \cdot \xi_j - 4\pi^2 t \xi_j^2}\, d\xi_j \\ &= \Pi_{j=1}^{n} e^{-\frac{x_j^2}{4t}} \frac{1}{\sqrt{4\pi t}} \\ &= e^{-\frac{|x|^2}{4t}} \frac{1}{(4\pi t)^{n/2}}.\end{aligned}$$

Thus the full solution

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy,$$

that we encountered already before.

7.3.2. *About the definition and extensions.* After seeing applications to PDEs we go back to the definition in $\mathbb{R}$.

Example 7.38 (Warning)**.** *The Fourier transform is well defined for $f \in L^1(\mathbb{R})$ because*

$$\left| f(x) e^{-2\pi i x \xi} \right| = |f(x)|$$

which is integrable. However, nothing guarantees that $\hat{f}(\xi)$ would be in $L^1(\mathbb{R})$. Indeed let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \chi_{(-1/2,1/2)}(x)$, which is in $L^1(\mathbb{R})$. Then for $\xi \neq 0$,

$$\begin{aligned}\hat{f}(\xi) &= \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi}\, dx \\ &= \int_{-1/2}^{1/2} e^{-2\pi i x \xi}\, dx \\ &= \int_{-1/2}^{1/2} \cos(2\pi x \xi)\, dx - i \underbrace{\int_{-1/2}^{1/2} \sin(2\pi x \xi)\, dx}_{=0} \\ &= \Big/_{-1/2}^{1/2} \frac{\sin(2\pi x \xi)}{2\pi \xi} \\ &= \frac{2\sin(\pi\xi)}{2\pi\xi} = \frac{\sin(\pi\xi)}{\pi\xi},\end{aligned}$$

but $\frac{\sin(\pi\xi)}{\pi\xi}$ is not integrable (the integral of the positive part $= \infty$ and similarly for the integral over the negative part over any interval (a, ∞)).

The problem described in the example above does not appear for the functions that are smooth and decay rapidly at the infinity, the Schwartz class that we defined before.

Next we state Plancherel's theorem. The theorem plays a central role, when extending the definition of the Fourier transform to the L^2-functions.

Theorem 7.39 (Plancherel)**.** *If $f \in S(\mathbb{R})$, then*

$$||f||_2 = ||\hat{f}||_2 = \left|\left|F^{-1}f\right|\right|_2. \tag{7.30}$$

It also holds that if $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $\hat{f} = F(f) \in L^2(\mathbb{R}^n)$ and $F^{-1}(f) \in L^2(\mathbb{R}^n)$ and the same norm equalities hold.

The last statement above after 'It also holds...' would also allow us to formulate the theory without Schwarz class by approximating $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ below.

Theorem 7.40. *Let $f \in L^2(\mathbb{R})$, and $\phi_j \in S(\mathbb{R}),\ j = 1, 2, \dots$ such that*

$$\lim_{j\to\infty} ||\phi_j - f||_2 = 0,$$

which exists. Then there exists a limit which we denote by $\hat{f}$ such that

$$\lim_{j\to\infty} ||\hat{\phi}_j - \hat{f}||_2 = 0.$$

The function $\hat{f}$ is called a Fourier transform of $f \in L^2(\mathbb{R})$. Moreover, $\hat{f}$ does not depend on the choice of an approximating sequence.

Proof. First of all, there exists a sequence $\phi_j \in S(\mathbb{R}),\ j = 1, 2, \dots$ such that

$$\lim_{j\to\infty} ||\phi_j - f||_2 = 0$$

because $S(\mathbb{R})$ is dense in $L^2(\mathbb{R})$: We have already seen that $C_0(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. On the other hand, if $f \in C_0(\mathbb{R})$ then $C_0^\infty(\mathbb{R}) \ni f * \phi_\varepsilon \to f$ in $L^2(\mathbb{R})$, where ϕ_ε is a standard mollifier, and we see that $C_0^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, which is contained in $S(\mathbb{R})$.

Then by Plancherel's theorem

$$||\hat{\phi}_j - \hat{\phi}_k||_2 = ||\phi_j - \phi_k||_2 \to 0$$

as $j, k \to \infty$ and thus $\hat{\phi}_j,\ j = 1, 2, \dots$ is a Cauchy sequence. Since $L^2(\mathbb{R})$ is complete, $\hat{\phi}_j$ converges to a limit, which we denote by $\hat{f}$.

Next we show that the limit is independent of the approximating sequence. Let φ_j be another sequence such that

$$\varphi_j \to f \quad \text{in} \quad L^2(\mathbb{R})$$

and let $g \in L^2(\mathbb{R})$ be the limit

$$\hat{\varphi}_j \to g \quad \text{in} \quad L^2(\mathbb{R}).$$

Then

$$0 \overset{\phi_j,\varphi_j \to f}{=} \lim_{j\to\infty} ||\varphi_j - \phi_j||_2 \overset{\text{Plancherel}}{=} \lim_{j\to\infty} ||\hat{\varphi}_j - \hat{\phi}_j||_2 = ||g - \hat{f}||_2. \qquad \square$$

Similarly we obtain a unique inverse Fourier transform of any L^2-function.

We state separately a result from the previous proof.

Corollary 7.41 (Plancerel in L^2)**.** *If $f \in L^2(\mathbb{R})$, then*

$$||f||_2 = ||\hat{f}||_2.$$

Remark 7.42.
- *Now, we could extend the basic properties of Lemma 7.28 to L^2 using approximations in the proofs.*
- *There are several versions of the Fourier-transform: for example*

$$F(u) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx,$$
$$F^{-1}(u) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} u(\xi)\, d\xi,$$

  in the Evans book. In each of the version, you will be having the 2π-popping up somewhere.
- *There are plenty of tables of Fourier transforms.*
- *There are other integral transforms like Laplace transform, $s \geq 0$*

$$u^{\#}(s) = \int_0^\infty e^{-st} u(t)\, dt$$

  that are useful in solving PDEs. This one for example for the heat equation taking Laplace transform wrt t that runs from $(0, \infty)$.

8. Matlab

8.1. Getting started.

- Readily installed on the university computers
- Once the command prompt is started, set 'Current directory' from the pull down menu, for example `U:\MATS230`. You can create a new directory also in Matlab writing `mkdir('U:\MATS230')`.
- Instead of using pull down menu, you can also write `cd U:\MATS230` to change the current directory.
- If you save data on Matlab, they are now saved to this directory, or if you write your own code put the file in this directory, so that Matlab can find it.
- Command `diary` will save command window input/output to a file.

8.2. Help.

- `doc`: open interactive manual
- `lookfor`: search for keyword in all help entries
- `help`: show function's help entry,

```
>> lookfor identity
EYE Identity matrix.
SPEYE  Sparse identity matrix.
>> help eye
 EYE Identity matrix.
     EYE(N) is the N-by-N identity matrix.
...
>> a=eye(2)
a =
     1     0
     0     1
```

8.3. Default variables.

Reserved variables:

`ans`	Answer of the most recent unassigned calculation.
`pi`	Value of π.
`i` or `j`	Imaginary unit.
`inf`	Positive infinity.
`nan`	Not-a-number.

8.4. Variables.

- `=` assigns to the variable named on the left the value on the right as we saw above

- Basic data type is matrix: scalar is 1×1 matrix, vectors $N \times 1$ (column) or $1 \times N$ (row)
- ; at the end of the line means that the result will not be displayed on the command prompt (, or <Enter> alone that it is displayed). **Remember to put ; when assigning for example big matrices!!!**
- <Enter> executes the command written on the line
- All the usual operations +, -, *, /, \, ^ are **matrix operations**!!!
- **Componentwise** operations by adding a dot .*, ./, .\, .^.
- ' on Hermitean transpose ja .' transpose (same for real matrices).
- . desimal point
- % comment (you may write notes in the code)
- [ ] creating matrix, collecting block matrix, assigning multible outputs on variables.
- whos Display all defined variables.
- clear Clear workspace variables.

```
>> a=[1 3; 2,4]
a =
     1     3
     2     4
>> x = [5;6]
x =
     5
     6
>> x2=a*x
x2 =
    23
    34
>> x2*x
??? Error using ==> mtimes
Inner matrix dimensions must agree.
>> x2+x
ans =
    28
    40
>> pi*a
ans =
    3.1416    9.4248
    6.2832   12.5664
```

- Plenty of ready functions : For example, sin, cos, tan, asin, acos, atan, sqrt, exp, log, abs, mod.
- See also: help elfun.

```
>> y=sin(x); % Lasketaan sini x:n alkioista
>> y'
ans =
   -0.9589   -0.2794
>> x2.*x
ans =
   115
   204
>> x2'*x
ans =
   319
>> a*[1+3i; 2.5]
ans =
    8.5000 + 3.0000i
   12.0000 + 6.0000i
```

8.5. **Indexing.**

- Matrix indexing: you can pick elements using brackets. **Indexing begins with 1 in Matlab, not with zero.**
- : alone creates a vector with consecutive integers or any specified interval. In indexing means all the entries in a row or column.

```
>> a=[1 4 7; 2 5 9; 3 6 9]
a =
     1     4     7
     2     5     8
     3     6     9
>> a(1,2)
ans =
     4
>> a(1,:)
ans =
     1     4    7
>> a(:,2:end)
ans =
     4     7
     5     8
     6     9
>> b = 1:3
b =
     1     2     3
>> a(b, [1 3])
ans =
```

```
     1     7
     2     8
     3     9
>> a(1:2,1:2) = eye(2)
a =
     1     0     7
     0     1     8
     3     6     9
```

- **Summary:**

`a=[1;2;3]`	Set the variable `a` to a column vector $(1, 2, 3)$.
`[1,2,3;4,5,6]`	Matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.
`[X Y]`	Block matrix $\begin{bmatrix} X & Y \end{bmatrix}$.
`2:5`	A vector $(2, 3, 4, 5)$.
`1:3:10`	A vector $(1, 4, 7, 10)$.
`a(2)`	2nd element of a vector.
`a(1,2)`	(1,2):th element of a matrix.
`a(1,:)`	first row of a matrix.
`a(:,3)=b`	Set the third column of matrix `a` to the value `b`.
`a(3:2:end,:)`	Matrix formed by every second rows from the third to the last row of `a`.

8.6. **Elementary matrix operations.** Some commands for creating matrices.

`linspace`	Linearly spaced vector.
`eye`	Identity matrix.
`diag`	Diagonal matrix or diagonal of a matrix.
`rand`	Random matrix with elements uniformly distributed in $(0, 1)$.
`zeros`	Matrix of zeros.
`ones`	Matrix of ones.
`length`	Length of a vector.
`size`	Size of a matrix.
`repmat`	Replicate matrix.
`find`	Find nonzero elements.

See also: `help elmat`.

```
>> a=rand(3)
a =
    0.9501    0.4860    0.4565
    0.2311    0.8913    0.0185
```

```
    0.6068    0.7621    0.8214
>> b=ones(3,1)
b =
     1
     1
     1
>> c=zeros(1,3)
c =
     0     0     0
>> d = [a b;c 42]
d =
    0.9501    0.4860    0.4565    1.0000
    0.2311    0.8913    0.0185    1.0000
    0.6068    0.7621    0.8214    1.0000
         0         0         0   42.0000
>> diag(d)
ans =
    0.9501
    0.8913
    0.8214
   42.0000
>> e = diag([pi exp(1)])
e =
    3.1416         0
         0    2.7183
```

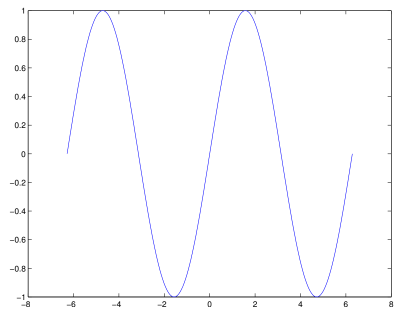
8.7. **Graphics.**

- One of MATLAB's strengths is visualization.
- Also here you create matrices or vectors and then just plot the values (examples below)
- Several ways of creating vectors that can be used in plots

```
>> x=0:0.1:0.5

x =

0   0.1000   0.2000   0.3000   0.4000   0.5000

>> x=linspace(0,0.5,5)

x =

0    0.1250    0.2500    0.3750    0.5000
```

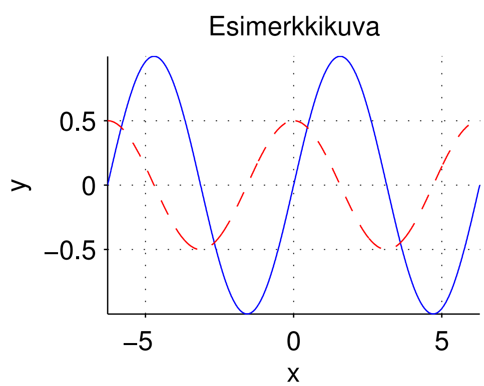
- Lets plot...

```
>> x = linspace(-2*pi,2*pi,1000);
>> y = sin(x);
>> plot(x,y)
```



- Lets add another curve and customise settings:

```
>> hold on
>> plot(x, 0.5*cos(x), 'r--')
>> axis tight, grid on, box off
>> xlabel 'x'; ylabel 'y'; title 'Esimerkkikuva'
>> set(gca,'tickdir','out');
>> print -depsc example.eps
```



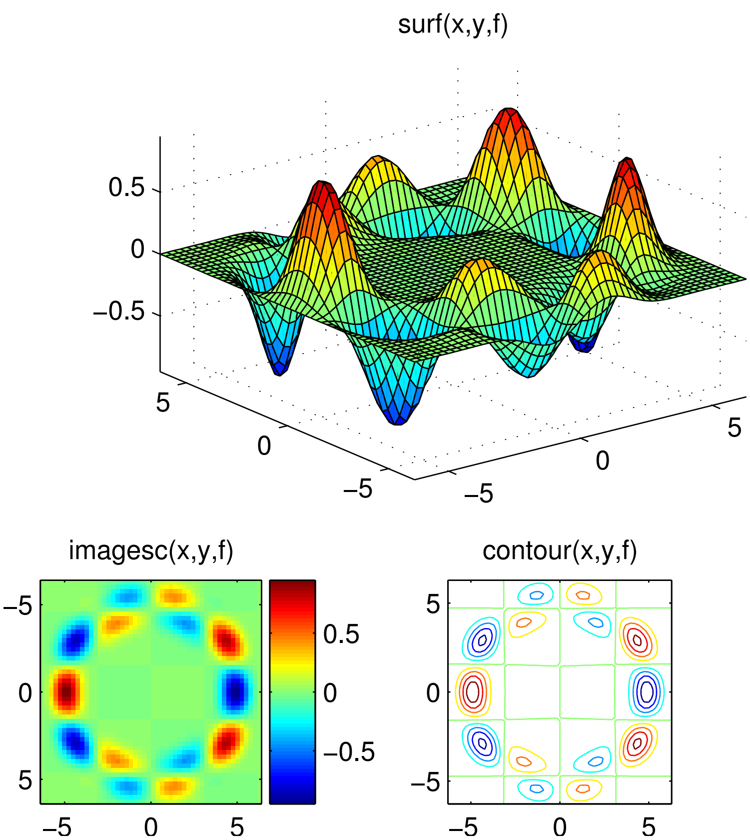
- Vizualization of 2D functions. First we need `meshgrid`:

```
x =  0     1     2     3
y =  8     9    10

>> [X,Y]=meshgrid(x,y)
X =
     0     1     2     3
     0     1     2     3
     0     1     2     3
Y =
     8     8     8     8
```

```
     9     9     9     9
    10    10    10    10
```

- Continue visualization...

```
>> x = linspace(-2*pi,2*pi,100); y=x;
>> [xx,yy] = meshgrid(x,y);
>> f =  sin(xx).*cos(yy) ...
.*exp(-(sqrt(xx.^2+yy.^2)-5).^2);
>> surf(x,y,f);
>> imagesc(x,y,f); colorbar
>> contour(x,y,f)
```



8.8. **M-files and own functions.**

- You can create "M-files" for example `mfile.m`, that includes a list of commands

```
N = 3;
a = ones(N,1);
x = [a, rand(N,N)]
```

- This corresponds to the typing the commands on the command prompt.
- 'Another type of 'M-file" is a functions meaning that you can call it with the parameters and it returns values.
- You execute the file calling it by name without .m so in this case *mfile*
- Function is saved in a file with the **same name**. For example function `foo`, we create `foo.m`.

```
>> edit foo
(write function on an editor and save it)
>> z = foo(15,4)
z =
   15.5242
```

- `foo.m` looks as:

```
function d = foo(x, y)
d = sqrt(x^2+y^2);
```

- Function needs to follow this syntax: function name on the first line and file name are the same.

8.9. Loops and logical expressions.

- MATLAB like other programming languages has loops and logical expressions
  - `if`: expression inside is executed if the condition is true: Condition is "true", if it is not zero.
  - `for`-loop goes through the values of a given vector.
  - `while`-loop goes on as long as the condition is true
- In logical conditions you may use for example `<`, `<=`, `==`, `~=`, `&`, `|` (`help relop`).

```
N = 6;
a = eye(N);
for k=1:N
  j = 1;
  while j<k
    if mod(k,j) == 0
      a(k,j) = 1;
    end
    j = j+1;
  end
end
```

- Logical operations also work for matrices, and can be used in indexing

```
>> a=1:5, b=rand(1,5)*5

a =

     1     2     3     4     5


b =

    4.0736    4.5290    0.6349    4.5669    3.1618

>> a<b

ans =



   1   1   0   1   0

>> find(a<b)

ans =

     1     2     4

>> a(a<b)=0

a =

     0     0     3     0     5

>> a=1:5

a =

     1     2     3     4     5

>> a(find(a<b))=0

a =

     0     0     3     0     5
```

21.11.2019

8.10. Programming strategy.

- For-loops are slow in Matlab! If possible replace with built-in matrix operations!
- Not like this:

```
n=1100;
x=linspace(0,2*pi,n);
y=linspace(pi,3*pi,n);

for i=1:n,
    for j=1:n,
        X(i,j)=x(i);
        Y(i,j)=y(j);
        Z(i,j)=sin(x(i))*cos(y(j));
    end
end
mesh(X,Y,Z)
```

- But like this:

```
n=1100;
x=linspace(0,2*pi,n);
y=linspace(pi,3*pi,n);
[X,Y]=meshgrid(x,y);
Z=sin(X).*cos(Y);
mesh(X,Y,Z)
```

8.11. Saving and loading data.

- Variables can be saved by a command `save`, and loaded by `load`

```
>> a=1:5;
>> save foo
>> clear
>> a
??? Undefined function or variable 'a'.

>> load foo
>> a
a =
     1     2     3     4     5
```

- You can also load data produced by other programs. For example in Excel you can save data in CSV-form (Comma Separated Values). Read data in MATLAB using `csvread` or `dlmread`.

9. Numerics

9.1. **Laplace equation.** Numerics has tight connections to the theory. For example many numerical methods for producing solutions are related to the existence methods. Convergence proofs for numerical approximations require regularity estimates etc. These are however mostly beyond our scope and we simply aim at producing (formally) approximating solutions using difference methods.

Let $u : \mathbb{R} \to \mathbb{R}$ be a smooth function. By Taylor's theorem

$$\begin{aligned} u(x+h) &= u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + \frac{1}{6}u'''(x)h^3 + O(h^4) \\ u(x-h) &= u(x) - u'(x)h + \frac{1}{2}u''(x)h^2 - \frac{1}{6}u'''(x)h^3 + O(h^4), \end{aligned}$$

the big oh notation here means $\left|O(h^4)\right| \le Ch^4$ as $h \to 0$. Subtracting, we get

$$u'(x) = \frac{1}{2h}(u(x+h) - u(x-h)) + O(h^2)$$

and summing up

$$u''(x) = \frac{1}{h^2}(u(x-h) - 2u(x) + u(x+h)) + O(h^2).$$

Next we divide the interval $[0,1]$ into

$$x_j = jh, \quad h = \frac{1}{m+1}, \quad j = 0, 1, \ldots, m+1,$$

and denote by u_j the appoximation for $u(x_j)$ that we are searching for.

Consider Poisson equation

$$\Delta u = f$$

in 1D. Thus dropping the error terms

$$\Delta u(x) = u''(x) \approx \frac{1}{h^2}(u(x+h) - 2u(x) + u(x-h)),$$

and

$$\begin{cases} \Delta u(x) = f(x), x \in (0,1) \\ u(0) = a \\ u(1) = b \end{cases}$$

can be discretized as

$$\frac{1}{h^2}\begin{pmatrix} h^2 & 0 & \dots & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \vdots \\ 0 & \dots & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & \dots & 0 & h^2 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_m \\ u_{m+1} \end{pmatrix} = \begin{pmatrix} a \\ f_1 \\ \vdots \\ f_m \\ b \end{pmatrix},$$

where we have encoded the boundary values on the first and the last row, and $f_j = f(x_j)$. If $u(0) = 0 = u(1)$ we may drop the first and the last row and column by observing

$$\begin{aligned} u''(1-h) &= \frac{1}{h^2}(u(1-2h) - 2u(1-h) + u(1)) \\ &= \frac{1}{h^2}(u(1-2h) - 2u(1-h)) \\ u''(h) &= \frac{1}{h^2}(u(0) - 2u(h) + u(2h)) \\ &= \frac{1}{h^2}(-2u(1-2h) + u(2h)). \end{aligned}$$

Thus the problem can be discretized as $\Delta_h \overline{u} = \overline{f}$, missä $\Delta_h \in \mathbb{R}^{m\times m}, \overline{u}, \overline{f} \in \mathbb{R}^m$

$$\Delta_h = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & \dots & 0 & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \vdots \\ 0 & \dots & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & 0 & 1 & -2 \end{pmatrix}, \quad \overline{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}, \text{ and } \quad \overline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}.$$

We want to solve $\overline{u}$ and know $\overline{f}$ so

$$\overline{u} = \Delta_h^{-1}\overline{f},$$

where Δ_h^{-1} is the inverse of the matrix Δ_h. To highlight that we are using Dirichlet condition on both ends, we denote $\Delta_h =: \Delta_{h,D-D}$. If we have nonzero Dirichlet condition $u(0) = a$, we get

$$u''(h) = \frac{1}{h^2}(a - 2u(h) + u(2h)) = \frac{1}{h^2}(-2u(h) + u(2h)) + \frac{a}{h^2}$$

and moving a/h^2 in $\overline{f}$, we can still write the equation in the form $\Delta_h u = (f_1 - a/h^2, f_2, \dots)^T$.

To obtain the Neumann boundary condition at 1, observe

$$u(1-h) = u(1) - u'(1)h + \frac{1}{2}u''(1)h^2 - \frac{1}{3!}u'''(1)h^3 + O(h^4)$$

$$-4u(1-h) = -4u(1) + 4u'(1)h - 4\frac{1}{2}u''(1)h^2 + 4\frac{1}{3!}u'''(1)h^3 + O(h^4)$$
$$u(1-2h) = u(1) - u'(1)2h + \frac{1}{2}u''(1)(2h)^2 - \frac{1}{3!}u'''(1)(2h)^3 + O(h^4)$$
$$3u(1) = 3u(1).$$

Summing up the last three equations we get

$$u(1-2h) - 4u(1-h) + 3u(1) = 0 + 2hu'(1) + 0 + O(h^3)$$

i.e.

$$\frac{1}{2h}(u(1-2h) - 4u(1-h) + 3u(1)) + O(h^2) = u'(1). \tag{9.31}$$

Thus

$$\begin{cases} \Delta u(x) = f(x), x \in (0,1) \\ u(0) = a \\ u_x(1) = b \end{cases}$$

corresponds to

$$\frac{1}{h^2}\begin{pmatrix} h^2 & 0 & \dots & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \vdots \\ 0 & \dots & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & \frac{h}{2} & -\frac{h}{2}4 & \frac{h}{2}3 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_m \\ u_{m+1} \end{pmatrix} = \begin{pmatrix} a \\ f_1 \\ \vdots \\ f_m \\ b \end{pmatrix}.$$

If $u'(1) = 0$ then by (9.31)

$$u(1) = \frac{1}{3}(4u(1-h) - u(1-2h)).$$

Using this in the second order expansion

$$\begin{aligned} u''(1-h) &\approx \frac{1}{h^2}(u(1-2h) - 2u(1-h) + u(1)) \\ &= \frac{1}{h^2}(u(1-2h) - 2u(1-h) + \frac{1}{3}(4u(1-h) - u(1-2h))) \\ &= \frac{1}{h^2}(\frac{2}{3}u(1-2h) - \frac{2}{3}u(1-h)). \end{aligned} \tag{9.32}$$

With $u(0) = u'(1) = 0$ using (9.32) we get the discretization of the Laplacian with Dirichlet-Neumann boundary conditions

$$\Delta_h = \Delta_{D-N,h} = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \vdots \\ 0 & \dots & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & 0 & \frac{2}{3} & -\frac{2}{3} \end{pmatrix},$$

$\Delta_h \in \mathbb{R}^{m\times m}$, and the equation again takes the form

$$\Delta_h \overline{u} = \overline{f}, \quad \overline{u} = \Delta_h^{-1}\overline{f},$$

$\overline{u}, \overline{f} \in \mathbb{R}^m$. If we have nonzero boundary condition, we may again move the excess terms into $\overline{f}$ and write the equation in the form $\Delta_h u = \ldots$.

9.1.1. *2 dimensional case.* Consider $\Omega = (0,1)\times(0,1)$. Set

$$m_1, m_2 \in \mathbb{N}, \quad h_1 = \frac{1}{m_1+1},\ h_2 = \frac{1}{m_2+1}.$$

Now it is convenient to write the approximative values $u(ih_1, jh_2) \approx u_{i,j}$ as

$$U := U_h = \begin{pmatrix} u_{1,1} & \ldots & u_{1,m_2} \\ \vdots & & \vdots \\ u_{m_1,1} & \ldots & u_{m_1,m_2} \end{pmatrix}, \quad i = 1,\ldots,m_1, \quad j = 1,\ldots,m_2.$$

Approximating again

$$u_{x_1x_1} \approx \frac{1}{h_1^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$
$$u_{x_2x_2} \approx \frac{1}{h_2^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}).$$

In matrix form

$$U_{x_1x_1} \approx \Delta_{h_1,D-D}U$$
$$U_{x_2x_2} \approx U\Delta_{h_2,D-D}^T.$$

Thus we get a discretization for the 2-dimensional Laplacian

$$\Delta u \approx \Delta_{h_1h_2}U := \Delta_{h_1,D-D}U + U\Delta_{h_2,D-D}^T.$$

We want to solve for U and therefore wish to write $\Delta_{h_1,D-D}U + U\Delta_{h_2,D-D}^T$ in the form matrix*vector since this is then easy to solve by inverting the matrix. To this end, let us then denote the vector

$$u_h = T(U) = (u_{1,1},\ldots,u_{m_1,1},u_{1,2},\ldots,u_{m_1,m_2})^T \in \mathbb{R}^{m_1m_2}$$

i.e. we write the matrix U as a vector by putting the colums at the top of each other (In MATLAB simply $U(:)$).

Lemma 9.1. *Let* $A \in \mathbb{R}^{m_1\times m_1}, U \in \mathbb{R}^{m_1\times m_2}, B \in \mathbb{R}^{m_2\times m_2}$

$$T(AUB^T) = (B\otimes A)T(U)$$

where

$$B\otimes A = \begin{pmatrix} b_{11}A & b_{12}A & \ldots & b_{1m_2}A \\ b_{21}A & b_{22}A & \ldots & b_{2m_2}A \\ \vdots & \vdots & & \vdots \\ b_{m_21}A & b_{m_22}A & \ldots & b_{m_2m_2}A \end{pmatrix}.$$

Proof that we omit is simply writing down carefully the both sides.

Using the lemma, we have

$$\begin{aligned} T(\Delta_{h_1 h_2} U) &= T(\Delta_{h_1,D-D} U + U \Delta_{h_2,D-D}^T) \\ &= T(\Delta_{h_1,D-D} U I_{m_2}^T + I_{m_1} U \Delta_{h_2,D-D}^T) \\ &\overset{lemma}{=} (I_{m_2} \otimes \Delta_{h_1} + \Delta_{h_2} \otimes I_{m_1}) u_h, \end{aligned}$$

where $I_{m_1} \in \mathbb{R}^{m_1 \times m_1}$, $I_{m_2} \in \mathbb{R}^{m_2 \times m_2}$ are identity matrices. Taking $h = h_1 = h_2$, we obtain

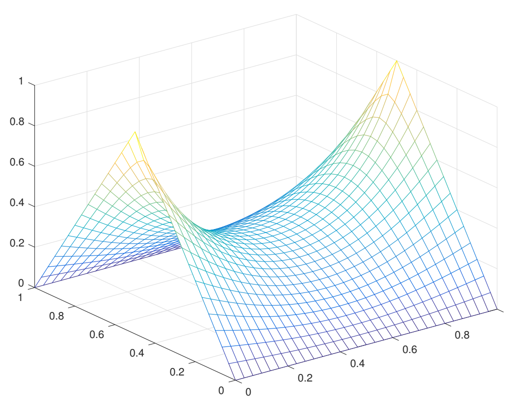
$$\Delta_h = \frac{1}{h^2} \begin{pmatrix} \begin{pmatrix} -4 & 1 & & & \\ 1 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 1 \\ & & & 1 & -4 \end{pmatrix} & I & & \\ I & \begin{pmatrix} -4 & 1 & & & \\ 1 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 1 \\ & & & 1 & -4 \end{pmatrix} & I & \\ & \ddots & \ddots & \ddots \\ & & I & \begin{pmatrix} -4 & 1 & & & \\ 1 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 1 \\ & & & 1 & -4 \end{pmatrix} \end{pmatrix}$$

Example 9.2. *Let us solve problem*

$$\begin{cases} \Delta u(x) = 0 & x = (x_1, x_2) \in \Omega = (0,1) \times (0,1) \\ u = f & \partial\Omega, \end{cases}$$

with

$$f(x) = 1 - |1 - 2x_1|.$$



```
% Dim2_d.m
% This solves           Laplace u(x)=0
% in unit square with the boundary values    u(x)=f(x).
```

```
%

n1=29;
n2=29;
h1=1/(n1+1);
h2=1/(n2+1);

D1=sparse(-diag(2*ones(n1,1))+diag(ones(n1-1,1),1)...
    +diag(ones(n1-1,1),-1))/h1^2;
D2=sparse(-diag(2*ones(n2,1))+diag(ones(n2-1,1),1)...
    +diag(ones(n2-1,1),-1))/h2^2;
M=sparse(kron(D2,eye(n1))+kron(eye(n2),D1));

b=zeros(n1*n2,1);
U=zeros(n1+2,n2+2);

for i=1:n1,
   k=i; s=f([i*h1;0]);                           % boundary values
   b(k)=b(k)-s/h2^2; U(i+1,1)=s;                 % added
   k=i+n1*(n2-1); s=f([i*h1;1]);                 % to b
   b(k)=b(k)-s/h2^2; U(i+1,n2+2)=s;              % and to
end                                              % matrix U
for j=1:n2 ,
   k=1+n1*(j-1); s=f([0;j*h2]);
   b(k)=b(k)-s/h1^2; U(1,j+1)=s;
   k=n1+n1*(j-1); s=f([1;j*h2]);
   b(k)=b(k)-s/h1^2; U(n1+2,j+1)=s;
end

u=M\b;                                       % solve equation

for j=1:n2 ,                                     % values to
   U(2:(n1+1),j+1)=u((j-1)*n1+1:j*n1);           % matrix U
end;
U(1,1)=f([0;0]); U(n1+2,1)=f([1;0]);            % and
U(1,n2+2)=f([0;1]); U(n1+2,n2+2)=f([1;1]);       % corners

mesh(linspace(0,1,n2+2),linspace(0,1,n1+2),U);
```

9.2. **Wave equation.** One dimensional case: we may discretize both in space and time. Let $\delta > 0$ be time discretization parameter and $h = 1/(1+m)$ the space discretization parameter. For the wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0, & \text{in } (0,1)\times(0,\infty) \\ u(x,0) = g(x), u_t(x,0) = h(x), & \text{on } (0,1)\times\{t=0\} \\ u(0,t) = u(1,t) = 0. \end{cases}$$

we get a discretized version

$$\frac{1}{\delta^2}(u_h^{k+1} - 2u_h^k + u_h^{k-1}) = \Delta_h u_h^k,$$

where k refers to the time step k i.e. $t = k\delta$, and

$$u_h^k = \begin{pmatrix} u_1^k \\ \vdots \\ u_m^k \end{pmatrix}, \quad g_h = \begin{pmatrix} g_1 \\ \vdots \\ g_m \end{pmatrix}, \quad h_h = \begin{pmatrix} h_1 \\ \vdots \\ h_m \end{pmatrix}.$$

We need two values u_h^0 and u_h^1 to start with the method. We get

$$\begin{aligned} u_h^0 &= g_h \\ u_h^1 &\approx u_h(0) + u_h'(0)\delta + \frac{1}{2}u_h''(0)\delta^2 + O(\delta^3) \\ &\approx u_h^0 + \delta h_h + \frac{1}{2}\Delta_h u_h(0)\delta^2 \\ &\approx g_h + \delta h_h + \frac{1}{2}\Delta_h g_h \delta^2, \end{aligned}$$

were we used initial conditions, Taylor's theorem and the equation, and $u_h(0)$ denotes vector obtained by disretizing in space but not time. This gives discretized problem

$$\begin{cases} u_h^{k+1} = 2u_h^k - u_h^{k-1} + \delta^2 \Delta_h u_h^k, \\ u_h^0 = g_h, \quad u_h^1 = (I + \frac{\delta^2}{2}\Delta_h)g_h + \delta h_h. \end{cases}$$

One could also discretize the heat equation along the similar lines but using the discretization for the first order time derivative.

We tested the method in the lectures with the wave equation and the code

```
% D1_hyp.m
%
% Solves 1-dim wave equation
%      u_tt=u_xx
% initial conditions u(x,0)=g(x), u_t(x,0)=h(x) and u(0)=u(1)=0
%

   n=99; h1=1/(n+1); dt=0.01; nt=200;
   D=(diag(-2*ones(n,1))+diag(ones(n-1,1),1)+diag(ones(n-1,1),-1))/h1^2;
   gh=0*D(:,1); hh=gh;
   for j=1:n , gh(j)=g(j*h1); hh(j)=h(j*h1);end
   uo=gh; u=gh+0.5*dt^2*(D*gh)+dt*hh;
   for k=2:nt ,
   un=2*u-uo+dt^2*(D*u); uo=u; u=un;
      plot([0;u;0])
      axis([1,n+2,-1.5,1.5]); drawnow
   end
```

In 2D we may use the 2D discretization of the Laplacian from above. Otherwise the methods remains the same. Here is an example that we tested in the lectures.

```
% D2_hyp.m
%
% Solves 2D wave eq
%      u_tt=Laplace(u)
% in (0,2)x(0,2) with the boundary conditions
%    u(0,x2,t)=u(2,x2,t)=0, u_{x2}(x1,0,t)=u_{x2}(x1,2,t)=0
```

```
% And initial conditions u(x,0)=f2(t), u_t(x,0)=g2(x) .
%

figure(1);
 n1=2*39; n2=2*39; h1=2/(n1+1); h2=2/(n2+1);  nt=200; dt=0.005;

 D1=spdiags(ones(n1,1)*[1,-2,1]/h1^2,[-1,0,1],n1,n1);
 D2=spdiags(ones(n2,1)*[1,-2,1]/h2^2,[-1,0,1],n2,n2);

 D2(1,1:2)=[-2/3,2/3]/h2^2;
 D2(n2,n2-1:n2)=[2/3,-2/3]/h2^2;                          % Neumann-condion
 D=sparse(kron(D2,eye(n1))+kron(eye(n2),D1));

 uo=zeros(n1*n2,1); gv=uo;
 for i=1:n1 , for j=1:n2 ,                                 % init. cond.
      uo(i+n1*(j-1))=f2([i*h1,j*h2]);
      gv(i+n1*(j-1))=g2([i*h1,j*h2]);
 end; end;
 u=uo+0.5*dt^2*(D*uo)+dt*gv; U=zeros(n1+2,n2+2);         % u(dt)

 for k=2:nt , un=2*u-uo+dt^2*(D*u); uo=u; u=un;          % time integration

   for j=1:n2 , U(2:(n1+1),j+1)=u((j-1)*n1+1:j*n1); end; % to U plot
   U(:,1)=(4*U(:,2)-U(:,3))/3;
   U(:,n2+2)=(4*U(:,n2+1)-U(:,n2))/3;                      % values on bdr
   mesh(U);
   axis([1,n1+2,1,n2+2,-1.5,1]); axis off;
   view(-32,24); drawnow,
 end;
```

10. Notes

These lecture notes are mostly based on Evans: Partial Differential Equations. Other references include the lecture notes Eirola: Osittaisdifferentiaaliyhtälöt, Kinnunen: Partial Differential Equations, and the books DiBenedetto: Partial Differential Equations, and Jost: Partial Differential Equations. I also thank Tero Kilpeläinen and Xiao Zhong for providing material at my disposal.

28.11.2019

Department of Mathematics and Statistics, University of Jyväskylä, PO Box 35, FI-40014 Jyväskylä, Finland

E-mail address: Please email typos/errors to:mikko.j.parviainen@jyu.fi