VISCOSITY THEORY
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Contents

1. Introduction

The theory of viscosity solutions provides a modern approach to partial differential equations and extend the classical concept of a solution. This concept was introduced and further studied in early 1980's by Michael G. Crandall, Lawrence C. Evans, Hitoshi Ishii, Pierre-Louis Lions and others.

The name viscosity theory is of historical reasons and refers the existence method of obtaining a solution by adding artificial viscosity term to the equation, and obtaining a solution by passing to a vanishing viscosity limit. This point of view is no longer central, but instead it has turned out that viscosity solutions fit naturally into the contexts of optimal control, differential and stochastic differential games as well as mathematical finance.

1.1. **Notations and basic definitions.**

$$
\begin{aligned}
\Omega \subset \mathbb{R}^n \quad & \text{open, bounded and connected set} \\
|x| &= \sqrt{x_1^2 + \ldots + x_n^2} \text{ for } x \in \mathbb{R}^n, \\
B_r(x) &= B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}, \\
\overline{B}_r(x) &= \overline{B}(x,r) = \{y \in \mathbb{R}^n : |x-y| \leq r\}, \\
2B_r &= B_{2r} \\
C(\Omega) &= \{f : f \text{ continuous in } \Omega\} \\
C^k(\Omega) &= \{f \in C(\Omega) : f \text{ is } k \text{ times continuously differentiable}\} \\
C^\infty(\Omega) &= \cap_{k=1}^\infty C^k(\Omega) = \text{smooth functions} \\
\frac{\partial^2 u}{\partial x_i \partial x_j} &= D_{ij}u = u_{ij} \text{ partial derivatives.}
\end{aligned}
$$

We mostly deal with $C^2(\Omega)$.

Remark 1.1. *Recall that*

$$u \in C^2(\Omega) \iff \frac{\partial^2 u}{\partial x_i \partial x_j} \in C(\Omega).$$

For $u : \Omega \to \mathbb{R}$, we define

$$Du = \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{pmatrix} \quad \text{(gradient)}$$

$$D^2u = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 u}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 u}{\partial x_n^2} \end{pmatrix} \quad \text{(second derivatives/ Hessian matrix)}$$

We may concentrate on symmetric $n \times n$ matrices denoted as S^n since whenever $u \in C^2$ it holds that $D^2u \in S^n$.

As usual I denotes identity matrix, and X' denotes the transpose of X. Further, for $X = [x_{ij}]_{i,j=1,2,\ldots,n}$

$$\operatorname{tr}(X) = \sum_{i=1}^{n} x_{ii},$$

and for $\xi, \eta \in \mathbb{R}^n$, we use a shorthand notation

$$\xi \otimes \eta = \begin{pmatrix} \xi_1\eta_1 & \ldots & \xi_1\eta_n \\ \vdots & \ddots & \vdots \\ \xi_n\eta_1 & \ldots & \xi_n\eta_n \end{pmatrix}.$$

We define a comparison in $X, Y \in S^n$ so that $X \leq Y$ means

$$\langle X\eta, \eta\rangle \leq \langle Y\eta, \eta\rangle$$

where $\langle X\eta, \eta\rangle := \eta' X \eta = \sum_{ij} x_{ij}\eta_i\eta_j$.

2. Motivation

At the course PDE2, we considered among other things (linear) equations in the divergence form, i.e.

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open set, $u : \Omega \to \mathbb{R}$ is the (a priori unknown) solution to the problem, and $g : \overline{\Omega} \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$. Here, L denotes a second order partial differential equation of the form

$$Lu(x) = -\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_ju(x)) + \sum_{i=1}^{n} b_i(x)D_iu(x) + c(x)u(x)$$

for given coefficients a_{ij}, b_i and c. The name divergence form is more apparent if we write $\mathcal{A}(x) = [a_{ij}]_{i,j=1,2,\dots,n}$

$$-\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u(x)) = -\operatorname{div}(\mathcal{A}(x)Du) = 0.$$

We assume that $\mathcal{A}$ is a symmetric matrix ie. $a_{ij} = a_{ji}$.

Example 2.1. *Let $\mathcal{A} = I$. Then*

$$-\operatorname{div}(\mathcal{A}(x)Du) = -\operatorname{div}(Du) = -\sum_{i=1}^{n} D_{ii}u = \Delta u$$

ie. we obtain Laplacian.

If we denote

$$Lu(x) = -\sum_{i,j=1}^{n} a_{ij}(x)D_{ij}u(x) + \sum_{i=1}^{n} b_i(x)D_i u(x) + c(x)u(x),$$

then the equation is in the non divergence form, and we cannot use the concept of a weak solution introduced in PDE2 but instead adopt another concept of a weak solution i.e. viscosity solutions (of course if a_{ij} is regular enough, then we can move between the different forms).

Example 2.2. *Many equations in optimal control or differential games are in non divergence form. Also observe that even if the examples above are linear, the viscosity theory also works in for fully nonlinear equations.*

As an example in the lectures we had optimal control problem with (x, t) as a starting point and time, $\alpha(s)$ a control, $x'(s) = f(x(s), \alpha(s))$ a trajectory, $r(x(s), \alpha(s))$ a running cost, $g(X(T))$ final cost. The value for this control problem is given by

$$u(x,t) = \inf_{\alpha} \int_t^T r(x(s), \alpha(s))\, ds + g(x(T)).$$

Questions

- *Optimal control α^*?*
- *Value of u?*

It turn out that u is a unique viscosity solution to Hamilton-Jacobi-Bellman first order PDE

$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^n \times (t, T) \\ u(x,T) = g(x) & \text{in } \mathbb{R}^n. \end{cases}$$

Thus finding u and α is transformed into a PDE question.

It is not always possible to move between the two concepts of solutions.

Example 2.3. *At PDE2, we essentially considered the following problem in divergence form:* $x \in (0,2) = \Omega$

$$a(x) = \begin{cases} 1, & x \in (0,1) \\ 2, & x \in [1,2) \end{cases}$$

Consider

$$\begin{cases} -(au')' = 1, & x \in \Omega \\ u(0) = 0 = u(2). \end{cases}$$

Then solving formally in $(0,1)$ *and* $[1,2)$ *as well as requiring suitable conditions in the middle , we obtain*

$$u(x) = \begin{cases} -\frac{x^2}{2} + \frac{5}{6}x & x \in (0,1) \\ -\frac{x^2}{4} + \frac{5}{12}x + \frac{1}{6}, & x \in [1,2). \end{cases}$$

This is not in C^2 *or even* C^1*. Nonetheless, this is a weak solution to the above problem in the sense of PDE2.*

Next consider the problem

$$\begin{cases} -au'' - 1 = 0, & x \in \Omega \\ u(0) = 0 = u(2). \end{cases}$$

which is in a non divergence form. The above function is not a solution to this problem in the viscosity sense. The function

$$u(x) = \begin{cases} -\frac{x^2}{2} + \frac{7}{8}x & x \in (0,1) \\ -\frac{x^2}{4} + \frac{3}{8}x + \frac{1}{4}, & x \in [1,2). \end{cases}$$

will be the viscosity solution. The discontinuity in the operator requires some work so we postpone the further analysis. This function is C^1 *but not* C^2*.*

The previous example was in a sense singular (discontinuous). Now we present a degenerate irregular example, common in the literature.

Example 2.4 (Eikonal equation, method of vanishing viscosity)**.** *Let* $\Omega = (-1,1)$*, and consider*

$$\begin{cases} |Du|^2 = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

There is no classical (that is C^1 *for first order equation) solution (Ex). One approach, which explains the name of the viscosity solutions, is to add viscosity term*

$$\begin{cases} -\varepsilon u_\varepsilon''(x) + (u_\varepsilon')^2 = 1 & \text{in } (-1,1) \\ u_\varepsilon(\pm 1) = 0. \end{cases}$$

The term $-\varepsilon u_\varepsilon''(x)$ is called vanishing viscosity term of when passing to a limit $\varepsilon \to 0$. We can solve this equation explicitly first by setting $v := u_\varepsilon'$ to obtain

$$-\varepsilon v' + v^2 = 1$$

and further

$$\frac{dv}{1-v^2} = -\frac{dx}{\varepsilon}.$$

Thus $tanh^{-1}(v) = -\frac{1}{\varepsilon}x + b$ i.e.

$$v = \tanh(b - \frac{1}{\varepsilon}x) = -\tanh(-b + \frac{1}{\varepsilon}x).$$

Moreover, the classical solution u for the original problem is unique (ex, this is uniformly elliptic PDE) if it exists. But by observing that $u_\varepsilon(x)$ and $u_\varepsilon(-x)$ are both solutions by plugging in into the original equation, we see that u_ε must be even, and thus $v(0) = u_\varepsilon'(0) = 0$. This implies that $b = 0$ so that

$$v(x) = -\tanh(\frac{1}{\varepsilon}x).$$

Thus

$$u_\varepsilon = \int v\,dx = -\varepsilon \log(\cosh(\frac{1}{\varepsilon}x)) + c$$

and recalling $u_\varepsilon(\pm 1) = 0$ we finally obtain

$$\begin{aligned} u_\varepsilon(x) &= -\varepsilon \log(\cosh(\frac{1}{\varepsilon}x)) + \varepsilon \log(\cosh(\pm\frac{1}{\varepsilon})) \\ &= -\varepsilon \log(\cosh(\frac{1}{\varepsilon}x)) + \varepsilon \log(\cosh(\frac{1}{\varepsilon})) \\ &= -\varepsilon \log(\frac{\cosh(\frac{1}{\varepsilon}x)}{\cosh(\frac{1}{\varepsilon})}). \end{aligned}$$

By passing to a limit $\varepsilon \to 0$, we obtain

$$u_\varepsilon \to 1 - |x|.$$

uniformly. (Ex)

Observation

- *This is not C^1, so generalized concept of a solution required-*
- *Using $\varepsilon u_\varepsilon''$ instead $-\varepsilon u_\varepsilon''$ would give $|x| - 1$. Would any 'zig zag' with $|u'| = 1$ a.e. do as a solution?*
  Important question: Is there uniqueness in the forthcoming theory?

3. Definition

Next we follow the idea in the theory of weak solutions (PDE2) and try to

- replace the derivatives on a solution as derivatives on a test function.
- integration by parts does not, in general, work as we cannot rely on the divergence form
- instead we try to use a sort of a max principle to move the derivatives on a test function.

Denote by the functions u_ε and u from Example 2.4. Choose $x_0 \in (-1,1)$ and $\varphi \in C^\infty(-1,1)$ such that φ touches u at x_0 from above. To be more precise

$$\varphi(x_0) = u(x_0)$$
$$\varphi(y) > u(y) \text{ when } y \neq x_0.$$

In particular, $u - \varphi$ has a strict max at x_0.

Lemma 3.1. *In the situation described above, there is a sequence x_ε such that $u_\varepsilon - \varphi$ has a local max at x_ε and*

$$x_\varepsilon \to x_0.$$

Proof. Since $u - \varphi$ has a strict max at x_0, for small enough $r > 0$

$$\max_{\partial B_r(x_0)} u - \varphi < u(x_0) - \varphi(x_0)$$

It holds that $u_\varepsilon \to u$ uniformly in $\overline{B}_r(x_0)$, and thus for small enough ε

$$\max_{\partial B_r(x_0)} u_\varepsilon - \varphi < u_\varepsilon(x_0) - \varphi(x_0)$$

and this means that $u_\varepsilon - \varphi$ has a local max in the interior of $B_r(x_0)$. Take a sequence $r \to 0$ to finish the proof. □

Since both u_ε and φ are smooth, it holds that (ex)

$$u_\varepsilon'(x_\varepsilon) = \varphi'(x_\varepsilon)$$
$$u_\varepsilon''(x_\varepsilon) \leq \varphi''(x_\varepsilon)$$

Thus

$$0 = -\varepsilon u_\varepsilon''(x_\varepsilon) + (u_\varepsilon'(x_\varepsilon))^2 - 1 \geq -\varepsilon \varphi''(x_\varepsilon) + (\varphi'(x_\varepsilon))^2 - 1.$$

Thus by $\varepsilon \to 0$

$$(\varphi'(x_0))^2 - 1 \leq 0.$$

Similarly if φ touches u at x_0 from below (whenever possible), we obtain

$$(\varphi'(x_0))^2 - 1 \geq 0.$$

The last two inequalities provide a candidate for a weak definition. Even if the method of vanishing viscosity does not in general work for the second order equations, the definition it inspires can also be used for them.

We consider second order PDEs of the form

$$F(x, Du, D^2u) = 0 \text{ in } \Omega$$

where $F : \Omega \times \mathbb{R}^n \times S^n \to \mathbb{R}$.

Example 3.2. *The linear equation*

$$-\sum_{i,j=1}^{n} a_{ij}(x)D_{ij}u(x) + \sum_{i=1}^{n} b_i(x)D_iu(x) - f(x)$$

with symmetric $\mathcal{A} = [a_{ij}]_{ij}$ can be written in this form

$$F(x, p, X) = -\operatorname{tr}(\mathcal{A}(x)X) + b(x) \cdot p - f(x).$$

Moreover, our standing assumption is that F is continuous in all the variables, and degenerate elliptic (in PDE2 we had uniform ellipticity which is different).

Definition 3.3 (Degenerate elliptic)**.** *F is degenerate elliptic if*

$$F(x, p, X) \leq F(x, p, Y)$$

for every $x \in \Omega, p \in \mathbb{R}^n, X, Y \in S^n$ for which

$$X \geq Y.$$

Example 3.4. *$-\Delta u$ is degenerate elliptic as simple calculation in the lectures showed. More generally,*

$$-\sum_{i,j=1}^{n} a_{ij}(x)D_{ij}u(x)$$

is degenerate elliptic whenever $\mathcal{A} = [a_{ij}]_{i,j} \geq 0$ ($\mathcal{A}$ always assumed to be symmetric) (ex).

The direction of the inequality in degenerate ellipticity reflected in the sign in these examples is a standard convention.

Definition 3.5. *We say that a function φ touches function u from above at $x \in \Omega$, if*

$$\varphi(x) = u(x) \quad \text{and} \quad u(y) < \varphi(y) \text{ when } y \neq x.$$

Definition 3.6. *We say that a function φ touches function u from below at $x \in \Omega$, if*

$$\varphi(x) = u(x) \quad \text{and} \quad u(y) > \varphi(y) \text{ when } y \neq x.$$

Definition 3.7 (Viscosity solution)**.** *A function $u \in C(\Omega)$ is a viscosity solution if whenever $\varphi \in C^2(\Omega)$ touches u at $x \in \Omega$ from below it holds that*

$$F(x, D\varphi(x), D^2\varphi(x)) \geq 0,$$

and whenever $\varphi \in C^2(\Omega)$ touches u at $x \in \Omega$ from above it holds that

$$F(x, D\varphi(x), D^2\varphi(x)) \leq 0.$$

Remark 3.8.
- *The vanishing viscosity method was purely motivational. From now on we 'forget' the vanishing viscosity method, and live with the above definition.*
- *Since $u \in C(\Omega)$, one cannot always find $\varphi \in C^2(\Omega)$ touching u from below (or above). If the set of test functions is empty, then that side of the definition is automatically satisfied.*

  For example, $u(x) : (-1, 1) \to \mathbb{R}$, $u(x) = 1 - |x|$ automatically satisfies the testing from below part at $x = 0$ since there are no test functions.

Example 3.9. *In the lectures, we verified directly from the definition that the functions in Examples 2.3 and 2.4 are viscosity solutions. Of course, strictly speaking because the discontinuity of a in Example 2.3, it is not within our assumption on F.*

Sometimes only one half of the definition is needed in the proofs. For this practical purpose, we introduce a terminology for each half.

Reminder: a function $u : \Omega \to (-\infty, \infty]$ is lower semicontinuous if the set $\{x \in \Omega : u(x) > \lambda\}$ is open for every $\lambda \in \mathbb{R}$. Equivalently, $u : \Omega \to (-\infty, \infty]$ is lower semicontinuous if

$$\liminf_{y \to x} u(y) := \lim_{r \to 0} \inf_{y \in B_r(x) \setminus \{x\}} u(y) \geq u(x).$$

23.1.2015

Analogously, $u : \Omega \to [-\infty, \infty)$ is upper semicontinuous if $\{x \in \Omega : u(x) < \lambda\}$ is open or equivalently

$$\limsup_{y \to x} u(y) \leq u(x).$$

The proofs for the next lemmas are left as exercises.

Lemma 3.10. *A function $u : \Omega \to (-\infty, \infty)$ is continuous if and only if it is upper and lower semicontinuous.*

Lemma 3.11. *If $u : \Omega \to (-\infty, \infty]$ is lower semicontinous, then there is a increasing sequence u_i of continuous functions such that*

$$u = \lim_{i \to \infty} u_i \quad \text{in } \Omega.$$

Moreover, if u_i are continuous functions then $\sup_i u$ is lower semicontinous.

Lemma 3.12. *A lower semicontinuous function attains its minimum on any compact set and is bounded below on a compact set.*

Definition 3.13 (Viscosity supersolution)**.** *A function $u : \Omega \to (-\infty, \infty]$ is a viscosity supersolution if*

(i) u is lower semicontinuous
(ii) $u \not\equiv \infty$ and
(iii) whenever $\varphi \in C^2(\Omega)$ touches u at $x \in \Omega$ from below, then

$$F(x, D\varphi(x), D^2\varphi(x)) \geq 0.$$

Definition 3.14 (Viscosity subsolution)**.** *A function $u : \Omega \to [-\infty, \infty)$ is a viscosity subsolution if*

(i) u is upper semicontinuous
(ii) $u \not\equiv -\infty$ and
(iii) whenever $\varphi \in C^2(\Omega)$ touches u at $x \in \Omega$ from above, then

$$F(x, D\varphi(x), D^2\varphi(x)) \leq 0.$$

We will prove that for C^2 function, it is enough to verify the definition for the function itself. This gives the following example.

Example 3.15. *Consider $\Omega = (-1, 1)$ and $F(x, Du, D^2u) = F(u'') = -u''$. Then $u(x) = ax + b$ is a solution*

$$F(u'') = -u''(x) = -(ax + b)'' = 0,$$

and $u(x) = -ax^2 + bx + c, a \geq 0$ is a supersolution ('super means above a solution')

$$F(u'') = -(-ax^2 + bx + c)'' = -(-2a) = 2a \geq 0,$$

and $u(x) = ax^2+bx+c, a > 0$ is a subsolution ('sub means below a solution')

$$F(u'') = -(ax^2 + bx + c)'' = -(2a) = -2a \leq 0.$$

3.1. **Equivalent definitions.**

Lemma 3.16. *Let $u \in C^2(\Omega)$. Then $F(x, Du(x), D^2u(x)) \geq 0$ for every $x \in \Omega$ if and only if u is a viscosity supersolution.*

Proof. '$\Rightarrow$'
It remains to verify that if $\varphi \in C^2(\Omega)$ touches u from below at x, then $F(x, D\varphi, D^2\varphi) \geq 0$. But similarly as in the earlier exercise, since $u - \varphi \in C^2(\Omega)$ has a minimum point at x, it holds that

$$Du(x) = D\varphi(x), \quad D^2u(x) \geq D^2\varphi(x). \tag{3.1}$$

Then

$$0 \overset{\text{assumption}}{\leq} F(x, Du(x), D^2u(x)) \overset{\text{deg ell}}{\leq} F(x, D\varphi(x), D^2\varphi(x)).$$

'⇐'
Since $u \in C^2(\Omega)$, it is always possible to touch it from below at $x \in \Omega$ with $\varphi \in C^2(\Omega)$. Indeed,

$$\varphi(y) = u(y) - |y-x|^4$$

touches u from below and $Du(x) = D\varphi(x),\ D^2u(x) = D^2\varphi(x)$.

$$\begin{aligned} 0 &\overset{\text{assumption}}{\leq} F(x, D\varphi(x), D^2\varphi(x)) \\ &\overset{Du(x)=D\varphi(x),\ D^2u(x)=D^2\varphi(x)}{=} F(x, Du(x), D^2u(x)). \end{aligned}$$

□

We may drop the assumptions that the test function is *strictly* and *globally* below (or above) the function u, and that φ touches u.

Lemma 3.17. *A function $u : \Omega \to (-\infty, \infty]$ is a viscosity supersolution if and only if*

(i) u is lower semicontinuous in Ω
(ii) $u \not\equiv \infty$ in Ω
(iii) whenever $\varphi \in C^2(U)$ for some open $U \ni x$ is such that $u - \varphi$ attains a local minimum at x it holds that

$$F(x, D\varphi(x), D^2\varphi(x)) \geq 0.$$

Proof. $' \Rightarrow '$
Sketch: Suppose that u is a viscosity supersolution, (i) and (ii) are clear. Fix x and let φ and U be as above in (iii). Then by setting $\phi(y) := u(x) - \varphi(x) + \varphi(y) - |y-x|^4 \in C^2(U)$ and touches u strictly locally from below at x in U. Observe that since u is lower semicontinuous, then there is $g \in C(\Omega)$ such that $u \geq g$ in Ω. Next choose $B_r(x) \Subset U$ such that $(u-\phi)(y) > (u-\phi)(x) = 0$ for $y \in \overline{B}_r(x), y \neq x$. Then let $\eta > 0$ and define

$$\begin{cases} \phi & \text{in } B_r(x) \\ \min(\inf_{B_r(x)} \phi, g - \eta) & \text{in } \Omega \setminus B_r(x). \end{cases}$$

Then mollify this function so that values are not changed in $B_{r/2}(x)$ (this also requires some care at the vicinity of $\partial\Omega$) , denote this by $\tilde{\varphi}$. It holds that $\tilde{\varphi} \in C^2(\Omega)$, $D\tilde{\varphi}(x) = D\varphi(x), D^2\tilde{\varphi}(x) = D^2\varphi(x)$ and $\tilde{\varphi}$ touches u from below at x in Ω, and thus since u is a viscosity supersolution, it follows that (iii) holds.

$' \Leftarrow '$
Choose any $\varphi \in C^2(\Omega)$ touching u from below. Since φ is admissible in (iii), also $F(x, D\varphi(x), D^2\varphi(x)) \geq 0$ and thus u is a viscosity supersolution. □

Suppose that $\varphi \in C^2(\Omega)$ touches u at x from below and recall the Taylor expansion

$$\begin{aligned} u(y) &\geq \varphi(y) \\ &= \varphi(x) + D\varphi(x)\cdot(y-x) \\ &\quad + \frac{1}{2}\langle D^2\varphi(x)(y-x),(y-x)\rangle + o(|y-x|^2), \end{aligned} \tag{3.2}$$

when $y \to x$, where

$$\lim_{y\to x}\left|\frac{o(|y-x|^2)}{|y-x|^2}\right| = 0.$$

This has an immediate consequence:

Lemma 3.18. *It is enough to use C^∞ test functions in the definition of a super (or subsolution).*

Proof. We are going to show that if $\varphi \in C^2(\Omega)$ touches u at x from below and the supersolution property fails i.e.

$$0 > F(x, D\varphi(x), D^2\varphi(x)), \tag{3.3}$$

then there is $\phi \in C^\infty(\Omega)$ touching u at x

$$0 > F(x, D\phi(x), D^2\phi(x)).$$

By Taylor's expansion

$$\begin{aligned} u(y) &\geq \varphi(y) \\ &= \varphi(x) + D\varphi(x)\cdot(y-x) + \frac{1}{2}\langle D^2\varphi(x)(y-x),(y-x)\rangle + o(|y-x|^2). \end{aligned}$$

But now

$$-\left|o(|y-x|^2)\right| \geq -\varepsilon\,|y-x|^2\,.$$

Thus, if we define

$$\Phi(y) := \varphi(x) + D\varphi(x)\cdot(y-x) + \frac{1}{2}\langle D^2\varphi(x)(y-x),(y-x)\rangle - \varepsilon\,|y-x|^2\,,$$

it holds that

$$\begin{aligned} \varphi(y) \geq \phi_\varepsilon(y), &\quad u(x) = \phi_\varepsilon(x) \\ D\varphi(x) = D\phi_\varepsilon(x), &\quad D^2\phi_\varepsilon(x) = D^2\varphi(x) - 2\varepsilon I \end{aligned}$$

Then

$$F(x, D\phi_\varepsilon(x), D^2\phi_\varepsilon(x)) = F(x, D\varphi(x), D^2\varphi(x) - 2\varepsilon I) < 0$$

for small enough $\varepsilon > 0$ by continuity of F and by (3.3). □

On the other hand, if there are $(p, X) \in \mathbb{R}^n \times S^n$ such that

$$u(y) \geq u(x) + p \cdot (y - x) + \frac{1}{2}\langle X(y-x), (y-x)\rangle + o(|y-x|^2), \tag{3.4}$$

we can use modification of the right hand side as a test function as shown below.

Sometimes the fact that only the terms up to the second order in the Taylor expansion matter in the definition of viscosity solutions is emphasized in the literature by terminology of *semi jets*. If $(p, X) \in \mathbb{R}^n \times S^n$ and (3.4) holds, then this is denoted as

$$(p, X) \in J^{2,-}u(x),$$

where $J^{2,-}u(x)$ is called a second order subjet of u at x. Analogously, if

$$u(y) \leq u(x) + p \cdot (y - x) + \frac{1}{2}\langle X(y-x), (y-x)\rangle + o(|y-x|^2)$$

then

$$(p, X) \in J^{2,+}u(x),$$

where $J^{2,+}u(x)$ is called a second order superjet of u at x. We can even take a sort of a closure of a jet. To be more precise, we define that

$$(p, X) \in \overline{J}^{2,-}u(x)$$

if there is a sequence $(p_i, X_i) \in J^{2,-}u(x_i)$ such that $(x_i, p_i, X_i) \to (x, p, X)$.

Naturally, $J^{2,-}u(x) \subset \overline{J}^{2,-}u(x)$, as well as

$$\overline{J}^{2,+}(-u(x)) = -\overline{J}^{2,-}u(x).$$

Proposition 3.19. *Let $u : \Omega \to (-\infty, \infty]$ be lower semicontinuous. The following are equivalent.*

(i) u is a viscosity supersolution
(ii) for every $x \in \Omega$ and $(p, X) \in J^{-,2}u(x)$ it holds that

$$F(x, p, X) \geq 0.$$

(iii) for every $x \in \Omega$ and $(p, X) \in \overline{J}^{-,2}u(x)$ it holds that

$$F(x, p, X) \geq 0.$$

Proof. (ii) $\Rightarrow$ (iii)
Let $(p, X) \in \overline{J}^{2,-}u(x)$. Then by definition, there is a sequence $(p_i, X_i) \in J^{2,-}u(x_i)$ such that $(x_i, p_i, X_i) \to (x, p, X)$. Thus

$$0 \overset{(ii)}{\leq} F(x_i, p_i, X_i) \overset{F \text{ cont}}{\to} F(x, p, X).$$

(iii) ⇒ *(i)*
Let $\varphi \in C^2(\Omega)$ touch u at x from below. Then by Taylor's expansion

$$\begin{aligned} u(y) &\geq \varphi(y) \\ &= \varphi(x) + D\varphi(x)\cdot(y-x) + \frac{1}{2}\langle D^2\varphi(x)(y-x),(y-x)\rangle + o(|y-x|^2), \end{aligned}$$

i.e. $(D\varphi(x), D^2\varphi(x)) \in J^{2,-}u(x) \subset \overline{J}^{2,-}u(x)$. Thus

$$0 \overset{\text{(iii)}}{\leq} F(x, D\varphi(x), D^2\varphi(x)).$$

(i) ⇒ *(ii)*
Let $(p,X) \in J^{2,-}u(x)$. We would like to use (i) by setting

$$u(y) \geq u(x) + p\cdot(y-x) + \frac{1}{2}\langle X(y-x),(y-x)\rangle + o(|x-y|^2) =: \varphi(y).$$

It remains to check that we can choose $o(|x-y|^2)$ so that the above $\varphi \in C^2(\Omega)$ and that it remains below u. Now $o(|y-x|^2)$ is fixed, and we may find a continuous nondecreasing $\omega : [0,\infty) \to [0,\infty]$, $\omega(0) = 0$ such that

$$-\left|o(|y-x|^2)\right| \geq -|y-x|^2\,\omega(|y-x|)$$

and thus $u(x)+p\cdot(y-x)+\frac{1}{2}\langle X(y-x),(y-x)\rangle - |y-x|^2\,\omega(|y-x|)$ remains below u, at least locally. We ensure C^2 by modifying the error term as

$$\begin{aligned} \psi(t) := \int_t^{\sqrt{3}t}\int_s^{2s}\omega(r)\,dr\,ds &\geq \int_t^{\sqrt{3}t}\omega(s)\int_s^{2s}dr\,ds \\ &\geq \omega(t)\frac{1}{2}((\sqrt{3}t)^2 - t^2) = t^2\omega(t). \end{aligned}$$

By adjusting ψ far away in the spirit of the proof of Lemma 3.17 if necessary, we see that for $\varphi(y) := u(x) + p\cdot(y-x) + \frac{1}{2}\langle X(y-x),(y-x)\rangle - \psi(|y-x|)$ touches u from below and that

$$\varphi \in C^2(\Omega),\ D\varphi(x) = p,\ D^2\varphi(x) = X.$$

Thus $F(x,p,X) \overset{\text{(i)}}{\geq} 0$. □

In the last step, by using continuity, we could have immediately observed that $u(y) \geq u(x) + p\cdot(y-x) + \frac{1}{2}\langle X(y-x),(y-x)\rangle - \varepsilon\,|x-y|^2 = \varphi(y)$.

Remark 3.20 (Connection to distributional solutions)**.** *In some cases (equation is in the divergence from...) it is a relevant question, if the definition of viscosity solution is equivalent to the distributional weak solutions. For example, for p-Laplacian $\Delta_p u = 0$, are the two concepts of solutions the same? The answer is yes, see*
P. Juutinen, P. Lindqvist, J.J. Manfredi : On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation, SIAM journal on mathematical analysis, 2001. In the proof, it is relatively easy to show that a weak solution is always a viscosity solution. Essentially, most of the

work then goes in showing that a viscosity solution is unique, which we will consider soon.

In the linear elliptic case, see H. Ishii: On the Equivalence of Two Notions of Weak Solutions, Viscosity Solutions and Distribution Solutions, 1995.

4. Comparison principles and uniqueness

In this section, we aim at showing for u subsolution and v supersolution that then roughly

$$v \geq u \text{ on } \partial\Omega \implies v \geq u \text{ on whole of } \overline{\Omega}.$$

Naturally this implies uniqueness in a sense that, if $u = v$ on $\partial\Omega$ are viscosity solutions, then $u \equiv v$. Making this precise requires some care with the boundary values.

Example 4.1 (Comparison for smooth functions)**.** *Let $B := B_1(0)$ and $u, v \in C^2(2B)$ such that*

$$\begin{cases} -\Delta u < 0 < -\Delta v & \text{in } B \\ u \leq v & \text{on } \partial B. \end{cases} \tag{4.5}$$

Suppose on the contrary that there is $x \in B$ such that

$$w(x) := u(x) - v(x) := \sup_B u - v > 0.$$

Then it holds that

$$Dw(x) = 0, \quad D^2 w(x) \leq 0$$

i.e.

$$Du(x) = Dv(x), \quad D^2 u(x) \leq D^2 v(x).$$

By taking traces, we have

$$0 \overset{(4.5)}{<} \Delta u = \operatorname{tr} D^2 u(x) \leq \operatorname{tr} D^2 v(x) = \Delta v \overset{(4.5)}{<} 0$$

a contradiction, and thus the comparison principle must hold.

There are two points in this example which will cause most of the work when following this path for viscosity solutions:

- *We took strict inequalities in (4.5). Excludes solutions!*
- *Viscosity solutions are not C^2 in general.*

We denote by $USC(\overline{\Omega})$, the space of lower semicontinuous functions on $\overline{\Omega}$. The definition is almost the same as before even if $\overline{\Omega}$ is not open just by intersecting $\overline{\Omega}$ whenever necessary.

We now start relaxing the assumptions of the previous example. First step is to prove a comparison principle between viscosity supersolutions and a C^2-subsolution for simplicity for $F = -\Delta$, without using linearity (property used in the proof is uniform ellipticity that we encounter later). To be more precise, we only assume smoothness for one of the functions, and drop strict inequalities by 'bending' v a little to guarantee the strict inequality.

Proposition 4.2. *Let $F = -\Delta$. Further, let $u \in USC(\overline{\Omega})$ be a viscosity subsolution, and $v \in LSC(\overline{\Omega}) \cap C^2(\Omega)$ for which $F(D^2 v) \geq 0$.*

$$\text{If } u \leq v \text{ on } \partial\Omega, \text{ then } u \leq v \text{ on } \overline{\Omega}.$$

Proof. We make a counter proposition, i.e.

$$\max_{\overline{\Omega}} u - v =: \theta > 0.$$

Let

$$\phi_\varepsilon(x) := \varepsilon e^{x_1}$$

We choose $\varepsilon > 0$ so small that always $\phi_\varepsilon \leq \theta/2$ in our bounded domain Ω. Choose x such that

$$(u - v + \phi_\varepsilon)(x) := \max_{\overline{\Omega}}(u - v + \phi_\varepsilon) \geq \theta,$$

for which $x \in \Omega$ since $u - v \leq 0$ on $\partial\Omega$ and $\phi_\varepsilon \leq \theta/2$. Denote $\varphi := v - \phi_\varepsilon \in C^2(\Omega)$, and that $u - \varphi$ has a max at x. Thus

$$\begin{aligned} 0 &\overset{u \text{ sub}}{\geq} F(x, D\varphi(x), D^2\varphi(x)) \\ &= F(x, D(v - \phi_\varepsilon)(x), D^2(v - \phi_\varepsilon)(x)) \\ &\overset{F = -\Delta}{=} -\Delta v + \varepsilon e^{x_1} \\ &\overset{-\Delta v \geq 0}{\geq} \varepsilon e^{x_1} > 0 \end{aligned}$$

a constradiction. □

Next we prove a preliminary first order result, where we deal with non smoothness of u and v. We also allow u dependence in the operator. The definition of viscosity (sub/super)solution is now

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \leq (\geq) 0$$

for similar test functions as before.

Proposition 4.3. *Let $\nu > 0$ and $F(x, u, Du) := \nu u + |Du|^2 - 1$. Further, let $u \in USC(\overline{\Omega})$ be a viscosity subsolution, and $v \in LSC(\overline{\Omega})$ a viscosity supersolution.*

$$\text{If } u \leq v \text{ on } \partial\Omega, \text{ then } u \leq v \text{ on } \overline{\Omega}.$$

Proof. As before, we make a counter proposition, i.e.

$$\max_{\overline{\Omega}} u - v =: \theta > 0.$$

Because of irregularity of v, the previous calculation does not work. Instead and this is very central technique in viscosity theory, we use **doubling of variables**, $\varepsilon > 0$

$$\Phi(x,y) := u(x) - v(y) - \varphi(x,y) := u(x) - v(y) - \frac{1}{2\varepsilon}|x-y|^2 .$$

There are $x_\varepsilon, y_\varepsilon \in \overline{\Omega} \times \overline{\Omega}$ such that

$$\Phi(x_\varepsilon, y_\varepsilon) := \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} \Phi(x,y).$$

The idea is that $x \mapsto u(x) - \varphi(x, y_\varepsilon)$ has a max at x_ε, i.e. $x \mapsto \varphi(x, y_\varepsilon)$ is a test function, and $y \mapsto v(y) + \varphi(x_\varepsilon, y)$ has a min at y_ε, i.e. $y \mapsto -\varphi(x_\varepsilon, y)$ is a test function. We have

$$-D_y\varphi(x_\varepsilon, y_\varepsilon) = \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), \quad D_x\varphi(x_\varepsilon, y_\varepsilon) = \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon).$$

If we can ensure that $x_\varepsilon, y_\varepsilon$ are not at the boundary, we may deduce

$$\begin{aligned} 0 &\le F(y_\varepsilon, v(y_\varepsilon), -D_y\varphi(x_\varepsilon, y_\varepsilon)) - F(x_\varepsilon, u(x_\varepsilon), D_x\varphi(x_\varepsilon, y_\varepsilon)) \\ &= \nu(v(y_\varepsilon) - u(x_\varepsilon)) + (|-D_y\varphi(x_\varepsilon, y_\varepsilon)|^2 - |D_x\varphi(x_\varepsilon, y_\varepsilon)|^2) \\ &= \nu(v(y_\varepsilon) - u(x_\varepsilon)) + \Big((\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon))^2 - (\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon))^2\Big) \\ &= \nu(v(y_\varepsilon) - u(x_\varepsilon)) \le -\nu\theta, \end{aligned}$$

a contradiction. Above the last inequality follows from 30.1.2015

$$u(x_\varepsilon) - v(y_\varepsilon) \ge \Phi(x_\varepsilon, y_\varepsilon) \ge \max_{x\in\overline{\Omega}} \Phi(x,x) \overset{\text{counter prop}}{=} \theta > 0. \tag{4.6}$$

It remains to check that neither of $x_\varepsilon, y_\varepsilon$ is at the boundary. First set

$$\max_{\overline{\Omega}} u - \min_{\overline{\Omega}} v =: M < \infty$$

using semicontinuity. By the last inequality in (4.6), we get an upper bound

$$\frac{1}{2\varepsilon}|x_\varepsilon - y_\varepsilon|^2 \le u(x_\varepsilon) - v(y_\varepsilon) - \theta \le M$$

for the penalty term. In other words,

$$|x_\varepsilon - y_\varepsilon|^2 \le 2\varepsilon M$$

and thus, since $\overline{\Omega}$ is compact, passing to a subsequence if necessary, there is $\hat{x} \in \overline{\Omega}$ such that $(x_\varepsilon, y_\varepsilon) \to (\hat{x}, \hat{x})$ as $\varepsilon \to 0$. By semicontinuity and the assumption $u \le v$ on $\partial\Omega$ it follows that

$$\hat{x} \in \Omega.$$

Indeed, otherwise if $\hat{x} \in \partial\Omega$, then $\theta \le \limsup_{\varepsilon\to 0} u(x_\varepsilon) - v(y_\varepsilon) \le \limsup_{y\to\hat{x}} u - \liminf_{y\to\hat{x}} v \le u(\hat{x}) - v(\hat{x}) \le 0$, a contradiction. But $\hat{x} \in \Omega$ implies that $x_\varepsilon, y_\varepsilon \in \Omega$ for ε small enough. □

Remark 4.4. *The previous proof actually contains that there is a sequence* $x_\varepsilon, y_\varepsilon \to \hat{x} \in \Omega$ *for which*

$$
\begin{aligned}
0 &\le \liminf_{\varepsilon} \frac{1}{\varepsilon} |x_\varepsilon - y_\varepsilon|^2 \\
&\le \limsup_{\varepsilon} \frac{1}{\varepsilon} |x_\varepsilon - y_\varepsilon|^2 \\
&\overset{above}{\le} \limsup (u(x_\varepsilon) - v(y_\varepsilon)) - \theta \\
&\le u(\hat{x}) - v(\hat{x}) - \theta \\
&\overset{\theta\ sup}{\le} \theta - \theta = 0.
\end{aligned}
$$

The previous proof does not apply to the eikonal equation since there is no νu, $\nu > 0$ term. Luckily, we can ensure the strict inequality perturbing the equation by a small constant by considering a sub- and a supersolution γu and v. These satisfy

$$
\begin{aligned}
|D(\gamma u)|^2 - \gamma^2 &\le 0 \\
|Dv|^2 - 1 &\ge 0.
\end{aligned}
$$

Proposition 4.5. *Let* $F(x, u, Du) := |Du|^2 - 1$*. Further, let* $u \in USC(\overline{\Omega})$ *be a viscosity subsolution, and* $v \in LSC(\overline{\Omega})$ *a viscosity supersolution.*

$$\text{If } u \le v \text{ on } \partial\Omega, \text{ then } u \le v \text{ on } \overline{\Omega}.$$

Proof. As before, we make a counter proposition, i.e.

$$\max_{\overline{\Omega}} \gamma u - v =: \theta > 0.$$

for some $\gamma < 1$ close enough 1. Then

$$\Phi(x, y) := \gamma u(x) - v(y) - \varphi(x, y) := \gamma u(x) - v(y) - \frac{1}{2\varepsilon} |x - y|^2 .$$

There are $x_\varepsilon, y_\varepsilon \in \overline{\Omega} \times \overline{\Omega}$ such that

$$\Phi(x_\varepsilon, y_\varepsilon) := \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} \Phi(x, y).$$

Analogously what we had before, the idea is that $x \mapsto \gamma u(x) - \varphi(x, y_\varepsilon)$ has a max at x_ε, i.e. $x \mapsto \varphi(x, y_\varepsilon)$ is a test function, and $y \mapsto v(y) + \varphi(x_\varepsilon, y)$ has a min at y_ε, i.e. $y \mapsto -\varphi(x_\varepsilon, y)$ is a test function. Other details are exactly as in the previous proof, so the contradiction follows from

$$
\begin{aligned}
0 &\le |-D_y\varphi(x_\varepsilon, y_\varepsilon)|^2 - 1 - (|D_x\varphi(x_\varepsilon, y_\varepsilon)|^2 - \gamma^2) \\
&= \gamma^2 - 1 < 0.
\end{aligned}
$$

a contradiction. □

Remark 4.6.
- *Eikonal equation has uniqueness within viscosity theory even if the uniqueness for relaxed solutions looked doubtful at the first glance.*

- *The techniques used above generalize to* $H(x, Du) - f(x) = 0$*, where* H *has suitable continuity assumption , scaling property* $\mu^\alpha H(x,p) = H(x,\mu p), \mu, \alpha > 0$ *and* $f \in C(\overline{\Omega}), \inf f > 0$*. Alternatively, if we have* $+\nu u$ *in the equation instead of the last two, the uniqueness also follows.*

Next we look at the second order case.

Example 4.7. *We start with*

$$F(x, u, Du, D^2u) = \nu u - \Delta u.$$

Lets try doubling of variables in order to prove comparison: To thrive for a contradiction, assume

$$\max_{\overline{\Omega}} u - v =: \theta > 0.$$

As above

$$\Phi(x, y) := u(x) - v(y) - \varphi(x, y) := u(x) - v(y) - \frac{1}{2\varepsilon} |x - y|^2 .$$

Then at the max $(x_\varepsilon, y_\varepsilon)$

$$\begin{aligned} -D_y\varphi(x_\varepsilon, y_\varepsilon) &= \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), \quad -D_{yy}\varphi(x_\varepsilon, y_\varepsilon) = -\frac{1}{\varepsilon}I \\ D_x\varphi(x_\varepsilon, y_\varepsilon) &= \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), \quad D_{xx}\varphi(x_\varepsilon, y_\varepsilon) = \frac{1}{\varepsilon}I. \end{aligned}$$

But now

$$\begin{aligned} 0 \le\ & F(y_\varepsilon, v(y_\varepsilon), -D_y\varphi(x_\varepsilon, y_\varepsilon), -D_{yy}\varphi(x_\varepsilon, y_\varepsilon)) \\ & \qquad - F(x_\varepsilon, u(x_\varepsilon), D_x\varphi(x_\varepsilon, y_\varepsilon), D_{xx}\varphi(x_\varepsilon, y_\varepsilon)) \\ =\ & \nu(v(y_\varepsilon) - u(x_\varepsilon)) + \frac{1}{\varepsilon}(n + n), \end{aligned}$$

and this provides no contradiction.

The problem above is the term $2n/\varepsilon$ *arising from the (too direct) application of the doubling of variables method. However, in the smooth case, Example 4.1, the corresponding term was* $-\Delta v(x) - (-\Delta u(x)) \overset{x \text{ max}}{\le} 0$ *so we should be able to do better. This is the content of the next result, called Ishii lemma, theorem of sums or max principle for semicontinuous functions.*

Theorem 4.8 (Theorem of sums)**.** *Let* $u \in USC(\overline{\Omega})$ *and* $v \in LSC(\overline{\Omega})$*. For* $\varphi \in C^2(\mathbb{R}^n \times \mathbb{R}^n)$*, suppose that there is* $(\hat{x}, \hat{y}) \in \Omega \times \Omega$ *such that*

$$u(\hat{x}) - v(\hat{y}) - \varphi(\hat{x}, \hat{y}) = \max_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} (u(x) - v(y) - \varphi(x, y)).$$

Then for each $\mu > 0$*, there are* $X = X(\mu)$ *and* $Y = Y(\mu)$ *such that*

$$(D_x\varphi(\hat{x}, \hat{y}), X) \in \overline{J}^{2,+} u(\hat{x}), \quad (-D_y\varphi(\hat{x}, \hat{y}), Y) \in \overline{J}^{2,-} v(\hat{y}),$$

and

$$-(\mu + ||D^2\varphi(\hat{x},\hat{y})||)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le D^2\varphi(\hat{x},\hat{y}) + \frac{1}{\mu}(D^2\varphi(\hat{x},\hat{y}))^2, \tag{4.7}$$

where

$$D^2\varphi(\hat{x},\hat{y}) = \begin{pmatrix} D_{xx}\varphi(\hat{x},\hat{y}) & D_{xy}\varphi(\hat{x},\hat{y}) \\ D_{yx}\varphi(\hat{x},\hat{y}) & D_{yy}\varphi(\hat{x},\hat{y}) \end{pmatrix}$$

and $||A|| = \max\{|\lambda_k| \,:\, \lambda_k$ *an eigenvalue of* $A\} = \max\{|\langle A\xi,\xi\rangle| \,:\, |\xi| \le 1\}$ *for* $A \in S^n$.

Remark 4.9. *If* $u, v \in C^2(\Omega)$*, then one can observe in the setup above (ex) that* $X = D^2u(\hat{x})$*,* $Y = D^2v(\hat{y})$

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le D^2\varphi(\hat{x},\hat{y}).$$

Thus Theorem of sums is a generalization of this with additional error term.

Example 4.10. *Let*

$$\varphi(x,y) = \frac{1}{2\varepsilon}|x-y|^2.$$

Then as above

$$D_y\varphi(x_\varepsilon,y_\varepsilon) = -\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), \quad D_{yx}\varphi(x_\varepsilon,y_\varepsilon) = -\frac{1}{\varepsilon}I, \quad D_{yy}\varphi(x_\varepsilon,y_\varepsilon) = \frac{1}{\varepsilon}I$$
$$D_x\varphi(x_\varepsilon,y_\varepsilon) = \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon), \quad D_{xy}\varphi(x_\varepsilon,y_\varepsilon) = -\frac{1}{\varepsilon}I, \quad D_{xx}\varphi(x_\varepsilon,y_\varepsilon) = \frac{1}{\varepsilon}I$$

so that

$$D^2\varphi(x_\varepsilon,y_\varepsilon) = \frac{1}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \quad (D^2\varphi(x_\varepsilon,y_\varepsilon))^2 = \frac{2}{\varepsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

Sometimes it is enough the use the last inequality in (4.7), we get

$$(D_x\varphi(x_\varepsilon,y_\varepsilon), X) \in \overline{J}^{2,+}u(x_\varepsilon), \quad (-D_y\varphi(x_\varepsilon,y_\varepsilon), Y) \in \overline{J}^{2,-}v(y_\varepsilon),$$

such that for any $\xi \in \mathbb{R}^n$ *it holds that*

$$\xi' X\xi - \xi' Y\xi = \begin{pmatrix} \xi \\ \xi \end{pmatrix}' \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}\begin{pmatrix} \xi \\ \xi \end{pmatrix} \le C\begin{pmatrix} \xi \\ \xi \end{pmatrix}'\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}\begin{pmatrix} \xi \\ \xi \end{pmatrix} = 0.$$

In other words, $X \le Y$*. Comparing to the smooth case, Example 4.1, we utilized* $-\Delta v(x) - (-\Delta u(x)) \overset{x\ max}{\le} 0$ *so now we have arrived to the corresponding result in the non smooth case.*

Continuing Example 4.7 for $F(x,u,Du,D^2u(x)) = \nu u - \Delta u$ *we now get*

$$\begin{aligned} 0 &\le F(y_\varepsilon, v(y_\varepsilon), -D_y\varphi(x_\varepsilon,y_\varepsilon), Y) \\ &\qquad - F(x_\varepsilon, u(x_\varepsilon), D_x\varphi(x_\varepsilon,y_\varepsilon), X) \\ &= \nu(v(y_\varepsilon) - u(x_\varepsilon)) \underbrace{- \operatorname{tr} Y - (-\operatorname{tr} X)}_{\le 0} \le -\nu\theta, \end{aligned}$$

where the last inequality follows from $u(x_\varepsilon) - v(y_\varepsilon) \ge \Phi(x_\varepsilon, y_\varepsilon) \ge \max_{x\in\overline{\Omega}} \Phi(x,x) \overset{\text{counter prop}}{=} \theta$. *This is a contradiction, and proves the comparison for* $F(x,u,Du,D^2u(x)) = \nu u - \Delta u$, $\nu > 0$.

We omit the details of the proof of Theorem 4.8 and only point out the idea:

- Alexandrov's differentiability result: a convex function is a.e. twice differentiable in Alexandrov's sense (has second order Taylor expansion). See Section 6 in Evans' and Gariepy's 'Measure Theory and Fine Properties of Functions'.
- Jensen's lemma: Collect all the points such that we can locally touch them by a plane from above at a vicinity of a strict max point of a semiconvex (i.e. there is λ such that $f(\xi) + \frac{\lambda}{2}|\xi|^2$ is convex) function. Techniques from ABP-max principle, that we will work through later, this has a positive measure.
- By replacing φ in the original statement of Theorem of sums by its Taylor approximation, we may reduce the situation.
- Semiconvexity approximation is fulfilled by observing that in Theorem of sums, the statement is close to forming so called sup/inf-convolutions, which are known to be semiconvex/semiconcave. Then we work out the proof for sup/inf-convolutions and pass back to the original functions by the known properties of sup/inf-convolutions.
- Thus it will be enough to consider

$$f(\xi) + \langle p, \xi\rangle - \langle B\xi, \xi\rangle - \delta|\xi|^2$$

  where f contains sup / inf convolutions of u and v. Then by Alexandrov's and Jensen's lemmas there is a point where f is twice differentiable and has its max at that point. Taking the second derivative we obtain the lower bound from the semi convexity of f, and upper bound from the max point property.

We postpone the discussion about the uniformly elliptic operators, and let the Laplacian $F(x,Du,D^2u) = -\Delta u$ represent the class.

Before that we state a useful lemma. Its proof is exercise.

Lemma 4.11. *Let* $v : \Omega \to (-\infty, \infty]$, $\psi \in C^2(\mathbb{R}^n)$ *and* $x \in \Omega$. *Then*

$$\overline{J}^{2,-}(v+\psi)(x) = (D\psi(x), D^2\psi(x)) + \overline{J}^{2,-}v(x).$$

The result for $\overline{J}^{2,+},\dots$ *is analogous.*

We will have to use both ideas introduced above: bend v to obtain a strict inequality, and use doubling of variables in combination with Theorem of sums to deal with the non smoothness of u and v.

Theorem 4.12. *Let* $F(x,u,Du,D^2u) := -\operatorname{tr}(D^2u) = -\Delta u$. *Further, let* $u \in USC(\overline{\Omega})$ *be a viscosity subsolution, and* $v \in LSC(\overline{\Omega})$ *a viscosity supersolution.*

$$\text{If } u \le v \text{ on } \partial\Omega, \text{ then } u \le v \text{ on } \overline{\Omega}.$$

Proof. We make a counter proposition, i.e.

$$\max_{\overline{\Omega}} u - v =: \theta > 0.$$

Double the variables

$$\begin{aligned}\Phi(x,y) &:= u(x) - (v(y) - \varepsilon e^{(y_\varepsilon)_1}) - \varphi(x,y)\\ &:= u(x) - (v(y) - \varepsilon e^{(y_\varepsilon)_1}) - \frac{1}{2\varepsilon}|x-y|^2.\end{aligned}$$

There are $x_\varepsilon, y_\varepsilon \in \overline{\Omega}\times\overline{\Omega}$ such that

$$\Phi(x_\varepsilon, y_\varepsilon) := \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} \Phi(x,y).$$

By Theorem of sums, Theorem 4.8, there are

$$(D_x\varphi(x_\varepsilon,y_\varepsilon),X) \in \overline{J}^{2,+}u(x_\varepsilon), \quad (-D_y\varphi(x_\varepsilon,y_\varepsilon),Y) \in \overline{J}^{2,-}(v(y_\varepsilon) - \varepsilon e^{(y_\varepsilon)_1})$$

that is by Lemma 4.11

$$\left(\varepsilon e^{(y_\varepsilon)_1} - D_y\varphi(x_\varepsilon,y_\varepsilon), Y + \underbrace{\begin{pmatrix}\varepsilon e^{(y_\varepsilon)_1} & 0\dots\\ 0 & \dots\\ \vdots & \ddots\end{pmatrix}}_{=:I_1}\right) \in \overline{J}^{2,-}v(y_\varepsilon),$$

where $X \le Y$ as before.

If $x_\varepsilon, y_\varepsilon$ are not at the boundary for small enough $\varepsilon > 0$, we may deduce

$$\begin{aligned}0 &\le F(y_\varepsilon, v(y_\varepsilon), \varepsilon e^{(y_\varepsilon)_1} - D_y\varphi(x_\varepsilon,y_\varepsilon), Y + I_1) - F(x_\varepsilon, u(x_\varepsilon), D_x\varphi(x_\varepsilon,y_\varepsilon), X)\\ &= -\operatorname{tr}(Y) - \varepsilon e^{(y_\varepsilon)_1} - (-\operatorname{tr}(X))\\ &\overset{X\le Y}{\le} -\varepsilon e^{(y_\varepsilon)_1} < 0,\end{aligned}$$

a contradiction.

It remains to check that x_ε or y_ε are not at the boundary. As before, let

$$M := \max_{\overline{\Omega}} u - \min_{\overline{\Omega}} v.$$

Since Ω is bounded, we can always choose $\varepsilon > 0$ small enough such that $\varepsilon \max_{\overline{\Omega}} e^{y_1} \leq \theta/2$. We can now estimate

$$
\begin{aligned}
u(x_\varepsilon) - v(y_\varepsilon) + \frac{\theta}{2} &\overset{\text{above choice}}{\geq} \Phi(x_\varepsilon, y_\varepsilon) \\
&= \max_{\overline{\Omega}\times\overline{\Omega}} \Phi(x,y) \geq \max_{x\in\overline{\Omega}} (u(x) - v(x) + \varepsilon e^{x_1}) \geq \theta.
\end{aligned}
\tag{4.8}
$$

Thus

$$
\frac{1}{2\varepsilon} |x_\varepsilon - y_\varepsilon|^2 \leq u(x_\varepsilon) - v(y_\varepsilon) + \varepsilon e^{(y_\varepsilon)_1} - \theta \leq M
$$

and using compactness of $\overline{\Omega}$ there exists $\hat{x} \in \overline{\Omega}$ such that $(x_\varepsilon, y_\varepsilon) \to (\hat{x}, \hat{x})$. Furthermore, making a counter proposition $\hat{x} \in \partial\Omega$, utilizing upper semicontinuity $\frac{\theta}{2} \leq \limsup_{\varepsilon\to 0} u(x_\varepsilon) - v(y_\varepsilon) \leq u(\hat{x}) - u(\hat{x}) \overset{\text{bdr values}}{\leq} 0$ we see that $\hat{x}$ and thus x_ε and y_ε cannot be at the boundary. □

4.1. **Parabolic case.** Next we briefly audit parabolic theory, and consider the problem

$$
u_t + F(x, t, Du, D^2u) = 0 \tag{4.9}
$$

in a cylinder $\Omega_T := \Omega \times (0, T)$. We still assume F to be continuous and degenerate elliptic.

Definition 4.13 (Parabolic viscosity solution)**.** *A function $u \in C(\Omega\times(0,T))$ is a viscosity solution to the above problem if whenever $\varphi \in C^2(\Omega \times (0,T))$ touches u at $(x,t) \in \Omega \times (0,T)$ from below it holds that*

$$
\varphi_t(x,t) + F(x,t,D\varphi,D^2\varphi) \geq 0,
$$

and whenever $\varphi \in C^2(\Omega \times (0,T))$ touches u at $(x,t) \in \Omega \times (0,T)$ from above it holds that

$$
\varphi_t(x,t) + F(x,t,D\varphi,D^2\varphi) \leq 0.
$$

Also the definition of supersolutions is analogous to the elliptic case. We exclude $\pm\infty$ values to make the discussion simpler.

Definition 4.14 (Parabolic viscosity supersolution)**.** *A function $u : \Omega \times (0,T) \to (-\infty, \infty)$ is a viscosity supersolution if*

- *(i) u is lower semicontinuous*
- *(ii) whenever $\varphi \in C^2(\Omega\times(0,T))$ touches u at $(x,t) \in \Omega\times(0,T)$ from below and*

$$
\varphi_t(x,t) + F(x,t,D\varphi,D^2\varphi) \geq 0.
$$

Remark 4.15.
- *The definition for subsolution is analogous*
- *There is no problem of stating the definitions in a more general domain $D \subset \mathbb{R}^{n+1}$ instead of $\Omega \times (0,T)$.*

- *We could also use the definition in terms of parabolic semi jets:* $(a,p,X) \in P^{2,+}u(x,t)$ *if*
$$u(y,s) \le u(x,t) + a(s-t) + p\cdot(y-x) + \frac{1}{2}\langle X(y-x), y-x\rangle + o(|t-s| + |y-x|^2).$$
The definitions of $P^{2,-}u(x,t)$, $\overline{P}^{2,-}u(x,t)$ *and* $\overline{P}^{2,+}u(x,t)$ *are then analogous to the elliptic case.*
- *Future does not affect: When touching* u *from below at* (x,t) *(analogously from above), it is enough to require that* $\varphi(x,t) = u(x,t)$ *and* $\varphi \le u$ *in* $(\Omega\times(0,T))\cap\{s : s<t\}$ *as long as comparison holds, see Juutinen, PAMS, 2001.*

We define the parabolic boundary
$$\partial_p(\Omega\times(0,T)) = (\overline{\Omega}\times\{0\}) \cup (\partial\Omega\times(0,T)) \tag{4.10}$$

Instead of working out the most general structural conditions, we again look at the Laplacian.

Theorem 4.16. *Consider* $u_t + F(x,Du,D^2u) := u_t - \Delta u = 0$. *Further, let* $u \in USC(\overline{\Omega}\times[0,T))$ *be a viscosity subsolution, and* $v \in LSC(\overline{\Omega}\times[0,T))$ *a viscosity supersolution.*

$$\textit{If } u \le v \textit{ on } \partial_p(\Omega\times(0,T)), \textit{ then } u\le v \textit{ on } \overline{\Omega}\times[0,T).$$

Proof. First, decreasing T if necessary we may assume that u is bounded from above and v is bounded from below. Counter proposition:
$$\sup_{\Omega_T}(u-v) =: \theta > 0.$$
We may assume that v is a strict supersolution i.e.
$$v_t - \Delta v > 0$$
in the viscosity sense by considering instead $v + \frac{\eta}{T-t}$ (ex), and $v(x,t)\to\infty$ as $t\to T$. Observe that we still have essentially the same counter proposition by choosing $\eta>0$ small enough. Let
$$\begin{aligned}\Phi(x,t,y,s) &= u(x,t) - v(y,s) - \varphi(x,t,y,s)\\ &:= u(x,t) - v(y,s) - \frac{1}{2\varepsilon}|x-y|^2 - \frac{1}{2\varepsilon}(t-s)^2, .\end{aligned}$$
There is $(x_\varepsilon,t_\varepsilon,y_\varepsilon,s_\varepsilon)\in\overline{\Omega}\times[0,T)\times\overline{\Omega}\times[0,T)$ such that
$$\Phi(x_\varepsilon,t_\varepsilon,y_\varepsilon,s_\varepsilon) = \max_{\overline{\Omega}\times[0,T)\times\overline{\Omega}\times[0,T)}\Phi(x,t,y,s).$$
Now, provided that $(x_\varepsilon,t_\varepsilon,y_\varepsilon,s_\varepsilon)$ is not at the boundary, then
$$(x,t)\mapsto u(x,t) - \varphi(x,t,y_\varepsilon,s_\varepsilon), \quad (y,s)\mapsto v(y,s) - (-\varphi(x_\varepsilon,t_\varepsilon,y,s))$$

have max at $(x_\varepsilon, t_\varepsilon)$ and min at $(y_\varepsilon, s_\varepsilon)$ respectively, and we have admissible test functions.

Using again Theorem of sums, we obtain

$$(\varphi_t, D_x\varphi, X) \in \overline{P}^{2,+}u(x_\varepsilon, t_\varepsilon), \quad (-\varphi_s, -D_y\varphi, Y) \in \overline{P}^{2,-}v(y_\varepsilon, s_\varepsilon),$$

with $X \leq Y$. Then

$$\begin{aligned} 0 &< (-\varphi_s) - \operatorname{tr}(Y) - (\varphi_t - \operatorname{tr}(X)) \\ &= \frac{t-s}{\varepsilon} - \frac{t-s}{\varepsilon} + \operatorname{tr}(X - Y) \leq 0, \end{aligned}$$

a contradiction.

It remains to verify that $(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon) \in \Omega \times (0,T) \times \Omega \times (0,T)$. To this end,

$$\begin{aligned} \frac{1}{2\varepsilon}\left|x_\varepsilon - y_\varepsilon\right|^2 + \frac{1}{2\varepsilon}(t_\varepsilon - s_\varepsilon)^2 &\leq \underbrace{\sup_{\overline{\Omega}\times[0,T)} u - \inf_{\overline{\Omega}\times[0,T)} v}_{=:M} - \theta \\ &\leq M \overset{\text{semicont}}{<} \infty. \end{aligned}$$

This then shows, passing to a subsequence if necessary, that $(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon) \to (\hat{x}, \hat{t}, \hat{x}, \hat{t})$ by using the above estimate and compactness of $\overline{\Omega} \times [0,T]$. It also holds that $(\hat{x}, \hat{t})$ is not on the (Euclidean) boundary of Ω_T. Obviously $(\hat{x}, \hat{t}) \notin \Omega \times \{T\}$ since $-v(x,t) \to -\infty$ when $t \to T$. If $(\hat{x}, \hat{t}) \in \partial_p\Omega_T$, then

$$\theta \leq \limsup_{\varepsilon} u(x_\varepsilon, t_\varepsilon) - v(y_\varepsilon, s_\varepsilon) \leq \limsup u - \liminf v \overset{\text{bdr cond, lsc usc}}{\leq} 0.$$

a contradition. □

6.2.2015

Remark 4.17. *As it will be useful for the parabolic equations having gradient singularity like normalized p-parabolic equation ($u_t = (\Delta u + (p-2)\langle D^2u\frac{Du}{|Du|}, \frac{Du}{|Du|}\rangle$), we point out how to take the gradient singularity into account, still in the case of the heat equations for simplicity. Let*

$$\begin{aligned} \Phi(x,t,y,s) &= u(x,t) - v(y,s) - \varphi(x,t,y,s) \\ &:= u(x,t) - v(y,s) - \frac{1}{q\varepsilon}\left|x-y\right|^q - \frac{1}{2\varepsilon}(t-s)^2, \end{aligned}$$

with $q > 2$, and max point $(x_\varepsilon, t_\varepsilon, y_\varepsilon, s_\varepsilon)$ as usual. There are two cases to be checked.

Case $x_\varepsilon \neq y_\varepsilon$**:** *This case is exactly similar to the case above, i.e. use Theorem of sums and obtain X and Y from the semi jets such that $X \leq Y$.*

Case $x_\varepsilon = y_\varepsilon$
Idea is that now the gradient is zero, but so are the second derivatives. Therefore operators of the type can still be defined consistently $\langle D^2u\frac{Du}{|Du|}, \frac{Du}{|Du|}\rangle = 0$.

The calculation for the heat equation: We have

$$(\varphi_t, D_x\varphi, D_{xx}\varphi) \in P^{2,+}u(x_\varepsilon, t_\varepsilon), \quad (-\varphi_s, -D_y\varphi, -D_{yy}\varphi) \in P^{2,-}v(y_\varepsilon, s_\varepsilon).$$

Here

$$\varphi_t = \frac{t-s}{\varepsilon} = -\varphi_s,\ D_x\varphi = \frac{1}{\varepsilon}\left|x-y\right|^{q-2}(x-y) = -D_y\varphi,$$
$$D_{xx}\varphi = D_{yy}\varphi = \frac{1}{\varepsilon}\left|x-y\right|^{q-2}((q-2)\frac{x-y}{|x-y|}\otimes\frac{x-y}{|x-y|} + I),$$

In this case $D_{xx}\varphi = -D_{yy}\varphi = 0$. Since u is sub and v is strict super, we have

$$\begin{aligned} 0 &< (-\varphi_s) - \operatorname{tr}(-D_{yy}\varphi) - (\varphi_t - \operatorname{tr}(D_{xx}\varphi)) \\ &= \frac{t-s}{\varepsilon} - \frac{t-s}{\varepsilon} + 0 = 0, \end{aligned}$$

a contradiction.

Remark 4.18.
- *Comparison for parabolic equations in unbounded domains with linear growth bounds? See Giga, Goto, Ishii, Sato: Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains, 1990.*
- *Comparison for parabolic equations of Hamilton-Jacobi-Bellman-Isaacs type in unbounded domains with more general growth bounds? See Buckdahn, Li: Stochastic differential games and viscosity solutions of Hamilton-Jacobi-Bellman-Isaacs type, 2008. (this is topic in Stochastic/PDE reading seminar at the time of this course).*

4.2. **p-Laplace case.** Next we consider nonlinear F that is p-Laplace operator for $p > 2$

$$-\Delta_p u := -\operatorname{div}(\left|Du\right|^{p-2} Du).$$

Whenever u is smooth and $Du \neq 0$, we may write after a short calculation

$$-\Delta_p u = -\left|Du\right|^{p-2}(\Delta u + (p-2)\Delta_\infty^N u) =: F(Du, D^2u).$$

where $\Delta_\infty^N u = \langle D^2u\frac{Du}{|Du|}, \frac{Du}{|Du|}\rangle$ is so called normalized infinity Laplacian. We also define $F(0, X) = 0$.

Remark 4.19. *Observe that the definition is always satisfied if $D\varphi? = 0$. Thus we can neglect such test functions.*

In order to prove the comparison, we need auxiliary lemmas. Idea is that we again want to guarantee the strict inequality, and to do this we use (distributional) weak solutions (recall PDE2). We will also use a fact that weak solution is always a viscosity solution:

Lemma 4.20. *Let $v \in W^{1,p}(\Omega)$ be p-harmonic (in the weak sense). Let v_ε be unique weak solution to*

$$\begin{cases} -\Delta_p v_\varepsilon = \varepsilon \\ v_\varepsilon = v & \text{on } \partial\Omega. \end{cases}$$

Then there is a subsequence $v_\varepsilon \to v$ locally uniformly.

Proof. Then in weak formulations

$$\int |Dv|^{p-2} Dv \cdot D\varphi \, dx = 0$$
$$\int |Dv_\varepsilon|^{p-2} Dv_\varepsilon \cdot D\varphi \, dx = \int \varepsilon\varphi \, dx,$$

we use $\varphi = v - v_\varepsilon \in W_0^{1,p}(\Omega)$ as a test function, and subtract the equations. Then

$$\begin{aligned} C \int |Dv - Dv_\varepsilon|^p \, dx &\overset{\text{known ineq}}{\leq} \int (|Dv|^{p-2} Dv - |Dv_\varepsilon|^{p-2} Dv_\varepsilon) \cdot D(v - v_\varepsilon) \, dx \\ &= \int \varepsilon(v - v_\varepsilon) \, dx \\ &\overset{\text{Sobo ie}}{\leq} C\varepsilon(\int |Dv - Dv_\varepsilon|^p \, dx)^{1/p}, \end{aligned}$$

where the proof for the first inequality can be found for example from 'Lindqvist: Notes on the p-Laplace equation'. This implies

$$\int_\Omega |Dv - Dv_\varepsilon|^p \, dx \leq C\varepsilon^{p/(p-1)}. \tag{4.11}$$

This and Sobolev's inequality implies that $v_\varepsilon \to v$ in $W^{1,p}(\Omega)$ as $\varepsilon \to 0$, where $K \Subset \Omega$.

We state without a proof that uniform local Hölder regularity estimates and Arzelà-Ascoli theorem we may choose a converging subsequence still denoted by v_ε such that

$$v_\varepsilon \to w$$

uniformly in K. But by (4.11), the it holds that $v = w$. □

We may restrict ourselves to smooth domains by exhaustion. Indeed, choose a smooth $D \Subset \Omega$ such that $u \leq v + \varepsilon$ in $\Omega \setminus D$. Then there is $\varphi \in C^\infty(\Omega)$ such that

$$u \leq \varphi \leq v + \varepsilon \text{ on } \partial D$$

which is possible by semicontinuity. Then we solve

$$\begin{cases} -\Delta_p h = 0 \\ h = \varphi & \text{on } \partial D \end{cases}$$

for the unique $h \in W^{1,p}(D) \cap C(\overline{D})$ (the smoothness of D quarantees that the boundary values are attained continuously).

Second, by Lemma 4.20 we may approximate $h_\varepsilon \to h$ locally uniformly in D. In order to prove comparison for $u \leq v$ it suffices to prove comparison for $u \leq h_\varepsilon$, the other side is analogous. Without a proof we also state that h_ε is a viscosity supersolution

Theorem 4.21. *Let F be as above. Further, let $u \in USC(\overline{\Omega})$ be a viscosity subsolution, and $v \in LSC(\overline{\Omega})$ a viscosity supersolution.*

$$\textit{If } u \leq v \textit{ on } \partial\Omega, \textit{ then } u \leq v \textit{ on } \overline{\Omega}.$$

Proof. According to what was explained above we may assume that v is a weak solution (which is also a viscosity supersolution as explained in the lecture) to

$$-\Delta_p v = \delta.$$

Counter proposition:

$$\sup_{\Omega}(u - v) =: \theta > 0.$$

Let

$$\begin{aligned} \Phi(x, y) &= u(x) - v(y) - \varphi(x, y) \\ &:= u(x) - v(y) - \frac{1}{2\varepsilon} |x - y|^2 , \end{aligned}$$

and max point $(x_\varepsilon, y_\varepsilon)$ as usual.

Then by theorem of sums there are

$$(D_x\varphi(x_\varepsilon, y_\varepsilon), X) \in \overline{J}^{2,+} u(x_\varepsilon), \quad (-D_y\varphi(x_\varepsilon, y_\varepsilon), Y) \in \overline{J}^{2,-} v(y_\varepsilon).$$

and $X \leq Y$. Thus we have

$$\begin{aligned} &\delta \leq F(-D_y\varphi(x_\varepsilon, y_\varepsilon), Y) - F(D_x\varphi(x_\varepsilon, y_\varepsilon), X) \\ &\overset{q=D_x\varphi}{=} - |q|^{p-2} (\mathrm{tr}(Y) + (p-2)\langle Y\frac{q}{|q|}, \frac{q}{|q|}\rangle) + |q|^{p-2} (\mathrm{tr}(X) + (p-2)\langle X\frac{q}{|q|}, \frac{q}{|q|}\rangle \\ &\leq |q|^{p-2} \mathrm{tr}(X - Y) + (p-2)\langle (X - Y)\frac{q}{|q|}, \frac{q}{|q|}\rangle \leq 0, \end{aligned}$$

a contradiction (the fact that $x_\varepsilon, y_\varepsilon$ are not at the boundary follows as before).

Observe that $x_\varepsilon \neq y_\varepsilon$ since otherwise $-D_y\varphi = 0$ and this contradict the strict inequality for v. $\square$

5. Existence

5.1. **Perron's method.** Denote by

$$u^*(x) = \lim_{r\to 0} \sup_{y\in B_r(x)\cap\overline{\Omega}} u(y)$$
$$u_*(x) = \lim_{r\to 0} \inf_{y\in B_r(x)\cap\overline{\Omega}} u(y).$$

the upper and lower semicontinuous envelopes of u. It holds that $u_* \le u \le u^*$. That they are USC/LSC is exercise.

Perron's method, which in this context is due to Ishii 1987. Idea

- Suppose that there are sub- and supersolution $u \in USC(\overline{\Omega})$ $v \in LSC(\overline{\Omega})$, $u \le v$ taking the same boundary values g in the sense that $v^* = g = u_*$.
- With
$$\mathcal{L} = \{w \,:\, w \text{ subsolution}, u \le w \le v\},$$
the function
$$h(x) = \sup\{w \,:\, w \in \mathcal{L}\}.$$
is a viscosity solution. This uses the following results: sup of subs is sub and sup of $\mathcal{L}$ is also super, when in both the cases the semicontinuous corrections are taken care of. Then comparison shows that corrections actually coincide.

Next we show that sup of sub solutions is a subsolution after upper semicontinuous correction (sup does not preserve upper semiontinuity) if finite.

Lemma 5.1 (Sup of subs is sub)**.** *Let S be a set of subsolutions and for $x \in \Omega$*

$$h(x) = \sup\{w \,:\, w \in S\}.$$

If $h^* < \infty$ in Ω, then h^* is a viscosity subsolution.

Proof. Let φ touch h^* from above at x. Let $B_{2r}(x) \subset \Omega$, and $\theta > 0$ such that

$$\max_{\partial B_r(x)} (h^* - \varphi) < \underbrace{(h^* - \varphi)(x)}_{=0} - \theta.$$

Then for any i choose $x_i \in B_r(x)$ such that $x_i \to x$ and

$$h^*(x) - \frac{1}{i} \overset{\text{def of } h^*}{\le} h(x_i)$$
$$|\varphi(x_i) - \varphi(x)| \le \frac{1}{i}.$$

Moreover, by def of sup, there is $h_i \in S$

$$h_i(x_i) + \frac{1}{i} \ge h(x_i).$$

Collecting the facts

$$h^*(x) - \frac{1}{i} \le h(x_i) \le h_i(x_i) + \frac{1}{i}.$$

Thus

$$\begin{aligned}\max_{\partial B_r(x)} (h_i - \varphi) &\le \max_{\partial B_r(x)} (h^* - \varphi) \\ &< (h^* - \varphi)(x) - \theta \\ &\le (h_i - \varphi)(x_i) - \theta + \frac{3}{i} \\ &< (h_i - \varphi)(x_i).\end{aligned}$$

This implies that there is a local max of $h_i - \varphi$ at some $x_r \in B_r(x)$, and thus

$$0 \ge F(x_r, D\varphi(x_r), D^2\varphi(x_r)) \overset{F\text{cont}}{\to} F(x, D\varphi(x), D^2\varphi(x))$$

by passing to a limit $r \to 0$. □

Lemma 5.2 (Bump construction)**.** *If $w \in \mathcal{L}$ and w_* is not a supersolution, then there is $\tilde{w} \in \mathcal{L}$ and z such that*

$$w(z) < \tilde{w}(z).$$

Proof. Thriving for a contradiction, suppose that w_* is not a supersolution. Thus we can choose φ touching w_* from below at a point x such that by continuity in $B_R(x)$ it holds

$$F(y, D\varphi, D^2\varphi) < 0. \tag{5.12}$$

In addition, considering $\varphi(y) - |x-y|^4$ if necessary we may assume that

$$\varepsilon + \varphi < w_* \text{ outside some } B_R(x)$$

where R may be chosen small by choosing smaller ε. Also observe that $\varphi(x) < v(x)$. Indeed otherwise φ would be admissible test function from below to v and this contradicts (5.12). Thus

$$\varphi + \varepsilon < v \text{ in } B_R(x)$$

for small enough ε. Thus

$$\tilde{w} := \max(\varphi + \varepsilon, w)$$

is the desired function. In particular, this is lower semicontinous and similarly as in the previous lemma this is seen to be subsolution. □

Theorem 5.3 (Perron's method)**.** *Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$, $u \le v$ be sub- and supersolutions such that $u_* > -\infty, v^* < \infty$ and $u_* = v^* = g$ on the boundary. Then there exists a viscosity solution h such that*

$$u \le h \le v.$$

Proof. Observe $u \in \mathcal{L} \neq \emptyset$, and define

$$h(x) = \sup\{w(x) \,:\, w \in \mathcal{L}\}.$$

By Lemma 5.1, h^* is a subsolution. On the boundary

$$g = u_* \leq h_* \leq h^* \leq v^* = g,$$

and thus all the inequalities above are equalities and

$$h_* = h = h^* \text{ on } \partial\Omega. \tag{5.13}$$

Thus since h^* is a subsolution and v a supersolution, we have

$$h^* \leq v$$

and thus $h^* \in \mathcal{L}$. But since h is a sup of such functions, we have

$$h^* = h \in \mathcal{L}.$$

Then if h_* is not a supersolution, there is $w \in \mathcal{L}$ and x such that $h(x) < w(x)$, but this contradicts sup in the definition of h. Thus h_* is a supersolution.

Also

$$h^* = h \overset{h_* \text{ super, comp, (5.13)}}{\leq} h_* \leq h^* \quad \text{on } \overline{\Omega},$$

and thus $h \in C(\overline{\Omega})$ is the desired solution. □

Example 5.4. *It is a nontrivial question, when we can find sub- and supersolutions for the Perron method. There are examples of the irregular domains where boundary values cannot be attained continuously. Here are examples when such solutions can be found.*

Consider

$$\begin{cases} |Du|^2 - f(x) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $f \in C(\overline{\Omega})$ and $\inf f > 0$. This equation satisfies a comparison principle by a similar proof as the eikonal equation. Then $u = 0$ is a subsolution since $-f(x) \leq 0$. Similarly as we have seen in 1D, $u(x) = \operatorname{dist}(x, \partial\Omega)$ is a solution to $|Du|^2 - 1 = 0$. Then choose $\gamma^2 \geq \sup_\Omega f$ and set $\gamma u(x) = \gamma \operatorname{dist}(x, \partial\Omega)$. We already know that γu solves $|D(\gamma u)|^2 - \gamma^2 = 0$ and thus γu is a supersolution to

$$|D(\gamma u)|^2 - f(x) \geq 0.$$

Now the assumptions of the Perron method are satisfies, and we obtain the existence.

13.2.2015

Example 5.5 ('Classical' barrier argument)**.** *Consider $F(D^2u) = -\Delta u$ and $f \in C(\overline{\Omega})$,* $\inf f > 0$

$$\begin{cases} F(D^2u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and that Ω satisfies the exterior ball type regularity condition: Let $R > 0$. For each $x_0 \in \partial\Omega$, there is a ball $B_R(x')$ at the complement of a domain touching Ω exactly at x_0. Then we construct a so called barrier:

$$v(x) = e^{-\alpha R^2} - e^{-\alpha r^2},$$

where $r = |x - x'|$. In particular,

$$Dv = 2\alpha e^{-\alpha r^2}(x - x'), \quad D^2v = 2\alpha e^{-\alpha r^2}(I - 2\alpha(x - x') \otimes (x - x')),$$

Thus

$$-\operatorname{tr}(D^2v) = -2\alpha e^{-\alpha r^2}(n - 2\alpha r^2) \geq \kappa$$

large enough α and some positive κ. Then by choosing $M > 0$ large enough such that $\kappa M \geq \sup f$, then Mv is a supersolution. Also $Mv(x_0) = 0$ but elsewhere the boundary values do not hold. However, for $\tilde{v}(x) = \inf_{x_0 \in \partial\Omega} v(x)$ they hold and this has the properties

- *$\tilde{v} \in C(\overline{\Omega})$ is a supersolution in Ω (each v is uniformly continuous)*
- *$\tilde{v}(x) = 0, \ x \in \partial\Omega$.*

Again $u = 0$ is suitable subsolution.

The method can also be extended to

$$\begin{cases} F(D^2u) = -\Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

for $g \in C(\overline{\Omega})$. For large enough α again the above barrier is a supersolution Then observe that for any ε we can choose large enough $M > 0$, such that

$$v_{\varepsilon,M}(x) := \varepsilon + g(x_0) + Mv(x)$$

and the supersolution can be constructed by taking $\inf$*s of such functions. By a similar construction from below for subsolutions we see that the assumptions of the Perron's method are satisfied.*

Example 5.6 (Heat equation in $\mathbb{R}^n \times (0,T)$)**.** *Let g be a Lipschitz continuous function with the Lipschitz constant L and bounded. In the reading seminar, we saw that the comparison holds for*

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u(x,0) = g(x) \end{cases}$$

say in the class of solutions obeying linear growth conditions at infinity. Let $y \in \mathbb{R}^n$. Consider $0 < \varepsilon < 1$, and let

$$v(x,t) = g(y) + \frac{A}{\varepsilon^{1/2}} t + 2L(|x-y|^2 + \varepsilon)^{1/2},$$
$$u(x,t) = g(y) - \frac{A}{\varepsilon^{1/2}} t - 2L(|x-y|^2 + \varepsilon)^{1/2}.$$

Then, we can choose A, independent of y, ε, so that v and u are viscosity super- and subsolutions. To see this, calculate

$$\begin{aligned} D_x u &= -L(|x-y|^2+\varepsilon)^{-1/2} 2(x-y), \\ D_{xx} u &= \frac{1}{2} L(|x-y|^2+\varepsilon)^{-3/2} 4(x-y) \otimes (x-y) - L(|x-y|^2+\varepsilon)^{-1/2} 2I \\ &= 2L(|x-y|^2+\varepsilon)^{-1/2} (L(|x-y|^2+\varepsilon)^{-1}(x-y)\otimes(x-y) - I) \\ u_t &= -\frac{A}{\varepsilon^{1/2}}. \end{aligned}$$

Then

$$\begin{aligned} \operatorname{tr}(D_{xx}u) &= 2L(|x-y|^2+\varepsilon)^{-1/2}(L(|x-y|^2+\varepsilon)^{-1}\left|x-y\right|^2 - n) \\ &\geq -2L(|x-y|^2+\varepsilon)^{-1/2} n \\ &\geq -2Ln \frac{1}{\varepsilon^{1/2}} \\ &\overset{A=2Ln}{\geq} -\frac{A}{\varepsilon^{1/2}} = u_t. \end{aligned}$$

By taking sups *over such solutions, one can obtain a suitable subsolution. The argument for supersolutions is analogous.*

5.2. **Stability and existence through stability principle.** Viscosity solutions have certain nice stability properties, as the following theorem shows.

Theorem 5.7. *Let $u_i \in C(\Omega)$ be viscosity solutions to $F_i(x_i, Du_i, D^2u_i) = 0$ and suppose that*

$$u_i \to u \quad \text{locally uniformly, and}$$
$$F_i(x_i, p_i, X_i) \to F(x,p,X) \quad \text{whenever} \quad (x_i,p_i,X_i) \to (x,p,X).$$

Then u is a viscosity solution to $F(x, Du, D^2u) = 0$.

Proof. Suppose that φ touches u from below at x. Then we know that there is $x_i \to x$ such that

$$u_i - \varphi \text{ has min at } x_i.$$

Thus

$$0 \leq F_i(x_i, D\varphi(x_i), D^2\varphi(x_i)) \overset{\text{assump}}{\to} F(x, D\varphi(x), D^2\varphi(x)).$$

We omit the analogous argument for the reverse inequality. □

So far we only consider continuous F, but next we consider bounded discontinuities for F. To be more precise assume that $(x, q, X) \mapsto F(x, q, X)$ has a discontinuity at $q = 0$ but it is continuous elsewhere. Since F is now (bounded) discontinuous at $Du = 0$, in order to define viscosity solutions, we will have to say something about this case. The standard way is to replace F by its semicontinuous envelopes

$$F^*(x, q, X) := \limsup_{(y,\tilde{q},Y)\to(x,q,X)} F(y, \tilde{q}, Y) := \lim_{r\to 0} \sup_{y\in B_r(x), |\tilde{q}-q|<r, ||X-Y||<r} F(y, \tilde{q}, Y),$$

$$F_*(x, q, X) = \liminf_{(y,\tilde{q},Y)\to(x,q,X)} F(y, \tilde{q}, Y).$$

Example 5.8. *For normalized p-Laplacian $p > 2$, simply*

$$F^*(q, X) = \begin{cases} F(q, X) & q \neq 0 \\ F^*(0, X) & q = 0 \end{cases} = \begin{cases} -(\mathrm{tr}(X) + (p-2)\langle X \frac{q}{|q|}, \frac{q}{|q|}\rangle) & q \neq 0 \\ -(\mathrm{tr}(X) + (p-2)\lambda_{min}(X)\rangle) & q = 0, \end{cases}$$

where $\lambda_{min}(X)$ denotes the minimum eigenvalue of X.

Definition 5.9 (Viscosity supersolution)**.** *A function $u : \Omega \to (-\infty, \infty]$ is a viscosity supersolution if*

(i) u is lower semicontinuous
(ii) $u \not\equiv \infty$ and
(iii) whenever $\varphi \in C^2(\Omega)$ touches u at $x \in \Omega$ from below and

$$F^*(x, D\varphi(x), D^2\varphi(x)) \geq 0.$$

Definition 5.10 (Viscosity subsolution)**.** *A function $u : \Omega \to [-\infty, \infty)$ is a viscosity subsolution if*

(i) u is upper semicontinuous
(ii) $u \not\equiv -\infty$ and
(iii) whenever $\varphi \in C^2(\Omega)$ touches u at $x \in \Omega$ from above and

$$F_*(x, D\varphi(x), D^2\varphi(x)) \leq 0.$$

A function $u : \Omega \to \mathbb{R}$ is a viscosity solution if it is both viscosity super- and a subsolution.

The next example motivates the above definition.

Example 5.11. *Let $p_i \to p$,*

$$F_i(u, Du, D^2u) = -(\Delta u + (p_i - 2)\langle D^2u \frac{Du}{|Du|}, \frac{Du}{|Du|}\rangle) = 0$$

and u_i a solution to this equation. Suppose that $C(\Omega) \ni u_i \to u$ uniformly which follows from the regularity theory and Arzela-Ascoli, passing to a subsequence, but we take it for granted now. Suppose that φ touches u from below at x.

First suppose that $D\varphi(x) \neq 0$. *Then we know that there is* $x_i \to x$ *such that*

$$u_i - \varphi \text{ has min at } x_i.$$

Moreover, for large enough i, *it holds that* $D\varphi(x_i) \neq 0$

$$\begin{aligned} 0 &\leq -(\Delta\varphi(x_i) + (p_i - 2)\langle D^2\varphi(x_i)\frac{D\varphi(x_i)}{|D\varphi(x_i)|}, \frac{Du(x_i)}{|Du(x_i)|}\rangle) \\ &\to -(\Delta\varphi(x) + (p-2)\langle D^2\varphi(x)\frac{D\varphi(x)}{|D\varphi(x)|}, \frac{Du(x)}{|Du(x)|}\rangle). \end{aligned}$$

Then suppose that $D\varphi(x) = 0$. *Then there are two cases:*
Case $D\varphi(x_j) \neq 0$ **for some large** j**:** *Then similarly as above for these* x_j*:s*

$$\begin{aligned} 0 &\leq \limsup_{x_j \to x} -(\Delta\varphi(x_i) + (p_i - 2)\langle D^2\varphi(x_i)\frac{D\varphi(x_i)}{|D\varphi(x_i)|}, \frac{Du(x_i)}{|Du(x_i)|}\rangle) \\ &\leq \limsup_{y \to x} F(y, D\varphi(y), D^2\varphi(y)) \\ &\leq F^*(x, D\varphi(x), D^2\varphi(x)). \end{aligned}$$

Case $D\varphi(x_j) = 0$ **for all large** j**:**

$$\begin{aligned} 0 &\leq \limsup_{x_j \to x}(F^*(x_j, 0, D^2\varphi(x_j))) \\ &\leq F^*(x, 0, D^2\varphi(x)), \end{aligned}$$

since also $\left|\left|D^2\varphi(y) - D^2\ \varphi(x)\right|\right| \to 0$ *as* $x \to y$. *Thus the limit is a solution.*

The same notion of a solution also works in the parabolic case.

Example 5.12. *To guarantee comparison boundedness assumptions are needed:*

$$-\infty < F_*(x, 0, 0) \leq F^*(x, 0, 0) < \infty.$$

Indeed, let $F(q, X) = -\langle Xq, q\rangle / |q|^{2+\alpha}$, $\alpha > 0$, *and consider*

$$\begin{cases} u_t + F(Du, D^2u) = 0 & \text{in } \mathbb{R}^n \times (0, 1) \\ u(x, 0) = 0 \end{cases}$$

in the above sense. Then $v = 0$ *is a (super)solution. In particular,* φ *from below*

$$\varphi_t + \underbrace{F^*(x, 0, 0)}_{=\infty} = \infty \geq 0.$$

On the other hand,

$$v = \begin{cases} 1 & t \geq \frac{1}{2} \\ 0 & t < \frac{1}{2}, \end{cases}$$

is a (sub)solution, contradicting the comparison.

Example 5.13 (Existence for infinity parabolic equation)**.** *Consider next* $u_t + F(x, D\varphi, D^2\varphi) := u_t - \Delta_\infty^N u = 0$ *in* $\Omega_T = \Omega \times (0,T)$ *where* $\Delta_\infty^N u = \langle D^2 u \frac{Du}{|Du|}, \frac{Du}{|Du|}\rangle$ *and where the solution is interpreted in the above sense at singularities.*

Next regularize the equation considering

$$u_t = \varepsilon \Delta u + \frac{1}{|Du|^2 + \delta^2} \langle D^2 u Du, Du\rangle$$

and estimate continuous boundary values by smooth ones.

The right hand side is continuous uniformly elliptic operator, and thus with smooth initial and boundary values, it has a unique solutions by the classical results (Ladyzhenskaja, Solonnikov, Uraltseva, Chapter 7) . Then idea is to derive regularity estimates independent of ε, δ *by the barrier arguments. Then pass to a limit*

$$\varepsilon, \delta \to 0$$

and use stability (see above) to verify that the limit is a solution. Then pass to the original boundary values and use again stability.

6. Regularity up to $C^{1,\alpha}$

6.1. **Uniformly elliptic equations.** In this section we consider the equations of the form

$$F(x, D^2 u) = f(x)$$

where f, F are continuous and sometimes

$$F(D^2 u) = 0.$$

Since we are interested in regularity properties, we drop semi continuities and assume continuity super(sub)solutions.

We adjust our sign convention to be consistent with Caffarelli-Cabre, i.e. now for example $F(D^2u) = \Delta u$.

Further, F is uniformly elliptic in the sense

Definition 6.1 (Uniformly elliptic)**.** F *is uniformly elliptic with ellipticity constant* $0 < \lambda \le \Lambda < \infty$*, if for any* $M \in S,\ x \in \Omega$ *it holds that*

$$\lambda \,||N|| \le F(x, M+N) - F(x,M) \le \Lambda \,||N|| \,,$$

for all $N \ge 0$*, where* $||N|| = \sup\{|\langle N\xi, \xi\rangle| \,:\, |\xi| \le 1\}$ *as before.*

Example 6.2. *Let* $F = \Delta u$*. Then*

$$F(M+N) - F(M) = \operatorname{tr}(N) = \sum \lambda_i \le n\,||N||\,.$$

Further, $\sum \lambda_i \ge ||N||$*.*

Example 6.3 (Uniformly elliptic is degenerate elliptic)**.** *After changing sign convention, we want to show $F(x,X) \leq F(x,Y)$ whenever $X \leq Y$. For any matrix $X \leq Y$, $X, Y \in S^n$, $Y = X + (Y - X)$ where $Y - X \geq 0$*

$$\begin{aligned} F(x,Y) &= F(x, X + (Y - X)) - F(x,X) + F(x,X) \\ &\geq \lambda \, ||X - Y|| + F(x,X) \\ &\geq F(x,X). \end{aligned}$$

Lemma 6.4. *F is uniformly elliptic if and only if*

$$\lambda' \operatorname{tr}(N) \leq F(x, M+N) - F(x,M) \leq \Lambda' \operatorname{tr}(N).$$

for $N \geq 0$.

Proof.

$$||N|| \leq \operatorname{tr}(N) \leq n \, ||N|| \, .$$

□

Lemma 6.5. *F is uniformly elliptic if and only if*

$$F(x, M+N) - F(x,M) \leq \Lambda \left|\left|N^+\right|\right| - \lambda \left|\left|N^-\right|\right|,$$

where $N = N^+ - N^-$ with $N^+, N^- \geq 0$.

Proof. $' \Leftarrow '$ If $N \geq 0$ it holds that $N = N^+$ and thus

$$F(x,M) - F(x, M+N) \leq \Lambda \, ||N||$$

Then let $M = \tilde{M} + \tilde{N}$ and $N = -\tilde{N}$, where $\tilde{N} \geq 0$ and observe

$$F(x,\tilde{M}) - F(\tilde{M} + \tilde{N}) = F(x, M+N) - F(x,M) \leq \underbrace{\Lambda \left|\left|N^+\right|\right|}_{=0} - \lambda \underbrace{\left|\left|N^-\right|\right|}_{=||-\tilde{N}||}.$$

Then multiply by -1 the last estimate.

'$\Rightarrow$' It holds that

$$\lambda \left|\left|N^-\right|\right| \leq F(M + N^+) - F(M + \underbrace{N}_{=N^+ - N^-}) \leq \Lambda \left|\left|N^-\right|\right|$$

ie

$$-\Lambda \left|\left|N^-\right|\right| \leq F(M+N) - F(M+N^+) \leq -\lambda \left|\left|N^-\right|\right|.$$

Also

$$\lambda \left|\left|N^+\right|\right| \leq F(M+N^+) - F(M) \leq \Lambda \left|\left|N^+\right|\right|$$

and by summing up

$$\begin{aligned} \lambda \left|\left|N^+\right|\right| - \Lambda \left|\left|N^-\right|\right| &\leq F(M+N) - F(M+N^+) + F(M+N^+) - F(M) \\ &\leq \Lambda \left|\left|N^+\right|\right| - \lambda \left|\left|N^-\right|\right|. \end{aligned}$$

□

Example 6.6. *Suppose that* $a_{ij} \in C^1(\Omega)$ *and consider the operator in divergence form*

$$Lu = \sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u(x)) = \sum_{i,j=1}^{n} D_i a_{ij}(x) D_j u(x) + a_{ij}(x) D_{ij} u(x).$$

L *is a divergence form operator. In PDE2 we said the divergence operator is uniformly elliptic if there exists constants*

$$0 < \lambda \leq \Lambda < \infty$$

such that the eigenvalues of the coefficient matrix $\mathcal{A}$ *are in* $[\lambda, \Lambda]$.

Next let

$$F(x, Du^2) = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u(x)$$

$$\begin{aligned} F(x, M+N) - F(x, M) &= \sum_{i,j=1}^{n} a_{ij}(x)(m_{ij} + n_{ij}) - \sum_{i,j=1}^{n} a_{ij}(x) m_{ij} \\ &= \sum_{i,j=1}^{n} a_{ij}(x) n_{ij} \\ &= \operatorname{tr}(\mathcal{A}N). \end{aligned}$$

$$\begin{aligned} \lambda \,||N|| &\overset{below}{\leq} \operatorname{tr}(\mathcal{A}N) \\ &\overset{\operatorname{tr}=\sum \lambda_i}{\leq} n\,||AN|| \leq n\Lambda \,||N|| \,. \end{aligned}$$

Choose v $|v| = 1$*, an eigenvector corresponding to max eigenvalue of* N *to see that* $\operatorname{tr}(AN) \geq ||AN|| \geq ANv = Av\lambda_{N,max} \geq \lambda \,||N||$*. Thus linear uniformly elliptic PDE is uniformly elliptic in the present sense.*

20.2.2015

Remark 6.7. *Observe that the first order operator is also degenerate elliptic but not uniformly elliptic second order operator.*

Definition 6.8 (Pucci operators)**.** *Let* $0 < \lambda \leq \Lambda < \infty$*. Denote by* $\mathcal{A}_{\lambda,\Lambda} \subset S^n$ *matrices whose eigenvalues belong to* $[\lambda, \Lambda]$*. Pucci operators for* $X \in S^n$ *are defined as*

$$\begin{aligned} P^+(X) &:= \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AX) \\ P^-(X) &:= \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AX) \end{aligned}$$

Similarly as for the linear operator, $P^{\pm}$ are uniformly elliptic with ellipticity constants λ and $n\Lambda$ (ex).

Pucci operators can be equivalently defined as

$$\begin{aligned} P^+(X) &= \lambda \sum_{\lambda_i<0} \lambda_i + \Lambda \sum_{\lambda_i>0} \lambda_i \\ P^-(X) &= \Lambda \sum_{\lambda_i<0} \lambda_i + \lambda \sum_{\lambda_i>0} \lambda_i, \end{aligned} \tag{6.14}$$

where λ_i are the eigenvalues of X.

Notation: if we want to emphasize the ellipticity constants, then we write $P^{\pm}_{\lambda,\Lambda}$.

Heurisitics: we will look at u such that

$$P^-(D^2u) \le f(x) \quad (super)$$

$$P^+(D^2u) \ge f(x) \quad (sub).$$

This is a rather large class: for example it covers all the equations $\mathrm{tr}(\mathcal{A}X) = f$ or $\sup_a \inf_b(\mathrm{tr}(\mathcal{A}(a,b)X) = f$ type equations when α satisfies the uniform ellipticity assumptions. Moreover, if u is a subsolution to the fully nonlinear uniformly elliptic equation, then it is also subsolution to a suitable Pucci problem.

Lemma 6.9. *Consider $F(x, D^2u) = f(x)$ fully nonlinear uniformly elliptic equation. Let u be a subsolution to this problem. Then u is a subsolution*

$$P^+_{\lambda/n,\Lambda}(D^2u) \ge f(x) - F(x, 0)$$

More generally if $\phi \in C^2$, then $u - \phi$ is a subsolution to

$$P^+_{\lambda/n,\Lambda}(D^2v) \ge f(x) - F(x, D^2\phi).$$

Proof. We only prove the second statement. Let φ touch $u - \phi$ from above. Then $\varphi + \phi$ touches u from above and

$$\begin{aligned} f(x) &\le F(x, D^2\varphi + D^2\phi) \\ &\overset{(6.5)}{\le} F(x, D^2\phi) + \Lambda \left|\left|(D^2\varphi)^+\right|\right| - \lambda \left|\left|(D^2\varphi)^-\right|\right| \\ &\le F(x, D^2\phi) + \Lambda \sum_{\lambda_i>0} \lambda_i + \frac{\lambda}{n} \sum_{\lambda_i<0} \lambda_i \\ &\overset{(6.14)}{=} F(x, D^2\phi) + P^+_{\lambda/n,\Lambda}(D^2\varphi) \end{aligned}$$

where λ_i denote the eigenvalues of $D^2\varphi$. □

Remark 6.10. *It is important to observe that what we actually accomplish in the proofs below is regularity at the first stage for u under the assumption*

$$\begin{aligned} P^-(D^2u) &\le f(x) \\ P^+(D^2u) &\ge f(x). \end{aligned}$$

Examples of equations whose solutions satisfy the above inequalities. Proofs are exercises.

- $P^{\pm}(D^2u) = f(x)$
- *Bellman equation:* $F(D^2u) = \inf_a(\mathrm{tr}(\mathcal{A}(a,x)X) = f(x)$ *where the eigenvalues of* $\mathcal{A}$ *are in* $[\lambda, \Lambda]$ *for any* x, a
- *Isaacs equation:* $F(D^2u) = \sup_a \inf_b(\mathrm{tr}(\mathcal{A}(a,b,x)X) = f(x)$ *where the eigenvalues of* $\mathcal{A}$ *are in* $[\lambda, \Lambda]$ *for any* x, a, b
- *Normalized p-Laplacian for* $p > 2$*:* $\Delta_p^N u := \Delta u + (p-2)\langle D^2 u \frac{Du}{|Du|}, \frac{Du}{|Du|}\rangle = f(x)$*. Observe that*

$$\Delta u + (p-2)\langle D^2 u \frac{Du}{|Du|}, \frac{Du}{|Du|}\rangle = \mathrm{tr}((I + (p-2)\frac{Du \otimes Du}{|Du|^2})D^2u)$$

and further that for any $\eta \in \mathbb{R}^n, |\eta| = 1$

$$\begin{aligned}\langle (I + (p-2)\frac{Du \otimes Du}{|Du|^2})\eta, \eta\rangle &= \eta^2 + (p-2)\frac{(\eta \cdot Du)^2}{|Du|^2} \\ &= 1 + (p-2)\frac{(\eta \cdot Du)^2}{|Du|^2} \in [1, p-1].\end{aligned}$$

Thus $I + (p-2)\frac{Du\otimes Du}{|Du|^2} \in \mathcal{A}_{\lambda,\Lambda}$ *with the constants* $1, p-1$*.*

6.2. **Harnack for viscosity solutions.** Very short history: Krylov-Safonov (1979) stochastic approach and Harnack for $\mathrm{tr}(A(x)D^2u) = 0$, Trudinger (1980) PDE proof, Caffarelli (1989) viscosity setting, fully nonlinear.

Denote $u \in S(\lambda, \Lambda, f)$ if

$$P^-(D^2u) \le f(x), \quad P^+(D^2u) \ge f(x),$$

or just S no confusion arises.

Our goal is to show:

Theorem 6.11 (Harnack's ie)**.** *Let* $0 \le u \in S$ *with* $f \in C(B_1)$ *and bounded. Then*

$$\sup_{B_{1/2}} u \le C(\inf_{B_{1/2}} u + ||f||_{L^n(B_1)}).$$

6.2.1. *ABP.* We will need Alexandrov-Bakelman-Pucci (ABP) estimate/max principle.

Theorem 6.12 (ABP, simple version)**.** *Let* $v \in C(\overline{B}_1)$ *be a solution to the uniformly elliptic problem* $F(D^2v) = \mathrm{tr}(D^2v) = f$ *with* $v \ge 0$ *on* ∂B_1*, and* $f \in C(B_1)$ *and bounded. Then*

$$\sup v_-^n \le C \int_D f_+^n \, dx,$$

where D *is the lower contact set*

$$D = \{x \in B_1 \ : \ \text{there is } p \text{ such that } v_-(y) \ge v_-(x) + \langle p, (y-x)\rangle \ \text{for all } y \in B_2\}.$$

Recall the area formula $g \in L^1(\mathbb{R}^n), \xi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ with $x = \xi(y)$

$$\int_{\xi(A)} |g(x)|\, dx \le \int_A |g(\xi(y))|\, |\det(D\xi(y))|\, dy$$

used in the proof.

For simplicity, we first assume that v is smooth solution. The proof consist of the following steps (recall the picture from the lecture):

(1) extend v by zero to B_2
(2) take the convex envelope to v from below. Denote it by

$$\begin{aligned}\Gamma(v) &= \sup_w \{w(x) \,:\, w \le v, w \text{ is convex in } B_2\} \\ &= \sup_L \{L(x) \,:\, a + \langle p, y - x\rangle =: L(y) \le v(y) \text{ in } B_2\}\end{aligned}$$

and the contact set by where $\Gamma(v) = v$ by D.
(3) then look at $x \mapsto D\Gamma(v)(x)$
(4) by the area formula

$$\text{Vol(image of the above function)} \le \int_D \det(D^2\Gamma(v)(x))\, dx$$

where the integration area is D because $\det(D^2\Gamma(v)(x))$ is zero outside D
(5) in the contact set D it holds that $0 \le D^2\Gamma(v)(x) \le D^2 v(x)$ and combining the previous estimates
(6)

$$\begin{aligned}\text{Vol(image)} &\le \int_D \det(D^2\Gamma(v)(x))\, dx \\ &\le \int_D \underbrace{\det(D^2 v(x))}_{=\prod \lambda_i \le \lambda_{\max}^n \le \operatorname{tr}(D^2 v)^n \le f_+^n} dx \\ &\le \int_D f_+^n\, dx\end{aligned}$$

(7) then lift from $-\infty$ up a plane ℓ with $|D\ell| < \sup v_-/4$ and thus this plane must touch $\Gamma(v)(x)$ somewhere within the B_1 and since ℓ is a tangent plane $D\ell$ is in the image. Since this holds for any plane with $|D\ell| < \sup v_-/4$, the image must contain a ball of radius $r = \sup v_-/4$ i.e.

$$(\sup v_-/4)^n \le C\text{Vol(image)},$$

and thus we have the ABP estimate.

Remark 6.13. *To extend the above reasoning for* $\operatorname{tr}(\mathcal{A}D^2 u) = f$ *where* $\mathcal{A} \in \mathcal{A}_{\lambda,\Lambda}$ *is uniformly elliptic observe that above we may estimate*

$$\det(D^2 v(x)) = \prod \lambda_i \le \lambda_{\max}^n \le \frac{1}{\lambda^n} \operatorname{tr}(\mathcal{A}D^2 v)^n \le \frac{f_+^n}{\lambda^n}$$

in D since $\operatorname{tr}(\mathcal{A}D^2v) \geq \lambda\lambda_{\max}$ where $\lambda_{\max}$ is the max eigenvalue of D^2v and $\lambda > 0$ the smallest eigenvalue of $\mathcal{A}$ as we have observed in Example 6.6.

Next we drop the regularity assumption and work with viscosity solutions. To this end, lets first consider regularizations of viscosity solutions. Let $\varepsilon > 0$ and

$$v_\varepsilon(x) = \inf_{y\in\Omega}(v(y) + \frac{1}{2\varepsilon}|x-y|^2)$$

be a so called inf-convolution. Suppose that v is lower semicontinuous and bounded by M. Denote

$$\Omega_\varepsilon = \{x \in \Omega \,:\, \operatorname{dist}(x, \partial\Omega) > 2(\varepsilon M)^{\frac{1}{2}}\}.$$

This convolution has the following properties

- $v_\varepsilon \nearrow v$
- $v_\varepsilon(x) - \frac{|x|^2}{2\varepsilon}$ is concave in Ω_ε
- v_ε is locally Lip
- The second Alexandrov derivatives exist a.e. (recall idea in the proof of Thm of sums)

Example 6.14. *Let $\Omega = (-1, 1)$,*

$$v(x) = \begin{cases} 0 & x \in (-1, 0] \\ 1 & x \in (0, 1). \end{cases}$$

Then

$$v_\varepsilon(x) = \begin{cases} 0, & x \in (-1, 0] \\ \frac{1}{2\varepsilon}|x|^2, & x \in (0, \sqrt{2\varepsilon}] \\ 1, & x \in (\sqrt{2\varepsilon}, 1). \end{cases}$$

Then

$$v_\varepsilon(x) - \frac{1}{2\varepsilon}x^2 = \begin{cases} -\frac{1}{2\varepsilon}x^2, & x \in (-1, 0] \\ 0, & x \in (0, \sqrt{2\varepsilon}] \\ 1 - \frac{1}{2\varepsilon}x^2, & x \in (\sqrt{2\varepsilon}, 1). \end{cases}$$

Lemma 6.15. *Let v be a bounded supersolution in Ω to $P^-(D^2v) \leq 0$. The functions v_ε are viscosity supersolutions to $P^-(D^2v_\varepsilon) \leq 0$ in $\Omega^\varepsilon = \{x \in \Omega \,:\, \operatorname{dist}(x, \partial\Omega) > 2(\varepsilon M)^{\frac{1}{2}}\}$ where $M = ||v||_{L^\infty(\Omega)}$.*

Proof. Fix $x \in \Omega^\varepsilon$. Observe that inf is attained at a point $\hat{x} \in \Omega$:

$$\frac{|x-\hat{x}|^2}{2\varepsilon} \leq \frac{|x-\hat{x}|^2}{2\varepsilon} + v(\hat{x}) + M = v_\varepsilon(x) + M \leq v(x) + M \leq 2M$$

i.e.

$$|x - \hat{x}| \leq 2(\varepsilon M)^{\frac{1}{2}}$$

but since $\operatorname{dist}(x, \partial\Omega) > 2(\varepsilon M)^{\frac{1}{2}}$, it follows that $\hat{x}$ is not at the boundary.

Next let $\varphi \in C^2(\Omega)$ touch v_ε from below at x. In particular,

$$\begin{aligned}
&\varphi(x) = v_\varepsilon(x) = v(\hat{x}) + \frac{1}{2\varepsilon}\left|x - \hat{x}\right|^2, \\
&\varphi(y) \le v_\varepsilon(y) \le v(z) + \frac{1}{2\varepsilon}\left|y - z\right|^2, \quad \text{for any } y, z.
\end{aligned} \tag{6.15}$$

Set

$$\phi(z) = \varphi(z + x - \hat{x}) - \frac{1}{2\varepsilon}\left|x - \hat{x}\right|^2$$

and observe that ϕ touches the original function v at $\hat{x}$ since

$$\begin{aligned}
\phi(\hat{x}) &= \varphi(\hat{x} + x - \hat{x}) - \frac{1}{2\varepsilon}\left|x - \hat{x}\right|^2 \\
&= \varphi(x) - \frac{1}{2\varepsilon}\left|x - \hat{x}\right|^2 \\
&\overset{(6.15)}{=} v(\hat{x}) + \frac{1}{2\varepsilon}\left|x - \hat{x}\right|^2 - \frac{1}{2\varepsilon}\left|x - \hat{x}\right|^2
\end{aligned}$$

and by choosing $y = z + x - \hat{x}$ in (6.15)

$$\begin{aligned}
\phi(z) &= \varphi(z + x - \hat{x}) - \frac{1}{2\varepsilon}\left|x - \hat{x}\right|^2 \\
&\overset{(6.15)}{\le} v(z) + \frac{1}{2\varepsilon}\left|z + x - \hat{x} - z\right|^2 - \frac{1}{2\varepsilon}\left|x - \hat{x}\right|^2 \\
&= v(z).
\end{aligned}$$

Thus

$$0 \ge P^-(D^2\phi(\hat{x})) = P^-(D^2\varphi(x))$$

and thus v_ε is a viscosity supersolution in Ω_ε. □

Observe that if we include the right hand side $f \in C(\Omega)$ into the above proof, then v_ε is a supersolution to

$$P^-(D^2 v_\varepsilon) \le f_\varepsilon$$

in Ω_ε with

$$f_\varepsilon(x) = \sup\{f(y) \,:\, |x - y| < 2(\varepsilon M)^{\frac{1}{2}}\}$$

By the known properties of the inf-convolution mentioned above, u_ε is semi-concave and thus twice differentiable a.e. (recall that idea of the proof of Theorem of Sums).

Idea in generalization of the result to the non smooth case: Set $v_\varepsilon^\delta = v_\varepsilon * \rho_\delta$ where ρ_δ is a standard mollifier. Then the above sketch applies to v_ε^δ and yields

$$(\sup(v_\varepsilon^\delta)_-/4)^n \le C \int_{D_\varepsilon^\delta} \det(D^2 v_\varepsilon^\delta(x))\, dx.$$

It can be shown, see Caffarelli, Crandall, Kocan, Swiech, 1996 or Koike's Guide for details, that

$$(\sup(v_\varepsilon)_-/4)^n \leq C \int_{D_\varepsilon} \det(D^2 v_\varepsilon(x))\, dx$$

. We also state that almost everywhere in D_ε we have $P^- v_\varepsilon \leq (f_\varepsilon)_+$, we further obtain

$$\begin{aligned}\det(D^2 v_\varepsilon(x)) &\leq \frac{1}{\lambda^n} \operatorname{tr}(\mathcal{A} D^2 v_\varepsilon)^n \\ &\leq \frac{1}{\lambda^n} \operatorname{tr}(\mathcal{A} D^2 v_\varepsilon)^n\end{aligned}$$

a.e. in D_ε for any $\mathcal{A} \in \mathcal{A}_{\lambda,\Lambda}$. Then taking infimum over all such $\mathcal{A}$, we have

$$\det(D^2 v_\varepsilon(x)) \leq C((f_\varepsilon)_+)^n.$$

a.e. in D_ε, and further

$$(\sup(v_\varepsilon)_-/4)^n \leq C \int_{D_\varepsilon} ((f_\varepsilon)_+)^n\, dx$$

and finally (consult again the above references for convergence properties of D_ε:

Theorem 6.16 (ABP). *Let $v \in C(\overline{B}_1)$ be a solution a viscosity supersolution to $P^-(D^2 u) \leq f$ with $v \geq 0$ on ∂B_1, and $f \in C(B_1)$ and bounded. Then*

$$\sup v_-^n \leq C \int_D f_+^n\, dx,$$

where D is the lower contact set

$$D = \{x \in B_1 \ : \ \text{there is } p \text{ such that } v(y) \geq v(x) + \langle p, (y-x)\rangle \ \text{ for all } y \in B_2\}.$$

6.2.2. *Proof of Harnack.* The proof of Harnack's inequality consists of two parts:

Step 1 Suppose that v is a nonnegative supersolution to $P^-(D^2 v) \leq 0$ (we again work out the simplest case and comment the case $P^+ \geq f$ etc at the end) and there is a pointwise control i.e. $v(0) = 1$, then we will show that v is also controlled in the measure sense.

Step 2 Suppose that u is a subsolution to $P^+(D^2 u) \geq 0$, and that there is a control in the measure sense. Then we will show that there is also a pointwise bound

$$\sup_Q u \leq M.$$

Combination The Harnack is for a solution i.e. $0 \leq u \in S$. By Step 1, there is a measurability bound for u. From Step 2, we get the desired sup bound.

27.2.2015 Then combination of these gives the Harnack.

In the proof of Step 1, we need the following Calderon-Zygmund type lemma.

Lemma 6.17 (CZ type lemma)**.** *Let $Q_1 \subset \mathbb{R}^n$ a unit cube, $A \subset B \subset Q_1$, $\delta < 1$ and suppose*

a) $|A| \leq \delta$
b) for any given dyadic cube $Q \subset Q_1$, denote Q_* its predecessor and assume that it holds that

$$\text{whenever } \frac{|A \cap Q|}{|Q|} > \delta, \text{ then } Q_* \subset B.$$

Then

$$|A| \leq \delta \, |B| \, .$$

Proof. We utilize Calderon-Zygmund decomposition:

(1) Recall that by assumptions $A \subset Q_1$ and that $|A| < \delta$.
(2) Split Q_1 into 2^n subcubes. Then we check if

$$\frac{|A \cap Q|}{|Q|} > \delta$$

in some of the cubes. Put such cubes 'aside' and continue splitting and putting 'aside' with other cubes. We end up putting 'aside' a sequence of cubes $Q^0, Q^1, Q^2, \ldots$.
(3) Since

$$\frac{|A \cap Q|}{|Q|} = \frac{1}{|Q|} \int_Q \chi_A(y) \, dy$$

and by Lebesgues differentiation $\frac{1}{|Q|} \int_{Q \ni x} \chi_A(y) \, dy \to 1$ for almost every $x \in A$ when the cubes shrink to x, it follows that, up to a measure zero,

$$A \overset{\text{a.e.}}{\subset} \cup_{i=0} Q^i,$$

(4) Denote the predecessor of Q^i by Q^i_*. Then

$$\frac{\left| A \cap Q^i \right|}{|Q^i|} > \delta, \qquad \frac{\left| A \cap Q^i_* \right|}{|Q^i_*|} \leq \delta. \tag{6.16}$$

We collect all the predecessors $Q^1_*, Q^2_*, \ldots$ and throw away those contained in a predecessor earlier in the sequence. Using the same notation we arrive at a disjoint covering such that

$$A \overset{\text{a.e.}}{\subset} \cup_{i=1} Q^i_*.$$

(5) Thus

$$|A| = \sum \left|A \cap Q_*^i\right| \overset{(6.16)}{\leq} \sum \delta \left|Q_*^i\right|$$

By the assumption of the lemma in combination with (6.16), it follows that $Q_i^* \subset B$ so that RHS above $\leq \delta\,|B|$.

□

Below we audit the idea of the proof of Harnack, and omit some of the technical details. Further details and how to make the ideas below precise can be found in Caffarelli-Cabre: Fully nonlinear elliptic equations.

Sketch of the proof of Step 1. Plan:

- ABP to pass from point to measure
- Localize
- Iterate using CZ to reach the level set estimate.

Suppose that $v \geq 0$ is a supersolution $v(0) = 1$ in Q_1. Then consider

$$\tilde{v}(x) = v(x) + 2(|x|^2 - 1) = v(x) + \phi(x).$$

Observe that $\phi = 2(|x|^2 - 1)$, $D^2\phi = 4I$ satisfies

$$P^+(D^2\phi) = \sup_{\mathcal{A} \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(A(4I)) = 4 \sup_{\mathcal{A} \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(A) = 4n\Lambda =: C.$$

Then $\tilde{v}$ is a supersolution (to make this rigorous left as ex) to

$$\begin{aligned} P^-(D^2\tilde{v}) &= \inf \operatorname{tr}(\mathcal{A}D^2(v + \phi)) \leq \inf \operatorname{tr}(\mathcal{A}D^2 v) + \sup \operatorname{tr}(\mathcal{A}D^2\phi) \leq 0 + C \\ &=: \tilde{f}. \end{aligned}$$

Also

$$\tilde{v}(0) = v(0) + 2(0 - 1) = 1 - 2 = -1$$

so that the ABP-estimate implies

$$1 \overset{\text{above}}{\leq} \sup \tilde{v}_- \overset{\text{ABP}}{\leq} C \int_D \underbrace{\tilde{f}_+^n}_{=C^n} \, dx \leq C\,|D|\,,$$

where $D \subset Q_1$ is the lower contact set for $\tilde{v}_-$ where necessarily $0 > \tilde{v} = v + 2(|x|^2 - 1)$ i.e.

$$2 \geq -2(|x|^2 - 1) > v$$

or in other words $D \subset \{x \in Q_1 \,:\, v < 2\}$. The last two estimates imply

$$\frac{1}{C} \leq |D| \leq |\{x \in Q_1 \,:\, v \leq 2\}|\,.$$

Then we further localize this result so that it is not only bounded in a set of nontrivial measure but that there is a nontrivial share in a nearby (for the exact sense of nearby, see the the CZ-context below where the resulting estimate is used) cube Q. This is done by construction such a barrier function ϕ which is supersolution $P^+(D^2\phi) \le 0$ outside Q and a quadratic polynomial (cf. the earlier case and recall the picture from the lecture) in Q, and $\phi(0) \le -2$. Then $\tilde{v} = u + \phi$ solves

$$P^-(D^2\tilde{v}) \le 0 + P^+(D^2\phi) \le C\chi_{D\cap Q} =: \tilde{f}$$

Thus, similarly as above,

$$1 \le \sup \tilde{v}_- \overset{\text{ABP}}{\le} C \int_D \chi_{D\cap Q_{1/8}}\, dx.$$

Then there is $N > 0$ such that

$$N \ge \phi > v$$

since touching can only happen where $0 > \tilde{v}$. We end up with

$$\theta\, |Q| < |\{x \in Q \, : \, v \le N\}| \, . \tag{6.17}$$

Then we use the CZ-type lemma for

$$A := A_k = \{v > N^k\} \cap Q_1, \quad B := A_{k-1} = \{v > N^{k-1}\} \cap Q_1, \quad \delta := 1 - \theta.$$

First the condition (a) in the CZ-lemma clearly holds by the above estimate. Then suppose that for some dyadic Q it holds that

$$\frac{|A_k \cap Q|}{|Q|} > 1 - \theta = \delta.$$

Then

$$\left| \{v \le N^k\} \cap Q \right| < \theta\, |Q|$$

i.e.

$$\left| \{\overline{v} := \frac{v}{N^{k-1}} \le N\} \cap Q \right| < \theta\, |Q| \, . \tag{6.18}$$

Then suppose that $Q_* \not\subset B$, where Q_* is the predecessor of Q. In other words, there is $x \in Q_*$ such that

$$v(x) \le N^{k-1} \Leftrightarrow \overline{v}(x) \le 1.$$

But using (6.17), this implies that $\theta\, |Q| < |\{\overline{v} \le N\} \cap Q|$, a contradiction with (6.18), and thus $Q_* \subset B$.

By the CZ-type lemma,

$$|A| = |A_k| = \left| \{v > N^k\} \right| \le \delta\, |B| = \delta\, |A_{k-1}| = \delta \left| \{v > N^{k-1}\} \right|$$

and iterating this $|A_k| \le (1 - \theta)^k\, |A_0|$

$$\left| \{v > N^k\} \cap Q \right| \le (1 - \theta)^k\, |\{v > 1\} \cap Q|$$

and this is the type of control in measure that we were after. □

Sketch of the proof of Step 2. Suppose that u is a subsolution such that

$$|\{x \in Q \,:\, u > \lambda\}| \leq C\lambda^{-\varepsilon}. \tag{6.19}$$

Observe that this is what we actually proved for supersolutions in the previous step since it yields by setting $N^k = \lambda$ that

$$|\{x \in Q \,:\, v > \lambda\}| \leq \delta^{\frac{\log \lambda}{\log N}} \leq C\lambda^{\frac{\log \delta}{\log N}} \overset{\varepsilon := -\frac{\log \delta}{\log N}}{=} C\lambda^{-\varepsilon}. \tag{6.20}$$

Thriving for a contradiction, suppose that

$$u(x_0) := \sup_{Q_{1/4}} u = M. \tag{6.21}$$

Then we choose $Q(x_0)$ (a cube centered at x_0) of suitable size, and intend to show that then for $\gamma > 0$ small enough

$$\sup_{Q(x_0)} u \geq (1+\gamma)M. \tag{6.22}$$

To establish this, we assume the opposite i.e. $\sup_{Q(x_0)} u < (1+\gamma)M$ and show that

$$\begin{aligned} |\{Q \,:\, u > M/2\}| &< \frac{1}{2}\,|Q| \\ |\{Q \,:\, u < M/2\}| &< \frac{1}{2}\,|Q| \end{aligned} \tag{6.23}$$

which is of course a contradiction. By (6.19), choosing $\lambda = \frac{M}{2}$, we have

$$\left|\{x \in Q \,:\, u > \frac{M}{2}\}\right| \leq CM^{-\varepsilon} \overset{\text{below}}{<} \frac{1}{2}\,|Q|$$

where at the last step we chose Q so that $|Q| = 3CM^{-\varepsilon}$. To establish the second inequality in (6.23), we choose $\gamma > 0$ suitably to be able to use Step 1 i.e. (6.20) for

$$\tilde{v} = \frac{(1+\gamma)M - u}{\gamma M}.$$

It has the following properties

- $\tilde{v}$ is a supersolution
- $\tilde{v}(x_0) = \frac{(1+\gamma)M - M}{\gamma M} = 1$
- $\tilde{v} \geq 0$ whenever $\sup u \leq (1+\gamma)M$ which was our counter proposition
- These are the assumptions of Step 1 and thus (6.20) holds (we can also write out the cube explicitly on the RHS)

$$
\begin{aligned}
|\{u < M/2\} \cap Q| &= \left|\left\{\tilde{v} = \frac{(1+\gamma)M - u}{\gamma M} > \frac{(1+\gamma)M - M/2}{\gamma M}\right\} \cap Q\right| \\
&= \left|\left\{\tilde{v} > \frac{(\frac{1}{2}+\gamma)}{\gamma}\right\} \cap Q\right| \\
&\le \left|\left\{\tilde{v} > \frac{1}{2\gamma}\right\} \cap Q\right| \\
&\overset{(6.20)}{\le} C(2\gamma)^{\varepsilon} |Q|,
\end{aligned}
$$

and by choosing small enough $\gamma > 0$, we have established the second inequality in (6.23). As a conclusion we have shown that if $u(x_0) = M$ and it is in weak L^{ε}, then

$$\sup_{Q_{3^{1/n} M^{-\varepsilon/n}}(x_0)} u \ge (1+\gamma)M,$$

where the cube was fixed above. Then we iterate using this: since

$$u(x_0) = M$$

then by the above result there is x_1 such that

$$u(x_1) \ge (1+\gamma)M, \qquad |x_1 - x_0| \le 3^{1/n} M^{-\varepsilon/n}.$$

Then using the above result again, we have

$$
\begin{aligned}
&u(x_2) \ge (1+\gamma)^2 M, \qquad |x_1 - x_0| \le 3^{1/n}((1+\delta)M)^{-\varepsilon/n} \\
&\quad \vdots \\
&u(x_k) \ge (1+\gamma)^k M, \qquad |x_k - x_{k-1}| \le 3^{1/n}((1+\delta)^{k-1}M)^{-\varepsilon/n} \\
&\quad \vdots
\end{aligned}
$$

Then it is enough to choose large enough M to begin with so that the sequence $x_0, x_1, \ldots$ converges in the cylinder $Q_{\frac{1}{2}}$, but since u is continuous and $u(x_i) \to \infty$, this is a contradiction. Thus there is upper bound C so that

$$\sup_{Q_{1/4}} u \le C.$$

□

Remark 6.18. *There are several possible further questions:*

- *Include RHS f.*
- *Guarantee the assumption $v(0) = 1$*
- *How to obtain our original claim from*

$$\sup_{Q_{1/4}} u \le C.$$

To extend the above the proof to the case $P^-(D^2u) \le |f|$, $P^+(D^2u) \ge -|f|$ *one needs to assume that* $||f||_{L^n}$ *is small enough. Then* f*-term in* $\int_D f + C\chi_D \, dx$ *can be embedded to the LHS.*

The assumption $v(0) = 1$ *can actually be replaced by* $\inf_{Q_{1/4}} v \le 1$ *(see the proof). All these can be guaranteed by looking at*

$$\frac{u}{\delta + \inf_{Q_{1/4}} u + ||f||_{L^n} / \varepsilon_0},$$

where $\delta > 0$ *takes into account the possibility* $\inf u = 0 = f$. *Then* $\sup_{Q_{1/4}} u \le C$ *gives us letting* $\delta \to 0$

$$\sup_{Q_{1/4}} u \le C(\inf_{Q_{1/4}} u + ||f||_{L^n}).$$

The statement in the original balls can be obtained by a covering argument.

Harnack's inequality implies Hölder continuity since we can subtract a constant from a (sub/super)solution (see lecture note of PDE2 or Caffarelli-Cabre) :

Corollary 6.19. *Let* $u \in S(\lambda, \Lambda, f)$ *in* Q_1. *Then* $\theta < 1$

$$\operatorname{osc}_{Q_{1/2}} u \le \theta \operatorname{osc}_{Q_{1/2}} u + 2 ||f||_{L^n(Q_1)}$$

and $u \in C^\alpha(\overline{Q}_{1/2})$ *with the estimate*

$$||u||_{C^\alpha(\overline{Q}_{1/2})} \le C(||u||_{L^\infty(Q_1)} + ||f||_{L^n(Q_1)}),$$

where $\alpha \in (0, 1)$ *and* $C = C(\lambda, \Lambda, n)$.

This (Krylov-Safonov) Harnack and Hölder theory is the counterpart for the non divergence equations of De Giorgi-Moser-Nash theory.

6.2.3. *Including the gradient term.* How to extend the above arguments to the Pucci equations with the gradient dependence:

$$\mathcal{P}^-(D^2u) - \mu |Du| \le f,$$
$$\mathcal{P}^+(D^2u) + \mu |Du| \ge f.$$

Consider

$$\Delta v \le f + |Du| \le f_+ + |Du|$$

and assume everything to be smooth. Set $\sup_{Q_1} v_- =: M$ and $||f_+||^n_{L^n(D)} =: \varepsilon$. Then

$$\begin{aligned}\log(\frac{M^n+\varepsilon}{\varepsilon}) &= \int_0^{M/4} \frac{r^{n-1}}{r^n+\varepsilon}\,dr = C\int_0^{M/4}\int_{\partial B_r} \frac{1}{\rho^n+\varepsilon}\,d\rho\,dr \\ &= C\int_{B_{M/4}(0)} \frac{1}{|p|^n+\varepsilon}\,dp \overset{\text{geom. arg}}{\leq} C\int_{\xi(B_1)} \frac{1}{|p|^n+\varepsilon}\,dp \\ &\overset{\xi(x)=p}{\leq} C\int_{B_1} \frac{|\det(D\xi(x))|}{|\xi(x)|^n+\varepsilon}\,dx \overset{\text{below}}{\leq} C\int_D \frac{\det D^2u}{|Du|^n+\varepsilon}\,dx \\ &\leq C\int_D \frac{\Delta u^n}{|Du|^n+\varepsilon}\,dp \leq C\int_D \frac{(|Du|+f_+)^n}{|Du|^n+\varepsilon}\,dp \\ &\leq C,\end{aligned}$$

where D is the lower contact set as before and above we used $\xi(x) = D(\Gamma(v))(x)$, $D^2\Gamma(v)(x) = 0$ outside D, $D^2u \geq D^2\Gamma(v)(x) \geq 0$ in D, and $Du(x) = D\Gamma(v)$ in D. The above implies $M^n \leq \varepsilon e^C + \varepsilon$ i.e.

$$\sup v_- \leq C\int_D f_+^n\,dx.$$

The rest of the proof of the Harnack does not explicitly use the equation but only the results: ABP, CZ-type lemma, the fact that if u is sub then $-u$ is super for the related Pucci operators and the fact that we can add a constant, and we are done.

6.3. $C^{1,\alpha}$**-regularity uniformly elliptic equation.** So far we only operated with Pucci operators but now we return to uniformly elliptic equations.

Theorem 6.20. *Let u and v be viscosity sub- and supersolutions, respectively, to $F(D^2w) = 0$ (no x-dependence) in Ω where F is uniformly elliptic. Then $w = u - v$ is a viscosity subsolution to*

$$P^+_{\lambda/n,\Lambda}(D^2w) \geq 0.$$

Proof. As we have observed before inf-convolution

$$v_\varepsilon(x) = \inf_{y\in\Omega}(v(y) + \frac{1}{2\varepsilon}|x-y|^2)$$

and sup-convolution (which would be shown in the similar manner)

$$u^\varepsilon(x) = \sup_{y\in\Omega}(u(y) - \frac{1}{2\varepsilon}|x-y|^2)$$

remain super- and subsolutions in slightly smaller domain Ω_ε, they are twice (Alexandrov) differentiable almost everywhere and a.e

$$F(D^2v_\varepsilon) \leq 0, \qquad F(D^2u^\varepsilon) \geq 0. \tag{6.24}$$

Let φ touch $u^\varepsilon - v_\varepsilon$ from above at x_0 and set

$$w = \underbrace{v_\varepsilon - u^\varepsilon + \varphi}_{\geq 0} + \delta \left|x - x_0\right|^2 - \delta r^2.$$

It holds that $w > 0$ on $\partial B_r(x_0)$ and $w(x_0) < 0$. Using ABP to w (and taking care of regularity) we obtain

$$0 < \int_D \det(\underbrace{D^2 w}_{\geq 0 (\text{conv})}) \, dx \tag{6.25}$$

where D is the lower contact set related to $-w_-$ continued as zero to $B_{2r}(x_0)$, and $\Gamma(w)(x)$ is the convex envelope. Denote by A the set where w is twice differentiable: in particular $|B_r(x_0) \setminus A| = 0$. By this fact and (6.25),

$$|D \cap A| > 0$$

6.3.2015 and thus there is $x_1 \in D \cap A$.

Now

$$\begin{aligned}
0 &\overset{(6.24)}{\leq} F(D^2 u^\varepsilon(x_1)) = F(D^2(v_\varepsilon - w + \varphi + \delta \left|x - x_0\right|^2)) \\
&= F(D^2 v_\varepsilon - D^2 w + D^2 \varphi + 2\delta I) \overset{\text{ell}}{\leq} F(D^2 v_\varepsilon - D^2 w + D^2 \varphi) + \Lambda 2\delta \\
&\overset{x_1 \in D, -D^2 w \leq 0, \text{deg ell}}{\leq} F(D^2 v_\varepsilon + D^2 \varphi) + \Lambda 2\delta \\
&\overset{\text{ell}}{\leq} F(D^2 v_\varepsilon) + \Lambda \left|\left|(D^2\varphi)^+\right|\right| - \lambda \left|\left|(D^2\varphi)^-\right|\right| + \Lambda 2\delta \\
&\overset{v_\varepsilon \text{ super}}{\leq} \Lambda \left|\left|(D^2\varphi)^+\right|\right| - \lambda \left|\left|(D^2\varphi)^-\right|\right| + \Lambda 2\delta \\
&\leq P^+_{\lambda/n, \Lambda}(D^2 \varphi(x_1)) + \Lambda 2\delta,
\end{aligned}$$

where at the last step we estimated $\Lambda \left|\left|(D^2\varphi)^+\right|\right| - \lambda \left|\left|(D^2\varphi)^-\right|\right| \leq \Lambda \sum_{\lambda_i > 0} \lambda_i + \frac{\lambda}{n} \sum_{\lambda_i < 0} \lambda_i$. The result follows by $\delta \to 0$ and $\varepsilon \to 0$ recalling from the exercises that Pucci subsolutions is a closed class (instance of a stability principle). □

The $C^{1,\alpha}$-regularity then follows by using the earlier C^α estimate to a solution (this is sol because of the previous result and translation invariance)

$$\frac{u(x + eh) - u(x)}{h}.$$

The estimate comes in two part: first iterate to show that u is Lip to control the RHS of the Hölder estimates and then use Hölder estimate to the above function. We omit the details.

The following formal calculation might add insight into the previous technique: Differentiate the equation $F(D^2 u) = 0$ to the direction α to have

$$\sum F_{ij}(D^2 u) D_{ij} u_\alpha = 0.$$

Uniform ellipticity implies (ex) $|\xi|^2 \lambda \leq \sum_{ij} F_{ij}\xi_i\xi_j \leq \Lambda |\xi|^2$, and thus u_α is a solution to the uniformly elliptic equation. Thus, if u_α is bounded, then by the C^δ-result, $u \in C^{1,\delta}$.

7. Applications: Control theory

We start by looking at the deterministic, finite time horizon, single controller setup.

We use the following terminology and notation

$$
\begin{aligned}
&(x,t) \in \mathbb{R}^n \times [0,T], \text{starting point}\\
&\alpha : [t,T] \to A, A \text{ a compact subset of } \mathbb{R}^n, \text{ admissible control}\\
&\mathcal{A}, \text{ the set of admissible controls}\\
&\begin{cases} x'(s) = f(x(s),\alpha(s)) & s \in [t,T] \\ x(t) = x \end{cases} \text{ dynamics given by } f\\
&f : \mathbb{R}^n \times A \to \mathbb{R}^n,\\
&r : \mathbb{R}^n \times A \to \mathbb{R}, \text{ running cost},\\
&g : \mathbb{R}^n \times A \to \mathbb{R}, \text{ terminal cost.}
\end{aligned}
$$

We assume that all the functions are continuous, Lip wrt x and bounded

$$
\begin{aligned}
&|f(x,\alpha)| \leq C,\ |r(x,\alpha)| \leq C,\ |g(x)| \leq C\\
&|f(x,\alpha) - f(y,\alpha)| \leq C\,|x-y|\,,\ |r(x,\alpha) - r(y,\alpha)| \leq C\,|x-y|\,,\\
&|g(x) - g(y)| \leq C\,|x-y|\,.
\end{aligned}
$$

The dynamics ODE has a unique solution $x(s)$ under these assumptions.

We are interested of the total costs and therefore define a value function

$$u(x,t) := \inf_{\alpha\in\mathcal{A}} C_{x,t}(\alpha) := \inf_{\alpha\in\mathcal{A}}\{\int_t^T r(x(s),\alpha(s))\,ds + g(x(T))\}.$$

We are going to show that u is a solution to a first order terminal value PDE of a type

$$\begin{cases} u_t + H(x,Du) = 0 & \text{in } \mathbb{R}^n \times (0,T)\\ u(x,T) = g(x) & \text{on } \mathbb{R}^n. \end{cases}$$

This is often called Hamilton-Jacobi-Bellman (HJB) equation, and can be seen as an infinitesimal version of so called dynamic programming principle or optimality condition:

Lemma 7.1 (DPP/optimality condition)**.** *For each $h > 0$ small enough so that $t + h \leq T$, we have*

$$u(x,t) = \inf_{\alpha\in\mathcal{A}}\{\int_t^{t+h} r(x(s),\alpha(s))\,ds + u(x(t+h),t+h)\},$$

where x is the trajectory with the control α.

To make sure that we remain within the uniqueness theory in unbounded domain, and make sure that we take the initial values continuously:

Theorem 7.2 (Estimates for value functions)**.** *The above value function u satisfies*

$$\begin{aligned} &|u(x,t)| \le C, \\ &\left|u(x,t) - u(\hat{x},\hat{t})\right| \le C\,|x-\hat{x}| + C\left|t-\hat{t}\right|, \end{aligned}$$

for $x, \hat{x} \in \mathbb{R}^n$, $0 \le t, \hat{t} \le T$.

13.3.2015

Next we connect the value function to the PDE. The PDE then provides (existence, uniqueness, solvers available...) us the way to access the value function and the optimal control.

The heuristics is that the PDE is an infinitesimal version of DPP. Formally supposing u is a smooth value we can substitute the Taylor expansion ($x(t) = x$ as before, some control α)

$$\begin{aligned} u(x(t+h), t+h) &= u(x,t) + Du(x,t)\cdot(x(t+h)-x) + u_t(x,t)h + \text{Error} \\ &\approx u(x,t) + Du(x,t)\cdot f(x,\alpha(t))h + u_t(x,t)h + \text{Error} \end{aligned}$$

(we don't comment yet how much we make an error when using $f(x,\alpha(t))h$ to estimate the displacement on $x(\cdot)$) to the DPP

$$\begin{aligned} u(x,t) &= \inf_{\alpha\in\mathcal{A}}\{\int_t^{t+h} r(x(s),\alpha(s))\,ds + u(x(t+h),t+h)\} \\ &= \inf_{\alpha\in\mathcal{A}}\{\int_t^{t+h} r(x(s),\alpha(s))\,ds \\ &\quad + u(x,t) + Du(x,t)\cdot f(x,\alpha(t))h + u_t(x,t)h + \text{Error}\}. \end{aligned}$$

Cancelling $u(x,t)$-terms, dividing by h, letting $h \to 0$ so that error term vanishes, we get

$$\begin{aligned} 0 &= \inf_{\alpha\in\mathcal{A}}\{r(x,\alpha(t)) + Du(x,t)\cdot f(x,\alpha(t)) + u_t(x,t)\} \\ &= \inf_{a\in A}\{r(x,a) + Du(x,t)\cdot f(x,a)\} + u_t(x,t) \\ &=: H(x, Du(x,t)) + u_t(x,t). \end{aligned}$$

This can be (and was at the lectures) proven rigorously.

Theorem 7.3. *Let u be the value for the control problem in the above setup. Then u is the unique viscosity solution to*

$$\begin{cases} u_t + H(x,Du) = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u(x,T) = g(x). \end{cases}$$

Observe that above we have a terminal value problem. By changing the direction of time by setting $w(x,t) = (x, T-t)$ we get a solution to the initial value problem.

The viscosity subsolution above means that, we touch u from above at (x,t) by $\varphi \in C^1$ and have

$$\varphi_t(x,t) + H(x, D\varphi(x,t)) \geq 0.$$

For the proofs (which were covered during the lectures), see for example Section 10.3 in Evans: Partial differential equations.

8. Higher regularity

We now have Hölder continuity of u, and $C^{1,\alpha}$. The higher regularity theory is called Evans-Krylov $C^{2,\alpha}$ theorem. It requires an additional assumption on F. Indeed, we assume that $M \mapsto F(x,M)$ is convex (concave) in M.

Definition 8.1. *F is convex if for all $N, M \in S^n$ and $\gamma \in [0,1]$ it holds that*

$$F(\gamma N + (1-\gamma)M) \leq \gamma F(N) + (1-\gamma)F(M)$$

Example 8.2.

- *Let $F(D^2u) = \Delta u = \operatorname{tr}(D^2u)$. Then*

$$\begin{aligned} F(\gamma N + (1-\gamma)M) &= \operatorname{tr}(\gamma N + (1-\gamma)M) \\ &= \gamma \operatorname{tr}(N) + (1-\gamma)\operatorname{tr}(M) \\ &= \gamma F(N) + (1-\gamma)F(M). \end{aligned}$$

- *Let $F(D^2u) = \sup_a \operatorname{tr}(\mathcal{A}(a)D^2u)$, $N, M \in S^n$, and $\gamma \in [0,1]$. Then*

$$\sup_a \operatorname{tr}(\mathcal{A}(a)(\gamma N + (1-\gamma)M)) \leq \gamma \sup_a \operatorname{tr}(\mathcal{A}(a)N) + (1-\gamma)\sup_a \operatorname{tr}(\mathcal{A}(a)M).$$

- *In contrast, the equation*

$$F(D^2u) = \sup_a \inf_b \operatorname{tr}(\mathcal{A}(a,b)D^2u)$$

are not convex nor concave.

Consider $F(D^2u) = 0$ where F is smooth and uniformly elliptic. Take directional derivative denoted by u_ν

$$\sum F_{ij}(D^2u)D_{ij}u_\nu = 0$$

where $F_{ij}(X) = \frac{\partial F}{\partial x_{ij}}$. This can be though as a linear equation

$$\sum a_{ij}(x)D_{ij}v = 0$$

with $u_\nu = v$ and no additional knowledge on regularity on the coefficients but only the bound on the eigenvalues of the coefficient matrix. Then task is to show that $F_{ij}(D^2u) \in C^\alpha$ because then by Schauder estimates (taken here for granted), we have $u_\nu \in C^{2,\alpha}(\Omega)$ so that $u \in C^{3,\alpha}$. But then $F_{ij}(D^2u) \in C^{1,\alpha}$, and by Schauder $u_\nu \in C^{3,\alpha}$ etc. so that key is to show

$$u \in C^{2,\alpha}(\Omega).$$

As a matter of fact, using global Schauder estimates it holds that

Theorem 8.3. *Let $u \in C^{2,\alpha}(\overline{\Omega})$ be a solution to $F(D^2u) = 0$ in Ω, $\partial\Omega \in C^\infty$, $u|_{\partial\Omega} \in C^\infty(\partial\Omega)$, and $F \in C^\infty(S^n)$. Then*

$$u \in C^\infty(\overline{\Omega}).$$

Let us begin our study of $C^{2,\alpha}$ regularity, and see why the concavity (convexity) is critical?

Differentiate the equation once more to have

$$\sum F_{ij}(D^2u)D_{ij}u_{\nu\nu} + \sum_{ij}\sum_{lk} D_{lk}u_\nu F_{ij,lk}(D^2u)D_{ij}u_\nu = 0, \tag{8.26}$$

where

$$F_{ij,lk}(X) = \frac{\partial^2 F}{\partial x_{ij}\partial x_{lk}}.$$

For smooth F is concave if and only if

$$\sum\sum F_{ij,lk}(M)n_{ij}n_{lk} \leq 0 \quad \text{for all N.}$$

The second term in (8.26) is negative if F is concave, and we get that $u_{\nu\nu}$

$$\sum F_{ij}(D^2u)D_{ij}u_{\nu\nu} \geq 0 \tag{8.27}$$

is a subsolution.

We may work with difference quotients: Suppose that $F \in C^1$ and that it is concave. Then

$$F(M) + \sum_{ij} F_{ij}(M)(n_{ij} - m_{ij}) \geq F(N).$$

The above gives

$$\sum_{ij} F_{ij}(D^2u(x))(D^2u(y) - D^2u(x))_{ij} \geq F(D^2u(y)) \overset{\text{sol}}{=} 0.$$

Further, choose $y = x \pm \nu h$

$$\sum_{ij} F_{ij}(D^2u(x))(D^2u(x+\nu h) - D^2u(x))_{ij} \geq 0$$
$$\sum_{ij} F_{ij}(D^2u(x))(D^2u(x-\nu h) - D^2u(x))_{ij} \geq 0$$

and sum up to obtain

$$\sum_{ij} F_{ij}(D^2u(x))(D^2u(x+\nu h) - 2D^2u(x) + D^2u(x-\nu h))_{ij} \geq 0.$$

Compare this to (8.27).

However, as we know, the subsolution property (8.27) alone is not enough for C^α. The extra information comes from the fact that originally $F(D^2u) = 0$, which in geometric terms says that the second derivatives are on the surface given by F.

We need the following lemmas.

Lemma 8.4. *If $F(M) = F(M+N) = 0$, then*

$$\left|\left|N^+\right|\right| \sim \left|\left|N^-\right|\right| \sim ||N||.$$

Proof. The uniform ellipticity

$$\lambda\left|\left|N^+\right|\right| - \Lambda\left|\left|N^-\right|\right| \leq \underbrace{F(M+N) - F(M)}_{=0} \leq \Lambda\left|\left|N^+\right|\right| - \lambda\left|\left|N^-\right|\right|.$$

implies

$$\lambda\left|\left|N^+\right|\right| \leq \Lambda\left|\left|N^-\right|\right| \leq \Lambda\left|\left|N^+\right|\right| + (\Lambda - \lambda)\left|\left|N^-\right|\right|.$$

The last inequality also implies $\lambda\,||N^-|| \leq \Lambda\,||N^+||$ and we are done since

$$||N|| = \max(\left|\left|N^+\right|\right|, \left|\left|N^-\right|\right|).$$ □

Remark 8.5 (Renormalization)**.** *Let u be a solution to $F(D^2u) = 0$. Then $v(x) := \frac{1}{\lambda}u(\mu x)$ is a solution to*

$$\frac{\mu^2}{\lambda}F(\frac{\lambda}{\mu^2}D^2v) = 0$$

where we put $\frac{\mu^2}{\lambda}$ in front to ensure that this satisfies the same uniform ellipticity assumptions as the original equation.

In particular if $\lambda = 1/\mu^2$ then we have a rescaled solution to the original equation.

Suppose that $\sup_{(x,y)\in B_1}\left|\left|D^2u(x) - D^2u(y)\right|\right| = \lambda_{\max}$ *(and that the quantities exists...). Then scaling $v(x) = \frac{1}{\lambda_{\max}}u(x)$ is a solution to*

$$G(D^2v) = \frac{1}{\lambda_{\max}}F(\lambda_{\max}D^2v) = 0$$

with $\sup_{(x,y)\in B_1}\left|\left|D^2v(x) - D^2v(y)\right|\right| = 1$.

We may also without loss of generality assume $F(0) = 0$. First, by uniform ellipticity and continuity, there exists $t \in \mathbb{R}$ such that $F(tI) = 0$ since

$$\lambda t + F(0) \leq F(tI) \leq t\Lambda + F(0), \quad t \geq 0$$
$$-\Lambda\,|t| + F(0) \leq F(tI) \leq -\lambda\,|t| + F(0), \quad t < 0.$$

It also holds by the above that $|t| \leq |F(0)|/\lambda$. Then

$$0 = F(D^2u) = F(D^2(u - \frac{t}{2}|x-x_0|^2) + tI) =: G(D^2(u - \frac{t}{2}|x-x_0|^2)).$$

Thus $G(0) = F(tI) = 0$ and $G(M+N) = F(M+N+tI) \leq F(M+tI) + \Lambda\,||N^+|| - \lambda\,||N^-|| = G(M) + \Lambda\,||N^+|| - \lambda\,||N^-||$. Moreover, if u is a solution to the F-problem, then $u - \frac{t}{2}|x-x_0|^2$ is a solution to G problem. Thus showing the regularity for G-problem, implies regularity for F-problem.

Lemma 8.6. *Let $u \le 0$ be a subsolution for $P^+(D^2u) \ge 0$ in B_1 such that*

$$\left|\{x \in B_{1/4} : u(x) \le -1\}\right| = \theta > 0.$$

Then $u(x) \le -C(\theta)$ in $B_{1/2}$.

Proof. Let $v \ge 0$ be a supersolution such that $v(x_0) = 1$. Then by the Step 1 in the proof of Harnack, we saw that v is in the weak L^ε.

Take u as above and set $v := -u \ge 0$ and thriving for a contradiction assume that for any choice of $C_i \to 0+$ there is $x_i \in B_{1/2}$ such that $v(x_i) < C_i$. Then $w_i = v/C_i$ satisfies the assumptions of Step 1 in the proof of Harnack , but the assumption in the lemma now reads as

$$\left|\{x \in B_{1/4} : w_i(x) > (C_i)^{-1}\}\right| = \theta$$

and this contradicts the uniform weak L^ε property. □

20.3.2015

Next we sketch the proof of $C^{2,\alpha}$-regularity for a concave F, $F(D^2u) = 0$. The idea is show that the oscillation of D^2u reduces. The path to $C^{2,\alpha}$, partly already accomplished, will be the following:

$$C \quad C^\alpha \quad C^{1,\alpha} \quad C^{1,1} \quad C^2 \quad C^{2,\alpha}$$

Remark 8.7. *The gap from $C^{1,\alpha}$ to C^2 uses the following ideas, which are commented in more detail later.*

- *What is said below about second derivatives can be formulated in terms of difference quotients*
- *$D_{\nu\nu}u$ are subsolutions which can be shown to be in weak L^ε, thus they are bounded by the Step 2 in Harnack*
- *Thus by the above lemma also $\left|\left|(D^2u)^+\right|\right| \sim \left|\left|(D^2u)\right|\right|^- \sim \left|\left|D^2u\right|\right|$ are all bounded*
- *These ideas give us $C^{1,1}$*
- *The second derivatives exist a.e. for such a function and the proof from C^2 to $C^{2,\alpha}$ can be adapted to a situation from $C^{1,1}$ to $C^{2,\alpha}$ by replacing* sup*s by* ess sup*s etc.*

Thus we want to prove

Theorem 8.8. *Let $u \in C^2$ be a solution to $F(D^2u) = 0$ in B_1, F uniformly elliptic and concave. Then $u \in C^{2,\alpha}(B_{1/2})$.*

Idea of the proof. Consider Hessian map $H : B_1 \to S^n$ $H(x) := D^2u(x)$. Without loss of generality, we may assume that $\operatorname{diam}(H(B_1)) = 1$ by using the renormalization of the previous remark . Indeed, set $\ell = \operatorname{diam} H(B_1)$, and consider $w = u/\ell$ which satisfies $\operatorname{diam} D^2w(B_1) = \operatorname{diam} D^2u(B_1)/\ell = 1$ and solves

$$G(D^2w) := \ell^{-1}F(D^2w\ell) = 0$$

where G satisfies the same ellipticity assumptions as F.

Then we show that shrinking the radius of the original set, the diameter of image falls below 1/2 i.e. $\operatorname{diam} H(B_{r_0}) \leq 1/2$. The idea of showing this is taking two coverings with radius δ and ε of the image of the Hessian, and then show that at least one ball is not needed to cover the image. The proof only uses the fact that the diameter is between $\frac{1}{2}$ and 1, and thus we may iterate this until we are below $\frac{1}{2}$.

Then idea is to iterate the argument by renormalizing the image so that the diameter is again 1 and repeat the above argument. This gives a geometric decay in diameters, and similarly as oscillation estimate in Corollary 6.19 implies C^α, this implies $C^{2,\alpha}$.

First cover the image $H(B_1) \subset \mathbb{R}^{n^2}$ by balls of radius $\delta > 0$,

$$\{B_\delta(M_i)\}.$$

We need at the most $C(\delta, n)$ such balls in the covering.

Then take the inverse image of the balls

$$H^{-1}(B_\delta(M_i)).$$

The number of balls is at the most $C(\delta, n)$ so for at least to one of the inverse images we must have

$$\left|H^{-1}(B_\delta(M_i))\right| \geq C(\delta, n)^{-1} |B_1|,$$

which we denote by $U_i = H^{-1}(B_\delta(M_i))$. Below $M_i = D^2 u(x_i)$ refers to the center of this chosen ball.

By the diameter condition there are $\tilde{M}_1, \tilde{M}_2$ such that $\left|\left|\tilde{M}_1 - \tilde{M}_2\right|\right| \geq \frac{1}{2}$ and by triangle inequality the distance to one of them, say $\tilde{M} := \tilde{M}_1$, is

$$\frac{1}{4} \leq \left|\left|\tilde{M} - M_i\right|\right| \overset{\text{Lemma 8.4}}{\sim} \left|\left|(\tilde{M} - M_i)^-\right|\right| \overset{\text{Lemma 8.4}}{\sim} \left|\left|(\tilde{M} - M_i)^+\right|\right|.$$

In particular this implies for each $x \in U_i$ that there is ν such that

$$(\tilde{M})_{\nu\nu} \geq D_{\nu\nu} u(x_i) + \frac{C}{4} \geq D_{\nu\nu} u(x) + \frac{C}{4}$$

since for $x \in U_i$ it holds that $\left|\left|D^2 u(x_i) - D^2 u(x)\right|\right| \leq \delta$. Thus

$$t := \sup_{B_1} D_{\nu\nu} u \geq (\tilde{M})_{\nu\nu} \geq D_{\nu\nu} u(x) + \frac{C}{4}$$

i.e.

$$-\frac{C}{4} \geq D_{\nu\nu} u(x) - t \text{ in } U_i.$$

In this way we would obtain

$$\left|\{x \in B_{1/4} \,:\, -\frac{C}{4} \geq D_{\nu\nu} u(x) - t\}\right| \geq C(n, \delta)^{-1} =: \theta.$$

Since $D_{\nu\nu}u(x) - t$ is a subsolutions to Pucci operator and thus, we have by Lemma 8.6,

$$D_{\nu\nu}u(x) - t \leq -\mathcal{C}(\theta) =: -3\varepsilon \quad \text{in} \quad B_{1/2}. \tag{8.28}$$

Then cover by $\{B_\varepsilon(N_i)\}$ the image $H(B_1)$. By (8.28), there is at least one ball in $\{B_\varepsilon(N_i)\}$ that is not needed to cover $H(B_{1/2})$. Indeed, let $\eta > 0$ small and $x_{\sup} \in B_1$ such that $D_{\nu\nu}u(x_{\sup}) \geq \sup_{B_1} D_{\nu\nu}u - \eta$. Then there is a ball such that

$$D^2u(x_{\sup}) \in B_\varepsilon(N_i).$$

On the other hand, by (8.28), the ball $B_\varepsilon(N_i)$ is not needed to cover $H(B_{1/2})$ since for every $D^2u(z) \in B_\varepsilon(N_i)$ it holds that

$$\left|\left|D^2u(x_{\sup}) - D^2u(z)\right|\right| < 2\varepsilon.$$

This implies for any $D^2u(z) \in B_\varepsilon(N_i)$

$$\begin{aligned}|D_{\nu\nu}u(z) - t| &= \left|D_{\nu\nu}u(z) - \sup_{B_1} D_{\nu\nu}u\right| \\ &\leq |D_{\nu\nu}u(z) - D_{\nu\nu}u(x_{\sup}) + D_{\nu\nu}u(x_{\sup}) - \sup_{B_1} D_{\nu\nu}u| \\ &< 2\varepsilon + \eta\end{aligned}$$

which by (8.28) shows that $z \notin B_{1/2}$.

Then we iterate the argument: Recall that $\operatorname{diam} D^2u(B_1) = 1$. Then by the previous argument, we can reduce the number of balls from the original, say, K to $K-1$ to cover $D^2u(B_{1/2})$. Then suppose that $\frac{1}{2} < \operatorname{diam} D^2u(B_{1/2}) \leq 1$, otherwise we terminate the iteration.

Consider

$$w(x) = 4u(x/2) \quad x \in B_1$$

for which it holds that

$$\begin{gathered}D^2w(x) = D^2u(\frac{x}{2}), \\ F(D^2w) = 0 \text{ in } B_1, \\ \frac{1}{2} < \operatorname{diam} D^2w(B_1) = \operatorname{diam} D^2u(B_{1/2}) \leq 1.\end{gathered}$$

Thus we may again use the above argument to w to discover that $D^2w(B_{1/2}) = D^2u(B_{1/4})$ can be covered by $K-2$ balls. Repeating in this way, we obtain that (since we cannot run out of balls in covering a nonempty set) for some $k = 1, 2, \ldots,$ such that $k < K$

$$\frac{1}{2} \geq \operatorname{diam} D^2u(B_{2^{-k}}) \geq \operatorname{diam} D^2u(B_{2^{-K}}).$$

□

Lemma 8.9. *Let F be concave and let u and v be viscosity subsolutions to $F(D^2w) = 0$. Then*

$$\frac{1}{2}(u + v)$$

is also a viscosity subsolution.

First observe that if $u, v \in C^2$, then by concavity

$$\begin{aligned} F(\frac{1}{2}D^2(u+v)) &= F(\frac{1}{2}(D^2u + D^2v)) \\ &\overset{\text{conc.}}{\geq} \frac{1}{2}F(D^2u) + \frac{1}{2}F(D^2v) \\ &\geq 0. \end{aligned}$$

Proof of Lemma 8.9. We use a sup-convolution and the stability property, i.e. it suffices to show that $F(\frac{1}{2}D^2(u^\varepsilon + v^\varepsilon)) \geq 0$ in the viscosity sense. To establish this, choose φ touching at x_0 from above. Then set

$$w(x) = \varphi(x) - \frac{1}{2}(u^\varepsilon(x) + v^\varepsilon(x)) + \delta\,|x - x_0|^2 - \delta r^2.$$

Then, as before, by using ABP and the fact that $\frac{1}{2}(u^\varepsilon(x)+v^\varepsilon(x))$ has second derivatives a.e., there is x_1 in $B_{2r}(x_0)$ such that $D^2w(x_1) \geq 0$ i.e. $D^2\varphi(x_1)+2\delta I \geq \frac{1}{2}D^2(u^\varepsilon(x_1) + v^\varepsilon(x_1))$, and $F(D^2u^\varepsilon(x_1)) \geq 0$ and $F(D^2v^\varepsilon(x_1)) \geq 0$. Thus

$$0 \overset{\text{conc.}}{\leq} F(\frac{1}{2}D^2(u^\varepsilon(x_1) + v^\varepsilon(x_1))) \overset{\text{deg. ell.}}{\leq} F(D^2\varphi(x_1) + 2\delta I) \overset{\delta\to 0, r\to 0}{\to} F(D^2\varphi(x_0)).$$

□

How to make ideas sketched in Remark 8.7 more rigorous, i.e. how to show $C^{1,1}$:
Assume that $F(0) = 0$. Since F is concave, there is a supporting hyperplane from above at 0, i.e. linear functional L such that

$$\begin{aligned} L(0) = 0, \quad L(M) \geq F(M), \quad M \in S^n, \\ L(M) = \sum a_{ij}m_{ij} = \operatorname{tr}(AM) \text{ where } A \in S^n. \end{aligned}$$

In addition, A has its eigenvalues between λ, Λ

$$\sum a_{ij}\xi_i\xi_j = L(\xi \otimes \xi) \geq F(\xi \otimes \xi) \overset{\text{unif. ell.}}{\geq} \underbrace{F(0)}_{=0} + \lambda\,||\xi \otimes \xi|| = \lambda\,|\xi|^2\,,$$

$$-\sum a_{ij}\xi_i\xi_j = L(-\xi \otimes \xi) \geq F(-\xi \otimes \xi) \overset{\text{unif. ell.}}{\geq} F(0) - \Lambda\,||\xi \otimes \xi|| = -\Lambda\,|\xi|^2\,.$$

In such a situation (see Remark 8.10), we may change the variables so that, without a loss of generality, we may look at the Laplacian

$$L(D^2u) = \Delta u = 0.$$

Further for any $\varphi \in C^2$

$$\Delta\varphi = L(D^2\varphi) \geq F(D^2\varphi) \geq 0,$$

and thus u is a viscosity subsolution to $\Delta u \geq 0$. Then

$$u(x_0) \leq \fint_{\partial B_\varepsilon(x_0)} u\, dy.$$

In order to see this, solve

$$\begin{cases} \Delta w = 0 & \text{in } B_\varepsilon(x_0) \\ w = u & \text{on } \partial B_\varepsilon(x_0). \end{cases}$$

Since u is a subsolution and w a solution

$$u(x_0) \overset{\text{comp.}}{\leq} w(x_0) = \fint_{\partial B_\varepsilon(x_0)} w\, dS = \fint_{\partial B_\varepsilon(x_0)} u\, dS.$$

Then let

$$\Delta_\varepsilon u(x) := \frac{1}{\varepsilon^2} \fint_{\partial B_\varepsilon(x)} u - u(x)\, dS \geq 0 \qquad \text{for } x \in B_{1/2},\ 0 < \varepsilon < \frac{1}{2}.$$

By the Taylor expansion and calculating the mean integrals

$$\Delta_\varepsilon u(x) = \frac{1}{\varepsilon^2} \fint_{\partial B_\varepsilon(x)} u - u(x)\, dS = \frac{1}{2n} \Delta u + o(\varepsilon).$$

It is rather easy to show L^1 estimate by multiplying by a bump function

$$\int_{B_{1/3}} |\Delta_\varepsilon u|\ dx \leq C(n).$$

The key point is to show that $v := \Delta_\varepsilon u$ satisfies in the viscosity sense

$$P^+_{\lambda/n,\Lambda}(D^2 v) \geq 0. \tag{8.29}$$

Lemma 8.9 generalizes to the integral $\fint_{\partial B_\varepsilon(x)} u\, dS$ since the class of subsolutions to $F(D^2u) \geq 0$ is stable under limits. Since u is a solution to $F(D^2u) = 0$, it is a supersolution to $P^-(D^2u) \leq 0$ and thus, by Lemma 6.20,

$$\Delta_\varepsilon u(x) := \frac{1}{\varepsilon^2} \fint_{\partial B_\varepsilon(x)} u - u(x)\, dS$$

is a subsolution as a difference of a subsolution and a supersolution i.e. (8.29) holds.

Then we are in a position of using Harnack to deduce that $\Delta_\varepsilon u$ is bounded. A short limit procedure implies that $||\Delta u||_{L^\infty} \leq C$ and thus of course also $\Delta u \in L^2$, and by L^2 regularity theory (see PDE2 for example) $\left|\left|D^2u\right|\right|_{L^2} \leq C$. From this we get an L^2-estimate for the second order difference quotient, and by passing to a limit to the second directional derivatives $D_{\nu\nu}u$. Using again the Harnack, we get that $D_{\nu\nu}u$ is bounded.

Choosing a direction of the largest positive eigenvalue and recalling Lemma 8.4, we get

$$\left|\left|D^2u(x)\right|\right| \le CD_{\nu\nu}u(x).$$

Thus we arrive at

$$u \in W^{2,\infty} = C^{1,1}$$

where $W^{2,\infty}$ denotes a second order Sobolev space and the last equality is up to a choice of a representative. 27.3.2015, Easter break

Remark 8.10. *Suppose that*

$$Lu = \sum a_{ij}D_{ij}u,$$

where $\mathcal{A}$ is a symmetric, **constant** *coefficient matrix has the eigenvalue bounds $0 < \lambda \le \Lambda < \infty$. Let P be a symmetric, constant, invertible matrix giving a linear transformation and change of variables*

$$y = Px, \quad \tilde{u}(y) := \tilde{u}(Px) := u(x) \text{ i.e. } \tilde{u}(x) = \tilde{u}(PP^{-1}x) = u(P^{-1}y),$$

so that

$$D_{x_i}u(x) = \sum_k D_{y_k}\tilde{u}(\underbrace{Px}_{y})p_{ki},$$

$$D_{x_ix_j}u(x) = \sum_{kl} D_{y_ky_l}\tilde{u}(y)p_{lj}p_{ki}$$

Thus

$$\sum_{ij} a_{ij}D_{ij}u(x) = \sum_{ijkl} a_{ij}D_{y_ky_l}\tilde{u}(y)p_{lj}p_{ki} =: \sum_{kl}\tilde{a}_{kl}D_{y_ky_l}\tilde{u}(y)$$

where $\tilde{a}_{kl} = p_l\mathcal{A}p_k'$ and p_l denotes a row of P. In other words, $\tilde{\mathcal{A}} = P\mathcal{A}P'$.

Since $\mathcal{A}$ is symmetric with eigenvalues on $[\lambda, \Lambda]$, it can be diagonalized by an orthogonal matrix P. In other words,

$$\tilde{\mathcal{A}} = P\mathcal{A}P' = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix},$$

where λ_is are the eigenvalues of $\mathcal{A}$. Thus, what we done so far is, we have diagonalized our system.

By doing another transformation (or directly $z = PDx$), we get the Laplacian. Set

$$D := \begin{pmatrix} \lambda_1^{-1/2} & & \\ & \ddots & \\ & & \lambda_n^{-1/2} \end{pmatrix}, \quad z = Dy, \ \overline{u}(z) := \tilde{u}(y)$$

Then similarly as above, we get

$$\sum_{kl} \tilde{a}_{kl} D_{y_k y_l} \tilde{u} = \sum_{ij} \overline{a}_{ij} D_{ij} \overline{u},$$

$\overline{\mathcal{A}} = D\tilde{\mathcal{A}}D' = I$ *i.e. the Laplacian.*

9. Further regularity results

In this section, we collect some further results without detailed proofs.

9.1. **Calderón-Zygmund type estimates, $W^{2,p}$.**

So far we had no x dependence in the equation when developing higher regularity theory. In contrast, estimates of this section apply to the viscosity solutions to

$$F(x, D^2u) = f(x).$$

We assume that F is uniformly elliptic and F, f are continuous as well as

$$F(x, 0) = 0.$$

If not consider $G(x, D^2u) := F(x, D^2u) - F(x, 0)$.

Theorem 9.1. *Let u be a bounded viscosity solution to*

$$F(x, D^2u) = f(x) \text{ in } B_1.$$

Assume that

- $F(x, 0) = 0$
- $F(x_0, D^2w) = 0$ *has* $C^{1,1}$ *for any* $x_0 \in B_1$
- $f \in L^p(B_1),\ n < p < \infty$.

Then there are β_0 and C such that if

$$\left(\fint_{B_r(x_0)} \left(\sup_{M \in S \setminus \{0\}} \frac{|F(x, M) - F(x_0, M)|}{||M||} \right)^n dx \right)^{1/n} \leq \beta_0$$

for all $B_r(x_0) \subset B_1$, then $u \in W^{2,p}(B_{1/2})$ and

$$||u||_{W^{2,p}(B_{1/2})} \leq C\, ||u||_{L^\infty(B_1)} + ||f||_{L^p(B_1)}\,.$$

Remark 9.2 (Reminder, Sobolev space)**.** *Define a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $|\alpha| := \alpha_1 + \ldots + \alpha_n \leq k$, and a derivative*

$$D^\alpha u := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

Let $u, v \in L^1_{loc}(\Omega)$ and α a multi-index. Then v is αth weak partial derivative of u if

$$\int_\Omega u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega v \varphi \, dx,$$

for every test function $\varphi \in C_0^\infty(\Omega)$. *We denote*

$$D^\alpha u := v.$$

Let $1 \le p \le \infty$ *and* $k \in \mathbb{N}$. *A function* $u : \Omega \to [-\infty, \infty]$ *belongs to a Sobolev space* $W^{k,p}(\Omega)$ *if* $u \in L^p(\Omega)$ *and its weak derivatives* $D^\alpha u$, $|\alpha| \le k$ *exist and belong to* $L^p(\Omega)$.

When we say above that $F(x_0, D^2 w) = 0$ has $C^{1,1}$, we mean that for any $x_0 \in B_1$ and $w_0 \in C(\partial B_1)$ there is a solution $w \in C^2(B_1) \cap C(\overline{B}_1)$ for

$$\begin{cases} F(x_0, D^2 w) = 0 & \text{in } B_1 \\ w = w_0 & \text{on } \partial B_1 \end{cases}$$

with the estimate

$$||w||_{C^{1,1}(\overline{B}_{1/2})} \le C_e \, ||w_0||_{L^\infty(\partial B_1)} \, .$$

This is the case, as we have seen, for example if $F(x_0, M)$ is concave.

Remark 9.3 (Schauder type estimates)**.** *If the frozen equation has* $C^{2,\alpha}$ *(or* $C^{1,\alpha}$*) estimates, then one can work out* $C^{2,\gamma}$ *(or* $C^{1,\gamma}$*),* $\gamma < \alpha$ *estimates for*

$$F(x, D^2 u) = f(x).$$

10. Additional regularity approaches

10.1. **Ishii-Lions method.** Next we look at other approaches to regularity. This approach is due to H Ishii, P.L Lions: 'Viscosity solutions of fully nonlinear second-order elliptic partial differential equations', JDE, 1990.

For simplicity we look at the Laplacian, but work out the proof in such a form that it applies to uniformly elliptic operators.

Theorem 10.1. *Let* $u \in C(B_4(0))$ *be a continuous viscosity solution to* $F(D^2 u) = \Delta u = 0$. *Then for* $\alpha \in (0,1)$ *there is* $C > 0$

$$|u(x) - u(y)| \le C\,|x-y|^\alpha \text{ for all } x, y \in B_{1/4}(0),$$

Proof. By considering $v := (u - \inf u)/\lambda$, $\lambda := \sup(u - \inf u)$, this satisfies $\frac{1}{\lambda} F(\lambda D^2 v) = 0$, an equation with a similar structure, we may assume

$$0 \le u \le 1.$$

Choose $z \in B_{1/4}$ and set

$$f(x,y) := \varphi(x,y) + 2\,|x-z|^2 := C\,|x-y|^\alpha + 2\,|x-z|^2 \, .$$

We want to show $u(x) - u(y) - f(x,y) \le 0$, and thriving for a contradiction suppose that there is $\theta > 0$ and $x, y \in B_1$ such that

$$u(x) - u(y) - f(x,y) = \sup_{(x',y') \in \overline{B}_1 \times \overline{B}_1} u(x') - u(y') - f(x',y') = \theta > 0.$$

We immediately observe that $x \neq y$ by the counter assumption. Also $(x, y) \notin \partial(B_1 \times B_1)$ since if $(x, y) \in \partial(B_1 \times B_1)$

$$C\left|x-y\right|^{\alpha}+2\left|x-z\right|^{2}\geq 1,$$

for C is large enough.

Thus we may use the theorem of sums to obtain

$$(D_x\varphi(x,y),X)\in \overline{J}^{2,+}(u(x)-2\left|x-z\right|^2),\quad (-D_y\varphi(x,y),Y)\in \overline{J}^{2,-}u(y)$$

i.e.

$$(D_x\varphi(x,y)+4(x-z),X+4I)\in \overline{J}^{2,+}u(x),\quad (-D_y\varphi(x,y),Y)\in \overline{J}^{2,-}u(y),$$

with the estimate

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq D^2\varphi+\frac{1}{\mu}(D^2\varphi)^2. \tag{10.30}$$

Now

$$D\varphi(x,y)=C\alpha\left|x-y\right|^{\alpha-2}(x-y,-(x-y)),\quad D^2\varphi(x,y)=\begin{pmatrix} M & -M \\ -M & M \end{pmatrix},$$

where $M=C\alpha\left|x-y\right|^{\alpha-2}\left((\alpha-2)\frac{x-y}{|x-y|}\otimes\frac{x-y}{|x-y|}+I\right)$. Further,

$$(D^2\varphi(x,y))^2=2\begin{pmatrix} M^2 & -M^2 \\ -M^2 & M^2 \end{pmatrix}.$$

where

$$\begin{aligned} M^2 &= C^2\alpha^2\left|x-y\right|^{2(\alpha-2)} \\ &\quad \cdot\left((\alpha-2)^2\frac{x-y}{|x-y|}\otimes\frac{x-y}{|x-y|}+2(\alpha-2)\frac{x-y}{|x-y|}\otimes\frac{x-y}{|x-y|}+I\right) \\ &= C^2\alpha^2\left|x-y\right|^{2(\alpha-2)}\Big(\underbrace{(\alpha^2-4\alpha+4+2\alpha-4)}_{=\alpha(\alpha-2)}\frac{x-y}{|x-y|}\otimes\frac{x-y}{|x-y|}+I\Big). \end{aligned}$$

In particular, all the eigenvalues of $X-Y$ are negative and since by using (10.30) we get respectively (by using $\begin{pmatrix} \xi \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \xi \end{pmatrix}$)

$$\begin{aligned} \xi' X\xi &\leq \xi'(M+\frac{2}{\mu}M^2)\xi \\ -\xi' Y\xi &\leq \xi'(M+\frac{2}{\mu}M^2)\xi. \end{aligned} \tag{10.31}$$

Next we observe

$$2\langle (M + 2\frac{1}{\mu}M^2)\frac{x-y}{|x-y|}, \frac{x-y}{|x-y|}\rangle = 2C\alpha\,|x-y|^{\alpha-2}(\alpha-2+1) + 4\frac{1}{\mu}C^2\alpha^2\,|x-y|^{2(\alpha-2)}(\alpha(\alpha-2)+1) \le C\alpha\,|x-y|^{\alpha-2}(\alpha-2+1)$$

by choosing $\frac{1}{\mu}$ such that

$$4\frac{1}{\mu}C^2\alpha^2\,|x-y|^{2(\alpha-2)}(\alpha(\alpha-2)+1) < -C\alpha\,|x-y|^{\alpha-2}(\alpha-2+1).$$

This together with (10.31) implies that one of the eigenvalues of $X - Y$ will have to be smaller than $C\alpha\,|x-y|^{\alpha-2}(\alpha-1)$. This together with non positivity of eigenvalues implies

$$0 \le \mathrm{tr}(X+4I) - \mathrm{tr}(Y) \le 4n + C\alpha\,|x-y|^{\alpha-2}(\alpha-1) < 0,$$

for large enough $C = C(n,\alpha)$. This contradiction completes the proof, by setting $x = z$. □

Next we look at the parabolic situation, see Imbert-Silvestre: 'Intro to fully nonlinear parabolic equations'.

We use a slightly modified theorem sums.

Theorem 10.2 (Modified theorem of sums)**.** *Let u and v be respectively continuous and bounded sub- and supersolutions in $\overline{B}_2 \times [-2,2]$ to a parabolic problem $-u_t + F(x,t,Du,D^2u) = 0$ where F is continuous and degenerate elliptic. Suppose that there is $(\hat{x},\hat{y},\hat{t}) \in B_2 \times B_2 \times (-2,2)$ such that*

$$u(\hat{x},\hat{t}) - v(\hat{y},\hat{t}) - \varphi(\hat{x}-\hat{y}) = \sup_{(x,y,t)\in B_2\times B_2\times(-2,2)} (u(x,t) - v(y,t) - \varphi(x-y)),$$

and that $\varphi \in C^2$ at the vicinity of $\hat{x}-\hat{y}$. Then for each $\mu > 1$, there are τ, X, Y such that

$$(\tau, p, X) \in \overline{J}^{2,+}u(\hat{x},\hat{t}), \quad (\tau, p, Y) \in \overline{J}^{2,-}v(\hat{y},\hat{t}),$$

where $p = D_x\varphi(\hat{x}-\hat{y}) = -D_y\varphi(\hat{x}-\hat{y})$ and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le D^2\varphi + \frac{1}{\mu}(D^2\varphi)^2 = \begin{pmatrix} M & -M \\ -M & M \end{pmatrix} + \frac{2}{\mu}\begin{pmatrix} M^2 & -M^2 \\ -M^2 & M^2 \end{pmatrix}, \tag{10.32}$$

where

$$M = D^2\varphi(\hat{x}-\hat{y}) = D_{xx}\varphi(\hat{x}-\hat{y}).$$

Theorem 10.3. *Let $u \in C(B_4(0)\times(-4,4))$ be a continuous viscosity solution to $-u_t + \Delta u = 0$. Then u is locally Hölder continuous in space.*

Proof. We again assume that $0 \le u \le 1$ and consider point with the same time (x,t) and (y,t).

Choose $(z,s) \in B_1 \times (-1,1)$ and set

$$f(x,y,t) := \varphi(x,y) + |x-z|^2 + (t-s)^2 := C\,|x-y|^\alpha + |x-z|^2 + (t-s)^2.$$

We want to show $u(x,t) - u(y,t) - f(x,y,t) \le 0$ in $B_2 \times B_2 \times (-2,2)$, and thriving for a contradiction suppose that there is $\theta > 0$ and $\hat{x}, \hat{y} \in B_2$ and $\hat{t} \in [-2,2]$ such that

$$u(\hat{x},\hat{t}) - u(\hat{y},\hat{t}) - f(\hat{x},\hat{y},\hat{t}) = \sup_{\overline{B}_2 \times \overline{B}_2 \times [-2,2]} u(x,t) - u(y,t) - f(x,y,t) = \theta > 0.$$

We immediately observe that $\hat{x} \ne \hat{y}$ by the counter assumption. Also $(\hat{x},\hat{y},\hat{t}) \notin \partial(B_2 \times B_2 \times (-2,2))$ since

$$C\,|\hat{x}-\hat{y}|^\alpha + |\hat{x}-z|^2 + \left|\hat{t}-s\right|^2 \ge 1,$$

for a suitable C and thus $u(x,t) - u(y,t) - f(x,y,t) \le 0$ on $\partial(B_2 \times B_2 \times (-2,2))$. Thus we may use the modified theorem of sums to

$$\underbrace{u(\hat{x},\hat{t}) - |\hat{x}-z|^2 - (\hat{t}-s)^2} - u(\hat{y},\hat{t}) - \varphi(\hat{x}-\hat{y})$$

in order to get

$$(\tau, p, X) \in \overline{P}^{2,+}(u(\hat{x},\hat{t}) - |\hat{x}-z|^2 - (\hat{t}-s)^2), \quad (\tau, p, Y) \in \overline{P}^{2,-} u(\hat{y},\hat{t}).$$

I.e.

$$(\tau + 2(\hat{t}-s), p + 2(\hat{x}-z), X + 2I) \in \overline{P}^{2,+} u(\hat{x},\hat{t}), \quad (\tau, p, Y) \in \overline{P}^{2,-} u(\hat{y},\hat{t}).$$

with the estimate

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le D^2\varphi + \frac{1}{\mu}(D^2\varphi)^2. \tag{10.33}$$

Again

$$D^2\varphi(x,y) = \begin{pmatrix} M & -M \\ -M & M \end{pmatrix},$$
$$M = C\alpha\,|x-y|^{\alpha-2}\Big((\alpha-2)\frac{x-y}{|x-y|} \otimes \frac{x-y}{|x-y|} + I\Big).$$

Since M is the same as in the elliptic version of the theorem, we have that eigenvalues of $X - Y$ are nonpositive and at least one of them will have to be smaller than $C\alpha\,|x-y|^{\alpha-2}(\alpha-1)$. This together with non positivity of eigenvalues implies

$$\begin{aligned} 0 &\le -\tau \underbrace{-2(\hat{t}-s)}_{\le L < \infty} + \operatorname{tr}(X + 4I) + \tau - \operatorname{tr}(Y) \\ &\le L + 4n + C\alpha\,|x-y|^{\alpha-2}(\alpha-1) < 0, \end{aligned}$$

for large enough $C = C(n, \alpha)$. This contradiction completes the proof by choosing $x = z, t = s$. □

10.2. **Berstein method.** Next we consider a method designed for showing boundedness of the gradient. By considering regularized equations, we may assume that the solution smooth, as long as our estimate does not depend on the smoothness.

Lemma 10.4. *Let F be concave, smooth, uniformly elliptic, $F(0) = 0$, $u \in C^3(\Omega)$ and*

$$F(D^2u) = 0 \text{ in } \Omega.$$

Then for

$$Lv := \sum a_{ij} D_{ij} v := \sum F_{ij}(D^2u) D_{ij} v$$

and $\nu \in \mathbb{R}^n$, $|\nu| = 1$ it holds that

$$Lu \leq 0, \quad Lu_\nu = 0 \text{ in } \Omega,$$

where u_ν is a directional derivative to direction ν.

Proof. Differentiate $F(D^2u) = 0$ to have

$$0 = \sum_{ij} F_{ij}(D^2u) D_{ij} u_\nu = Lu_\nu,$$

to obtain the second claim.

To obtain the first claim, consider

$$\psi(t) = F((1-t)D^2u(x) + t \cdot 0)$$

and observe

$$\psi(1) = F(0) = 0, \quad \psi(0) = F(D^2u(x)) = 0.$$

Thus by concavity, $\psi(t) \geq 0$ for $t \in [0, 1]$ and thus

$$0 \leq \psi'(0) = \sum F_{ij}(D^2u)(-D_{ij}u) = -Lu.$$

□

10.4.2015

Theorem 10.5 (Bernstein's method)**.** *Let F be concave, smooth, $F(0) = 0$. Let $u \in C^3(\mathbb{R}^n)$ satisfy*

$$F(D^2u) = 0 \text{ in } B_1.$$

Then there is $C > 0$ such that

$$||Du||_{L^\infty(B_{1/2})} \leq C \, ||u||_{L^\infty(B_1)} .$$

Proof. We use a bumb function

$$\varphi \in C_0^\infty(B_1), \quad 0 \le \varphi \le 1, \quad \varphi = 1 \text{ in } \overline{B}_{1/2}$$

such that

$$\varphi + |D\varphi| + \left|\left|D^2\varphi\right|\right| \le C.$$

Let $M := \sup_{\overline{B}_1} u$. The idea in the Bernstein method is to consider an auxiliary quantity

$$h = \delta(M-u)^2 + \varphi^2 |Du|^2 ,$$

show that this is a subsolution to the linearized problem of the previous lemma i.e. $Lh \ge 0$ and use max principle which says that the max within the domain can be bounded by a max at the boundary. This along with the definition of the bumb function gives an estimate of the gradient in terms of the function itself.

In this proof, we drop for brevity the sum signs which is a rather usual convention in the literature: sum over each free index you see below. Thus $Lu = a_{ij}(x)D_{ij}u = a_{ij}u_{ij}$ and observe that

$$\begin{aligned} L(vw) &= a_{ij}D_{ij}(vw) = a_{ij}D_i(wD_jv + vD_jw) \\ &= a_{ij}(D_iwD_jv + wD_{ij}v + D_ivD_jw + vD_{ij}w) \\ &= vLw + 2a_{ij}D_ivD_jw + wLv \\ &= wLv + 2a_{ij}v_iw_i + vLw. \end{aligned}$$

Then observe that

$$L(u_k^2) = 2u_kLu_k + 2a_{ij}u_{ki}u_{kj}$$

and further

$$\begin{aligned} Lh &= L(\delta(M-u)^2 + \varphi^2 |Du|^2) \\ &= 2\delta(M-u)(-Lu) + 2\delta a_{ij}u_iu_j \\ &+ L(\varphi^2)\,|Du|^2 + 2\varphi^2 u_kLu_k + 2\varphi^2 a_{ij}u_{ki}u_{kj} + 8a_{ij}\varphi\varphi_i u_k u_{kj}. \end{aligned}$$

The last term results from $2a_{ij}(\varphi^2)_i(u_k^2)_j$.

By the previous lemma, $Lu \le 0$ and $Lu_k = 0$, and a_{ij} has the ellipticity constants $0 < \lambda \le \Lambda < \infty$. To estimate the last term, we set

$$v_j = \varphi u_{kj} \text{ and } w_i = \varphi_i u_k$$

and assume without loss of generality that the resulting vectors are nonzero

$$\begin{aligned}|8a_{ij}\varphi\varphi_i u_k u_{kj}| &= |8a_{ij}v_j w_i| \\ &= 8\left|\langle\mathcal{A}\frac{v}{|v|},\frac{w}{|w|}\rangle\right| |v|\,|w| \\ &= 8\Lambda\,|v|\,|w| \\ &\overset{\text{Young}}{\leq} 2\lambda\,|v|^2 + C\,|w|^2 \\ &= 2\lambda\sum_j \varphi^2 u_{kj}^2 + C\sum_i \varphi_i^2 u_k^2.\end{aligned}$$

Collecting the facts we get

$$\begin{aligned}Lh &\geq 0 + 2\delta\lambda\,|Du|^2 \\ &\quad - C\,|Du|^2 + 0 + 2\lambda\varphi^2\sum_j (u_{kj})^2 \\ &\quad - 2\lambda\varphi^2\sum_j u_{kj}^2 - C\,|Du|^2 \\ &\geq 2\delta\lambda\,|Du|^2 - C\,|Du|^2\,.\end{aligned}$$

Thus by choosing δ large enough we see that $Lh \geq 0$. Then by the max principle (ex)

$$\sup_{\overline{B}_1} h \leq \sup_{\partial B_1} h.$$

It follows that

$$\sup_{B_{1/2}} |Du|^2 \leq \sup_{B_{1/2}} h \leq \sup_{\partial B_1} h = \sup_{\partial B_1} \delta(M-u)^2 \leq \delta(\underbrace{\operatorname{osc}_{B_1}}_{:=\sup u - \inf u} u)^2,$$

which implies the claim. □

10.3. **Recent developments.** $C^{2,\alpha}$ regularity was a longstanding open problem for uniformly elliptic fully nonlinear operators without the convexity/concanicty assumption on F which we studied above, until Nadirashvili reported in dimensions $n = 12$ a counterexample if no convexity is assumed on F, see Nonclassical solutions of fully nonlinear elliptic equations, Nadirashvili and Vlăduţ In some cases it is possible to work with different assumptions, see for example O. Savin, Small perturbation solutions for elliptic equations, Comm. Partial Differential Equations 32 (2007) 557–578 (flatness assumption) and Cabré, X., Caffarelli, L.A., Interior $C^{2,\alpha}$ regularity theory for a class of nonconvex fully nonlinear elliptic equations, J. Math. Pures Appl. 82, 2003, 573–612 as well as references therein.

11. Differential games

In Section 7 we discussed one controller situation. Now we briefly discuss a two controller, which are referred as players, situation. The corresponding PDE is called Isaacs' equation (Isaacs, Differential games, 1965).

We consider zero sum games, which means that one of the players, say, Player II pays Player I at the end of the game; losses and gains sum up to zero.

We use the following terminology and notation akin to Section 7

$$
\begin{aligned}
&(x,t) \in \mathbb{R}^n \times [0,T], \text{starting point}\\
&K \text{ a compact subsets of } \mathbb{R}^n\\
&a : [0,T] \to K, \text{ admissible control}\\
&b : [0,T] \to K, \text{ admissible control}\\
&\mathcal{A} = \{a : [0,T] \to K \,:\, a \text{ is measurable}\} \text{ the set of admissible controls}\\
&\begin{cases} x'(s) = f(x(s), a(s), b(s)) & s \in [t,T] \\ x(t) = x \end{cases} \text{ dynamics given by } f\\
&f : \mathbb{R}^n \times K \times K \to \mathbb{R}^n,\\
&r : \mathbb{R}^n \times K \times K \to \mathbb{R}, \text{ running cost,}\\
&g : \mathbb{R}^n \times K \times K \to \mathbb{R}, \text{ terminal cost.}
\end{aligned}
$$

Later we use some other time intervals as well and might emphasize it by denoting for example $\mathcal{A}_{t,T}$. Also observe, that admissible controls defined on $[t, t+h]$ $[t+h, T]$ define a control on $[t,T]$, point $t+h$ has a measure zero and thus we may take this from any of the controls. When clear front the context, we drop the time intervals.

Above we take the control spaces to be the same for both the players for simplicity.

We assume that all the functions are contiuous, Lip wrt x and bounded

$$
\begin{aligned}
&|f(x,a,b)| \le C, \; |r(x,a,b)| \le C, \; |g(x)| \le C\\
&|f(x,a,b) - f(y,a,b)| \le C\,|x-y|\,, \; |r(x,a,b) - r(y,a,b)| \le C\,|x-y|\,,\\
&|g(x) - g(y)| \le C\,|x-y|\,.
\end{aligned}
$$

The dynamics ODE has a unique Lipschitz continuous solution $x(s)$ under these assumptions.

We are interested of the total costs

$$
C_{x,t}(a,b) := \int_t^T r(x(s), a(s), b(s))\,ds + g(x(T)).
$$

However, now we need to decide who chooses first. This is often done so that one of the players chooses a control and another a strategy

$$\alpha : \mathcal{A} \to \mathcal{A}, \quad \beta : \mathcal{A} \to \mathcal{A}.$$

We identify two controls that coincide a.e. and denote $a = b$ (we do not care sets of measure zero). We work under the progressive strategies: if $a = b$ on [0,s], then $\alpha(a) = \alpha(b)$ on $[0, s]$, i.e. the staregy does not foresee the future, and denote such strategies by $\mathcal{S}$

The one choosing strategy has advantage, so there are two natural definition of values..

Definition 11.1 (Game values)**.**

$$\begin{aligned} V_-(x,t) &= \sup_{a\in\mathcal{A}} \inf_{\beta\in\mathcal{S}} C_{x,t}(a, \beta(a)) \\ &:= \sup_a \inf_\beta \{\int_t^T r(x(s), a(s), \beta(a)(s))\, ds + g(x(T))\}, \\ V_+(x,t) &= \sup_{\alpha\in\mathcal{S}} \inf_{b\in\mathcal{A}} C_{x,t}(\alpha(b), b) \\ &:= \sup_\alpha \inf_b \{\int_t^T r(x(s), \alpha(b)(s), b(s))\, ds + g(x(T))\}. \end{aligned}$$

Remark 11.2. *In the above formulation, it does not matter which order we put* $\sup\inf$ *(ex).*

This is of course not to be taken as a general statement about $\inf\sup = \sup\inf$*. One direction holds in general:*

$$\sup_{x\in X} \inf_{y\in Y} f(x,y) \le \inf_{y\in Y} \sup_{x\in X} f(x,y).$$

This follows from the definition of $\inf$ *and* $\sup$ *as*

$$\begin{aligned} f(x,y) &\le \sup_{x\in X} f(x,y) \text{ for all } x, y \\ \inf_{y\in Y} f(x,y) &\le \sup_{x\in X} f(x,y) \\ \sup_{x\in X} \inf_{y\in Y} f(x,y) &\le \sup_{x\in X} f(x,y) \\ \sup_{x\in X} \inf_{y\in Y} f(x,y) &\le \inf_{y\in Y} \sup_{x\in X} f(x,y). \end{aligned}$$

Example 11.3. *Define* $f : [0,1] \times [0,1] \to \mathbb{R}$

$$f(x,y) = \begin{cases} 1 & x = y \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sup_{x\in[0,1]} \inf_{y\in[0,1]} f(x,y) = \sup_{x\in[0,1]} 0 = 0$$

and

$$\inf_{y\in[0,1]} \sup_{x\in[0,1]} f(x,y) = \inf_{y\in[0,1]} 1 = 1.$$

There is no problem of modifying this example to be continuous either.

The dynamic programming principle reads as

Lemma 11.4 (DPP)**.** *For each $h > 0$ small enough so that $t+h \le T$ setting $V := V_+$, we have*

$$V(x,t) = \sup_\alpha \inf_b \{ \int_t^{t+h} r(x(s), \alpha(b)(s), b(s))\, ds + V(x(t+h), t+h)\},$$

where

$$\begin{cases} x'(s) = f(s, \alpha(b)(s), b(s)), & t \le s \le t+h \\ x(t) = x. \end{cases}$$

Proof 11.4. Let $\eta > 0$ and choose by the definion of sup, α_0 such that

$$\begin{aligned} \sup_\alpha \inf_b \{ \int_t^{t+h} & r(x(s), \alpha(b)(s), b(s))\, ds + V(x(t+h), t+h)\} - \eta \\ & \le \inf_b \{ \int_t^{t+h} r(x(s), \alpha_0(b)(s), b(s))\, ds + V(x(t+h), t+h)\} \\ & \le \int_t^{t+h} r(x(s), \alpha_0(b)(s), b_0(s))\, ds + V(x_{\alpha_0, b_0}(t+h), t+h), \end{aligned}$$

for any b_0 (at the last step we emphasized to which control and strategy the trajectory is related; mostly this is not explicitly denoted but will have to be deduced from the context). Similarly choose α_1 such that for any b_1 over the interval $[t+h, T]$ (where the choices of the previous step determine the starting point $x(t+h) = x_{\alpha_0, b_0}(t+h)$)

$$\begin{aligned} V(x_{\alpha_0,b_0}(t+h), t+h) &= \sup_\alpha \inf_b \{ \int_{t+h}^T r(x(s), \alpha(b)(s), b(s))\, ds + g(x(T))\} - \eta \\ &\le \int_{t+h}^T r(x(s), \alpha_1(b)(s), b_1(s))\, ds + g(x(T)). \end{aligned}$$

The above strategies and any fixed control b define a control $\alpha_{0,1}(b)$ over the whole time interval $[t,T]$ and we have

$$\begin{aligned}
&\sup_\alpha \inf_b \int_t^T r(x(s),\alpha(b)(s),b(s))\,ds + g(x(T))\\
&\geq \inf_b \int_t^T r(x(s),\alpha_{0,1}(b)(s),b(s))\,ds + g(x(T))\\
&\overset{\text{choose } b}{\geq} \Big(\int_t^{t+h} r(x(s),\alpha_{0,1}(b)(s),b(s))\,ds\\
&\quad + \int_{t+h}^T r(x(s),\alpha_{0,1}(b)(s),b(s))\,ds + g(x(T))\Big) - \eta\\
&\overset{\text{above}}{\geq} -3\eta + \sup_\alpha \inf_b \{\int_t^{t+h} r(x(s),\alpha(b)(s),b(s))\,ds + V(x(t+h),t+h)\}\\
&\quad - V(x_{\alpha_0,b}(t+h),t+h) + \sup_\alpha \inf_b \{\int_{t+h}^T r(x(s),\alpha(b)(s),b(s))\,ds + g(x(T))\},
\end{aligned}$$

where at the last step we used the above estimates. This proves the first direction, since the right hand side

$$\begin{aligned}
&\sup_\alpha \inf_b \{\int_t^{t+h} r(x(s),\alpha(b)(s),b(s))\,ds + V(x(t+h),t+h)\}\\
&\qquad - V(x_{\alpha_0,b}(t+h),t+h) + V(x_{\alpha_0,b}(t+h),t+h) - 3\eta.
\end{aligned}$$

To prove the opposite inequality, write

$$\begin{aligned}
&\sup_\alpha \inf_b \{\int_t^{t+h} r(x(s),\alpha(b)(s),b(s))\,ds + V(x(t+h),t+h)\}\\
&\qquad \geq \inf_b \{\int_t^{t+h} r(x(s),\alpha(b)(s),b(s))\,ds + V(x(t+h),t+h)\}\\
&\qquad \overset{\text{choose } b_0}{\geq} \int_t^{t+h} r(x_{\alpha,b_0}(s),\alpha(b_0)(s),b_0(s))\,ds + V(x_{\alpha,b_0}(t+h),t+h) - \eta,
\end{aligned}$$

for any fixed α and this particular b_0. Using the staring point given by these choices $x_{\alpha,b_0}(t+h)$, we get

$$\begin{aligned}
&V(x_{\alpha,b_0}(t+h),t+h)\\
&\quad = \sup_\alpha \inf_b \int_{t+h}^T r(x(s),\alpha(b)(s),b(s))\,ds + g(x(T))\\
&\quad \geq \inf_b \int_{t+h}^T r(x(s),\alpha(b_1)(s),b(s))\,ds + g(x(T))\\
&\quad \geq \int_{t+h}^T r(x_{\alpha,b_1}(s),\alpha(b_1)(s),b_1(s))\,ds + g(x_{\alpha,b_1}(T)) - \eta.
\end{aligned}$$

again for any α and for this particular b_1. Combining the estimates

$$\begin{aligned}
&\sup_\alpha \inf_b \{\int_t^{t+h} r(x(s),\alpha(b)(s),b(s))\,ds + V(x(t+h),t+h)\} \\
&\qquad \geq \int_t^{t+h} r(x_{\alpha,b_0}(s),\alpha(b_0)(s),b_0(s))\,ds + V(x_{\alpha,b_0}(t+h),t+h) - \eta \\
&\qquad \geq \int_t^{t+h} r(x_{\alpha,b_0}(s),\alpha(b_0)(s),b_0(s))\,ds \\
&\qquad + \int_{t+h}^{T} r(x_{\alpha,b_1}(s),\alpha(b_1)(s),b_1(s))\,ds + g(x_{\alpha,b_1}(T)) - 2\eta \\
&\qquad \geq \inf_{b\in\mathcal{A}_{t,T}} \int_t^{T} r(x(s),\alpha(b)(s),b(s))\,ds + g(x(T)) - 2\eta.
\end{aligned}$$

Then take $\sup_{\mathcal{A}\in\mathcal{S}_{t,T}}$ to complete the proof. □

To make sure that we remain within the uniqueness theory in unbounded domain, and make sure that we take the initial values continuously:

Theorem 11.5 (Estimates for value functions)**.** *The value function $V := V_+$ (or V_- respectively) is bounded and Lipschitz in (x,t).*

It holds that $V := V_+$ (or V_- respectively) is a solution to a first order terminal value Isaacs' PDE of a type

$$\begin{cases} V_t + H(x,DV) = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ V(x,T) = g(x) & \text{on } \mathbb{R}^n. \end{cases}$$

The next derivation is formal. Use Taylor's expansion to $V := V_+$

$$\begin{aligned}
&V(x(t+h),t+h) \\
&\quad = V(x,t) + DV(x,t)\cdot(x(t+h)-x) + V_t(x,t)h + \text{Error} \\
&\quad \approx V(x,t) + DV(x,t)\cdot f(x,a(t),b(t))h + V_t(x,t)h + \text{Error}
\end{aligned}$$

in the DPP

$$\begin{aligned}
V(x,t) &= \sup_\alpha \inf_b \{\int_t^{t+h} r(x(s),\alpha(b)(s),b(s))\,ds + V(x(t+h),t+h)\} \\
&\approx \sup_\alpha \inf_b \{\int_t^{t+h} r(x(s),\alpha(b)(s),b(s))\,ds \\
&\quad + V(x,t) + DV(x,t)\cdot f(x,\alpha(b)(t),b(t))h + V_t(x,t)h + \text{Error}\}.
\end{aligned}$$

Cancelling $V(x,t)$-terms, dividing by h, letting $h \to 0$, we get

$$\begin{aligned}
0 &= \sup_{\alpha} \inf_{b} \{ r(x, \alpha(b)(t), b(t)) + DV(x,t) \cdot f(x, \alpha(b)(t), b(t)) + V_t(x,t) \} \\
&= \inf_{\tilde{b} \in K} \sup_{\tilde{a} \in K} \{ r(x, \tilde{a}, \tilde{b}) + DV(x,t) \cdot f(x, \tilde{a}, \tilde{b} \} + V_t(x,t) \\
&=: H_+(x, DV(x,t)) + V_t(x,t) \\
&=: H(x, DV(x,t)) + V_t(x,t).
\end{aligned}$$

Theorem 11.6. *Let $V := V_+$ (respectively $V := V_-$) be the value for the game in the above setup. Then V is the unique viscosity solution to*

$$\begin{cases} V_t + H(x, DV) = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ V(x,T) = g(x), \end{cases}$$

$$H(x, DV) := \inf_{\tilde{b} \in K} \sup_{\tilde{a} \in K} \{ r(x, \tilde{a}, \tilde{b}) + DV(x,t) \cdot f(x, \tilde{a}, \tilde{b}) \},$$

(or with $H_-(x, DV) := \sup_{\tilde{a} \in K} \inf_{\tilde{b} \in K} \{ r(x, \tilde{a}, \tilde{b}) + DV(x,t) \cdot f(x, \tilde{a}, \tilde{b}) \}$ respectively).

As before the viscosity subsolution above means that, we touch u from above at (x,t) by $\varphi \in C^1$ and have

$$\varphi_t(x,t) + H(x, D\varphi(x,t)) \geq 0.$$

Definition 11.7 (Isaacs' condition)**.** *Whenever*

$$\begin{aligned}
&\inf_{\tilde{b} \in K} \sup_{\tilde{a} \in K} \{ r(x, \tilde{a}, \tilde{b}) + DV(x,t) \cdot f(x, \tilde{a}, \tilde{b}) \} \\
&\qquad = \sup_{\tilde{a} \in K} \inf_{\tilde{b} \in K} \{ r(x, \tilde{a}, \tilde{b}) + DV(x,t) \cdot f(x, \tilde{a}, \tilde{b}) \}
\end{aligned}$$

holds, we say that Isaacs' condition holds.

Corollary 11.8. *If the Isaacs' condition hold, then*

$$V_-(x,t) = V_+(x,t).$$

Reason: each of the values is unique viscosity solution to the related PDE. If Isaacs' condition hold, then the PDEs are the same.

Remark 11.9. *There are several possible directions to change the setup:*

- *Bounded domain*
- *Bdd domain, and infinite time horizon:*

$$\sup_{\alpha} \inf_{b} \int_0^\infty r(x(s), \alpha(b)(s), b(s)) e^{-rs} \, ds.$$

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