AN ASYMPTOTIC MEAN VALUE CHARACTERIZATION FOR A CLASS OF NONLINEAR PARABOLIC EQUATIONS RELATED TO TUG-OF-WAR GAMES

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Abstract. We characterize solutions to the homogeneous parabolic p-Laplace equation $u_t = |\nabla u|^{2-p}\,\Delta_p u = (p-2)\Delta_\infty u + \Delta u$ in terms of an asymptotic mean value property. The results are connected with the analysis of tug-of-war games with noise in which the number of rounds is bounded. The value functions for these game approximate a solution to the PDE above when the parameter that controls the size of the possible steps goes to zero.

1. Introduction.

Mean value properties for solutions to elliptic and parabolic partial differential equations are useful tools for the study of their qualitative properties. The classical mean value property for harmonic functions states that u solves $\Delta u = 0$ if and only if it satisfies

$$u(x) = \frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} u(y)\,dy = \fint_{B_\varepsilon(x)} u(y)\,dy.$$

In fact, as remarked in [MPR], we can relax this condition by requiring that it holds asymptotically

$$u(x) = \fint_{B_\varepsilon(x)} u(y)\,dy + o(\varepsilon^2),$$

as $\varepsilon \to 0$. This result follows easily for classical C^2 solutions by using the Taylor expansion and for continuous functions by using the theory of viscosity solutions. In addition, a weak asymptotic mean value formula holds for elliptic problems in some nonlinear cases as well. In [MPR] the authors characterized p-harmonic functions by means of asymptotic mean value properties that hold in a weak sense, that we call viscosity sense (see Definition 2.3 below). In fact, the asymptotic expansion

$$u(x) = \frac{\alpha}{2}\left\{\max_{\overline{B}_\varepsilon(x)} u + \min_{\overline{B}_\varepsilon(x)} u\right\} + \beta \fint_{B_\varepsilon(x)} u(y)\,dy + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0,$$

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holds for all x in a domain Ω in the viscosity sense if and only if

$$\Delta_p u(x) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)(x) = 0,$$

in Ω in the viscosity sense, where α and β are given by

$$\alpha = \frac{p-2}{p+n} \qquad \text{and} \qquad \beta = \frac{2+n}{p+n}. \tag{1.1}$$

Our main objective in this paper is to extend this analysis to parabolic problems and to study parabolic tug-of-war games with noise.

To begin with, let us consider the heat equation. We observe that a function u solves

$$u_t(x,t) = \Delta u(x,t)$$

if and only if

$$u(x,t) = \fint_{t-\varepsilon^2/(n+2)}^{t} \fint_{B_\varepsilon(x)} u(y,s)\, dy\, ds + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0.$$

In the case $p \neq 2$ our results are easier to state if we rescale the time variable so that we consider viscosity solutions u to the equation,

$$(n+p)u_t(x,t) = |\nabla u|^{2-p}\, \Delta_p u(x,t). \tag{1.2}$$

These are characterized by the asymptotic mean value formula

$$\begin{aligned} u(x,t) = \frac{\alpha}{2} \fint_{t-\varepsilon^2}^{t} &\left\{ \max_{y\in \overline{B}_\varepsilon(x)} u(y,s) + \min_{y\in \overline{B}_\varepsilon(x)} u(y,s) \right\} ds \\ &+ \beta \fint_{t-\varepsilon^2}^{t} \fint_{B_\varepsilon(x)} u(y,s)\, dy\, ds + o(\varepsilon^2), \quad \text{as} \quad \varepsilon \to 0, \end{aligned}$$

that should hold in the viscosity sense. Here, as before, α and β are given by (1.1).

These mean value formulas are related to the Dynamic Programming Principle (DPP) satisfied by the value functions of parabolic tug-of-war games with noise. The DPP is precisely the mean value formula without the correction term $o(\varepsilon^2)$. We call functions that satisfy the DPP (p,ε)-*parabolic*. For elliptic counterparts see [LG], [LGA], and [MPR2]. It turns out that (p,ε)-parabolic equations have interesting properties making them interesting on their own, but in addition, they approximate solutions to the corresponding parabolic equation.

Le Gruyer and Archer [LGA, LG] used a mean value approach to solve the infinity Laplace equation and related problems. Oberman [O] implemented various convergent difference schemes for infinity harmonic functions using mean values. Kohn and Serfaty [KS] studied a deterministic game theoretic approach to general parabolic equations. They consider a large class of fully nonlinear parabolic equations including the mean curvature flow. Barron, Evans, and Jensen [BEJ] considered various generalizations of L^∞-variational problems. In particular, they obtained a version of our results in the case $p = \infty$, see Theorem 4.13 below. Finally, (1.2) has desirable properties in image processing, see Does [KD].

2. An asymptotic mean value characterization. Recall that for $1 < p < \infty$ we have

$$|\nabla u|^{2-p} \Delta_p u = (p-2)\Delta_\infty u + \Delta u, \tag{2.1}$$

where

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

denotes the p-Laplacian and

$$\Delta_\infty u = |\nabla u|^{-2} \langle D^2 u\, \nabla u, \nabla u\rangle = |\nabla u|^{-2} \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$$

the 1-homogeneous infinity Laplacian. Observe that in equation (1.2) we get

$$u_t = \Delta_\infty u$$

when $p \to \infty$, and

$$(n+2)u_t = \Delta u$$

when $p = 2$.

Let $T > 0$, and $\Omega \subset \mathbb{R}^n$ be an open set, and let $\Omega_T = \Omega \times (0,T)$ be a space-time cylinder with the parabolic boundary

$$\partial_p \Omega_T = \{\partial\Omega \times [0,T]\} \cup \{\Omega \times \{0\}\}.$$

We denote the mean value integral with the usual notation

$$\fint_B f(y)\, dy = \frac{1}{|B|} \int_B f(y)\, dy.$$

The parabolic equation (1.2) is singular when the gradient vanishes. We recall the definition of viscosity solution based on semicontinuous extensions of the operator, and refer the reader to Chen-Giga-Goto [CGG], Evans-Spruck [ES], and Giga's monograph [G]. Below we denote by $\lambda_{\max}((p-2)D^2\phi(x,t))$, and $\lambda_{\min}((p-2)D^2\phi(x,t))$ the largest, and the smallest of the eigenvalues to the symmetric matrix $(p-2)D^2\phi(x,t) \in \mathbb{R}^{n\times n}$ for a smooth test function ϕ. We write $\lambda_{\max}((p-2)D^2\phi(x,t))$ instead of $(p-2)\lambda_{\max}(D^2\phi(x,t))$ to give a unified treatment for the cases $p \geq 2$ and $1 < p < 2$.

Definition 2.1. *A function $u : \Omega_T \to \mathbb{R}$ is a viscosity solution to* (1.2) *if u is continuous and whenever $(x_0,t_0) \in \Omega_T$ and $\phi \in C^2(\Omega_T)$ is such that*

i) $u(x_0,t_0) = \phi(x_0,t_0)$,
ii) $u(x,t) > \phi(x,t)$ *for* $(x,t) \in \Omega_T,\ (x,t) \neq (x_0,t_0)$,

then we have at the point (x_0,t_0)

$$\begin{cases} (n+p)\phi_t \geq (p-2)\Delta_\infty \phi + \Delta\phi, & \text{if} \quad \nabla\phi(x_0,t_0) \neq 0, \\ (n+p)\phi_t \geq \lambda_{min}((p-2)D^2\phi) + \Delta\phi, & \text{if} \quad \nabla\phi(x_0,t_0) = 0. \end{cases}$$

Moreover, we require that when touching u with a test function from above all the inequalities are reversed and $\lambda_{min}((p-2)D^2\phi)$ is replaced by $\lambda_{max}((p-2)D^2\phi)$.

It will become useful to observe that we can further reduce the number of test functions in the definition of a viscosity solution. Indeed, if the gradient of a test function vanishes we may assume that $D^2\phi = 0$, and thus $\lambda_{\max} = \lambda_{\min} = 0$. Nothing is required if $\nabla\phi = 0$ and $D^2\phi \neq 0$. The proof follows the ideas in [ES], see also [CGG] and Lemma 3.2. in [JK] for $p = \infty$. For the convenience of the reader we provide the details.

LEMMA 2.2. *A function $u : \Omega_T \to \mathbb{R}$ is a viscosity solution to* (1.2) *if u is continuous and whenever $(x_0, t_0) \in \Omega_T$ and $\phi \in C^2(\Omega_T)$ is such that*

i) $u(x_0, t_0) = \phi(x_0, t_0)$,
ii) $u(x, t) > \phi(x, t)$ *for* $(x, t) \in \Omega_T,\ (x, t) \neq (x_0, t_0)$,

then at the point (x_0, t_0) we have

$$\begin{cases} (n+p)\phi_t \;\geq\; (p-2)\Delta_\infty\phi + \Delta\phi, & \textit{if} \quad \nabla\phi(x_0,t_0) \neq 0, \\ \phi_t(x_0,t_0) \;\geq\; 0, & \textit{if} \quad \nabla\phi(x_0,t_0) = 0, \textit{ and } D^2\phi(x_0,t_0) = 0. \end{cases}$$

Moreover, we require that when testing from above all the inequalities are reversed.

Proof. The proof is by contradiction: We assume that u satisfies the conditions in the statement but still fails to be a viscosity solution in the sense of Definition 2.1. If this is the case, we must have $\phi \in C^2(\Omega_T)$ and $(x_0, t_0) \in \Omega_T$ such that

i) $u(x_0, t_0) = \phi(x_0, t_0)$,
ii) $u(x, t) > \phi(x, t)$ for $(x, t) \in \Omega_T,\ (x, t) \neq (x_0, t_0)$,

for which $\nabla\phi(x_0, t_0) = 0,\ D^2\phi(x_0, t_0) \neq 0$ and

$$(n+p)\phi_t(x_0,t_0) < \lambda_{\min}((p-2)D^2\phi(x_0,t_0)) + \Delta\phi(x_0,t_0), \tag{2.2}$$

or the analogous inequality when testing from above (in this case the argument is symmetric and we omit it). Let

$$w_j(x,t,y,s) = u(x,t) - \Big(\phi(y,s) - \frac{j}{4}|x-y|^4 - \frac{j}{2}|t-s|^2\Big)$$

and denote by (x_j, t_j, y_j, s_j) the minimum point of w_j in $\overline{\Omega}_T \times \overline{\Omega}_T$. Since (x_0, t_0) is a local minimum for $u - \phi$, we may assume that

$$(x_j, t_j, y_j, s_j) \to (x_0, t_0, x_0, t_0), \quad \text{as} \quad j \to \infty$$

and $(x_j, t_j), (y_j, s_j) \in \Omega_T$ for all large j, similarly to [JK].

We consider two cases: either $x_j = y_j$ infinitely often or $x_j \neq y_j$ for all j large enough. First, let $x_j = y_j$, and denote

$$\varphi(y,s) = \frac{j}{4}|x_j - y|^4 + \frac{j}{2}(t_j - s)^2.$$

Then

$$\phi(y,s) - \varphi(y,s),$$

has a local maximum at (y_j, s_j). By (2.2) and continuity of

$$(x,t) \mapsto \lambda_{\min}((p-2)D^2\phi(x,t)) + \Delta\phi(x,t),$$

we have

$$(n+p)\phi_t(y_j,s_j) < \lambda_{\min}((p-2)D^2\phi(y_j,s_j)) + \Delta\phi(y_j,s_j)$$

for j large enough. As $\phi_t(y_j,s_j) = \varphi_t(y_j,s_j)$ and $D^2\phi(y_j,s_j) \le D^2\varphi(y_j,s_j)$, we have by the previous inequality

$$\begin{aligned} 0 &< -(n+p)\varphi_t(y_j,s_j) + \lambda_{\min}((p-2)D^2\varphi(y_j,s_j)) + \Delta\varphi(y_j,s_j) \\ &= -(n+p)j(t_j - s_j), \end{aligned} \tag{2.3}$$

where we also used the fact that $y_j = x_j$ and thus $D^2\varphi(y_j,s_j) = 0$.

Next denote

$$\psi(x,t) = -\frac{j}{4}\left|x - y_j\right|^4 - \frac{j}{2}(t-s_j)^2.$$

Similarly,

$$u(x,t) - \psi(x,t)$$

has a local minimum at (x_j,t_j), and thus since $D^2\psi(x_j,t_j) = 0$, our assumptions imply

$$0 \le (p+n)\psi_t(x_j,t_j) = (p+n)j(t_j - s_j), \tag{2.4}$$

for j large enough. Summing up (2.3) and (2.4), we get

$$0 < -(n+p)j(t_j-s_j) + (p+n)j(t_j-s_j) = 0,$$

a contradiction.

Next we consider the case $y_j \neq x_j$. For the following notation, we refer to [CIL], [OS], and [JLM]. We also use the parabolic theorem of sums for w_j which implies that there exists symmetric matrices X_j, Y_j such that $X_j - Y_j$ is positive semidefinite and

$$\begin{aligned} &\Big(j(t_j-s_j),\, j\left|x_j-y_j\right|^2(x_j-y_j),\, Y_j\Big) \in \overline{\mathcal{P}}^{2,+}\phi(y_j,s_j) \\ &\Big(j(t_j-s_j),\, j\left|x_j-y_j\right|^2(x_j-y_j),\, X_j\Big) \in \overline{\mathcal{P}}^{2,-}u(x_j,t_j). \end{aligned}$$

Using (2.2) and the assumptions on u, we get

$$\begin{aligned} 0 &= (n+p)j(t_j-s_j) - (n+p)j(t_j-s_j) \\ &< (p-2)\langle Y_j\frac{(x_j-y_j)}{|x_j-y_j|}, \frac{(x_j-y_j)}{|x_j-y_j|}\rangle + \operatorname{tr}(Y_j) \\ &\quad - (p-2)\langle X_j\frac{(x_j-y_j)}{|x_j-y_j|}, \frac{(x_j-y_j)}{|x_j-y_j|}\rangle - \operatorname{tr}(X_j) \\ &= (p-2)\langle (Y_j-X_j)\frac{(x_j-y_j)}{|x_j-y_j|}, \frac{(x_j-y_j)}{|x_j-y_j|}\rangle + \operatorname{tr}(Y_j-X_j) \\ &\le 0, \end{aligned}$$

because $Y_j - X_j$ is negative semidefinite. If $1 < p < 2$, the last inequality follows from the calculation

$$
\begin{aligned}
(p-2)\langle (Y_j - X_j)\frac{(x_j-y_j)}{|x_j-y_j|}, \frac{(x_j-y_j)}{|x_j-y_j|}\rangle + \operatorname{tr}(Y_j - X_j) \\
&\le (p-2)\lambda_{\min} + \sum_{i=1}^{n} \lambda_i \\
&= (p-1)\lambda_{\min} + \sum_{\lambda_i \neq \lambda_{\min}} \lambda_i \\
&\le 0,
\end{aligned}
$$

where λ_i, $\lambda_{\min}$, and $\lambda_{\max}$ denote the eigenvalues of $Y_j - X_j$. This provides the desired contradiction. □

Similarly to in the elliptic case in [MPR], the asymptotic mean value formulas hold in a *viscosity sense*. We test the mean value formulas for u with a test function touching u from above or below.

Definition 2.3. *A continuous function u satisfies the asymptotic mean value formula*

$$
\begin{aligned}
u(x,t) = \frac{\alpha}{2} \fint_{t-\varepsilon^2}^{t} \left\{ \max_{y\in \overline{B}_\varepsilon(x)} u(y,s) + \min_{y\in \overline{B}_\varepsilon(x)} u(y,s) \right\} ds \\
+ \beta \fint_{t-\varepsilon^2}^{t} \fint_{B_\varepsilon(x)} u(y,s)\, dy\, ds + o(\varepsilon^2), \quad \text{as} \quad \varepsilon \to 0,
\end{aligned}
\tag{2.5}
$$

in the viscosity sense *at $(x,t) \in \Omega_T$ if for every ϕ as in Lemma 2.2, we have*

$$
\begin{aligned}
\phi(x,t) \ge \frac{\alpha}{2} \fint_{t-\varepsilon^2}^{t} \left\{ \max_{y\in \overline{B}_\varepsilon(x)} \phi(y,s) + \min_{y\in \overline{B}_\varepsilon(x)} \phi(y,s) \right\} ds \\
+ \beta \fint_{t-\varepsilon^2}^{t} \fint_{B_\varepsilon(x)} \phi(y,s)\, dy\, ds + o(\varepsilon^2), \quad \text{as} \quad \varepsilon \to 0,
\end{aligned}
\tag{2.6}
$$

and analogously when testing from above. Observe that the asymptotic mean value formula is free of gradients, and, in particular, that the case $\nabla\phi(x,t) = 0$, $D^2\phi(x,t) = 0$ is included. Next we characterize viscosity solutions to $(n+p)u_t = |\nabla u|^{2-p}\Delta_p u$.

Theorem 2.4. *Let $1 < p \le \infty$ and let u be a continuous function in Ω_T. The asymptotic mean value formula*

$$
\begin{aligned}
u(x,t) = \frac{\alpha}{2} \fint_{t-\varepsilon^2}^{t} \left\{ \max_{y\in \overline{B}_\varepsilon(x)} u(y,s) + \min_{y\in \overline{B}_\varepsilon(x)} u(y,s) \right\} ds \\
+ \beta \fint_{t-\varepsilon^2}^{t} \fint_{B_\varepsilon(x)} u(y,s)\, dy\, ds + o(\varepsilon^2), \quad \text{as} \quad \varepsilon \to 0,
\end{aligned}
$$

holds for every $(x,t) \in \Omega_T$ in the viscosity sense if and only if u is a viscosity solution to

$$
(n+p)u_t(x,t) = |\nabla u|^{2-p}\Delta_p u(x,t).
$$

Above

$$\alpha = \frac{p-2}{p+n}, \qquad \beta = \frac{2+n}{p+n}.$$

Observe that $\alpha \geq 0, \beta \geq 0$, $\alpha + \beta = 1$, and that if $p = 2$, then $\alpha = 0$, and $\beta = 1$ and if $p = \infty$, then $\alpha = 1$ and $\beta = 0$. Thus, as a special case of the above theorem, we obtain an asymptotic mean value formula for the parabolic infinity Laplacian. This equation was recently studied in [JK] and [J].

THEOREM 2.5. *Let u be a continuous function in Ω_T. The asymptotic mean value formula*

$$u(x,t) = \frac{1}{2} \fint_{t-\varepsilon^2}^{t} \left\{ \max_{y \in \overline{B}_\varepsilon(x)} u(y,s) + \min_{y \in \overline{B}_\varepsilon(x)} u(y,s) \right\} ds + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0,$$

holds for every $(x,t) \in \Omega_T$ in the viscosity sense if and only if u is a viscosity solution to

$$u_t(x,t) = \Delta_\infty u(x,t).$$

3. Proof of Theorem 2.4 . We divide the proof in three parts: First, we consider the cases $p = 2$ and $p = \infty$ separately, and then combine the results to obtain Theorem 2.4 for any $1 < p \leq +\infty$.

THE HEAT EQUATION: Let us first consider the smooth case.

PROPOSITION 3.1. *Let u be a smooth function in Ω_T. The asymptotic mean value formula*

$$u(x,t) = \fint_{t-\varepsilon^2/(n+2)}^{t} \fint_{B_\varepsilon(x)} u(y,s)\, dy\, ds + o(\varepsilon^2), \quad \text{as } \varepsilon \to 0,$$

holds for all $(x,t) \in \Omega_T$ if and only if

$$u_t(x,t) = \Delta u(x,t)$$

in Ω_T.

Proof. Let $(x,t) \in \Omega_T$ and let u be a smooth function. We use the Taylor expansion

$$\begin{aligned} u(y,s) &= u(x,t) + \nabla u(x,t) \cdot (y-x) + \frac{1}{2} \langle D^2 u(x,t)(y-x), (y-x) \rangle \\ &\quad + u_t(x,t)(s-t) + o(|y-x|^2 + |s-t|) \\ &= u(x,t) + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} (y-x)_i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} (y-x)_i (y-x)_j \\ &\quad + u_t(x,t)(s-t) + o(|y-x|^2 + |s-t|). \end{aligned} \tag{3.1}$$

Averaging both sides, we get

$$\begin{aligned}
&\fint_{t-\varepsilon^2/(n+2)}^{t} \fint_{B_\varepsilon(x)} u(y,s)\,dy\,ds \\
&\qquad = u(x,t) + \fint_{B_\varepsilon(x)} \nabla u(x,t)\cdot(y-x)\,dy \\
&\qquad\quad + \frac{1}{2}\fint_{B_\varepsilon(x)} \langle D^2u(x,t)(y-x),(y-x)\rangle\,dy \\
&\qquad\quad + u_t(x,t)\fint_{t-\varepsilon^2/(n+2)}^{t} (s-t)\,ds + o(\varepsilon^2).
\end{aligned} \tag{3.2}$$

Because of symmetry, the first integral on the right hand side vanishes and the second can be simplified as in [MPR] to get

$$\frac{1}{2}\fint_{B_\varepsilon(x)} \langle D^2u(x,t)(y-x),(y-x)\rangle\,dy = \frac{\varepsilon^2}{2(n+2)}\Delta u(x,t).$$

Finally,

$$\fint_{t-\varepsilon^2/(n+2)}^{t} (s-t)\,ds = -\frac{\varepsilon^2}{2(n+2)},$$

and thus (3.2) implies

$$\begin{aligned}
&\fint_{t-\varepsilon^2/(n+2)}^{t} \fint_{B_\varepsilon(x)} u(y,s)\,dy\,ds \\
&= u(x,t) + \frac{\varepsilon^2}{2(n+2)}\left(\Delta u(x,t) - u_t(x,t)\right) + o(\varepsilon^2).
\end{aligned} \tag{3.3}$$

This holds for any smooth function.

If u is a solution to the heat equation, then (3.3) immediately implies that u satisfies the asymptotic mean value property. According to classical results, a solution to the heat equation is smooth and thus smoothness assumption is not restrictive here.

Next we assume that a smooth u satisfies the asymptotic mean value formula and show that then u is a solution to the heat equation. According to the assumption and (3.3), we have

$$\begin{aligned}
u(x,t) &= \fint_{t-\varepsilon^2/(n+2)}^{t} \fint_{B_\varepsilon(x)} u(y,s)\,dy\,ds + o(\varepsilon^2) \\
&= u(x,t) + \frac{\varepsilon^2}{2(n+2)}\left(\Delta u(x,t) - u_t(x,t)\right) + o(\varepsilon^2).
\end{aligned}$$

Dividing by ε^2 and passing to the limit $\varepsilon \to 0$ implies

$$0 = \Delta u(x,t) - u_t(x,t).$$

This finishes the proof. □

In the space-time cylinders $B_\varepsilon(x)\times(t-\varepsilon^2,t)$, the asymptotic mean value formula characterizes solutions to the rescaled heat equation

$$(n+2)u_t(x,t)=\Delta u(x,t).$$

In this case, (3.3) takes the form

$$\begin{aligned}&\fint_{t-\varepsilon^2}^{t}\fint_{B_\varepsilon(x)} u(y,s)\,dy\,ds\\ &=u(x,t)+\frac{\varepsilon^2}{2(n+2)}\left(\Delta u(x,t)-(n+2)u_t(x,t)\right)+o(\varepsilon^2).\end{aligned}\tag{3.4}$$

Alternatively, the same argument shows that solutions to the heat equation are also characterized by asymptotic mean value formula

$$u(x,t)=\fint_{B_\varepsilon(x)} u\left(y,t-\frac{\varepsilon^2}{2(n+2)}\right)\,dy+o(\varepsilon^2),\quad \text{as } \varepsilon\to 0.$$

The parabolic infinity Laplacian: Next we turn our attention to the homogeneous parabolic infinity Laplacian. We show that the asymptotic mean value formula

$$u(x,t)=\frac{1}{2}\fint_{t-\varepsilon^2}^{t}\left\{\max_{y\in\overline{B}_\varepsilon(x)} u(y,s)+\min_{y\in\overline{B}_\varepsilon(x)} u(y,s)\right\}ds+o(\varepsilon^2),\quad \text{as } \varepsilon\to 0,$$

characterizes the viscosity solutions to

$$u_t=\Delta_\infty u.$$

The proof employs the Taylor expansion (3.1) and uses the fact that the minimum and maximum of the test function ϕ over the ball $\overline{B}_\varepsilon(x)$ at a fixed time is approximately obtained at the points

$$x-\varepsilon\frac{\nabla\phi}{|\nabla\phi|}\quad\text{and}\quad x+\varepsilon\frac{\nabla\phi}{|\nabla\phi|}.$$

The integration over a time interval takes care of the term that involves time derivatives.

Proof of Theorem 2.5 To begin with, choose a point $(x,t)\in\Omega_T$, $\varepsilon>0$, $s\in(t-\varepsilon^2,t)$ and any smooth ϕ. Denote by $x_1^{\varepsilon,s}$ a point in which ϕ attains its minimum over a ball $\overline{B}_\varepsilon(x)$ at time s, that is,

$$\phi(x_1^{\varepsilon,s},s)=\min_{y\in\overline{B}_\varepsilon(x)}\phi(y,s).$$

Evaluating the Taylor expansion (3.1) for ϕ at $y=x_1^{\varepsilon,s}$, we get

$$\begin{aligned}\phi(x_1^{\varepsilon,s},s)=\ &\phi(x,t)+\nabla\phi(x,t)\cdot(x_1^{\varepsilon,s}-x)\\ &+\frac{1}{2}\langle D^2\phi(x,t)(x_1^{\varepsilon,s}-x),(x_1^{\varepsilon,s}-x)\rangle\\ &+\phi_t(x,t)(s-t)+o(\varepsilon^2+|s-t|),\end{aligned}$$

as $\varepsilon \to 0$. Evaluating the Taylor expansion at $y = \tilde{x}_1^{\varepsilon,s}$, where $\tilde{x}_1^{\varepsilon,s}$ is the symmetric point of $x_1^{\varepsilon,s}$ with respect to x, given by

$$\tilde{x}_1^{\varepsilon,s} = 2x - x_1^{\varepsilon,s},$$

we obtain

$$\begin{aligned}\phi(\tilde{x}_1^{\varepsilon,s}, s) = \quad & \phi(x,t) - \nabla\phi(x,t)\cdot(x_1^{\varepsilon,s} - x) \\ & + \frac{1}{2}\langle D^2\phi(x,t)(x_1^{\varepsilon,s} - x), (x_1^{\varepsilon,s} - x)\rangle \\ & + \phi_t(x,t)(s-t) + o(\varepsilon^2 + |s-t|).\end{aligned}$$

Adding the expressions, we get

$$\begin{aligned}\phi(\tilde{x}_1^{\varepsilon,s}, s) + \phi(x_1^{\varepsilon,s}, s) - 2\phi(x,t) = {}& \langle D^2\phi(x,t)(x_1^{\varepsilon,s} - x), (x_1^{\varepsilon,s} - x)\rangle \\ & + 2\phi_t(x,t)(s-t) + o(\varepsilon^2 + |s-t|).\end{aligned}$$

As $x_1^{\varepsilon,s}$ is the point where the minimum of $\phi(\cdot, s)$ on $\overline{B}_\varepsilon(x)$ is attained, it follows that

$$\phi(\tilde{x}_1^{\varepsilon,s}, s) + \phi(x_1^{\varepsilon,s}, s) - 2\phi(x,t) \le \max_{y\in\overline{B}_\varepsilon(x)} \phi(y,s) + \min_{y\in\overline{B}_\varepsilon(x)} \phi(y,s) - 2\phi(x,t),$$

and thus

$$\begin{aligned}& \max_{y\in\overline{B}_\varepsilon(x)} \phi(y,s) + \min_{y\in\overline{B}_\varepsilon(x)} \phi(y,s) - 2\phi(x,t) \\ & \qquad \ge \langle D^2\phi(x,t)(x_1^{\varepsilon,s} - x), (x_1^{\varepsilon,s} - x)\rangle + 2\phi_t(x,t)(s-t) + o(\varepsilon^2 + |s-t|).\end{aligned}$$

Integration over the time interval implies

$$\begin{aligned}& \frac{1}{2}\fint_{t-\varepsilon^2}^{t} \left\{ \max_{y\in\overline{B}_\varepsilon(x)} \phi(y,s) + \min_{y\in\overline{B}_\varepsilon(x)} \phi(y,s) \right\} ds - \phi(x,t) \\ & \ge \frac{\varepsilon^2}{2}\left(\fint_{t-\varepsilon^2}^{t} \left\langle D^2\phi(x,t)\frac{x_1^{\varepsilon,s} - x}{\varepsilon}, \frac{x_1^{\varepsilon,s} - x}{\varepsilon} \right\rangle ds - \phi_t(x,t) \right) + o(\varepsilon^2).\end{aligned} \tag{3.5}$$

This inequality holds for any smooth function. By considering a point where ϕ attains its maximum, we could derive a reverse inequality.

Because ϕ is smooth, if $\nabla\phi(x,t) \neq 0$, so is $\nabla\phi(x,s)$ for $t - \varepsilon^2 \le s \le t$ and for small enough $\varepsilon > 0$ and thus $x_1^{\varepsilon,s} \in \partial B_\varepsilon(x)$ for small ε. We deduce

$$\lim_{\varepsilon\to 0} \frac{x_1^{\varepsilon,s} - x}{\varepsilon} = -\frac{\nabla\phi}{|\nabla\phi|}(x,t).$$

Moreover, we get the limit

$$\begin{aligned}& \lim_{\varepsilon\to 0} \fint_{t-\varepsilon^2}^{t} \left\langle D^2\phi(x,t)\frac{x_1^{\varepsilon,s} - x}{\varepsilon}, \frac{x_1^{\varepsilon,s} - x}{\varepsilon} \right\rangle ds \\ & \qquad = \left\langle D^2\phi(x,t)\frac{\nabla\phi}{|\nabla\phi|}(x,t), \frac{\nabla\phi}{|\nabla\phi|}(x,t) \right\rangle = \Delta_\infty \phi(x,t).\end{aligned} \tag{3.6}$$

Next we assume that u satisfies the asymptotic mean value formula in the viscosity sense and show that then u satisfies the definition of a viscosity solution whenever $\nabla\phi \neq 0$. In particular, we have

$$0 \geq -\phi(x,t) + \frac{1}{2} \fint_{t-\varepsilon^2}^{t} \left\{ \max_{y\in\overline{B}_\varepsilon(x)} \phi(y,s) + \min_{y\in\overline{B}_\varepsilon(x)} \phi(y,s) \right\} ds + o(\varepsilon^2),$$

for any smooth ϕ touching u at $(x,t) \in \Omega_T$ from below. By the previous inequality, the left hand side of (3.5) is bounded above by $o(\varepsilon^2)$. It follows from this fact dividing (3.5) by ε^2, passing to a limit, and using (3.6) that

$$0 \geq \Delta_\infty \phi(x,t) - \phi_t(x,t).$$

To prove a reverse inequality, we first derive a reverse inequality to (3.5) by considering the maximum point of ϕ, and then choose a function ϕ that touches u from above.

To prove the reverse implication, assume that u is a viscosity solution. Let ϕ, $\nabla\phi \neq 0$, be a smooth test function touching u from above at $(x,t) \in \Omega_T$. We have

$$\Delta_\infty \phi(x,t) - \phi_t(x,t) \geq 0. \tag{3.7}$$

It suffices to prove

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \left(-\phi(x,t) + \frac{1}{2} \fint_{t-\varepsilon^2}^{t} \left\{ \max_{y\in\overline{B}_\varepsilon(x)} \phi(y,s) + \min_{y\in\overline{B}_\varepsilon(x)} \phi(y,s) \right\} ds \right) \geq 0.$$

This again follows from (3.5). Indeed, divide (3.5) by ε^2, use (3.6), and deduce from (3.7) that the limit on the right hand side is bounded from below by zero. The argument for the reverse inequality is analogous.

Finally, let $\nabla\phi(x,t) = 0$, and suppose that ϕ touches u at (x,t) from below. According to Lemma 2.2, we may also assume that $D^2\phi(x,t) = 0$, and thus the Taylor expansion implies

$$\phi(y,s) - \phi(x,t) = \phi_t(x,t)(s-t) + o(\varepsilon^2)$$

in the space-time cylinder. Thus supposing that the asymptotic mean value formula holds at (x,t), we deduce

$$\begin{aligned} 0 &\geq \frac{1}{2} \fint_{t-\varepsilon^2}^{t} \left\{ \max_{y\in\overline{B}_\varepsilon(x)} \Big(\phi(y,s) - \phi(x,t)\Big) + \min_{y\in\overline{B}_\varepsilon(x)} \Big(\phi(y,s) - \phi(x,t)\Big) \right\} ds \\ &\quad + o(\varepsilon^2) \\ &= \fint_{t-\varepsilon^2}^{t} \phi_t(x,t)(s-t)\, ds + o(\varepsilon^2) \\ &= -\frac{\varepsilon^2}{2} \phi_t(x,t) + o(\varepsilon^2). \end{aligned}$$

Dividing by ε^2, and passing to a limit, we get $0 \leq \phi_t(x,t)$. Lemma 2.2 and an analogous calculation when testing from above shows that u is a viscosity solution.

Suppose then that u is a viscosity solution and ϕ is a test function with $\nabla\phi(x,t) = 0, D^2\phi(x,t) = 0$ that touches u at (x,t) from below. Then a similar calculation as above implies

$$\begin{aligned}\fint_{t-\varepsilon^2}^{t} \left\{ \max_{y\in \overline{B}_\varepsilon(x)} \phi(y,s) + \min_{y\in \overline{B}_\varepsilon(x)} \phi(y,s) \right\} ds - 2\phi(x,t) \\ = -\varepsilon^2\phi_t(x,t) + o(\varepsilon^2).\end{aligned}$$

By Lemma 2.2, $\phi_t(x,t) \geq 0$. Thus, dividing the above equality by ε^2 and passing to the limit shows that the asymptotic expansion holds. ∎

A similar proof also shows that u is a viscosity solution to

$$u_t(x,t) = \Delta_\infty u(x,t)$$

if and only if

$$u(x,t) = \frac{1}{2}\left\{ \max_{y\in \overline{B}_\varepsilon(x)} u\left(y, t-\frac{\varepsilon^2}{2}\right) + \min_{y\in \overline{B}_\varepsilon(x)} u\left(y, t-\frac{\varepsilon^2}{2}\right)\right\} + o(\varepsilon^2) \text{ as } \varepsilon \to 0$$

in the viscosity sense.

THE p-LAPLACIAN: Next we combine the asymptotic mean value formulas from the previous sections. The main point is that, formally, adding the equations

$$(n+2)u_t = \Delta, u$$

and

$$(p-2)u_t = (p-2)\Delta_\infty u$$

we obtain

$$(n+p)u_t = \Delta u + (p-2)\Delta_\infty u;$$

that is,

$$(n+p)u_t = |\nabla u|^{2-p}\,\Delta_p u.$$

Proof of Theorem 2.4 Assume first that $p \geq 2$ so that $\alpha \geq 0$. Multiplying (3.4) by β and (3.5) by α, and adding, we obtain

$$\begin{aligned}
&\frac{\alpha}{2}\fint_{t-\varepsilon^2}^{t} \left\{ \max_{y\in \overline{B}_\varepsilon(x)} \phi(y,s) + \min_{y\in \overline{B}_\varepsilon(x)} \phi(y,s) \right\} ds \\
&\qquad + \beta \fint_{t-\varepsilon^2}^{t}\fint_{B_\varepsilon(x)} \phi(y,s)\,dy\,ds - \phi(x,t) \\
&\geq \frac{\alpha\varepsilon^2}{2}\left(\fint_{t-\varepsilon^2}^{t} \left\langle D^2\phi(x,t)\frac{x_1^{\varepsilon,s}-x}{\varepsilon}, \frac{x_1^{\varepsilon,s}-x}{\varepsilon}\right\rangle ds - \phi_t(x,t)\right) \\
&\qquad + \frac{\beta\varepsilon^2}{2(n+2)}\left(\Delta\phi(x,t) - (n+2)\phi_t(x,t)\right) + o(\varepsilon^2) \\
&= \frac{\beta\varepsilon^2}{2(n+2)}\Bigg((p-2)\fint_{t-\varepsilon^2}^{t} \left\langle D^2\phi(x,t)\frac{x_1^{\varepsilon,s}-x}{\varepsilon}, \frac{x_1^{\varepsilon,s}-x}{\varepsilon}\right\rangle ds \\
&\qquad + \Delta\phi(x,t) - (n+p)\phi_t(x,t)\Bigg) + o(\varepsilon^2).
\end{aligned} \tag{3.8}$$

Notice that this again holds for any smooth function, and (3.6) still holds whenever $\nabla\phi \neq 0$. The rest of the proof follows closely the proof of Theorem 2.5. Further, by considering the maximum point instead of the minimum point $x_1^{\varepsilon,s}$, we can derive a reverse inequality to (3.8).

If $p < 2$, it follows that $\alpha < 0$ and the inequality (3.8) is reversed. On the other hand, so is the reverse inequality that can be obtained by considering the maximum point instead of the minimum point $x_1^{\varepsilon,s}$. Thus we still have the both inequalities, and we can repeat the same argument. ∎

An analogous proof also shows that u is a solution to

$$(n+p)u_t(x,t) = |\nabla u|^{2-p}\,\Delta_p u(x,t)$$

in the viscosity sense if and only if

$$\begin{aligned} u(x,t) = \frac{\alpha}{2}\left\{\max_{y\in\overline{B}_\varepsilon(x)} u\Big(y,t-\frac{\varepsilon^2}{2}\Big) + \min_{y\in\overline{B}_\varepsilon(x)} u\Big(y,t-\frac{\varepsilon^2}{2}\Big)\right\} \\ + \beta \fint_{B_\varepsilon(x)} u\Big(y,t-\frac{\varepsilon^2}{2}\Big)\,dy + o(\varepsilon^2), \quad \text{as } \varepsilon\to 0. \end{aligned} \tag{3.9}$$

We will take this formulation as a starting point when studying the tug-of-war games with limited number of rounds in the next section.

4. (p,ε)**-parabolic functions and Tug-of-war games.** Motivated by the asymptotic mean value theorems, we next study the functions satisfying the mean value property (3.9) without the correction term $o(\varepsilon^2)$ for $p \geq 2$. We call these functions (p,ε)*-parabolic.* It turns out that (p,ε)-parabolic functions have interesting properties to be studied in their own right, but in addition they approximate solutions to (1.2), and are value functions of a tug-war-game with noise when the number of rounds is limited.

Recall that $\Omega_T \subset \mathbb{R}^{n+1}$ is an open set. To prescribe boundary values, we denote the boundary strip of width ε by

$$\Gamma_\varepsilon = \Big(S^\varepsilon \times (-\frac{\varepsilon^2}{2}, T]\Big) \cup \Big(\Omega \times (-\frac{\varepsilon^2}{2}, 0]\Big),$$

where

$$S^\varepsilon = \{x \in \mathbb{R}^n \setminus \Omega \,:\, \operatorname{dist}(x,\partial\Omega) \leq \varepsilon\}.$$

Below $F : \Gamma_\varepsilon \to \mathbb{R}$ denotes a bounded Borel function.

Definition 4.1. *The function u_ε is (p,ε)-parabolic, $2 \leq p \leq \infty$, in Ω_T with boundary values F* [1] *if*

$$\begin{aligned} u_\varepsilon(x,t) &= \frac{\alpha}{2}\left\{\sup_{y\in\overline{B}_\varepsilon(x)} u_\varepsilon\Big(y,t-\frac{\varepsilon^2}{2}\Big) + \inf_{y\in\overline{B}_\varepsilon(x)} u_\varepsilon\Big(y,t-\frac{\varepsilon^2}{2}\Big)\right\} \\ &\quad + \beta \fint_{B_\varepsilon(x)} u_\varepsilon\Big(y,t-\frac{\varepsilon^2}{2}\Big)\,dy \qquad \text{for every } (x,t)\in\Omega_T \\ u_\varepsilon(x,t) &= F(x,t), \qquad \text{for every} \quad (x,t)\in\Gamma_\varepsilon, \end{aligned}$$

[1] Note added: For measurability questions, see Luiro, Parviainen, Saksman: 'On the existence and uniqueness of p-harmonious functions'. However, one should replace $\overline{B}_\varepsilon(x)$ by $B_\varepsilon(x)$.

where

$$\alpha = \frac{p-2}{p+n}, \quad \beta = \frac{n+2}{p+n}.$$

The reason for using the boundary strip Γ_ε instead of simply using the parabolic boundary $\partial_p \Omega_T$ is the fact that $\overline{B}_\varepsilon(x) \times \{t - \frac{\varepsilon^2}{2}\}$ is not necessarily contained in Ω_T.

Next we study the tug-of-war game with noise studied in [MPR2], and in a different form in Peres-Sheffield [PS]. See also Peres-Schramm-Sheffield-Wilson [PSSW]. It is a zero-sum-game between two players, Player I and Player II. In this paper, there are two key differences: the game has a preset maximum number of rounds and boundary values may change with time.

To be more precise, at the beginning we fix the maximum number of rounds to be N and place a token at a point $x_0 \in \Omega$. The players toss a biased coin with probabilities α and β, $\alpha + \beta = 1$. If they get heads (probability α), they play a tug-of-war game, that is, a fair coin is tossed and the winner of the toss decides a new game position $x_1 \in \overline{B}_\varepsilon(x_0)$. On the other hand, if they get tails (probability β), the game state moves according to the uniform probability density to a random point in the ball $B_\varepsilon(x_0)$. They continue playing the game until either the token hits the boundary strip S^ε or the number of rounds reaches N. We denote by $\tau_N \in \{0, 1, \ldots, N\}$ the hitting time of S^ε or N, whichever comes first, and by $x_{\tau_N} \in \Omega \cup S^\varepsilon$ the end point of the game. When no confusion arises, we simply write τ. At the end of the game Player I earns $\mathcal{F}(x_{\tau_N}, \tau_N)$ while Player II earns $-\mathcal{F}(x_{\tau_N}, \tau_N)$. Here

$$\mathcal{F} : (S^\varepsilon \times \{0, \ldots, N\}) \cup (\Omega \times \{N\}) \to \mathbb{R}$$

is a given payoff function.

Denote by $H = \Omega \cup S^\varepsilon$. A run of the game is a sequence

$$\omega = (\omega_0, \omega_1, \ldots, \omega_N) \in H^{N+1}.$$

We define random variables

$$x_k(\omega) = \omega_k, \quad x_k : H^{N+1} \to \mathbb{R}^n, \ k = 0, 1, \ldots, N,$$

and

$$\tau_N(\omega) = \min\{N, \inf\{k \, : \, x_k(\omega) \in S^\varepsilon, k = 0, 1, \ldots, N\}\}.$$

A strategy S_{I} for Player I is a function which gives the next game position

$$S_{\mathrm{I}}(x_0, x_1, \ldots, x_k) = x_{k+1} \in \overline{B}_\varepsilon(x_k)$$

if Player I wins the coin toss. Similarly, Player II plays according to a strategy S_{II}.

The fixed starting point x_0, the number of rounds N, the domain Ω and the strategies S_{I} and S_{II} determine a unique probability measure $\mathbb{P}^{x_0,N}_{S_I,S_{\mathrm{II}}}$ in H^{N+1}. This measure is built by using the initial distribution $\delta_{x_0}(A)$, and the family of transition probabilities

$$\begin{aligned}\pi_{S_{\mathrm{I}},S_{\mathrm{II}}}(x_0(\omega), \ldots, x_k(\omega), A) &= \pi_{S_{\mathrm{I}},S_{\mathrm{II}}}(\omega_0, \ldots, \omega_k, A) \\ &= \beta \frac{|A \cap B_\varepsilon(\omega_k)|}{|B_\varepsilon(\omega_k)|} + \frac{\alpha}{2} \delta_{S_{\mathrm{I}}(\omega_0, \ldots, \omega_k)}(A) + \frac{\alpha}{2} \delta_{S_{\mathrm{II}}(\omega_0, \ldots, \omega_k)}(A).\end{aligned}$$

For more details, we refer to [MPR2, MPR3, PSSW].

The expected payoff, when starting from x_0 with the maximum number of rounds N, and using the strategies $S_{\mathrm{I}}, S_{\mathrm{II}}$, is

$$\mathbb{E}^{x_0,N}_{S_{\mathrm{I}},S_{\mathrm{II}}}[\mathcal{F}(x_{\tau_N},\tau_N)] = \int_{H^{N+1}} \mathcal{F}(x_{\tau_N}(\omega),\tau_N(\omega))\, d\mathbb{P}^{x_0,N}_{S_{\mathrm{I}},S_{\mathrm{II}}}(\omega).$$

The *value of the game for Player I* when starting at x_0 with the maximum number of rounds N is given by

$$u^{\varepsilon,N}_{\mathrm{I}}(x_0,0) = \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0,N}_{S_{\mathrm{I}},S_{\mathrm{II}}}[\mathcal{F}(x_{\tau_N},\tau_N)]$$

while the *value of the game for Player II* is given by

$$u^{\varepsilon,N}_{\mathrm{II}}(x_0,0) = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}^{x_0,N}_{S_{\mathrm{I}},S_{\mathrm{II}}}[\mathcal{F}(x_{\tau_N},\tau_N)].$$

More generally, we define the value of the game when starting at x and playing for $h = N - k$ rounds to be

$$u^{\varepsilon,N}_{\mathrm{I}}(x,k) = \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}^{x,h}_{S_{\mathrm{I}},S_{\mathrm{II}}}[\mathcal{F}(x_{\tau_h},k+\tau_h)]$$

while the *value of the game for Player II* is given by

$$u^{\varepsilon,N}_{\mathrm{II}}(x,k) = \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}^{x,h}_{S_{\mathrm{I}},S_{\mathrm{II}}}[\mathcal{F}(x_{\tau_h},k+\tau_h)].$$

Here $\tau_h \in \{0,1,\dots,h\}$ is the hitting time of the boundary $(S^\varepsilon \times \{0,\dots,N\}) \cup (\Omega \times \{N\})$. In order to accommodate for time dependent boundary values, we need to keep track of the number k of rounds played. The values $u^{\varepsilon,N}_{\mathrm{I}}(x,k)$ and $u^{\varepsilon,N}_{\mathrm{II}}(x,k)$ are the expected outcomes the each player can guarantee when the game starts at x with maximum number of rounds $N-k$.

The next lemma states the *Dynamic Programming Principle* (DPP) for the tug-of-war game with a maximum number of rounds. For a detailed proof in the elliptic case see [MPR3]. The parabolic case turns out to be easier since backtracking can be directly implemented. See Chapter 3 in [MS2] and [MS].

Lemma 4.2 (DPP). *The value function for Player I satisfies*

$$\begin{aligned} u^{\varepsilon,N}_I(x,k) =& \frac{\alpha}{2}\left\{ \sup_{\overline{B}_\varepsilon(x)} u^{\varepsilon,N}_I(y,k+1) + \inf_{\overline{B}_\varepsilon(x)} u^{\varepsilon,N}_I(y,k+1) \right\} \\ &+ \beta \fint_{B_\varepsilon(x)} u^{\varepsilon,N}_I(y,k+1)\,dy, \quad \text{if} \quad x\in\Omega \text{ and } k<N, \\ u^{\varepsilon,N}_I(x,k) =& \mathcal{F}(x,k), \quad \text{if} \quad x \in S^\varepsilon \text{ or } k = N. \end{aligned}$$

The value function for Player II, $u^{\varepsilon,N}_{II}$, satisfies the same equation. The expectation is obtained by summing up the expectations of three possible outcomes for the next step with the corresponding probabilities, Player I chooses the next position (probability $\alpha/2$), Player II chooses (probability $\alpha/2$) and the next position is random (probability β). This is the heuristic background for the DPP.

Next we describe the change of time scale that relates values of the tug-of-war games with noise and (p,ε)-parabolic functions. The definition of (p,ε)-parabolic function u_ε, Definition 4.1, refers to a forward-in-time parabolic equation. The values $u_\varepsilon(\cdot,t)$ at time t are determined by the values $u_\varepsilon(\cdot,t-\frac{\varepsilon^2}{2})$. In contrast, in Lemma 4.2 above, the values at step k are determined by the values at step $k+1$.

For $0<t<T$ let $N(t)$ be the integer defined by

$$\frac{2t}{\varepsilon^2} \le N(t) < \frac{2t}{\varepsilon^2}+1.$$

We use the shorthand notation $N(t)=\lceil 2t/\varepsilon^2\rceil$. Set $t_0=t$ and $t_{k+1}=t_k-\varepsilon^2/2$ for $k=0,1,\dots,N(t)-1$; that is,

$$t_k=\varepsilon^2\frac{N(t)-k}{2}+t_{N(t)}.$$

Observe that $t_{N(t)}\in(-\frac{\varepsilon^2}{2},0]$. When no confusion arises, we simply write N for $N(t)$.

Given $F:\Gamma_\varepsilon\to\mathbb{R}$ a boundary value function, define a payoff function $\mathcal{F}_t:\{S^\varepsilon\times\{0,\dots,N\}\}\cup\{\Omega\times\{N\}\}\to\mathbb{R}$ by

$$\mathcal{F}_t(x_\tau,\tau)=F(x_\tau,\varepsilon^2(N-\tau)/2+t_N)=F(x_\tau,t_\tau). \tag{4.1}$$

It might be instructive to think of a parabolic cylinder $\Omega\times(0,t)$ when t and ε are given determining N and t_N. The game begins at $k=0$ corresponding to $t_0=t$ in the time scale. When we play one round $k\to k+1$, the clock steps $\varepsilon^2/2$ backwards, $t_{k+1}=t_k-\varepsilon^2/2$, and we play until we get outside the cylinder when $k=\tau$ corresponding to t_τ in the time scale.

Next we define

$$u_{\mathrm{I}}^\varepsilon(x,t)=u_{\mathrm{I}}^{\varepsilon,N(t)}(x,0). \tag{4.2}$$

This equation defines values of $u_{\mathrm{I}}^\varepsilon(x,t)$ for every instant $t\in(0,T)$. For these functions, the DPP takes the form

$$\begin{aligned} u_{\mathrm{I}}^\varepsilon(x,t)&=\frac{\alpha}{2}\left\{\sup_{y\in\overline{B}_\varepsilon(x)}u_{\mathrm{I}}^\varepsilon\Big(y,t-\frac{\varepsilon^2}{2}\Big)+\inf_{y\in\overline{B}_\varepsilon(x)}u_{\mathrm{I}}^\varepsilon\Big(y,t-\frac{\varepsilon^2}{2}\Big)\right\}\\ &\quad+\beta\fint_{B_\varepsilon(x)}u_{\mathrm{I}}^\varepsilon\Big(y,t-\frac{\varepsilon^2}{2}\Big)\,dy \qquad \text{for every } (x,t)\in\Omega_T\\ u_{\mathrm{I}}^\varepsilon(x,t)&=F(x,t), \qquad \text{for every} \quad (x,t)\in\Gamma_\varepsilon, \end{aligned}$$

which agrees with Definition 4.1.

Comparison and convergence: The (p,ε)-parabolic functions satisfy comparison principle and are unique. The proofs are based on martingale arguments similar to those in [MPR2] recalling (4.2) and the fact that the relevant stopping time is now bounded.

We start with a comparison principle for the value functions. The connection of boundary values in different formulations is given in (4.1) and to simplify the notation we will use F in both formulations.

THEOREM 4.3. *If v_ε is a (p, ε)-parabolic function in Ω_T with boundary values F_{v_ε} in Γ_ε such that $F_{v_\varepsilon} \geq F_{u_I^\varepsilon}$, then $v_\varepsilon \geq u_I^\varepsilon$.*

Proof. Player I follows any strategy and Player II follows a strategy S_{II}^0 such that at $x_{k-1} \in \Omega$ he chooses to step to a point that almost minimizes $v_\varepsilon(\cdot, t_k)$, that is, to a point $x_k \in \overline{B}_\varepsilon(x_{k-1})$ such that

$$v_\varepsilon(x_k, t_k) \leq \inf_{y \in \overline{B}_\varepsilon(x_{k-1})} v_\varepsilon(y, t_k) + \eta 2^{-k}$$

for some fixed $\eta > 0$.

Choose $(x_0, t_0) \in \Omega_T$, and set $N = \lceil 2t_0/\varepsilon^2 \rceil$. It follows that

$$\begin{aligned}
&\mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}^0}^{x_0, N}[v_\varepsilon(x_k, t_k) + \eta 2^{-k} \,|\, x_0, \ldots, x_{k-1}] \\
&\leq \frac{\alpha}{2} \left\{ \inf_{y \in \overline{B}_\varepsilon(x_{k-1})} v_\varepsilon(y, t_k) + \eta 2^{-k} + \sup_{y \in \overline{B}_\varepsilon(x_{k-1})} v_\varepsilon(y, t_k) \right\} \\
&\qquad\qquad\qquad\qquad + \beta \fint_{B_\varepsilon(x_{k-1})} v_\varepsilon(y, t_k)\, dy + \eta 2^{-k} \\
&\leq v_\varepsilon(x_{k-1}, t_{k-1}) + \eta 2^{-(k-1)},
\end{aligned}$$

where we have estimated the strategy of Player I by sup and used the fact that v_ε is (p, ε)-parabolic. Thus

$$M_k = v_\varepsilon(x_k, t_k) + \eta 2^{-k}$$

is a supermartingale. Since $F_{v_\varepsilon} \geq F_{u_{\mathrm{I}}^\varepsilon}$ at Γ_ε, we deduce

$$\begin{aligned}
u_{\mathrm{I}}^\varepsilon(x_0, t_0) &= \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}} \mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}}^{x_0, N}[F_{u_{\mathrm{I}}^\varepsilon}(x_\tau, t_\tau)] \leq \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}^0}^{x_0, N}[F_{v_\varepsilon}(x_\tau, t_\tau) + \eta 2^{-\tau}] \\
&= \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}^0}^{x_0, N}[v_\varepsilon(x_\tau, t_\tau) + \eta 2^{-\tau}] \\
&\leq \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}^0}^{x_0, N}[M_0] = v_\varepsilon(x_0, t_0) + \eta,
\end{aligned}$$

where the fact that τ is a bounded stopping time allowed us to use the optional stopping theorem for M_k. Since η was arbitrary this proves the claim. ∎

Similarly, we can prove that $u_{\mathrm{II}}^\varepsilon$ is the largest (p, ε)-parabolic function: Player II follows any strategy and Player I always chooses to step to the point where v_ε is almost maximized. This implies that $v_\varepsilon(x_k) - \eta 2^{-k}$ is a submartingale.

Next we show that the game has a value. This together with the previous comparison principle proves the uniqueness of (p, ε)-parabolic functions with given boundary values.

THEOREM 4.4. *With a given payoff function, the game has a value; that is, we have the equality*

$$u_I^\varepsilon = u_{II}^\varepsilon.$$

Proof. It always holds that $u_{\mathrm{I}}^{\varepsilon} \le u_{\mathrm{II}}^{\varepsilon}$ so it remains to show $u_{\mathrm{II}}^{\varepsilon} \le u_{\mathrm{I}}^{\varepsilon}$. To see this we use the same argument as in the previous theorem: Player II follows a strategy S_{II}^0 such that at $x_{k-1} \in \Omega$, he always chooses to step to a point that almost minimizes $u_{\mathrm{I}}^{\varepsilon}$, that is, to a point x_k such that

$$u_{\mathrm{I}}^{\varepsilon}(x_k, t_k) \le \inf_{y \in \overline{B}_\varepsilon(x_{k-1})} u_{\mathrm{I}}^{\varepsilon}(y, t_k) + \eta 2^{-k},$$

for a fixed $\eta > 0$. We start from the point (x_0, t_0) so that $N = \lceil 2t_0/\varepsilon^2 \rceil$. It follows that from the choice of strategies and the dynamic programming principle for $u_{\mathrm{I}}^{\varepsilon}$ that

$$\begin{aligned}
&\mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}^0}^{x_0, N}[u_{\mathrm{I}}^{\varepsilon}(x_k, t_k) + \eta 2^{-k} \,|\, x_0, \dots, x_{k-1}] \\
&\le \frac{\alpha}{2} \left\{ \inf_{y \in \overline{B}_\varepsilon(x_{k-1})} u_{\mathrm{I}}^{\varepsilon}(y, t_k) + \eta 2^{-k} + \sup_{y \in \overline{B}_\varepsilon(x_{k-1})} u_{\mathrm{I}}^{\varepsilon}(y, t_k) \right\} \\
&\qquad\qquad + \beta \fint_{B_\varepsilon(x_{k-1})} u_{\mathrm{I}}^{\varepsilon}(y, t_k)\, dy + \eta 2^{-k} \\
&\le u_{\mathrm{I}}^{\varepsilon}(x_{k-1}, t_{k-1}) + \eta 2^{-(k-1)}.
\end{aligned}$$

Thus

$$M_k = u_{\mathrm{I}}^{\varepsilon}(x_k, t_k) + \eta 2^{-k}$$

is a supermartingale. According to the optional stopping theorem

$$\begin{aligned}
u_{\mathrm{II}}^{\varepsilon}(x_0, t_0) &= \inf_{S_{\mathrm{II}}} \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}}^{x_0, N}[F(x_\tau, t_\tau)] \le \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}^0}^{x_0, N}[F(x_\tau, t_\tau) + \eta 2^{-\tau}] \\
&= \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}^0}^{x_0, N}[u_{\mathrm{I}}^{\varepsilon}(x_\tau, t_\tau) + \eta 2^{-\tau}] \\
&\le \sup_{S_{\mathrm{I}}} \mathbb{E}_{S_{\mathrm{I}}, S_{\mathrm{II}}^0}^{x_0, N}[u_{\mathrm{I}}^{\varepsilon}(x_0, t_0) + \eta] = u_{\mathrm{I}}^{\varepsilon}(x_0, t_0) + \eta. \qquad \square
\end{aligned}$$

Theorems 4.3 and 4.4 imply uniqueness for (p, ε)-parabolic functions.

THEOREM 4.5. *There exists a unique (p, ε)-parabolic function with given boundary values F, and it coincides with the value of the game by virtue of* (4.2).

Proof. Due to the dynamic programming principle, the values of the games are (p, ε)-parabolic functions. This proves the existence part of the theorem. Theorems 4.3 and 4.4 together with the remark after Theorem 4.3 imply the uniqueness. $\square$

This theorem together with Theorem 4.3 gives the comparison principle for (p, ε)-parabolic functions.

THEOREM 4.6. *If v_ε and u_ε are (p, ε)-parabolic functions with boundary values $F_{v_\varepsilon} \ge F_{u_\varepsilon}$, then $v_\varepsilon \ge u_\varepsilon$ in Ω_T.*

Next, we show that (p, ε)-parabolic functions approximate solutions to

$$(n + p) u_t(x, t) = |\nabla u|^{2-p} \, \Delta_p u(x, t).$$

To prove the convergence, we use the Arzela-Ascoli type compactness lemma. Note that (p, ε)-parabolic functions are, in general, discontinuous. Nevertheless, their oscillation is controlled at scale ε. Therefore, the Arzela-Ascoli lemma has to be modified accordingly. For the proof of the lemma below, the reader can consult for example [MPR2].

LEMMA 4.7. *Let* $\{u_\varepsilon : \overline{\Omega}_T \to \mathbb{R},\ \varepsilon > 0\}$ *be a set of functions such that*

1. *there exists* $C > 0$ *so that* $|u_\varepsilon(x,t)| < C$ *for every* $\varepsilon > 0$ *and every* $(x,t) \in \overline{\Omega}_T$,
2. *given* $\eta > 0$ *there are constants* r_0 *and* ε_0 *such that for every* $\varepsilon < \varepsilon_0$ *and any* $(x,t),(y,s) \in \overline{\Omega}$ *with* $|x-y| + |t-s| < r_0$ *it holds*
$$|u_\varepsilon(x,t) - u_\varepsilon(y,s)| < \eta.$$

Then, there exists a uniformly continuous function $u : \overline{\Omega}_T \to \mathbb{R}$ *and a subsequence still denoted by* $\{u_\varepsilon\}$ *such that*

$$u_\varepsilon \to u \qquad \text{uniformly in} \quad \overline{\Omega}_T,$$

as $\varepsilon \to 0$.

First we recall the estimate for the stopping time of a random walk from [MPR2]. In this lemma, there is no bound for the maximum number of rounds.

LEMMA 4.8. *Let us consider an annular domain* $B_R(z) \setminus \overline{B}_\delta(z)$ *and a random walk such that when at* x_{k-1}, *the next point* x_k *is chosen according to a uniform probability distribution at* $B_\varepsilon(x_{k-1}) \cap B_R(z)$. *Let*

$$\tau^* = \inf\{k \,:\, x_k \in \overline{B}_\delta(z)\}.$$

Then

$$\mathbb{E}^{x_0}(\tau^*) \le \frac{C(R/\delta)\,\mathrm{dist}(\partial B_\delta(z), x_0) + o(1)}{\varepsilon^2},$$

for $x_0 \in B_R(z) \setminus \overline{B}_\delta(z)$. *Above* $o(1) \to 0$ *as* $\varepsilon \to 0$.

Next we derive an estimate for the asymptotic uniform continuity of a family $\{u_\varepsilon\}$ of (p,ε)-parabolic functions with fixed boundary values.

We assume that Ω satisfies an exterior sphere condition: For each $y \in \partial\Omega$, there exists $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$ with $\delta > 0$ such that $y \in \partial B_\delta(z)$. Below δ is always chosen small enough according to this condition.

We also assume that F satisfies

$$|F(x,t_x) - F(y,t_y)| \le L\left(|x-y| + |t_x - t_y|^{1/2}\right) \tag{4.3}$$

in Γ_ε. First, we consider the case where (y,t_y) is a point at the lateral boundary strip.

LEMMA 4.9. *Let* F *and* Ω *be as above. The* (p,ε)*-parabolic function* u_ε *with the boundary data* F *satisfies*

$$\begin{aligned} &|u_\varepsilon(x,t_x) - u_\varepsilon(y,t_y)| \\ &\quad \le C \min\left\{|x-y|^{1/2} + o(1),\, t_x^{1/2} + \varepsilon\right\} + L\,|t_x - t_y|^{1/2} + 2L\delta \end{aligned} \tag{4.4}$$

for every $(x,t_x) \in \Omega$, *and* $y \in S^\varepsilon$. *The constant* C *depends on* δ, n, L *and the diameter of* Ω. *In the above inequality* $o(1)$ *is taken relative to* ε.

Proof. Suppose for the moment that $t_x = t_y$, denote $t_0 = t_x = t_y$, and set $x_0 = x$ as well as $N = \lceil 2t_x/\varepsilon^2 \rceil$. By the exterior sphere condition, there exists $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$ such that $y \in \partial B_\delta(z)$. Player I chooses a strategy of pulling towards z, denoted by S_I^z. Then the calculation

$$\begin{aligned}
&\mathbb{E}^{x_0,N}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[|x_k - z| \,|\, x_0, \ldots, x_{k-1}] \\
&\le \frac{\alpha}{2} \left\{ |x_{k-1} - z| + \varepsilon + |x_{k-1} - z| - \varepsilon \right\} + \beta \fint_{B_\varepsilon(x_{k-1})} |x - z| \, dx \\
&\le |x_{k-1} - z| + C\varepsilon^2
\end{aligned} \tag{4.5}$$

implies that

$$M_k = |x_k - z| - C\varepsilon^2 k$$

is a supermartingale for some C independent of ε. The first inequality follows from the choice of the strategy, and the second from the estimate

$$\fint_{B_\varepsilon(x_{k-1})} |x - z| \, dx \le |x_{k-1} - z| + C\varepsilon^2.$$

The optional stopping theorem and Jensen's inequality then gives

$$\begin{aligned}
\mathbb{E}^{x_0,N}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[|x_\tau - z| + |t_\tau - t_0|^{1/2}] &= \mathbb{E}^{x_0,N}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}} \left[|x_\tau - z| + \varepsilon \left(\frac{\tau}{2} \right)^{1/2} \right] \\
&\le |x_0 - z| + C\varepsilon \left(\mathbb{E}^{x_0,N}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[\tau] \right)^{1/2}.
\end{aligned} \tag{4.6}$$

In formula (4.5), the expected distance of the pure tug-of-war is bounded by $|x_{k-1} - z|$ whereas the expected distance of the pure random walk is slightly larger. Therefore, we can bound from above the stopping time of our process by a stopping time of the random walk in the setting of Lemma 4.8 by choosing $R > 0$ such that $\Omega \subset B_R(z)$. Thus, we obtain

$$\begin{aligned}
\mathbb{E}^{x_0,N}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[\tau] &\le \min \left\{ \mathbb{E}^{x_0,N}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[\tau^*],\, N \right\} \\
&\le \min \left\{ C(R/\delta)(\operatorname{dist}(\partial B_\delta(z), x_0) + o(1))/\varepsilon^2,\, N \right\}.
\end{aligned}$$

Since $y \in \partial B_\delta(z)$, we have

$$\operatorname{dist}(\partial B_\delta(z), x_0) \le |y - x_0|,$$

and together with (4.6) this gives

$$\begin{aligned}
\mathbb{E}^{x_0,N}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[|x_\tau - z| + |t_\tau - t_0|^{1/2}] \le& \min \left\{ C(R/\delta)(|x_0 - y| + o(1)),\, C\varepsilon^2 N \right\}^{1/2} \\
&+ |x_0 - z|.
\end{aligned}$$

Thus, we end up with

$$\begin{aligned}
&F(z, t_0) - L \left(\min \left\{ C(R/\delta)(|x_0 - y| + o(1)),\, C\varepsilon^2 N \right\}^{1/2} + |x_0 - z| \right) \\
&\quad \le \mathbb{E}^{x_0,N}_{S_{\mathrm{I}}^z,S_{\mathrm{II}}}[F(x_\tau, t_\tau)] \\
&\quad \le F(z, t_0) + L \left(\min \left\{ C(R/\delta)(|x_0 - y| + o(1)),\, C\varepsilon^2 N \right\}^{1/2} + |x_0 - z| \right),
\end{aligned}$$

which implies

$$\begin{aligned}
&\sup_{S_{\mathrm{I}}}\inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0,N}_{S_{\mathrm{I}},S_{\mathrm{II}}}[F(x_\tau,t_\tau)] \\
&\quad\geq \inf_{S_{\mathrm{II}}} \mathbb{E}^{x_0,N}_{S^z_{\mathrm{I}},S_{\mathrm{II}}}[F(x_\tau,t_\tau)] \\
&\quad\geq F(z,t_0) - L\left(\min\left\{C(R/\delta)(|x_0-y|+o(1)),\, C\varepsilon^2 N\right\}^{1/2} + |x_0-z|\right) \\
&\quad\geq F(y,t_0) - 2L\delta - L\min\left\{C(R/\delta)(|x_0-y|+o(1)),\, C\varepsilon^2 N\right\}^{1/2}.
\end{aligned}$$

The upper bound can be obtained by choosing for Player II a strategy where he points to z, and thus (4.4) follows.

Finally, if $t_x \neq t_y$, then we utilize the above estimate and obtain

$$\begin{aligned}
|u_\varepsilon(x,t_x)-u_\varepsilon(y,t_y)| &\leq |u_\varepsilon(x,t_x)-u_\varepsilon(y,t_x)| + |u_\varepsilon(y,t_x)-u_\varepsilon(y,t_y)| \\
&\leq 2L\delta + \min\left\{C(R/\delta)(|x-y|+o(1)),\, C\varepsilon^2 N\right\}^{1/2} + L\,|t_x-t_y|^{1/2},
\end{aligned}$$

and the proof is completed by recalling that $N = \lceil 2t_x/\varepsilon^2\rceil$. □

Next we consider the case when the boundary point (y,t_y) lies at the initial boundary strip.

Lemma 4.10. *Let F and Ω be as in Lemma 4.9. The (p,ε)-parabolic function u_ε with the boundary data F satisfies*

$$|u_\varepsilon(x,t_x)-u_\varepsilon(y,t_y)| \leq C\left(|x-y| + t_x^{1/2} + \varepsilon\right), \tag{4.7}$$

and for every $(x,t_x)\in\Omega_T$ and $(y,t_y)\in\Omega\times(-\varepsilon^2/2,0]$. *Proof.* Set $x_0 = x$, and $N=\lceil 2t_x/\varepsilon^2\rceil$. Player I pulls to y. Then

$$M_k = |x_k-y|^2 - Ck\varepsilon^2$$

is a supermartingale. Indeed,

$$\begin{aligned}
&\mathbb{E}^{x_0,N}_{S^y_{\mathrm{I}},S_{\mathrm{II}}}[|x_k-y|^2\,|\,x_0,\ldots,x_{k-1}] \\
&\leq \frac{\alpha}{2}\left\{(|x_{k-1}-y|+\varepsilon)^2 + (|x_{k-1}-y|-\varepsilon)^2\right\} + \beta \fint_{B_\varepsilon(x_{k-1})} |x-y|^2\,dx \\
&\leq \alpha\left\{|x_{k-1}-y|^2+\varepsilon^2\right\} + \beta\left(|x_{k-1}-y|^2 + C\varepsilon^2\right) \leq |x_{k-1}-y|^2 + C\varepsilon^2.
\end{aligned}$$

According to optional stopping theorem,

$$\mathbb{E}^{x_0,N}_{S^y_{\mathrm{I}},S_{\mathrm{II}}}[|x_\tau-y|^2] \leq |x_0-y|^2 + C\varepsilon^2\mathbb{E}^{x_0,N}_{S^y_{\mathrm{I}},S_{\mathrm{II}}}[\tau],$$

and since the stopping time is bounded by $\lceil 2t_x/\varepsilon^2\rceil$, this implies

$$\mathbb{E}^{x_0,N}_{S^y_{\mathrm{I}},S_{\mathrm{II}}}[|x_\tau-y|^2] \leq |x_0-y|^2 + C(t_x+\varepsilon^2).$$

Finally, Jensen's inequality gives

$$\begin{aligned}
\mathbb{E}^{x_0,N}_{S^y_{\mathrm{I}},S_{\mathrm{II}}}[|x_\tau-y|\,] &\leq \left(|x_0-y|^2 + C(t_x+\varepsilon^2)\right)^{1/2} \\
&\leq |x_0-y| + C(t_x^{1/2}+\varepsilon).
\end{aligned}$$

The rest of the argument is similar to the one used in the previous proof. In particular, we obtain the upper bound by choosing for Player II a strategy where he points to y. We end up with

$$|u_\varepsilon(x,t_x) - u_\varepsilon(y,t_y)| \le C\left(|x-y| + t_x^{1/2} + \varepsilon\right). \qquad \square$$

Next we will show that (p,ε)-parabolic functions are asymptotically uniformly continuous.

Lemma 4.11. *Let F and Ω be as in Lemma 4.9. Let $\{u_\varepsilon\}$ be a family of (p,ε)-parabolic functions. Then this family satisfies the conditions in Lemma 4.7.*

Proof. It follows from the definition of (p,ε)-parabolic function that

$$|u_\varepsilon| \le \sup_{\Gamma_\varepsilon} F$$

and we can thus concentrate on the second condition of Lemma 4.7. Observe that the case $x,y \in \Gamma_\varepsilon$ readily follows from the uniform continuity of F, and thus we can concentrate on the cases $x \in \Omega,\ y \in S^\varepsilon$, and $x,y \in \Omega$.

Choose any $\eta > 0$. By (4.4) and (4.7), there exists $\varepsilon_0 > 0$, $\delta > 0$, and $r_0 > 0$ so that

$$|u_\varepsilon(x,t_x) - u_\varepsilon(y,t_y)| < \eta$$

for all $\varepsilon < \varepsilon_0$ and for any $(x,t_x) \in \Omega_T,\ (y,t_y) \in \Gamma_\varepsilon$ such that $|x-y|^{1/2} + |t_x - t_y|^{1/2} \le r_0$.

Next we consider a slightly smaller domain

$$\tilde\Omega_T = \{(z,t) \in \Omega_T \,:\, \mathrm{d}((z,t),\, \partial_p\Omega_T) > r_0/3\}$$

with

$$\mathrm{d}((z,t),\, \partial_p\Omega_T) = \inf\{|z-y|^{1/2} + |t-s|^{1/2} \,:\, (y,s) \in \partial_p\Omega\},$$

and the boundary strip

$$\tilde\Gamma = \{(z,t) \in \overline{\Omega}_T \,:\, \mathrm{d}((z,t),\, \partial_p\Omega_T) \le r_0/3\}.$$

Suppose then that $x,y \in \Omega_T$ with $|x-y|^{1/2} + |t_x - t_y|^{1/2} < r_0/3$. First, if $x,y \in \tilde\Gamma$, then we can estimate

$$|u_\varepsilon(x,t_x) - u_\varepsilon(y,t_y)| \le 3\eta$$

for $\varepsilon < \varepsilon_0$ by comparing the values at x and y to the nearby boundary values and using the previous step. Finally, a translation argument finishes the proof. Let $(x,t_x),(y,t_y) \in \tilde\Omega_T$. Without loss of generality we may assume that $t_x > t_y$. Define

$$\tilde F(z,t_z) = u_\varepsilon(z-x+y,\, t_z + t_y - t_x) + 3\eta \quad \text{for} \quad (z,t_z) \in \tilde\Gamma.$$

We have

$$\tilde F(z,t_z) \ge u_\varepsilon(z,t_z) \quad \text{in} \quad \tilde\Gamma$$

by the reasoning above. Solve the (p,ε)-parabolic function $\tilde u_\varepsilon$ in $\tilde\Omega_T$ with the boundary values $\tilde F$ in $\tilde\Gamma$. By the comparison principle Theorem 4.6, and the uniqueness Theorem 4.5, we deduce

$$u_\varepsilon(x,t_x)\le \tilde u_\varepsilon(x,t_x)=u_\varepsilon(x-x+y,t_x-t_x+t_y)+3\eta=u_\varepsilon(y,t_y)+3\eta \quad \text{in} \quad \tilde\Omega_T.$$

The lower bound follows by a similar argument. ∎

Corollary 4.12. *Let F satisfy the continuity condition* (4.3) *and Ω satisfy the exterior sphere condition. Let $\{u_\varepsilon\}$ be a family of (p,ε)-parabolic functions with boundary values F. Then there exists a uniformly continuous u and a subsequence still denoted by $\{u_\varepsilon\}$ such that*

$$u_\varepsilon\to u \qquad \textit{uniformly in} \quad \overline{\Omega}$$

as $\varepsilon\to 0$.

Theorem 4.13. *Let F satisfy the continuity condition* (4.3) *and Ω satisfy the exterior sphere condition. Then, the uniform limit*

$$u=\lim_{\varepsilon\to 0}u_\varepsilon$$

of (p,ε)-parabolic functions obtained in Corollary 4.12 is a viscosity solution to the equation

$$(n+p)u_t(x,t)=|\nabla u|^{2-p}\,\Delta_p u(x,t)$$

with boundary values F.

Proof. First, clearly $u=F$ on $\partial\Omega$, and we can focus attention on showing that u is a viscosity solution. Similarly as in (3.8), we can derive for any $\phi\in C^2$ an estimate

$$\begin{aligned}
&\frac{\alpha}{2}\left\{\max_{y\in\overline{B}_\varepsilon(x)}\phi\Big(y,t-\frac{\varepsilon^2}{2}\Big)+\min_{y\in\overline{B}_\varepsilon(x)}\phi\Big(y,t-\frac{\varepsilon^2}{2}\Big)\right\}\\
&\qquad\qquad+\beta\fint_{B_\varepsilon(x)}\phi\Big(y,t-\frac{\varepsilon^2}{2}\Big)\,dy-\phi(x,t)\\
&\quad\ge\frac{\beta\varepsilon^2}{2(n+2)}\Bigg((p-2)\left\langle D^2\phi(x,t)\frac{x_1^{\varepsilon,t-\varepsilon^2/2}-x}{\varepsilon},\frac{x_1^{\varepsilon,t-\varepsilon^2/2}-x}{\varepsilon}\right\rangle\\
&\qquad\qquad+\Delta\phi(x,t)-(n+p)\phi_t(x,t)\Bigg)+o(\varepsilon^2),
\end{aligned}\tag{4.8}$$

where

$$\phi\Big(x_1^{\varepsilon,t-\varepsilon^2/2},t-\frac{\varepsilon^2}{2}\Big)=\min_{y\in\overline{B}_\varepsilon(x)}\phi\Big(y,t-\frac{\varepsilon^2}{2}\Big).$$

Suppose then that ϕ touches u at (x,t) from below. By the uniform convergence, there exists sequence $\{(x_\varepsilon,t_\varepsilon)\}$ converging to (x,t) such that $u_\varepsilon-\phi$ has an approximate minimum at $(x_\varepsilon,t_\varepsilon)$, that is, for $\eta_\varepsilon>0$, there exists $(x_\varepsilon,t_\varepsilon)$ such that

$$u_\varepsilon(y,s)-\phi(y,s)\ge u_\varepsilon(x_\varepsilon,t_\varepsilon)-\phi(x_\varepsilon,t_\varepsilon)-\eta_\varepsilon,$$

in the neighborhood of $(x_\varepsilon, t_\varepsilon)$. Further, set $\tilde\phi = \phi + u_\varepsilon(x_\varepsilon, t_\varepsilon) - \phi(x_\varepsilon, t_\varepsilon)$, so that

$$u_\varepsilon(x_\varepsilon, t_\varepsilon) = \tilde\phi(x_\varepsilon, t_\varepsilon) \quad \text{and} \quad u_\varepsilon(y,s) \geq \tilde\phi(y,s) - \eta_\varepsilon.$$

Thus, by recalling the fact that u_ε is (p,ε)-parabolic, we obtain

$$\begin{aligned}\eta_\varepsilon \geq & -\tilde\phi(x_\varepsilon, t_\varepsilon) + \beta \fint_{B_\varepsilon(x_\varepsilon)} \tilde\phi\Big(y, t_\varepsilon - \frac{\varepsilon^2}{2}\Big)\, dy \\ & + \frac{\alpha}{2}\left\{ \sup_{y\in \overline{B}_\varepsilon(x_\varepsilon)} \tilde\phi\Big(y, t_\varepsilon - \frac{\varepsilon^2}{2}\Big) + \inf_{y\in \overline{B}_\varepsilon(x_\varepsilon)} \tilde\phi\Big(y, t_\varepsilon - \frac{\varepsilon^2}{2}\Big)\right\}.\end{aligned} \tag{4.9}$$

According to (4.8), choosing $\eta_\varepsilon = o(\varepsilon^2)$, and observing $\nabla\phi = \nabla\tilde\phi$, $D^2\tilde\phi = D^2\phi$, we have

$$\begin{aligned}0 \geq \frac{\beta\varepsilon^2}{2(n+2)} \Bigg(& (p-2)\left\langle D^2\phi(x_\varepsilon, t_\varepsilon)\frac{x_1^{\varepsilon, t-\varepsilon^2/2} - x_\varepsilon}{\varepsilon}, \frac{x_1^{\varepsilon, t-\varepsilon^2/2} - x_\varepsilon}{\varepsilon}\right\rangle \\ & + \Delta\phi(x_\varepsilon, t_\varepsilon) - (n+p)\phi_t(x_\varepsilon, t_\varepsilon)\Bigg) + o(\varepsilon^2).\end{aligned}$$

Suppose that $\nabla\phi(x,t) \neq 0$. Dividing by ε^2 and letting $\varepsilon \to 0$, we get

$$0 \geq \frac{\beta}{2(n+2)}\big((p-2)\Delta_\infty\phi(x) + \Delta\phi(x) - (n+p)\phi_t(x,t)\big).$$

To verify the other half of the definition of a viscosity solution, we derive a reverse inequality to (4.8) by considering the maximum point of the test function and choose a function ϕ which touches u from above. The rest of the argument is analogous.

Now we consider the case $\nabla\phi(x,t) = 0$. By Lemma 2.2, we can also assume that $D^2\phi(x,t) = 0$ and it suffices to show that

$$\phi_t(x,t) \geq 0.$$

In this case, (4.8) takes the form

$$\begin{aligned}& \frac{\alpha}{2}\left\{ \max_{y\in \overline{B}_\varepsilon(x)} \phi\Big(y, t - \frac{\varepsilon^2}{2}\Big) + \min_{y\in \overline{B}_\varepsilon(x)} \phi\Big(y, t - \frac{\varepsilon^2}{2}\Big)\right\} \\ & \qquad\qquad + \beta \fint_{B_\varepsilon(x)} \phi\Big(y, t - \frac{\varepsilon^2}{2}\Big)\, dy - \phi(x,t) \\ & \geq -\frac{\beta\varepsilon^2(n+p)}{2(n+2)}\phi_t(x,t) + o(\varepsilon^2).\end{aligned}$$

Since (4.9) still holds, we can repeat the argument above.

Finally, we conclude that also the original sequence converges to a unique viscosity solution. To this end, observe that by above any sequence $\{u_\varepsilon\}$ contains a subsequence that converges uniformly to some viscosity solution u. By [CGG] (see also [ES] and [GGIS]), viscosity solutions to (1.2) are uniquely determined by their boundary values. Hence we conclude that the whole original sequence converges. □

Observe that the above theorem also gives a proof of the existence of viscosity solutions to (1.2) using probabilistic arguments.

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