EXERCISE SET 1 A
VISCOSITY THEORY, 2015
DEADLINE 27.02.2015

1. Show that
$$u(x) = \begin{cases} -\frac{x^2}{2} + \frac{5}{6}x & x \in (0,1] \\ -\frac{x^2}{4} + \frac{5}{12}x + \frac{1}{6}, & x \in (1,2). \end{cases}$$
is not a viscosity solution to
$$\begin{cases} F(x,u',u'') = -au'' - 1 = 0, & x \in (0,2) \\ u(0) = 0 = u(2). \end{cases} \qquad a(x) = \begin{cases} 1 & x \in (0,1) \\ 2 & x \in [1,2) \end{cases}.$$
Instruction: We required continuity for F in the lecture note, do not care it here.
2. Show that if $u, \varphi \in C^2(\mathbb{R})$, and $u - \varphi$ has a maximum at x, then
$$u'(x) = \varphi'(x)$$
$$u''(x) \le \varphi''(x).$$
3. Show that there is no classical (i.e. C^1 solution) to
$$\begin{cases} (u')^2 = 1 & \text{in } (-1,1) \\ u(\pm 1) = 0. \end{cases}$$
4. Find/guess (and show) what is a viscosity solution to $F(x,u',u'') = -(u')^2 + 1 = 0$.
5. Show that
$$-\varepsilon \log\left(\frac{\cosh(\frac{1}{\varepsilon}x)}{\cosh(\frac{1}{\varepsilon})}\right) \to 1 - |x|$$
uniformly on $[-1,1]$ when $\varepsilon \to 0$.
6. Let $h, \eta \in \mathbb{R}^n, n \ge 2$. Show that $h'(\eta \otimes \eta)h = (h \cdot \eta)^2$.
7. Let $u(x) = |x|^\alpha$, $x \in \mathbb{R}^n, \alpha \in (0,1)$, $x \neq 0$. Calculate Du and D^2u and use $\otimes$ notation to shorten the notation whenever possible.
8. Let $u(x) = |x|^\alpha$ as above. Show that u is a viscosity subsolution to $-\Delta u = -\operatorname{tr} D^2 u = 0$ in $B_1(0)$.
9. Let $x \in \mathbb{R}^n, n \ge 3$. Show that $u(x) = |x|^{-n+2}$ is a viscosity supersolution to $F = -\Delta$.
10. Show that $\operatorname{tr}(\mathcal{A}X) = \operatorname{tr}((\mathcal{A}^{\frac{1}{2}})'X\mathcal{A}^{\frac{1}{2}}) = \sum_i (\mathcal{A}^{\frac{1}{2}})_i' X (\mathcal{A}^{\frac{1}{2}})_i$ where $\mathcal{A}, X \in S^n$, $\mathcal{A} \ge 0$ and $\mathcal{A}^{\frac{1}{2}} \in S^n$ is its positive semidefinite symmetric matrix square root and $(\mathcal{A}^{\frac{1}{2}})_i$ the ith column of $\mathcal{A}^{\frac{1}{2}}$.

11. Show that
$$-\sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u(x)$$
which can written in the form
$$F(x, r, p, X) = -\operatorname{tr}(\mathcal{A}(x)X),$$
is degenerate elliptic, where $\mathcal{A}, X \in S^n$, and for the coefficient matrix it holds that $\mathcal{A} \geq 0$.
12. Show the equivalence of two definitions of lower semicontinuity.
13. Show that if $u : \Omega \to (-\infty, \infty]$ is lower semicontinous, then there is a increasing sequence u_i of continuous functions such that
$$u = \lim_{i\to\infty} u_i \quad \text{in } \Omega.$$
Moreover, if u_i are continuous functions then $\sup_i u$ is lower semicontinous.
14. [1] Show that if there is a solution $u \in C^2(-1,1) \cap C([-1,1])$ of
$$\begin{cases} -\varepsilon u_\varepsilon''(x) + (u_\varepsilon')^2 = 1 & \text{in } (-1,1) \\ u_\varepsilon(\pm 1) = 0, \end{cases}$$
then this is the only solution with this regularity.

[1] Bonus: You can earn an extra point, but this exercise is not required i.e. not taken into account when calculating grade limits. In other words, if you complete all exercises, your round 1A points are 14/13.