EXERCISE SET 2B
VISCOSITY THEORY, 2015
DEADLINE 10.4.2015

1. (2 points) Show that if $u \in C(\Omega)$ is a bounded viscosity subsolution to $\Delta u \geq 0$ in Ω, then sup-convolution u^ε of u is still a subsolution in Ω_ε.
2. Let u and u^ε be as in the previous exercise. Show that $u^\varepsilon \searrow u$.
3. (2 points) Let u^ε be as above. Show that there is C such that $u^\varepsilon(x)+C\left|x\right|^2$ is convex.
4. Show that if u is a viscosity subsolution to $P^+(D^2u) \geq f$, then $v := -u$ is a viscosity supersolution to $P^-(D^2v) \leq -f$.

   Does it hold that if F is degenerate elliptic and u a subsolution to $F(D^2u) \geq 0$, then is $v =: -u$ is a supersolution to $F(D^2v) \leq 0$? How about if F uniformy elliptic?
5. Show that if $v \in C(\Omega)$ is a viscosity supersolution to $P^-(D^2v) \leq 0$, then
$$\tilde{v}(x) = v(x) + 2(\left|x\right|^2 - 1)$$
   is a supersolution to $P^-(D^2\tilde{v}) \leq C$.
6. Show that $\nu'(D^2u)\nu = g''(t) =: D_{\nu\nu}u$ where $g(t) := u(x+\nu t)$.
7. (Bonus) Show that if F is uniformly elliptic and smooth, then
$$\left|\xi\right|^2 \tilde{\lambda} \leq \sum_{ij} F_{ij}\xi_i\xi_j \leq \tilde{\Lambda}\left|\xi\right|^2,$$
   for suitable constants $0 < \tilde{\lambda} \leq \tilde{\Lambda} < \infty$. Above $F_{ij}(X) = \partial F(X)/(\partial x_{ij})$.