EXERCISE SET 1
PARTIAL DIFFERENTIAL EQUATIONS 2, 2013
DEADLINE 22.2.2013

1. Recall concepts of partial derivative and differentiability. Does it follow from the continuity of partial derivatives that function is differentiable or vice versa? Does it follow from the existence of partial derivatives that function is differentiable or vice versa? Whenever the answer is negative, give a counterexample.
2. Show that for
$$\varphi : \mathbb{R}^n \to \mathbb{R}, \quad \varphi(x) = \begin{cases} e^{1/(|x|^2-1)} & |x| < 1 \\ 0, & |x| \geq 1. \end{cases}$$
it holds $\varphi \in C_0^\infty(\mathbb{R}^n)$.
3. Prove Young's inequality in the form (ie. ε-version) given in the lecture note.
4. Prove Hölder's inequality.
5. Find a weak derivative (and show that it is a weak derivative) of
$$u : (-1, 1) \to \mathbb{R}, \quad u(x) = \begin{cases} 0, & -1 < x \leq 0 \\ \sqrt{x}, & 0 < x < 1. \end{cases}$$
6. (a) Show that $u \notin W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ where
$$u : \mathbb{R} \to \mathbb{R}, \quad u(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x > 0. \end{cases}$$
   (b) Show that $\int_{\mathbb{R}} u\varphi' \, dx = -\varphi(0)$ for every $\varphi \in C_0^\infty(\mathbb{R})$.
7. (a) Show that for $a, b \geq 0$ it holds that $(a+b)^p \leq C_1(a^p + b^p)$ and $a^p + b^p \leq C_2(a+b)^p$ for $p > 0$ where C_1, C_2 are independent of a, b.
   (b) Show that $||u||_{W^{k,p}(\Omega)}$ is equivalent with the norm
$$\sum_{|\alpha| \leq k} \Big(\int_\Omega |D^\alpha u|^p \, dx \Big)^{1/p} \quad \text{if } 1 \leq p \leq \infty.$$
8. (a) Show that $\eta_\varepsilon * f = f * \eta_\varepsilon$.
   (b) Let $\varphi \in C_0^\infty(\Omega)$ and $f \in L^1_{\mathrm{loc}}(\Omega)$. Show starting from the definition that $\varphi * f \in C(\mathbb{R}^n)$.
9. Show that if $f \in W^{k,p}_{\mathrm{loc}}(\Omega)$ for $1 \leq p \leq \infty$, $k \in \mathbb{N}$, then
$$D^\alpha f_\varepsilon = \eta_\varepsilon * D^\alpha f \quad \text{in } \Omega_\varepsilon.$$
   Tips: $D^\alpha f_\varepsilon = f * D^\alpha \eta_\varepsilon$. Use this together with the definition of a weak derivative.

10. If $f \in W^{k,p}_{\text{loc}}(\Omega)$, for $1 \le p < \infty$, $k \in \mathbb{N}$, then
$$f_\varepsilon \to f \quad \text{in } W^{k,p}_{\text{loc}}(\Omega).$$
Tips: Previous exercise. Also show that this does not hold for $p = \infty$.
11. Let $u_i \to u$ in $L^p(\Omega)$. Show that there exists a subsequence (still denoted by u_i) such that $u_i \to u$ a.e. in Ω.
12. Show by a counterexample, that the previous theorem does not hold without passing to a subsequence.
13. Show that
$$\frac{\partial f(u)}{\partial x_j} = f'(u)\frac{\partial u}{\partial x_j}, \quad j = 1, \dots, n$$
a.e. in Ω when $p = 1$. Specify appropriate conditions for f.
14. Show that the following holds: Let $u \in W^{1,p}(\Omega)$ and Ω bounded. Then for
$$u_\lambda := \min(\max(u, -\lambda), \lambda)) = \begin{cases} \lambda & \{x \in \Omega \,:\, u(x) \ge \lambda\} \\ u & \{x \in \Omega \,:\, \lambda < u(x) < \lambda\} \\ -\lambda & \{x \in \Omega \,:\, u(x) \le \lambda\} \end{cases}$$
we have
$$u_\lambda \to u \quad \text{in } W^{1,p}(\Omega)$$
when $\lambda \to \infty$.
15. Let $u_i, v_i \in W^{1,p}(\Omega)$ and
$$u_i \to u, v_i \to v \quad \text{in } W^{1,p}(\Omega).$$
Show that
$$\min(u_i, v_i) \to \min(u, v) \quad \text{in } W^{1,p}(\Omega)$$
16. Show that if $u, v \in W^{1,p}_0(\Omega)$ then $\min(u, v) \in W^{1,p}_0(\Omega)$.
17. Show that if $0 \le u \le v$, $u \in W^{1,p}(\Omega)$ and $v \in W^{1,p}_0(\Omega)$, then $u \in W^{1,p}_0(\Omega)$.
18. Show that if $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, then $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, and
$$\frac{\partial(uv)}{\partial x_j} = \frac{\partial u}{\partial x_j}v + u\frac{\partial v}{\partial x_j}$$
almost everywhere in Ω. Does this hold without $L^\infty(\Omega)$ in the assumptions?
19. Show that for
$$\frac{1}{p_1} + \ldots + \frac{1}{p_m} = 1$$
and $u_1 \in L^{p_1}(\Omega), \dots, u_m \in L^{p_m}(\Omega)$, it holds
$$\int_\Omega |u_1 \cdot \ldots \cdot u_m| \, dx \le \prod_{i=1}^m \Big(\int_\Omega |u_i|^{p_i} \, dx \Big)^{1/p_i}.$$
20. Show that $(\int_{\mathbb{R}} |u|^q \, dx)^{1/q} \le C(n, q) \int_{\mathbb{R}} |u'| \, dx$ fails in $W^{1,1}(\mathbb{R})$.

21. Let $p = n > 1$. Show that then there exists $C = C(n)$ such that
$$\Big(\int_{B(x,r)} |u|^q\, dy\Big)^{1/q} \le C r^{p/q} \Big(\int_{B(x,r)} |Du|^p\, dy\Big)^{1/p},$$
for any positive $q < \infty$.
22. $W_0^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$.
23. $W_0^{1,p}(B(0,1)) \neq W^{1,p}(B(0,1))$.
24. Let $\Omega \subset \mathbb{R}^n$ be open and connected domain, $u \in W^{1,p}(\Omega)$. Show that if $Du = 0$ a.e. in Ω, then $u = c(= \mathit{constant})$ a.e. in Ω.
25. Let $\Omega \subset \mathbb{R}^n$ be open and connected domain, $u \in W_0^{1,p}(\Omega)$. Show that if $Du = 0$ a.e. in Ω, then $u = 0$ a.e. in Ω.
26. [1] Show that Cantor function is not in $W^{1,1}(0,1)$.

[1]Bonus: You can earn an extra point, but this exercise is not required/taken into account when calculating grade limits