Nonlinear balayage on metric spaces

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Abstract. We develop a theory of balayage on complete doubling metric measure spaces supporting a Poincaré inequality. In particular, we are interested in continuity and p-harmonicity of the balayage. We also study connections to the obstacle problem. As applications, we characterize regular boundary points and polar sets in terms of balayage.

Key words and phrases: Balayage, boundary regularity, continuity, doubling measure, metric space, nonlinear, obstacle problem, Perron solution, p-harmonic, polar set, Poincaré inequality, potential theory, superharmonic.

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1. Introduction

Balayage is one of the most useful tools in linear potential theory and has been used to obtain many important results therein. Heinonen, Kilpeläinen and Martio were the first to use nonlinear balayage for studying $\mathcal{A}$-harmonic functions on $\mathbf{R}^n$ in [22], [23] and [24]. The purpose of this paper is to develop the nonlinear balayage theory on metric spaces.

Analysis and nonlinear potential theory on metric measure spaces have undergone a rapid development during the last decade, see e.g. Hajłasz [19], Heinonen–Koskela [25], Koskela–MacManus [34], Hajłasz–Koskela [20], Cheeger [16], Shanmugalingam [37], Kinnunen–Martio [28], [29] and more recently Keith–Zhong [26].

Using upper gradients, which were introduced by Heinonen and Koskela in [25], it is possible to define (Newtonian) Sobolev-type spaces on general metric spaces. Variational inequalities can then be used to define p-harmonic and superharmonic functions (see Section 3). In our generality there are no corresponding partial differential equations, which causes some difficulties. Nevertheless, under rather mild assumptions on the metric space, a large part of the theory of p-harmonic and superharmonic functions on weighted $\mathbf{R}^n$ has been extended to metric spaces, see, e.g., Shanmugalingam [38], [39], Kinnunen–Martio [30], Kinnunen–Shanmugalingam [32], [33], Björn–Björn–Shanmugalingam [9], [10] and Björn–Björn [5]. Examples of spaces satisfying our assumptions include weighted $\mathbf{R}^n$, manifolds, Heisenberg groups and more general Carnot groups and Carnot–Carathéodory spaces, see, e.g., [5], [10] and Hajłasz–Koskela [20].

Balayage is a regularized infimum of the family of superharmonic functions lying above a given obstacle. First, we use the fundamental convergence theorem from Björn–Björn–Parviainen [8] to show that regularizing changes the infimum only on a set of capacity zero and that the resulting function is superharmonic. This makes it possible for us to develop the theory of balayage in a way different from Heinonen–Kilpeläinen–Martio [24], where a substantial part of the balayage theory was developed before proving that the infimum only needs to be regularized on a set of capacity zero. We generalize the balayage results from [22], [23] and [24] to metric spaces, but in most cases our proofs are different.

Sets of capacity zero in potential theory correspond to sets of measure zero in the study of L^p-spaces and can sometimes be disregarded. In linear potential theory there are two ways of defining the balayage, depending on if sets of capacity zero are ignored or not, and it is almost immediate that they are equivalent. This equivalence is then used to obtain many important consequences. In the nonlinear case it is not known whether the two definitions, which we call R- and Q-balayage, see Section 4, always coincide. A partial result on their equality was obtained in Heinonen–Kilpeläinen [23] in $\mathbf{R}^n$. We extend this result to metric spaces and also provide other sufficient conditions for when the two types of balayage coincide. This is particularly useful in our characterizations of polar sets by means of barriers in Section 8.

We develop the theories of R- and Q-balayage in parallel, proving results for both types of balayage where possible. In most cases we are able to obtain results for the Q-balayage, but in connection with Perron solutions we can only obtain some parts for the R-balayage.

On metric spaces, obstacle problems have earlier been used instead of balayage to prove various results in nonlinear potential theory. In Section 5, we study the relationship between balayage and obstacle problems. We also study the continuity of balayage and show that even for irregular obstacles, the balayage is p-harmonic in the set where it lies strictly above the obstacle, see Section 6.

As an application of the theory of balayage, in Section 7 we provide two types of characterizations of regular boundary points in terms of balayage. These complement the large number of characterizations obtained in Björn–Björn [5]. Finally we use balayage for calculating capacities. Our results are also used in Mäkäläinen [35] to obtain a characterization of removable singularities for Hölder continuous Cheeger p-harmonic functions on metric spaces.

Many of our results are new also in $\mathbf{R}^n$. The results and proofs given in this paper hold also for Cheeger p-harmonic functions, as discussed in e.g. Björn–MacManus–Shanmugalingam [14] and Björn–Björn–Shanmugalingam [9], and for $\mathcal{A}$-harmonic functions as defined on pp. 56–57 of Heinonen–Kilpeläinen–Martio [24].

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2. Preliminaries

We assume throughout the paper that $1 < p < \infty$ and that $X = (X, d, \mu)$ is a complete metric space endowed with a metric d and a positive complete Borel measure μ which is *doubling*, i.e. there exists a constant $C_\mu \geq 1$ such that for all balls $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ in X,

$$0 < \mu(2B) \leq C_\mu \mu(B),$$

where $\lambda B = B(x_0, \lambda r)$. It follows that X is proper, i.e. that closed bounded sets are compact.

In this paper, a *path* in X is a rectifiable nonconstant continuous mapping from a compact interval. A path can thus be parametrized by arc length ds.

We follow Heinonen–Koskela [25] introducing upper gradients as follows (they called them very weak gradients).

Definition 2.1. A nonnegative Borel function g on X is an *upper gradient* of an extended real-valued function f on X if for all paths $\gamma : [0, l_\gamma] \to X$,

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g\, ds \tag{2.1}$$

whenever both $f(\gamma(0))$ and $f(\gamma(l_\gamma))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. If g is a nonnegative measurable function on X and if (2.1) holds for p-a.e. path, then g is a *p-weak upper gradient* of f.

By saying that (2.1) holds for p-a.e. path, we mean that it fails only for a path family with zero p-modulus, see Definition 2.1 in Shanmugalingam [37]. It is implicitly assumed that $\int_\gamma g\, ds$ is defined (with a value in $[0, \infty]$) for p-a.e. path.

The p-weak upper gradients were introduced in Koskela–MacManus [34]. They also showed that if $g \in L^p(X)$ is a p-weak upper gradient of f, then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of f such that $g_j \to g$ in $L^p(X)$. If f has an upper gradient in $L^p(X)$, then it has a *minimal p-weak upper gradient* $g_f \in L^p(X)$ in the sense that for every p-weak upper gradient $g \in L^p(X)$ of f, $g_f \leq g$ a.e., see Corollary 3.7 in Shanmugalingam [38].

Next we define a version of Sobolev spaces on the metric space X due to Shanmugalingam [37]. Cheeger [16] gave an alternative definition which leads to the same space, when $p > 1$.

Definition 2.2. Whenever $u \in L^p(X)$, let

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

Definition 2.3. The *capacity* of a set $E \subset X$ is the number

$$C_p(E) = \inf \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on E.

By truncation, the infimum can be taken over u such that $0 \le u \le 1$. The capacity is countably subadditive. For this and other properties as well as equivalent definitions of the capacity we refer to Kilpeläinen–Kinnunen–Martio [27], Kinnunen–Martio [28], [29], and Björn–Björn [7].

We say that a property holds *quasieverywhere* (q.e.) if the set of points for which the property does not hold has capacity zero. The capacity is the correct gauge for distinguishing between two Newtonian functions. Indeed, if $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ q.e. in X. Moreover, if $u, v \in N^{1,p}(X)$ and $u = v$ a.e., then $u \sim v$.

The following consequence of Mazur's lemma will be useful. For a proof see Björn–Björn–Parviainen [8].

Lemma 2.4. *Assume that* $\{u_i\}_{i=1}^\infty$ *is bounded in* $N^{1,p}(X)$ *and that* $u_i \to u$ *q.e. Then* $u \in N^{1,p}(X)$ *and*

$$\int_X g_u^p \, d\mu \le \liminf_{i\to\infty} \int_X g_{u_i}^p \, d\mu.$$

We assume further that X supports a *weak* $(1,p)$*-Poincaré inequality*, i.e. there exist constants $C > 0$ and $\lambda \ge 1$ such that for all balls $B \subset X$, all integrable functions f on X and for all upper gradients g of f,

$$⨍_B |f - f_B| \, d\mu \le C(\operatorname{diam} B) \biggl(⨍_{\lambda B} g^p \, d\mu \biggr)^{1/p}, \tag{2.2}$$

where $f_B := ⨍_B f \, d\mu := \int_B f \, d\mu / \mu(B)$.

By the Hölder inequality, it is easy to see that if X supports a weak $(1,p)$-Poincaré inequality, then it supports a weak $(1,q)$-Poincaré inequality for every $q > p$. A deep theorem of Keith and Zhong [26] shows that X even supports a weak $(1,\overline{p})$-Poincaré inequality for some $\overline{p} < p$, which was earlier a standard assumption for the theory of p-harmonic functions on metric spaces. In the definition of the Poincaré inequality we can equivalently assume that g is a p-weak upper gradient.

Under these assumptions, Lipschitz functions are dense in $N^{1,p}(X)$, and the functions in $N^{1,p}(X)$ are quasicontinuous, see Shanmugalingam [37] and Björn–Björn–Shanmugalingam [11]. This means that in the Euclidean setting, $N^{1,p}(\mathbf{R}^n)$ is the refined Sobolev space.

We need a Newtonian space with zero boundary values defined as follows for an open set $\Omega \subset X$,

$$N_0^{1,p}(\Omega) = \{f|_\Omega : f \in N^{1,p}(X) \text{ and } f = 0 \text{ in } X \setminus \Omega\}.$$

One can replace the assumption "$f = 0$ in $X \setminus \Omega$" with "$f = 0$ q.e. in $X \setminus \Omega$" without changing the obtained space. We say that $f \in N_{\mathrm{loc}}^{1,p}(\Omega)$ if for every $x \in \Omega$ there is r_x such that $f \in N^{1,p}(B(x, r_x))$. This is clearly equivalent to saying that $f \in N^{1,p}(V)$ for every open $V \Subset \Omega$. By saying that $V \Subset \Omega$ we mean that $\overline{V}$ is a compact subset of Ω.

3. Minimizers and superharmonic functions

Let us recall that we assume that X is a complete metric space supporting a weak $(1,p)$-Poincaré inequality and that μ is doubling. Assume also from now on that Ω is a nonempty open set which is either unbounded or is such that $C_p(X \setminus \Omega) > 0$. (See Section 9 for the exceptional case when X is bounded and $C_p(X \setminus \Omega) = 0$.)

Definition 3.1. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *minimizer* in Ω if for all $\varphi \in N^{1,p}_0(\Omega)$ we have that

$$\int_{\varphi \neq 0} g_u^p \, d\mu \le \int_{\varphi \neq 0} g_{u+\varphi}^p \, d\mu. \tag{3.1}$$

A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *superminimizer* in Ω if (3.1) holds for all nonnegative $\varphi \in N^{1,p}_0(\Omega)$.

By Proposition 3.2 in A. Björn [3] it is enough to test (3.1) with (all and nonnegative, respectively) $\varphi \in \mathrm{Lip}_c(\Omega)$.

We follow Björn–Björn [7] in the definition of the obstacle problem. This definition is a special case of the definition used by Farnana [18] for the double obstacle problem.

Definition 3.2. Let $V \subset X$ be a nonempty bounded open set with $C_p(X \setminus V) > 0$. Let $f \in N^{1,p}(V)$ and $\psi : V \to [-\infty, \infty]$. Then we define

$$\mathcal{K}_{\psi,f}(V) = \{v \in N^{1,p}(V) : v - f \in N^{1,p}_0(V) \text{ and } v \ge \psi \text{ q.e. in } V\}.$$

Furthermore, a function $u \in \mathcal{K}_{\psi,f}(V)$ is a *solution of the $\mathcal{K}_{\psi,f}(V)$-obstacle problem* if

$$\int_V g_u^p \, d\mu \le \int_V g_v^p \, d\mu \quad \text{for all } v \in \mathcal{K}_{\psi,f}(V).$$

We also let $\mathcal{K}_{\psi,f} = \mathcal{K}_{\psi,f}(\Omega)$.

Kinnunen–Martio [30] made a similar definition but with "q.e." replaced by "a.e.", which was sufficient for their purposes. Classical Sobolev functions in Euclidean spaces are defined only up to a.e. equivalence classes, so the a.e. obstacle problem is the only reasonable interpretation in that case. On the other hand, Newtonian functions are defined up to q.e. equivalence classes and correspond to the fine representatives of Sobolev functions. Hence, the q.e. definition is more natural for them.

If $\psi \in N^{1,p}_{\mathrm{loc}}(\Omega)$, then the two types of obstacle problems coincide, but more generally there are differences, see the discussion in Farnana [18]. In particular, if $E \subset \Omega$ has zero measure but positive capacity, then our definition of the obstacle problem leads to the capacitary potential of E in Ω, whereas solutions of the a.e. obstacle problem are trivial. In several of our results, e.g. in Theorem 5.3 and Proposition 5.6, it will be important that we work with the definition above.

In nonlinear potential theory, even in the Euclidean case, obstacle problems and Sobolev spaces are a useful tool. In the classical linear theory, these notions, being essentially replaced by potentials, are often not visible at all, cf. Armitage–Gardiner [1] or Doob [17].

We shall use the $\operatorname{ess\,lim\,inf}$*-regularization*

$$u^*(x) = \operatorname*{ess\,lim\,inf}_{y \to x} u(y) := \lim_{R \to 0} \operatorname*{ess\,inf}_{B(x,R)} u. \tag{3.2}$$

It is easily verified that u^* is indeed lower semicontinuous.

If Ω is bounded and $\mathcal{K}_{\psi,f} \neq \varnothing$, then there is a solution u of the $\mathcal{K}_{\psi,f}$-obstacle problem, and the solution is unique up to equivalence in $N^{1,p}(\Omega)$. The proof of this fact is slightly more involved than the proof of Theorem 3.2 in [30] for the a.e.-obstacle problem, see either Farnana [18] or Björn–Björn [7]. Moreover $u^* = u$ q.e. and u^* is the unique $\operatorname{ess\,lim\,inf}$-regularized solution of the $\mathcal{K}_{\psi,f}$-obstacle problem.

A function u is a superminimizer if and only if it is a solution of the $\mathcal{K}_{u,u}(\Omega')$-obstacle problem for every nonempty open subset $\Omega' \Subset \Omega$. On the other hand, if Ω is bounded, then a solution of the $\mathcal{K}_{\psi,f}$-obstacle problem is a superminimizer, and a superminimizer $u \in N^{1,p}(\Omega)$ is a solution of the $\mathcal{K}_{u,u}$-obstacle problem. Moreover, if u is a superminimizer then $u = u^*$ q.e. and u^* is superharmonic (see below). If $\psi \equiv -\infty$, the obstacle problem reduces to the usual Dirichlet problem.

By Proposition 3.8 and Corollary 5.5 in Kinnunen–Shanmugalingam [32] a minimizer can be modified on a set of capacity zero so that it becomes locally Hölder continuous. A *p-harmonic function* is a continuous minimizer. For $f \in N^{1,p}(V)$, we define $H_V f$ to be the continuous solution of the $\mathcal{K}_{-\infty,f}(V)$-obstacle problem.

Definition 3.3. A function $u : \Omega \to (-\infty, \infty]$ is *superharmonic* in Ω if

(i) u is lower semicontinuous;
(ii) u is not identically ∞ in any component of Ω;
(iii) for every nonempty open set $V \Subset \Omega$ and all functions $v \in \operatorname{Lip}(X)$, we have that $H_V v \le u$ in Ω' whenever $v \le u$ on ∂V.

A function $u : \Omega \to [-\infty, \infty)$ is *subharmonic* if $-u$ is superharmonic.

This definition is equivalent to the definitions used in Heinonen–Kilpeläinen–Martio [24] and Kinnunen–Martio [30], see A. Björn [2].

If u and v are superharmonic, $\alpha > 0$ and $\beta \in \mathbf{R}$, then $\alpha u + \beta$ and $\min\{u, v\}$ are superharmonic, but in general $u + v$ is not. Superharmonic functions are $\operatorname{ess\,lim\,inf}$-regularized, and a function in $N^{1,p}_{\mathrm{loc}}(\Omega)$ is superharmonic if and only if it is an $\operatorname{ess\,lim\,inf}$-regularized superminimizer. However, there are superharmonic functions not belonging to $N^{1,p}_{\mathrm{loc}}(\Omega)$, and thus they are not superminimizers, see also a discussion in Björn–Björn–Parviainen [8]. A superharmonic function u satisfies the *strong minimum principle*: If u attains its minimum in Ω at some point $x \in \Omega$, then u is constant in the component containing x. For the facts above on superminimizers, superharmonic functions and obstacle problems we refer to [30].

The following comparison lemma is proved for the a.e.-obstacle problem in Björn–Björn [5], Lemma 5.4, the proof is the same in our case, see also Farnana [18], where the corresponding result is proved for the more general double (q.e.)-obstacle problem.

Lemma 3.4. *Assume that Ω is bounded. Let $\psi_j : \Omega \to \overline{\mathbf{R}}$ and $f_j \in N^{1,p}(\Omega)$ be such that $\mathcal{K}_{\psi_j,f_j} \neq \varnothing$, and let u_j be the* $\operatorname{ess\,lim\,inf}$*-regularized solution of the $\mathcal{K}_{\psi_j,f_j}$-obstacle problem, $j = 1, 2$. Assume that $\psi_1 \le \psi_2$ q.e. in Ω and that $(f_1 - f_2)_+ \in N^{1,p}_0(\Omega)$, then $u_1 \le u_2$ in Ω.*

We will need two results for superminimizers and superharmonic functions from Björn–Björn–Parviainen [8].

Proposition 3.5. *If u is superharmonic in Ω and bounded from above by an $N^{1,p}_{\mathrm{loc}}(\Omega)$-function, then u is a superminimizer.*

For the second result, called the fundamental convergence theorem, we first need to define the $\liminf$*-regularization* of a function $f : \Omega \to \overline{\mathbf{R}}$ as

$$\hat{f}(x) = \lim_{r \to 0} \inf_{\Omega \cap B(x,r)} f, \quad x \in \Omega.$$

It follows that $\hat{f} \le f$, and it is easy to show that $\hat{f}$ is lower semicontinuous.

Theorem 3.6. (The fundamental convergence theorem) *Let $\mathcal{F}$ be a nonempty family of superharmonic functions in Ω. Assume that there is a function $f \in N^{1,p}_{\mathrm{loc}}(\Omega)$ such that $u \ge f$ a.e. in Ω for all $u \in \mathcal{F}$. Let $w = \inf \mathcal{F}$. Then the following are true*:

(a) *$\widehat{w}$ is superharmonic*;
(b) *$\widehat{w} = w^*$ in Ω*;
(c) *$\widehat{w} = w$ q.e. in Ω.*

We will also use Choquet's topological lemma. We say that a family of functions $\mathcal{U}$ is *downward directed* if for each $u, v \in \mathcal{U}$ there is $w \in \mathcal{U}$ with $w \leq \min\{u, v\}$.

Lemma 3.7. (Choquet's topological lemma) *Let* $\mathcal{U} = \{u_\gamma : \gamma \in I\}$ *be a nonempty family of functions* $u_\gamma : \Omega \to \overline{\mathbf{R}}$. *Let* $u = \inf \mathcal{U}$. *If* $\mathcal{U}$ *is downward directed, then there is a decreasing sequence of functions* $v_j \in \mathcal{U}$ *with* $v = \lim_{j\to\infty} v_j$ *such that* $\hat{v} = \hat{u}$.

Proof. The proof of Lemma 8.3 in Heinonen–Kilpeläinen–Martio [24] generalizes directly to metric spaces. Just remember that our metric space X is separable. See also Björn–Björn [7]. □

One way of solving the Dirichlet problem for p-harmonic functions is by using the Perron method, which was studied in Björn–Björn–Shanmugalingam [10] in the metric space setting.

Definition 3.8. Assume that Ω is bounded. Let $f : \partial\Omega \to \overline{\mathbf{R}}$. Let $\mathcal{U}_f$ be the set of all superharmonic functions u on Ω bounded from below such that

$$\liminf_{\Omega \ni y \to x} u(y) \geq f(x) \quad \text{for all } x \in \partial\Omega.$$

The *upper Perron solution* of f is defined by

$$\overline{P} f(x) = \inf_{u \in \mathcal{U}_f} u(x), \quad x \in \Omega.$$

Similarly, we define $\mathcal{L}_f$ to be the set of all subharmonic functions u on Ω bounded from above such that

$$\limsup_{\Omega \ni y \to x} u(y) \leq f(x) \quad \text{for all } x \in \partial\Omega,$$

and the *lower Perron solution* of f is

$$\underline{P} f(x) = \sup_{u \in \mathcal{L}_f} u(x), \quad x \in \Omega.$$

If $\overline{P} f = \underline{P} f$, then we set $P_\Omega f = P f = \overline{P} f$, and f is said to be *resolutive*.

In Theorem 6.1 in Björn–Björn–Shanmugalingam [10], it is shown that if $f \in C(\Omega)$, then f is resolutive. Moreover, if $f \in N^{1,p}(X)$, then f is resolutive and $Pf = Hf$, by Theorem 5.1 in [10].

Definition 3.9. Assume that Ω is bounded. A point $x_0 \in \partial\Omega$ is *regular* if

$$\lim_{\Omega \ni y \to x_0} P f(y) = f(x_0) \quad \text{for all } f \in C(\partial\Omega).$$

The set Ω is *regular*, if all $x_0 \in \partial\Omega$ are regular. If $x_0 \in \partial\Omega$ is not regular, then it is *irregular*.

In Theorems 4.2 and 6.1 in Björn–Björn [5], regular boundary points were characterized in several ways by means of barriers and obstacle problems. We recall the characterizations in order to analyze the Poisson modification in the irregular points. Contrast to the Euclidean case, even the balls may not be regular in the metric spaces as pointed out below. In Section 7 we obtain some other characterizations in terms of balayage.

Theorem 3.10. *Assume that* Ω *is bounded. Let* $x_0 \in \partial\Omega$, $\delta > 0$ *and* $B = B(x_0, \delta)$. *Then the following are equivalent*:

(a) *The point x_0 is a regular boundary point.*
(b) *The point x_0 is regular with respect to $B \cap \Omega$.*
(c) *It is true that*

$$\lim_{\Omega \ni y \to x_0} \overline{P} f(y) = f(x_0)$$

for all bounded $f : \partial\Omega \to \mathbf{R}$ which are continuous at x_0.
(d) *It is true that*

$$\limsup_{\Omega \ni y \to x_0} \overline{P} f(y) \le f(x_0)$$

for all functions $f : \partial\Omega \to \mathbf{R}$ which are bounded from above on $\partial\Omega$ and upper semicontinuous at x_0.

Proof. (a) ⇒ (d) This is Proposition 7.1 in Björn–Björn–Shanmugalingam [10].
(d) ⇒ (c) This was shown in the proof of Corollary 7.2 in [10].
(a) ⇔ (c) This is part of Theorem 4.2 in Björn–Björn [5]
(a) ⇔ (b) This is part of Theorem 6.1 in [5] □

We will also use the Kellogg property which was obtained in Björn–Björn–Shanmugalingam [9], Theorem 3.9, to analyze the irregular points in a context of the Poisson modification.

Theorem 3.11. (The Kellogg property) *Assume that Ω is bounded. Then it is true that*

$$C_p(\{x \in \partial\Omega : x \text{ is irregular}\}) = 0.$$

We will use the following two pasting lemmas for superminimizers and superharmonic functions, respectively. Lemma 3.12 was proved for quasisuperminimizers in Björn–Martio [12].

Lemma 3.12. *Assume that $\Omega' \subset \Omega$ is open, and that u and u' are superminimizers in Ω and Ω', respectively. Let*

$$v = \begin{cases} \min\{u, u'\}, & \text{in } \Omega' \\ u, & \text{in } \Omega \setminus \Omega'. \end{cases}$$

If $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$, then v is a superminimizer in Ω.

Lemma 3.13. *Assume that $\Omega' \subset \Omega$ is open, and that u and u' are superharmonic in Ω and Ω', respectively. Let*

$$v = \begin{cases} \min\{u, u'\}, & \text{in } \Omega' \\ u, & \text{in } \Omega \setminus \Omega'. \end{cases}$$

If v is lower semicontinuous, then it is superharmonic in Ω.

Proof. As our definition of superharmonicity is equivalent to the one used in Heinonen–Kilpeläinen–Martio [24], the proof of Lemma 7.9 in [24] generalizes directly to metric spaces. □

Next we introduce the Poisson modification, which is used in the proofs of Theorem 5.8 and Corollary 6.2. In both cases, in view of Theorem 1.1 in Björn–Björn [6], we could have done the Poisson modifications with respect to regular sets. We have refrained from this and our proofs therefore do not depend on approximations by regular sets. (Actually, we do use this in the proof of Lemma 3.13, but in Björn–Björn [7] there is a proof without this ingredient.)

Note that in metric spaces, balls need not be regular, for a simple example see Example 3.1 in [6]. There even exist metric spaces satisfying our assumptions, in which no balls are regular. More precisely, Proposition 7 in Capogna–Garofalo [15] shows that the complement of the Carnot–Carathéodory ball $B(0,r)$ in the Heisenberg group $\mathbf{H}^n$, $n \ge 1$, near each of its poles $(0, \pm r^2)$ is contained in a Euclidean cone. Theorem 3.4 in Hansen–Hueber [21] then shows that if $n \ge 2$ and $p = 2$, then the Euclidean cone is thin at its vertex, i.e. its complement (and thus also the Carnot–Carathéodory ball $B(0,r)$) is not regular. Due to the group structure this means that in $\mathbf{H}^n$, $n \ge 2$, $p = 2$, there do not exist any regular balls, and in particular no base of regular balls. On the other hand, by Corollary 1.2 in Björn–Björn [6] there always exists a base of regular sets.

Proposition 3.14. (Poisson modification for superharmonic functions) *Assume that u is superharmonic in Ω and let $G \Subset \Omega$ be open. Let further*

$$v = \begin{cases} u, & \text{in } \Omega \setminus G, \\ \underline{P}_G u, & \text{in } G. \end{cases}$$

Then

$$v^*(x) = \begin{cases} u(x), & x \in \Omega \setminus \overline{G}, \\ \underline{P}_G u(x), & x \in G, \\ \min\Bigl\{ u(x), \liminf_{G \ni y \to x} \underline{P}_G u(y) \Bigr\}, & x \in \partial G. \end{cases} \tag{3.3}$$

Moreover, v^* is superharmonic in Ω and p-harmonic in G, and $v^* \le v \le u$ in Ω.

Let $E = \{x \in \partial G : x \text{ is irregular with respect to } G\}$. Then $v^* = v$ in $\Omega \setminus E$, in particular $v^* = v$ q.e. in Ω.

For u locally bounded from above, part of this result was given in Lemma 4.2 in Björn–Björn–Shanmugalingam [10]. See also Theorem 9.1 in Kinnunen–Martio [31].

Observe that in general, v is not lower semicontinuous, and hence not superharmonic. However, if G is regular, then $E = \varnothing$ and $v^* = v$.

Proof. As u is superharmonic, it is lower semicontinuous and does not take the value $-\infty$. Hence u is bounded from below on $\overline{G}$. Thus $u \in \mathcal{U}_u(G)$ and $u \ge \overline{P}_G u \ge \underline{P}_G u$ in G. Therefore $v \le u$ in Ω. As v is $\operatorname{ess\,lim\,inf}$-regularized in $\Omega \setminus \partial G$, it is easy to see that v^* is given by (3.3) and that $v^* \le v$ in Ω.

Next we want to show that v^* is superharmonic. Let $u_k = \min\{u, k\}$ and

$$v_k = \begin{cases} u_k, & \text{in } \Omega \setminus G, \\ \underline{P}_G u_k, & \text{in } G. \end{cases}$$

Then v_k^* is given by an expression similar to (3.3). Functions in $\mathcal{L}_u(G)$ are bounded from above, from which it follows that $\underline{P}_G u_k \to \underline{P}_G u$ in G, and thus $v_k \to v$ in Ω, as $k \to \infty$.

By Corollary 7.8 in Kinnunen–Martio [30], $u_k \in N^{1,p}_{\rm loc}(\Omega)$. Thus $\underline{P}_G u_k = H_G u_k$, by Theorem 5.1 in Björn–Björn–Shanmugalingam [10], from which it follows that $v_k \in N^{1,p}_{\rm loc}(\Omega)$. Since $v_k \le u_k$ (in the same way as $v \le u$) it follows from Lemma 3.12 that v_k is a superminimizer in Ω.

Thus v_k^* is superharmonic in Ω and $v_k = v_k^*$ q.e. in Ω. It follows that $v = \lim_{k\to\infty} v_k^*$ q.e. in Ω. By Lemma 7.1 in [30], $\lim_{k\to\infty} v_k^*$ is superharmonic in Ω and thus $\operatorname{ess\,lim\,inf}$-regularized. Hence, $v^* = \lim_{k\to\infty} v_k^*$ everywhere in Ω. That v^* is p-harmonic in G follows from Theorem 4.1 in [10].

As u is lower semicontinuous and bounded from below on $\overline{G}$, Theorem 3.10(d) applied to $-u$ shows that

$$\liminf_{G \ni y \to x} \underline{P}_G u(y) = u(x) \quad \text{for } x \in \partial G \setminus E.$$

Thus, $v^* = u = v$ in $\Omega \setminus E$ and, by the Kellogg property (Theorem 3.11), $v^* = v$ q.e. in Ω. □

4. Balayage

The balayage is roughly speaking the smallest superharmonic function lying above a given obstacle function. Before analyzing its connection to the obstacle problem, we develop the basic theory of balayage on metric spaces. In particular, we prove that sets of capacity zero can sometimes be neglected. This is useful in many applications of the theory.

Definition 4.1. Let

$$\begin{aligned} \Phi^\psi &= \Phi^\psi(\Omega) = \{u : u \text{ is superharmonic in } \Omega \text{ and } u \ge \psi \text{ in } \Omega\}, \\ \Psi^\psi &= \Psi^\psi(\Omega) = \{u : u \text{ is superharmonic in } \Omega \text{ and } u \ge \psi \text{ q.e. in } \Omega\}, \\ R^\psi &= R^\psi(\Omega) = \inf \Phi^\psi, \\ Q^\psi &= Q^\psi(\Omega) = \inf \Psi^\psi. \end{aligned}$$

The lim inf-regularizations $\widehat{R}^\psi$ and $\widehat{Q}^\psi$ are called the R- and Q-*balayages* of ψ in Ω, respectively. If $\Phi^\psi = \varnothing$, then we set $\widehat{R}^\psi = \infty$ and similarly for $\widehat{Q}^\psi$.

In this paper, we always assume that the obstacle function ψ in the definition of the balayage is bounded from below by an $N^{1,p}_{\mathrm{loc}}(\Omega)$-function.

Clearly, $\widehat{Q}^\psi \le \widehat{R}^\psi$. As superharmonic functions are lim inf-regularized it follows directly that $\widehat{R}^\psi \le R^\psi$ and $\widehat{Q}^\psi \le Q^\psi$. The two definitions of balayage are known to be equivalent in the linear theory, see Theorem 5.7.3(ii) in Armitage–Gardiner [1], but in the nonlinear case this is still an open problem, even in $\mathbf{R}^n$. A partial result was obtained in Heinonen–Kilpeläinen [23], which we here generalize to metric spaces in Theorem 4.10. See also Section 10 for comments on the linear case.

We start this section by deriving a number of rather basic conclusions about the balayage. We prove several different results on when the R- and Q-balayages coincide and end the section with some convergence results.

Definition 4.2. For $E \subset \Omega$ we define

$$\Phi^\psi_E = \Phi^{\psi\chi_E}, \quad R^\psi_E = R^{\psi\chi_E}, \quad \Psi^\psi_E = \Psi^{\psi\chi_E} \quad \text{and} \quad Q^\psi_E = Q^{\psi\chi_E},$$

where χ_E is the characteristic function of E.

Proposition 4.3. (a) *If $\psi_1 \le \psi_2$ and $\Omega_1 \subset \Omega_2$, then $\widehat{R}^{\psi_1}(\Omega_1) \le \widehat{R}^{\psi_2}(\Omega_2)$ and $\widehat{Q}^{\psi_1}(\Omega_1) \le \widehat{Q}^{\psi_2}(\Omega_2)$ in Ω_1.*
(b) *If $E \subset F \subset \Omega$ and $\psi \ge 0$, then $\widehat{R}^\psi_E \le \widehat{R}^\psi_F$ and $\widehat{Q}^\psi_E \le \widehat{Q}^\psi_F$.*
(c) *If $\lambda > 0$ and $\mu \in \mathbf{R}$, then $\widehat{R}^{\lambda\psi+\mu} = \lambda \widehat{R}^\psi + \mu$ and $\widehat{Q}^{\lambda\psi+\mu} = \lambda \widehat{Q}^\psi + \mu$.*

Proof. These are easy observations following directly from the definition. □

As a consequence of the fundamental convergence theorem we obtain the following result rather easily. Compare this to Section 8 in Heinonen–Kilpeläinen–Martio [24], where the theory of the R-balayage is developed in a different order. Observe also that the Q-balayage gives the right representative even without the regularization as shown below.

Theorem 4.4. *It is true that*

$$\widehat{R}^\psi = (R^\psi)^* \text{ in } \Omega \quad \text{and} \quad \widehat{R}^\psi = R^\psi \ge \psi \text{ q.e. in } \Omega. \tag{4.1}$$

Moreover, if $\Phi^\psi \neq \varnothing$*, then the balayage* $\widehat{R}^\psi$ *is superharmonic.*

Similarly,

$$\widehat{Q}^\psi = (Q^\psi)^* = Q^\psi \ \textit{in}\ \Omega \quad \textit{and} \quad \widehat{Q}^\psi \geq \psi \ \textit{q.e. in}\ \Omega, \tag{4.2}$$

and if $\Psi^\psi \neq \varnothing$*, then* $\widehat{Q}^\psi$ *is superharmonic.*

Proof. The identities (4.1) and (4.2) are trivial in the cases when $\Phi^\psi = \varnothing$ and $\Psi^\psi = \varnothing$, respectively. We thus assume that $\Phi^\psi \neq \varnothing$ and $\Psi^\psi \neq \varnothing$, respectively, in the rest of the proof.

The result now follows directly from Theorem 3.6 with one exception. Trivially $R^\psi \geq \psi$ everywhere in Ω, and thus, by Theorem 3.6, $\widehat{R}^\psi = R^\psi \geq \psi$ q.e. in Ω.

For the Q-balayage the corresponding inequality is a little more subtle. By Choquet's topological lemma (Lemma 3.7) there is a decreasing sequence of superharmonic functions $v_j \in \Psi^\psi$ with $v = \lim_{j\to\infty} v_j$ such that $\hat{v} = \widehat{Q}^\psi$. As $v_j \geq \psi$ q.e., it follows that $v \geq \psi$ q.e. in Ω. By Theorem 3.6, we have that $\widehat{Q}^\psi = \hat{v} = v \geq \psi$ q.e. in Ω. As $\widehat{Q}^\psi$ is superharmonic, we have $\widehat{Q}^\psi \in \Psi^\psi$ and hence $\widehat{Q}^\psi \geq Q^\psi$ everywhere in Ω. The converse inequality is trivial. □

The following corollary immediately follows from the previous theorem by using the ess lim inf-regularization.

Corollary 4.5. *It is true that*

$$\widehat{R}^\psi \geq \widehat{Q}^\psi \geq \psi^* \quad \textit{in}\ \Omega.$$

We say that ψ is *essentially lower semicontinuous* if $\psi^* \geq \psi$.

Proposition 4.6. *If* ψ *is essentially lower semicontinuous, in particular if* ψ *is lower semicontinuous, then*

$$\widehat{R}^\psi = \widehat{Q}^\psi \geq \psi^* \geq \psi \quad \textit{in}\ \Omega.$$

Proof. In view of Corollary 4.5 we only need to show that $\widehat{R}^\psi \leq \widehat{Q}^\psi$. But as $\widehat{Q}^\psi \geq \psi$ we have that $\widehat{Q}^\psi \in \Phi^\psi$, and hence $\widehat{R}^\psi \leq R^\psi \leq \widehat{Q}^\psi$. □

This in particular shows that if $\psi \geq 0$ is superharmonic in Ω and $E \subset \Omega$ (which is the situation often considered in linear balayage) then $\widehat{R}^\psi_E = R^\psi_E = \widehat{Q}^\psi_E$, provided that E is open, cf. Theorem 5.3.4(v) in Armitage–Gardiner [1].

Proposition 4.7. *If* ψ *is superharmonic, then*

$$\widehat{R}^\psi = \widehat{Q}^\psi = \psi \quad \textit{in}\ \Omega.$$

Proof. We have that $\widehat{R}^\psi = \widehat{Q}^\psi \geq \psi$ by Proposition 4.6. On the other hand, as $\psi \in \Phi^\psi$ we have that $\widehat{R}^\psi \leq \psi$. □

Proposition 4.8. *If* $\psi \geq 0$ *is superharmonic and* $E \subset \Omega$*, then*

$$\widehat{R}^\psi_E = \widehat{Q}^\psi_E = \psi \quad \textit{q.e. in } E \textit{ and everywhere in } \operatorname{int} E.$$

In particular,

$$\widehat{R}^1_E = \widehat{Q}^1_E = 1 \quad \textit{q.e. in}\ E.$$

Proof. As $\psi \in \Psi^\psi_E$ we have that $\widehat{Q}^\psi_E \leq \widehat{R}^\psi_E \leq \hat{\psi} = \psi$. On the other hand, by Theorem 4.4, $\psi = \psi\chi_E \leq \widehat{Q}^\psi_E$ q.e. in E. Thus $\widehat{Q}^\psi_E = \psi$ q.e. in E.

As $\widehat{Q}^\psi_E$ and ψ are both ess lim inf-regularized they must coincide in $\operatorname{int} E$. □

Proposition 4.9. *If $\widehat{R}^\psi \in N^{1,p}_{\rm loc}(\Omega)$, in particular if ψ is bounded, then $\widehat{R}^\psi$ is an* $\operatorname{ess\,lim\,inf}$*-regularized superminimizer in Ω.*

Similarly, if $\widehat{Q}^\psi \in N^{1,p}_{\rm loc}(\Omega)$, in particular if ψ is bounded, then $\widehat{Q}^\psi$ is an $\operatorname{ess\,lim\,inf}$*-regularized superminimizer in Ω.*

Proof. The proofs for $\widehat{R}^\psi$ and $\widehat{Q}^\psi$ are similar, we give the proof for $\widehat{Q}^\psi$. By Theorem 4.4, $\widehat{Q}^\psi$ is superharmonic, and hence $\operatorname{ess\,lim\,inf}$-regularized. If ψ is bounded, then so is $\widehat{Q}^\psi$, and as a bounded superharmonic function, $\widehat{Q}^\psi$ is a superminimizer in $N^{1,p}_{\rm loc}(\Omega)$, by Corollary 7.8 in Kinnunen–Martio [30]. If $\widehat{Q}^\psi$ merely belongs to $N^{1,p}_{\rm loc}(\Omega)$ we instead use Corollary 7.9 in [30] to deduce that $\widehat{Q}^\psi$ is a superminimizer. $\square$

Next we show that if $\widehat{Q}^\psi \in N^{1,p}(\Omega)$, then the R- and Q-balayages coincide. The proof is similar to the corresponding proof for unweighted $\mathbf{R}^n$ in Heinonen–Kilpeläinen [23], Lemma 2.1. For the reader's convenience we include a proof here with the necessary references to the metric space literature. Later in Corollary 5.4 and Corollary 6.3, we provide conditions in terms of the obstacle ψ.

The idea in the proof is to add a correction term to $\widehat{Q}^\psi$ so that the resulting function lies above the obstacle also in the exceptional set $\{x \in \Omega : \widehat{Q}^\psi(x) < \psi(x)\}$. Then we use the corresponding solutions of the obstacle problem and the fact that the exceptional set is of capacity zero to show that $\widehat{R}^\psi \le \widehat{Q}^\psi$. This suffices to prove the claim since the converse inequality follows by definition.

Theorem 4.10. *Assume that Ω is bounded. If $\widehat{Q}^\psi \in N^{1,p}(\Omega)$, then $\widehat{Q}^\psi = \widehat{R}^\psi$.*

Proof. Let $E = \{x \in \Omega : \widehat{Q}^\psi(x) < \psi(x)\}$. Theorem 4.4, implies that $C_p(E) = 0$. By Corollary 1.3 in Björn–Björn–Shanmugalingam [11], C_p is an outer capacity, i.e. there exists, for $j = 1, 2, ...,$ an open set $G_j \supset E$ with $C_p(G_j) < 2^{-jp}$ and thus a nonnegative $\varphi_j \in N^{1,p}(X)$ such that $\|\varphi_j\|_{N^{1,p}(X)} < 2^{-j}$ and $\varphi_j \ge \chi_{G_j}$.

Let $\varphi = \sum_{j=1}^\infty \varphi_j$ and let w be the $\operatorname{ess\,lim\,inf}$-regularized solution of the $\mathcal{K}_{\varphi,\varphi}$-obstacle problem. Then $0 \le w \in N^{1,p}(\Omega)$ is a lower semicontinuous function and $w = \infty$ in E. Let

$$\psi_j = \widehat{Q}^\psi + \frac{w}{j}$$

and let v_j be the $\operatorname{ess\,lim\,inf}$-regularized solution of the $\mathcal{K}_{\psi_j,\psi_j}$-obstacle problem. We have, as ψ_j is lower semicontinuous, that

$$v_j(x) = \operatorname*{ess\,lim\,inf}_{y\to x} v_j(y) \ge \operatorname*{ess\,lim\,inf}_{y\to x} \psi_j(y) \ge \psi_j(x) \ge \psi(x) \quad \text{for all } x \in \Omega.$$

Now

$$v := \lim_{j\to\infty} v_j \ge R^\psi \ge \widehat{R}^\psi.$$

On the other hand, $\psi_j \to \widehat{Q}^\psi$ in $N^{1,p}(\Omega)$, and $\widehat{Q}^\psi$ is a solution of the $\mathcal{K}_{\widehat{Q}^\psi,\widehat{Q}^\psi}$-obstacle problem since it is superharmonic. It follows from Proposition 3.2 in Kinnunen–Shanmugalingam [33], which also appears as Proposition 5.5 in Björn–Björn–Shanmugalingam [10], that $v = \widehat{Q}^\psi$ q.e. in Ω. Hence

$$\widehat{Q}^\psi(x) = \operatorname*{ess\,lim\,inf}_{y\to x} v(y) \ge \operatorname*{ess\,lim\,inf}_{y\to x} \widehat{R}^\psi(y) = \widehat{R}^\psi(x) \quad \text{for all } x \in \Omega. \qquad \square$$

For the Q-balayage, we have the following convergence result for increasing sequences. It is not known if the corresponding result for the R-balayage holds. In fact, if the corresponding result would hold for the R-balayage, then it would follow that the R- and Q-balayages coincide for all functions.

Proposition 4.11. *Let $\Omega_1 \subset \Omega_2 \subset ... \subset \Omega = \bigcup_{j=1}^{\infty} \Omega_j$ be open sets and $\psi_j : \Omega_j \to (-\infty, \infty]$ be a sequence of functions such that for each j there exists a function $f_j \in N^{1,p}_{\mathrm{loc}}(\Omega_j)$ so that $\psi_{j+1} \geq \psi_j \geq f_j$ in Ω_j. Let further $\psi = \lim_{j\to\infty} \psi_j$ and assume that $\Psi^\psi \neq \varnothing$. Then*

$$\lim_{j\to\infty} \widehat{Q}^{\psi_j}(\Omega_j) = \widehat{Q}^{\psi}(\Omega).$$

Proof. By Proposition 4.3, we have that $\{\widehat{Q}^{\psi_k}(\Omega_k)\}_{k=j}^{\infty}$ is a nondecreasing sequence of superharmonic functions in Ω_j, $j = 1, 2, \ldots$. Let $v = \lim_{j\to\infty} \widehat{Q}^{\psi_j}(\Omega_j)$. Clearly, $\widehat{Q}^{\psi_j}(\Omega_j) \leq \widehat{Q}^{\psi}$ for all j, and thus the inequality $v \leq \widehat{Q}^{\psi}$ is true. Moreover $\widehat{Q}^{\psi}$ is superharmonic and hence not identically ∞ in any component of Ω. By Lemma 7.1 in Kinnunen–Martio [30], v is superharmonic in Ω_j for every j and thus in Ω. On the other hand, $\widehat{Q}^{\psi_j}(\Omega_j) \geq \psi_j$ q.e. in Ω_j for all j. It follows that $v \geq \psi$ q.e. in Ω, and thus $v \geq \widehat{Q}^{\psi}$ in Ω. □

Proposition 4.12. *Assume that $\psi_j \to \psi$ uniformly in Ω. Then $\widehat{Q}^{\psi_j} \to \widehat{Q}^{\psi}$ and $\widehat{R}^{\psi_j} \to \widehat{R}^{\psi}$ uniformly in Ω.*

Proof. This follows directly from the fact that if $\psi - \varepsilon \leq \psi_j \leq \psi + \varepsilon$, then $\widehat{Q}^{\psi} - \varepsilon \leq \widehat{Q}^{\psi_j} \leq \widehat{Q}^{\psi} + \varepsilon$ and $\widehat{R}^{\psi} - \varepsilon \leq \widehat{R}^{\psi_j} \leq \widehat{R}^{\psi} + \varepsilon$. □

This result can be applied using the following observation, which follows directly from Proposition 4.12.

Proposition 4.13. *Assume that $\psi_j \to \psi$ uniformly in Ω, and that $\widehat{Q}^{\psi_j} = \widehat{R}^{\psi_j}$ for all j. Then $\widehat{Q}^{\psi} = \widehat{R}^{\psi}$.*

Remark 4.14. A function $u : \Omega \to (-\infty, \infty]$ is *hyperharmonic* if it is lower semicontinuous and satisfies (iii) of Definition 3.3. It follows that a function is hyperharmonic if and only if in every component it is either superharmonic or identically ∞.

In the definition of the balayage we could have used hyperharmonic functions lying above ψ (everywhere or q.e.). The difference would, of course, only be in how the balayage is defined for ψ such that $\Phi^\psi = \varnothing$ or $\Psi^\psi = \varnothing$ (still using superharmonic functions in the definitions of Φ^ψ and Ψ^ψ). To be more precise, it would be possible for $\widehat{R}^{\psi}$ (and $\widehat{Q}^{\psi}$) to be identically ∞ in some component while still superharmonic in some other component.

Using hyperharmonic functions in the definition we would directly find that if G is a component of Ω, then

$$\widehat{R}^{\psi}(\Omega)|_G = \widehat{R}^{\psi}(G) \quad \text{and} \quad \widehat{Q}^{\psi}(\Omega)|_G = \widehat{Q}^{\psi}(G),$$

without any condition on ψ (at present we need to require that $\Phi^\psi \neq \varnothing$ and $\Psi^\psi \neq \varnothing$, respectively). Also Proposition 4.11 would hold without assuming that $\Psi^\psi \neq \varnothing$.

5. Balayage, obstacle problem and continuity

In this section, we show that the balayage has a connection to the obstacle problem. This fact has many immediate consequences. Indeed, we apply the balayage in calculating capacities and consider the continuity of the balayage. We also derive a condition for the p-harmonicity of the solution of the obstacle problem. This result provides a starting point for the analysis of the p-harmonicity of the balayage in the next section.

In view of the next lemma, the solution of the obstacle problem can be approximated by solutions of obstacle problems in smaller sets.

Lemma 5.1. *Assume that Ω is bounded. Let $\Omega_1 \subset \Omega_2 \subset ... \subset \Omega = \bigcup_{j=1}^{\infty} \Omega_j$ be open sets. Let also $\psi : \Omega \to [-\infty, \infty]$ and $f \in N^{1,p}(\Omega)$ be such that $f \geq \psi$ q.e. in Ω and there exists $U \Subset \Omega$ so that $\psi = f$ q.e. in $\Omega \setminus U$. Let v_j be the* ess lim inf*-regularized solution of the $\mathcal{K}_{\psi,f}(\Omega_j)$-obstacle problem, $j = 1, 2, ...$. Then $v = \lim_{j\to\infty} v_j$ is the* ess lim inf*-regularized solution of the $K_{\psi,f}(\Omega)$-obstacle problem.*

Proof. We can assume that $U \subset \Omega_1$ and extend each v_j by f to Ω. Since v_{j+1} is an ess lim inf-regularized solution of the $\mathcal{K}_{v_{j+1},v_{j+1}}(\Omega_j)$-obstacle problem, and $v_{j+1} \geq \psi$ q.e. in Ω and $v_{j+1} \geq f = v_j$ q.e. in $\Omega \setminus \Omega_j$, the comparison Lemma 3.4 implies that $v_{j+1} \geq v_j$ for all $j = 1, 2, ...$. Hence $v = \lim_{j\to\infty} v_j$ exists everywhere in Ω and again by the comparison lemma $v \leq u$, where u is the ess lim inf-regularized solution of the $\mathcal{K}_{\psi,f}$-obstacle problem. As v is an increasing limit of superharmonic functions in each Ω_j, it is itself superharmonic in each Ω_j, and hence in Ω, by Theorem 7.1 in Kinnunen–Martio [30]. In particular v is ess lim inf-regularized.

As $v_1 \in \mathcal{K}_{\psi,f}(\Omega_j)$ for all $j = 1, 2, ...$, we have that

$$\int_\Omega g_{v_j}^p \, d\mu \leq \int_\Omega g_{v_1}^p \, d\mu,$$

i.e. the sequence v_j is bounded in $N^{1,p}(\Omega)$ and Lemma 2.4 shows that $v \in N^{1,p}(\Omega)$. Hence v is a solution of the $\mathcal{K}_{v,v}$-obstacle problem. As $v \geq \psi$ q.e. in Ω and $v \geq \psi = f$ q.e. in $\Omega \setminus U$, the comparison Lemma 3.4 shows that $v \geq u$, and thus $v = u$. □

Definition 5.2. The variational capacity of $E \Subset \Omega$ with respect to Ω is

$$\operatorname{cap}_p(E, \Omega) = \inf_u \int_\Omega g_u^p \, d\mu,$$

where the infimum is taken over all $u \in N_0^{1,p}(\Omega)$ such that $u \geq 1$ on E.

Under our assumptions the two capacities (cap_p and C_p) are more or less equivalent, see J. Björn [13], Lemma 3.3. In particular they have the same sets of zero capacity.

Next we show that the balayage coincides with the solution of the obstacle problem. Locally, this is a straightforward consequence of the comparison principle: Clearly, the Q-balayage is the smallest superharmonic function *q.e.* above the obstacle and, on the other hand, the comparison lemma implies the converse inequality. This is the content of Proposition 5.6.

Globally, we only know that the balayage is in $N_{\mathrm{loc}}^{1,p}(\Omega)$ in the setting of Theorem 5.3. Therefore, in order to use the comparison lemma, we approximate the domain from inside. Theorem 5.3 immediately shows that the balayage is a capacitary function.

Theorem 5.3. *Assume that Ω is bounded. Assume further that there exist $f \in N^{1,p}(\Omega)$ and $U \Subset \Omega$ such that $f \geq \psi$ q.e. in Ω and $\psi = f$ q.e. in $\Omega \setminus U$. Then $\widehat{Q}^\psi$ is the* ess lim inf*-regularized solution of the $\mathcal{K}_{\psi,f}$-obstacle problem. Moreover $\widehat{Q}^\psi = \widehat{R}^\psi$.*

In particular, if $E \Subset \Omega$ then

$$\operatorname{cap}_p(E, \Omega) = \int_\Omega g_w^p \, d\mu,$$

where $w = \widehat{Q}_E^1 = \widehat{R}_E^1$.

Corollary 5.4. *Assume that Ω is bounded. If $\psi \in N^{1,p}(\Omega)$, then $\widehat{Q}^\psi = \widehat{R}^\psi$.*

Corollary 5.5. *Assume that Ω is bounded and that $E \Subset \Omega$. If there is a function $f \in N^{1,p}_{\rm loc}(\Omega)$ such that $\psi \le f$ in E, in particular if ψ is bounded in E, then $\widehat{Q}^\psi_E = \widehat{R}^\psi_E$.*

Proof of Theorem 5.3. Choose open sets $\Omega_1 \subset \Omega_2 \subset ... \Subset \Omega = \bigcup_{j=1}^\infty \Omega_j$ such that $U \subset \Omega_1$. Let v_j be the $\operatorname{ess\,lim\,inf}$-regularized solution of the $\mathcal{K}_{\psi,f}(\Omega_j)$-obstacle problem, $j = 1, 2, ...$, extended by f to the whole of Ω. Then $v_j \to v$ by Lemma 5.1, where v is the $\operatorname{ess\,lim\,inf}$-regularized solution of the $\mathcal{K}_{\psi,f}$-obstacle problem.

Let $u = \widehat{Q}^\psi$. As v is superharmonic in Ω and $v \ge \psi$ q.e. in Ω, we have that $u \le v$. Since u is superharmonic, it follows from Proposition 3.5 that u is a superminimizer, and hence $u \in N^{1,p}_{\rm loc}(\Omega)$.

As u is the $\operatorname{ess\,lim\,inf}$-regularized solution of the $\mathcal{K}_{u,u}(\Omega_j)$-obstacle problem, and $u \ge \psi = f$ q.e. in $\Omega_j \setminus U$ and $u \ge \psi$ q.e. in Ω_j, the comparison Lemma 3.4 shows that $u \ge v_j$ in Ω_j. Letting $j \to \infty$ shows that $u \ge v$ in Ω, and thus $u = v \in N^{1,p}(\Omega)$.

It follows from Theorem 4.10 that $\widehat{Q}^\psi = \widehat{R}^\psi$. □

Proposition 5.6. *Assume that $V \subset \Omega$ is open and bounded and that $\widehat{Q}^\psi \in N^{1,p}(V)$. Then $\widehat{Q}^\psi$ is the* $\operatorname{ess\,lim\,inf}$*-regularized solution of the $\mathcal{K}_{\psi,\widehat{Q}^\psi}(V)$-obstacle problem.*

Proof. Fix k for the moment and let $\psi_k = \min\{\psi, k\}$. Now $u_k := \widehat{Q}^{\psi_k}$ belongs to $N^{1,p}_{\rm loc}(\Omega)$ and by Proposition 4.9 it is an $\operatorname{ess\,lim\,inf}$-regularized superminimizer in Ω. Similarly, by Corollary 7.9 in Kinnunen–Martio [30], $u := \widehat{Q}^\psi$ is an $\operatorname{ess\,lim\,inf}$-regularized superminimizer in V. Let v_k and v be the $\operatorname{ess\,lim\,inf}$-regularized solutions of the $\mathcal{K}_{\psi_k,u_k}(V)$- and $\mathcal{K}_{\psi,u}(V)$-obstacle problems, respectively. As u_k and u are the $\operatorname{ess\,lim\,inf}$-regularized solutions of the $\mathcal{K}_{u_k,u_k}(V)$- and $\mathcal{K}_{u,u}(V)$-obstacle problems, respectively, the comparison Lemma 3.4 shows that $v_k \le u_k$ and $v_k \le v \le u$ in V. Then

$$w = \begin{cases} u_k, & \text{in } \Omega \setminus V, \\ v_k, & \text{in } V, \end{cases}$$

is a superminimizer in Ω, by Lemma 3.12 (that $w \in N^{1,p}_{\rm loc}(\Omega)$ follows from the fact that $u_k - v_k \in N^{1,p}_0(V)$). As $w^* \ge \psi_k$ q.e. in Ω, we have that $w^* \in \Psi^{\psi_k}$, and hence $w^* \ge u_k$. In particular, in V we obtain that $u_k \le w^* = v_k \le u_k$.

We conclude, by Proposition 4.11, that

$$v \ge \lim_{k\to\infty} v_k = \lim_{k\to\infty} u_k = u,$$

and thus that $v = u$. □

The following result about obstacle problems is a generalization of Theorem 5.5 in Kinnunen–Martio [30] and will be used later.

Theorem 5.7. *Assume that Ω is bounded and that $\mathcal{K}_{\psi,f} \neq \varnothing$. Let u be the* $\operatorname{ess\,lim\,inf}$*-regularized solution of the $\mathcal{K}_{\psi,f}$-obstacle problem. Then u is continuous at all points in*

$$E := \{x \in \Omega : u(x) \ge \psi(x) \neq \infty \text{ and } \psi \text{ is upper semicontinuous at } x\}.$$

Moreover, u is p-harmonic in

$$G := \bigcup_{j=1}^\infty \operatorname{int}\{x \in \Omega : u(x) > \psi(x) + 1/j\}.$$

In particular, u is p-harmonic (and continuous) in

$$G' := \operatorname{int}\{x \in \Omega : u(x) > \psi(x) \text{ and } \psi \text{ is upper semicontinuous at } x\}.$$

Proof. The proof of Theorem 5.5 in [30] shows that u is continuous at points in E.

Let $\varphi \in \operatorname{Lip}_c(G)$. By compactness, there is $j > 0$ such that $u > \psi + 1/j$ in $\operatorname{supp} \varphi$. Therefore there is $0 < t < 1$ such that

$$w := (1-t)u + t(u+\varphi) = u + t\varphi \geq \psi \quad \text{in } \Omega,$$

and thus $w \in \mathcal{K}_{\psi,f}(\Omega)$. Hence, using convexity,

$$\begin{aligned}\int_{\varphi\neq 0} g_u^p \, d\mu \leq \int_{\varphi\neq 0} g_w^p \, d\mu &\leq \int_{\varphi\neq 0} ((1-t)g_u + t g_{u+\varphi})^p \, d\mu \\ &\leq (1-t)\int_{\varphi\neq 0} g_u^p \, d\mu + t\int_{\varphi\neq 0} g_{u+\varphi}^p \, d\mu.\end{aligned}$$

Subtracting the first term in the right-hand side and dividing by t shows that

$$\int_{\varphi\neq 0} g_u^p \, d\mu \leq \int_{\varphi\neq 0} g_{u+\varphi}^p \, d\mu.$$

Thus, u is an $\operatorname{ess\,lim\,inf}$-regularized minimizer, i.e. p-harmonic, in G.

Finally, if $y \in G'$ then there exists a neighbourhood V of y such that $u > \psi$ and ψ is upper semicontinuous in V. We have shown that u is continuous in V and it follows that $u > \psi + 1/j$ in V' for some neighbourhood $V' \subset V$ of x and some nonnegative j. Hence $V' \subset G$, which concludes the proof. □

As a corollary to Proposition 5.6 and Theorem 5.7, we prove a counterpart of Theorem 5.7 for the balayage in Theorem 6.5 below.

Next we consider the continuity of the balayage. The upper semicontinuity of the obstacle function turns out to be the essential condition here.

Theorem 5.8. *Assume that $\Psi^\psi \neq \varnothing$. Then $\widehat{Q}^\psi$ is continuous at all points in*

$$E := \{x \in \Omega : \widehat{Q}^\psi(x) \geq \psi(x) \neq \infty \text{ and } \psi \text{ is upper semicontinuous at } x\}.$$

We state the obvious consequence of this result for continuous functions in Corollary 6.9.

Proof. Let $x_0 \in E$ and $\varepsilon > 0$. By the lower and upper semicontinuity of $\widehat{Q}^\psi$ and ψ, respectively, and the condition $\widehat{Q}^\psi(x_0) \geq \psi(x_0) \neq \infty$, it follows that there is a ball $B = B(x_0, r) \Subset \Omega$ such that

$$\widehat{Q}^\psi + 2\varepsilon \geq \psi(x_0) + \varepsilon \geq \psi \quad \text{on } \overline{B}.$$

Let $u = \widehat{Q}^\psi + 2\varepsilon$ and

$$v(x) = \begin{cases} u(x), & x \in \Omega \setminus \overline{B}, \\ \underline{P}_B u(x), & x \in B, \\ \min\Bigl\{u(x), \liminf\limits_{B \ni y \to x} \underline{P}_B u(y)\Bigr\}, & x \in \partial B, \end{cases}$$

be the Poisson modification of u. By Proposition 3.14, $v \leq u$ is superharmonic in Ω and p-harmonic in B. Also, as $\psi(x_0) + \varepsilon \in \mathcal{L}_u(B)$, we have that

$$v \geq \psi(x_0) + \varepsilon \geq \psi \quad \text{on } \overline{B}.$$

On the other hand $v = \widehat{Q}^\psi + 2\varepsilon \geq \psi$ q.e. in $\Omega \setminus \overline{B}$. Thus $v \in \Psi^\psi$ and $\widehat{Q}^\psi \leq v$ in Ω. Therefore

$$\limsup_{x\to x_0} \widehat{Q}^\psi(x) \leq \lim_{x\to x_0} v(x) = v(x_0) \leq \widehat{Q}^\psi(x_0) + 2\varepsilon.$$

Letting $\varepsilon \to 0$ shows that $\widehat{Q}^\psi$ is upper semicontinuous at x_0, and as $\widehat{Q}^\psi$ is lower semicontinuous (everywhere) in Ω the continuity of $\widehat{Q}^\psi$ at x_0 follows. □

6. p-harmonicity of balayage

In this section, we consider the p-harmonicity of the balayage. Generally speaking, the balayage is p-harmonic if the obstacle is subharmonic or the balayage is strictly above the obstacle. Furthermore, as a corollary, we obtain another condition for the equivalence of the R- and Q-balayages in Corollary 6.3. The main tools are the Poisson modification and the connection to the obstacle problem from the previous section.

Theorem 6.1. *Assume that ψ is subharmonic in some open set $U \subset \Omega$. If $\Phi^\psi \neq \varnothing$, then $\widehat{R}^\psi$ is p-harmonic in U and $R^\psi = \widehat{R}^\psi \geq \psi$ in U.*

Similarly, if $\Psi^\psi \neq \varnothing$, then $\widehat{Q}^\psi$ is p-harmonic in U and $Q^\psi = \widehat{Q}^\psi \geq \psi$ in U.

Corollary 6.2. *If $E \subsetneq \Omega$ is relatively closed then both $\widehat{R}^\psi_E$ and $\widehat{Q}^\psi_E$ are p-harmonic in $\Omega \setminus E$, provided that they are not identically ∞.*

Proof of Theorem 6.1. We prove the result for the Q-balayage $\widehat{Q}^\psi$. The proof for $\widehat{R}^\psi$ is similar. By Choquet's topological lemma, there exists a decreasing sequence of functions $v_i \in \Psi^\psi$ such that $v = \lim_{i\to\infty} v_i$ and

$$\hat{v} = \widehat{Q}^\psi \quad \text{in } \Omega. \tag{6.1}$$

Let $V \Subset U$ be open, and let s_i be the Poisson modification of v_i in V given by

$$s_i(x) = \begin{cases} v_i(x), & \text{if } x \in \Omega \setminus \overline{V}, \\ \underline{P}_V v_i(x), & \text{if } x \in V, \\ \min\Big\{v_i(x), \liminf\limits_{V \ni y \to x} \underline{P}_V v_i(y)\Big\}, & \text{if } x \in \partial V. \end{cases}$$

By Proposition 3.14, s_i is p-harmonic in V and superharmonic in Ω. By the comparison principle we have that $s_{i+1} \leq s_i \leq v_i$.

Let W be open and such that $V \Subset W \Subset U$. Let $m = \sup_{\overline{W}} \psi$ which is finite as ψ is upper semicontinuous and does not take the value ∞ in U. Let $v_i' = \min\{v_i, m\}$ and $s_i' = \min\{s_i, m\}$. Since s_i' is the $\operatorname{ess\,lim\,inf}$-regularized solution of the $\mathcal{K}_{s_i',v_i'}(W)$-obstacle problem and $H_W v_i'$, by definition, is the $\operatorname{ess\,lim\,inf}$-regularized solution of the $\mathcal{K}_{-\infty,v_i'}(W)$-obstacle problem, the comparison Lemma 3.4 shows that $H_W v_i' \leq s_i'$ in W. Similarly $\psi \leq H_W v_i'$ in W (apply Lemma 3.4 to $-\psi$ and $-v_i'$). Therefore $\psi \leq s_i' \leq s_i$ in W and q.e. in Ω. Thus

$$Q^\psi \leq s := \lim_{i\to\infty} s_i \leq v$$

and hence by (6.1), it follows that $\widehat{Q}^\psi = \hat{s}$. As $\widehat{Q}^\psi$ is superharmonic in Ω it is bounded from below on V, and thus by Harnack's convergence theorem (Proposition 5.1 in Shanmugalingam [39]) s is p-harmonic in V. It follows that $s = \hat{s}$ in V and consequently

$$\psi \leq s = \hat{s} = \widehat{Q}^\psi \leq Q^\psi \leq s \quad \text{in } V,$$

i.e. $\widehat{Q}^\psi \geq \psi$ is p-harmonic in V, and as V was arbitrary also in U. □

Corollary 6.3. *Assume that $\Psi^\psi \neq \varnothing$, that ψ is subharmonic in some open $U \subset \Omega$, and that ψ is essentially lower semicontinuous in $\Omega \setminus U$. Then $\widehat{Q}^\psi = Q^\psi = \widehat{R}^\psi = R^\psi \geq \psi$ in Ω.*

Proof. By Theorem 6.1, $\widehat{Q}^\psi \geq \psi$ in U. On the other hand, by Corollary 4.5, $\widehat{Q}^\psi \geq \psi^* \geq \psi$ in $\Omega \setminus U$. Hence $\widehat{Q}^\psi \in \Phi^\psi$ and $\widehat{R}^\psi \leq R^\psi \leq \widehat{Q}^\psi \leq \widehat{R}^\psi$. The inequalities $\widehat{Q}^\psi \leq Q^\psi \leq R^\psi$ complete the proof. □

The sheaf property is open for p-harmonic functions on metric spaces (with our assumptions), i.e. if u is p-harmonic in two open sets U and V it is not known if u is p-harmonic in $U \cup V$. (For Cheeger p-harmonic functions this is known.) The sheaf property is also open for sub- and superharmonic functions. For this reason it may be worth pointing out the following slight generalization of Corollary 6.3.

Corollary 6.4. *Let U_j, $j = 1, 2, \ldots$, be open sets. Assume that $\Psi^\psi \neq \varnothing$, that ψ is subharmonic in U_j for each j, and that ψ is essentially lower semicontinuous in $\Omega \setminus \bigcup_{j=1}^\infty U_j$. Then $\widehat{Q}^\psi = Q^\psi = \widehat{R}^\psi = R^\psi \geq \psi$ in Ω.*

The proof is similar to the proof of Corollary 6.3.

Theorem 6.5. *Assume that $\widehat{Q}^\psi \in N^{1,p}_{\mathrm{loc}}(\Omega)$. Then $\widehat{Q}^\psi$ is p-harmonic in*

$$G := \bigcup_{j=1}^\infty \operatorname{int}\{x \in \Omega : \widehat{Q}^\psi(x) > \psi(x) + 1/j\}.$$

Proof. Let $V \Subset G$. By Proposition 5.6, $\widehat{Q}^\psi$ is the ess lim inf-regularized solution of the $\mathcal{K}_{\psi,\widehat{Q}^\psi}(V)$-obstacle problem. Theorem 5.7 then yields that $\widehat{Q}^\psi$ is p-harmonic in V, and hence in G. □

The following example shows that we cannot replace G by $\operatorname{int}\{x \in \Omega : \widehat{Q}^\psi(x) > \psi(x)\}$ above. Nevertheless, for upper semicontinuous obstacles, and in particular for continuous obstacles, this is possible, as we later show in Theorem 6.8 and Corollary 6.9.

Example 6.6. Let $\Omega = (0, 1) \subset X = \mathbf{R}$, $1 < p < \infty$, and let $f(x) = 1 - \bigl(x - \frac{1}{2}\bigr)^2$. Observe that the p-harmonic functions on Ω are functions of the form $x \mapsto ax + b$, and that a function is superharmonic if and only if it is concave (the situation is the same for all p).

We enumerate the dyadic rational numbers as $x_{j,k} = (2j + 1 - 2^k)/2^k$ and let $\psi(x_{j,k}) = f(x_{j,k}) - 2^{-k}$ for $2^{k-1} \leq j < 2^k$, $k = 1, 2, \ldots$. Let further $\psi(x) = -\infty$ for all other x. It is now easy to see that the least concave function on Ω lying above ψ is f. Hence $R^\psi = \widehat{R}^\psi = Q^\psi = \widehat{Q}^\psi = f$. However $f > \psi$ in Ω and f is not p-harmonic in Ω.

Next we deduce a generalization of Theorem 6.5. It turns out that if the set in which the obstacle is near the balayage is of capacity zero, then it can be neglected.

Theorem 6.7. *Assume that $\widehat{Q}^\psi \in N^{1,p}_{\mathrm{loc}}(\Omega)$. Then $\widehat{Q}^\psi$ is p-harmonic in*

$$G := \{x \in \Omega : C_p(\{y \in B(x,r) : \widehat{Q}^\psi(y) \leq \psi(y) + \delta\}) = 0 \textit{ for some positive } r \textit{ and } \delta\}.$$

Proof. Let $A_j = \{x \in \Omega : \widehat{Q}^\psi(x) > \psi(x) + 1/j\}$ and

$$A'_j = \{x \in \Omega : C_p(B(x,r) \setminus A_j) = 0 \text{ for some } r > 0\}.$$

Then $G = \bigcup_{j=1}^\infty A'_j$ and G is open. Let now

$$\varphi = \begin{cases} -\infty, & \text{in } \bigcup_{j=1}^\infty (A'_j \setminus A_j), \\ \psi, & \text{otherwise.} \end{cases}$$

The separability of X implies that $\varphi = \psi$ q.e., and hence $\widehat{Q}^\varphi = \widehat{Q}^\psi$. Moreover

$$G = \bigcup_{j=1}^\infty \operatorname{int}\{x \in \Omega : \widehat{Q}^\varphi(x) > \varphi(x) + 1/j\}.$$

Thus $\widehat{Q}^\varphi$ is p-harmonic in G, by Theorem 6.5. □

Theorem 6.8. *Assume that $\Psi^\psi \neq \varnothing$. Then $\widehat{Q}^\psi$ is p-harmonic in*

$$G := \operatorname{int}\{x \in \Omega : \widehat{Q}^\psi(x) > \psi(x) \text{ and } \psi \text{ is upper semicontinuous at } x\}.$$

In particular, if ψ is upper semicontinuous in all of Ω, then $\widehat{Q}^\psi$ is p-harmonic in the open set $\{x \in \Omega : \widehat{Q}^\psi(x) > \psi(x)\}$.

Proof. Let $\psi_j = \min\{\psi, j\}$, $j \in \mathbf{Z}$. By Proposition 4.11, $\widehat{Q}^{\psi_j} \to \widehat{Q}^\psi$ in Ω, as $j \to \infty$. Thus for each $x \in G$ we can find m_x such that $\widehat{Q}^{\psi_{m_x}}(x) > \psi(x)$. As $\widehat{Q}^{\psi_{m_x}}$ is lower semicontinuous and ψ is upper semicontinuous at x it follows that there is a neighbourhood $B_x \ni x$ such that $\widehat{Q}^{\psi_{m_x}} > \psi$ in B_x.

Let $V \Subset G$. As $\overline{V}$ is compact we can find a finite set $\{x_1, \dots, x_n\}$ such that $\overline{V} \subset \bigcup_{j=1}^n B_{x_j}$. Let $k \geq \max_{1 \leq j \leq n} m_{x_j}$. Then $\widehat{Q}^{\psi_k} > \psi \geq \psi_k$ in $\overline{V}$. By Proposition 5.6, $\widehat{Q}^{\psi_k}$ is the ess lim inf-regularized solution of the $\mathcal{K}_{\psi_k, \widehat{Q}^{\psi_k}}(V)$-obstacle problem. Since ψ_k is upper semicontinuous in G, Theorem 5.7 implies that $\widehat{Q}^{\psi_k}$ is p-harmonic in V.

As $\widehat{Q}^{\psi_k}$ is bounded from below on V it follows from Harnack's convergence theorem (Proposition 5.1 in Shanmugalingam [39]) that $\widehat{Q}^\psi = \lim_{k\to\infty} \widehat{Q}^{\psi_k}$ is p-harmonic in V, and as V was arbitrary also in G. □

Corollary 6.9. *Let $\psi \in C(\Omega)$ be such that $\Psi^\psi \neq \varnothing$. Then $\widehat{R}^\psi = \widehat{Q}^\psi \geq \psi$ is continuous everywhere in Ω and p-harmonic in*

$$G := \{x \in \Omega : \widehat{Q}^\psi(x) > \psi(x)\}.$$

Proof. By Proposition 4.6, $\widehat{R}^\psi = \widehat{Q}^\psi \geq \psi$. The continuity follows from Theorem 5.8. As for the p-harmonicity, the set G is open by the continuity of ψ and the lower semicontinuity of $\widehat{Q}^\psi$. Thus Theorem 6.8 implies that $\widehat{Q}^\psi$ is p-harmonic in G. □

As an application, we derive a global version of Theorem 7.7 in Kinnunen–Martio [30] by using the balayage. See also Theorem 8.15 in Heinonen–Kilpeläinen–Martio [24].

Theorem 6.10. *Assume that $u : \Omega \to (-\infty, \infty]$ is superharmonic. Then there is an increasing sequence of continuous bounded superminimizers $v_i : \Omega \to \mathbf{R}$ such that $\lim_{i\to\infty} v_i = u$ everywhere in Ω. If $u \geq 0$, then we can choose v_i, $i = 1, 2, \dots$, nonnegative.*

Proof. Since u is lower semicontinuous, there is an increasing sequence of continuous functions $\psi_i : \Omega \to \mathbf{R}$ such that $\lim_{i\to\infty} \psi_i = u$. If $u \geq 0$, then we require that $\psi_i \geq 0$. Replacing ψ_i by $\min\{\psi_i, i\}$, we may assume that each ψ_i is bounded from above. By Corollary 6.9, the functions $\widehat{R}^{\psi_i}$ are continuous superminimizers and since $u \in \Phi^{\psi_i}$, we get that $\psi_i \leq \widehat{R}^{\psi_i} \leq u$. Thus $\lim_{i\to\infty} \widehat{R}^{\psi_i} = u$ and since $\psi_{i+1} \geq \psi_i$, we obtain that $\widehat{R}^{\psi_{i+1}} \geq \widehat{R}^{\psi_i}$. □

7. Boundary regularity

Our aim in this section is to give characterizations of regularity for boundary points in terms of balayage. Recall the discussion on boundary regularity in Section 3.

In this section we add the additional assumption that Ω is bounded. Recall that we also assume that Ω is nonempty, open and such that $C_p(X \setminus \Omega) > 0$.

The following lemma relates the balayage to the Perron solutions. We recall that R_E^u denotes the infimum of superharmonic functions in Ω above $u\chi_E$, $\widehat{R}_E^u$ the lower semicontinuous regularization of R_E^u, and $\overline{P}_{\Omega\setminus E} u$ the upper Perron solution in $\Omega \setminus E$ with boundary values u.

Theorem 7.1. *Assume that $E \subsetneq \Omega$ is relatively closed, $u \ge 0$ in Ω, $u = 0$ on $\partial\Omega$ and $\Phi_E^u \ne \varnothing$. Then the following hold:*

(a) *The balayage $\widehat{R}_E^u$ is p-harmonic in $\Omega \setminus E$ and*

$$\widehat{R}_E^u \ge \overline{P}_{\Omega\setminus E} u \quad \textit{in } \Omega \setminus E.$$

(b) *If u is superharmonic in Ω, then*

$$R_E^u = \widehat{R}_E^u = \overline{P}_{\Omega\setminus E} u \quad \textit{in } \Omega \setminus E.$$

(c) *If $E \subset \Omega$ is compact and u is a superminimizer in Ω (not necessarily* ess lim inf*-regularized), then $\widehat{R}_E^u = \widehat{Q}_E^u$ in Ω and*

$$\widehat{R}_E^u = P_{\Omega\setminus E} u \quad \textit{in } \Omega \setminus E.$$

Proof. (a) Since by Corollary 6.2, $\widehat{R}_E^u$ is p-harmonic in $\Omega \setminus E$, it remains to show that $\widehat{R}_E^u \ge \overline{P}_{\Omega\setminus E} u$ in $\Omega \setminus E$. This is a consequence of the lower semicontinuity of superharmonic functions and the continuity of the upper Perron solution. Indeed, if $w \in \Phi_E^u$ and we extend w by zero on $\partial\Omega$, then by lower semicontinuity we have that $\liminf_{\Omega \ni y \to x} w(y) \ge w(x) \ge u(x)$ for every $x \in \partial(\Omega \setminus E)$. Hence $w \ge \overline{P}_{\Omega\setminus E} u \ge 0$ and thus $R_E^u \ge \overline{P}_{\Omega\setminus E} u$ in $\Omega \setminus E$. Theorem 4.1 in Björn–Björn–Shanmugalingam [10] shows that $\overline{P}_{\Omega\setminus E} u$ is p-harmonic, and in particular continuous, in $\Omega \setminus E$. By the continuity of $\overline{P}_{\Omega\setminus E} u$, it follows that $\widehat{R}_E^u \ge \overline{P}_{\Omega\setminus E} u$ in $\Omega \setminus E$.

(b) If u is superharmonic in Ω, then the converse inequality is a consequence of the pasting lemma. More precisely, choose $u' \in \mathcal{U}_u(\Omega \setminus E)$. Then $\liminf_{\Omega\setminus E \ni y \to x} u'(y) \ge u(x)$ for all $x \in \partial(\Omega \setminus E)$. An application of the pasting Lemma 3.13 shows that

$$v = \begin{cases} \min\{u, u'\}, & \text{in } \Omega' := \Omega \setminus E, \\ u, & \text{in } \Omega \setminus \Omega' = E, \end{cases}$$

is superharmonic in Ω and hence $v \in \Phi_E^u$. Taking infimum over all $u' \in \mathcal{U}_u(\Omega \setminus E)$ shows that $\overline{P}_{\Omega\setminus E} u \ge R_E^u \ge \widehat{R}_E^u$ in $\Omega \setminus E$.

(c) The crucial point here is that $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and that both the Q-balayage and the Newtonian space ignore sets of capacity zero. Indeed, choose $\eta \in \operatorname{Lip}_c(\Omega)$ such that $\chi_E \le \eta \le 1$. Then $u = u\eta$ on $\partial(\Omega \setminus E)$ and $u\eta \in N^{1,p}(X)$. It follows from Theorem 5.1 in Björn–Björn–Shanmugalingam [10] that u is resolutive with respect to $\Omega \setminus E$.

By Corollary 5.5 we get that $\widehat{R}_E^u = \widehat{Q}_E^u$ in Ω. Since $u = u^*$ q.e. and u^* is superharmonic, this together with (b) and the resolutivity of u and u^* implies that

$$\widehat{R}_E^u = \widehat{Q}_E^u = \widehat{Q}_E^{u^*} = \widehat{R}_E^{u^*} = P_{\Omega\setminus E} u^* = P_{\Omega\setminus E} u \quad \text{in } \Omega \setminus E,$$

where the last equality follows from Theorem 5.1 in Björn–Björn–Shanmugalingam [10] and the fact that $u\eta$ and $u^*\eta$ belong to the same equivalence class in $N^{1,p}(X)$. □

The following theorem gives a sufficient condition on the balayage to guarantee that a boundary point is regular.

Theorem 7.2. *Assume that $x_0 \in \partial\Omega$. If*

$$\widehat{R}^1_{V\setminus\Omega}(V)(x_0) = 1 \quad \textit{for all bounded open sets } V \ni x_0,$$

or

$$\widehat{Q}^1_{V\setminus\Omega}(V)(x_0) = 1 \quad \textit{for all bounded open sets } V \ni x_0,$$

then x_0 is a regular boundary point.

Proof. Since $\widehat{Q}^1_{V\setminus\Omega}(V) \le \widehat{R}^1_{V\setminus\Omega}(V) \le 1$, it is enough to prove the result for the R-balayage. Let $f \in C(\partial\Omega)$ and $\varepsilon > 0$ be arbitrary. We can assume that $f(x_0) = 0$. By continuity, there exists an open set $V \ni x_0$ such that $|f| < \varepsilon$ in $\partial\Omega \cap V$. Let $E = V \setminus \Omega$ and $M = \max_{\partial\Omega} |f|$. Then $f \le \varepsilon + M(1 - \chi_E)$ on $\partial\Omega$ and hence

$$Pf = \underline{P}f \le \varepsilon + M(1 - \overline{P}\chi_E) \quad \text{in } \Omega. \tag{7.1}$$

Theorem 7.1 (b) applied to $\Omega' = \Omega \cup V$ and $u = \chi_{\Omega'}$ together with Proposition 4.3 yields

$$\overline{P}\chi_E = \widehat{R}^u_E(\Omega') = \widehat{R}^1_E(\Omega') \ge \widehat{R}^1_E(V) \quad \text{in } V \cap \Omega.$$

Inserting this into (7.1) gives

$$Pf \le \varepsilon + M(1 - \widehat{R}^1_E(V)) \quad \text{in } V \cap \Omega,$$

and hence by the lower semicontinuity of $\widehat{R}^1_E(V)$,

$$\limsup_{\Omega \ni y \to x_0} Pf(y) \le \varepsilon + M\Big(1 - \liminf_{\Omega \ni y \to x_0} \widehat{R}^1_E(V)(y)\Big) \le \varepsilon + M(1 - \widehat{R}^1_{V\setminus\Omega}(V)(x_0)) = \varepsilon.$$

Applying the same argument with f replaced by $-f$, and letting $\varepsilon \to 0$ implies that

$$\lim_{\Omega \ni y \to x_0} Pf(y) = 0 = f(x_0)$$

as desired. □

Remark 7.3. In Heinonen–Kilpeläinen–Martio [24], an analogue of Theorem 7.2 was used to prove the Kellogg property. For metric spaces, the Kellogg property was proved by a different method in Björn–Björn–Shanmugalingam [9], Theorem 3.9.

We are now ready to prove a balayage characterization for regular boundary points. In Theorem 7.7 we give another type of balayage characterization for regular boundary points.

Theorem 7.4. *Assume that $x_0 \in \partial\Omega$. Then the following are equivalent*:

(a) *The point x_0 is regular*;

(b) *For all bounded open sets $V \ni x_0$,*

$$\widehat{R}^1_{V\setminus\Omega}(V)(x_0) = 1;$$

(c) *For all bounded open sets $V \ni x_0$,*

$$\widehat{Q}^1_{V\setminus\Omega}(V)(x_0) = 1;$$

(d) *It is true that*

$$\widehat{R}^u_{\overline{U}\setminus\Omega}(V)(x_0) = u(x_0),$$

whenever $U \Subset V$ are bounded open sets with $x_0 \in U$ and $u \ge 0$ is superharmonic in V;

(e) *It is true that*

$$\widehat{Q}^u_{\overline{U}\setminus\Omega}(V)(x_0) = u(x_0),$$

whenever $U \Subset V$ are bounded open sets with $x_0 \in U$ and $u \ge 0$ is superharmonic in V.

Proof. Since the Q-balayage is majorized by the R-balayage and both u and the constant function 1 are competing in the definition of the R-balayage, the implications (c)⇒(b) and (e)⇒(d) are trivial.

(d)⇒(b) and (e)⇒(c) Fix an arbitrary open set $U \Subset V$ such that $x_0 \in U$. Since constant functions are superharmonic, we have that

$$1 \geq \widehat{R}^1_{V\setminus\Omega}(V)(x_0) \geq \widehat{R}^1_{\overline{U}\setminus\Omega}(V)(x_0) = 1$$

and similarly for the Q-balayage.

(b)⇒(a) This follows from Theorem 7.2.

(a)⇒(e) Theorem 6.10 provides us with an increasing sequence $\{f_j\}_{j=1}^\infty$ of nonnegative, bounded, and continuous superminimizers in V such that $\lim_{j\to\infty} f_j = u$ on $\overline{U}$. Let $f_j = 0$ on ∂V. Theorem 7.1 with $E = \overline{U} \setminus \Omega \Subset V$ implies that for all $j = 1, 2, ...,$

$$P_{V\setminus E} f_j = \widehat{R}^{f_j}_E(V) = R^{f_j}_E(V) \quad \text{in } V \setminus E.$$

By Corollary 4.4 in Björn–Björn [5], x_0 is regular for $V \setminus E$, and thus by Theorem 3.10(c),

$$f_j(x_0) = \lim_{V\setminus E \ni y\to x_0} P_{V\setminus E} f_j(y) = \liminf_{V\setminus E\ni y\to x_0} R^{f_j}_E(V)(y).$$

At the same time, $f_j \leq R^{f_j}_E$ in E and hence

$$f_j(x_0) \leq \liminf_{E\ni y\to x_0} R^{f_j}_E(V)(y).$$

The last two inequalities and Theorem 7.1(c) imply that

$$f_j(x_0) \leq \liminf_{y\to x_0} R^{f_j}_E(V)(y) = \widehat{R}^{f_j}_E(V)(x_0) = \widehat{Q}^{f_j}_E(V)(x_0) \leq \widehat{Q}^u_E(V)(x_0).$$

Letting $j \to \infty$ yields $u(x_0) \leq \widehat{Q}^u_E(V)(x_0)$ and the converse inequality follows from the definition of the balayage. □

Before studying connections between balayage and barriers, we show that at a regular boundary point the balayage attains the boundary value given by a continuous function, cf. Theorem 9.26 in Heinonen–Kilpeläinen–Martio [24].

Theorem 7.5. *Assume that $\Phi^\psi \neq \varnothing$ and that there exists an open set $U \Subset \Omega$, such that ψ is bounded in $\Omega \setminus U$. If $x_0 \in \partial\Omega$ is a regular boundary point and ψ is continuous at x_0 (in the sense that the limit $\psi(x_0) := \lim_{\Omega\ni y\to x_0} \psi(y)$ exists), then*

$$\lim_{\Omega\ni y\to x_0} \widehat{Q}^\psi(y) = \lim_{\Omega\ni y\to x_0} \widehat{R}^\psi(y) = \psi(x_0).$$

As in the proof of Theorem 7.4 we intend to use Theorem 3.10(c). To do so we first need to construct a suitable bounded function.

Proof. We may assume that $\psi(x_0) = 0$. Let $\varepsilon > 0$ be arbitrary and find a ball $B \ni x_0$ such that $|\psi| < \varepsilon$ in $B \cap \Omega \subset \Omega \setminus \overline{U}$. Let also V be an open set such that $U \Subset V \Subset \Omega \setminus \overline{B}$. Let $M = \sup_{\Omega\setminus U} |\psi|$ and fix some $u \in \Phi^\psi$. Since $u \geq \psi \geq -M$ in $\Omega \setminus U$, the lower semicontinuity of u shows that it is bounded from below in Ω. By adding a constant to u, we can assume that u is nonnegative.

Let $w = \widehat{R}^u_U$. By Proposition 4.8, $w = u$ in U. Theorem 6.1 shows that w is p-harmonic in $\Omega \setminus \overline{U}$ and hence bounded on ∂V. We next want to show that $w = \widehat{R}^w_{\overline{V}}$. Indeed, $\widehat{R}^w_{\overline{V}} \leq w$ as $w \in \Phi^w_{\overline{V}}$. On the other hand, as $u\chi_U = w\chi_U \leq w\chi_{\overline{V}}$, Proposition 4.3 implies that $w \leq \widehat{R}^w_{\overline{V}}$. This and Theorem 7.1 show that $w = P_{\Omega\setminus\overline{V}} w$

in $\Omega \setminus \overline{V}$, where we let $w = 0$ on $\partial\Omega$. In particular we see that w is bounded in $\Omega \setminus \overline{V}$. Let further

$$v = \begin{cases} w + M, & \text{in } \Omega, \\ 0, & \text{on } \partial\Omega. \end{cases}$$

Clearly, $v \geq M \geq \psi$ in $\Omega \setminus U$, and $v = u + M \geq \psi$ in U. Hence $\psi \leq \varepsilon + v\chi_{\Omega\setminus B}$ in Ω and it follows from Theorem 7.1 that

$$\widehat{R}^{\psi} \leq \varepsilon + \widehat{R}^{v}_{\Omega\setminus B} = \varepsilon + \overline{P}_{B\cap\Omega} v \quad \text{in } B \cap \Omega. \tag{7.2}$$

By Theorem 3.10(b), x_0 is regular for $B \cap \Omega$. Since v is bounded on $\partial(B \cap \Omega)$ and zero on $\partial\Omega$, Theorem 3.10(c) together with (7.2) implies that

$$\limsup_{\Omega \ni y \to x_0} \widehat{R}^{\psi}(y) \leq \varepsilon + \limsup_{\Omega \ni y \to x_0} \overline{P}_{B\cap\Omega} v(y) \leq \varepsilon.$$

At the same time, $\widehat{Q}^{\psi} \geq \psi > -\varepsilon$ q.e. in $B \cap \Omega$, and hence, as $\widehat{Q}^{\psi}$ is ess lim inf-regularized, $\widehat{Q}^{\psi} \geq -\varepsilon$ everywhere in $B \cap \Omega$. It follows that

$$\liminf_{\Omega \ni y \to x_0} \widehat{Q}^{\psi}(y) \geq -\varepsilon$$

and letting $\varepsilon \to 0$ finishes the proof. □

We can now give some further characterizations of regular boundary points. To do so we will use the concept of barriers.

Definition 7.6. A function u is a *barrier* (with respect to Ω) at $x_0 \in \partial\Omega$ if
- (a) u is superharmonic in Ω;
- (b) $\liminf_{\Omega \ni y \to x} u(y) > 0$ for every $x \in \partial\Omega \setminus \{x_0\}$;
- (c) $\lim_{\Omega \ni y \to x_0} u(y) = 0$.

Theorem 7.7. *Let $x_0 \in \partial\Omega$. Assume that $\Phi^{\psi} \neq \varnothing$, that there is an open set $U \Subset \Omega$ such that ψ is bounded in $\Omega \setminus U$, and that*

$$\lim_{\Omega \ni y \to x_0} \psi(y) = 0 \quad \textit{and} \quad \liminf_{\Omega \ni y \to x} \psi(y) > 0 \textit{ for all } x \in \partial\Omega \setminus \{x_0\}.$$

Then the following are equivalent:
- (a) *x_0 is regular*;
- (b) *there is a barrier at x_0*;
- (c) $$\lim_{\Omega \ni y \to x_0} \widehat{R}^{\psi}(y) = 0;$$
- (d) $$\lim_{\Omega \ni y \to x_0} \widehat{Q}^{\psi}(y) = 0.$$

Proof. (a) ⇔ (b) This is part of Theorem 4.2 in Björn–Björn [5]

(a) ⇒ (c) and (a) ⇒ (d) This follows directly from Theorem 7.5.

(d) ⇒ (b) and (c) ⇒ (b) Let $x \in \partial\Omega \setminus \{x_0\}$. As $\liminf_{\Omega \ni y \to x} \psi(y) > 0$, there exist $\varepsilon > 0$ and a ball $B \ni x$ such that $\psi \geq \varepsilon$ in $B \cap \Omega$. Theorem 4.4 implies that $\widehat{Q}^{\psi} \geq \varepsilon$ q.e. in $B \cap \Omega$. As $\widehat{Q}^{\psi}$ is ess lim inf-regularized, $\widehat{R}^{\psi} \geq \widehat{Q}^{\psi} \geq \varepsilon$ everywhere in $B \cap \Omega$. Hence,

$$\liminf_{\Omega \ni y \to x} \widehat{R}^{\psi}(y) \geq \liminf_{\Omega \ni y \to x} \widehat{Q}^{\psi}(y) \geq \varepsilon > 0.$$

As $\widehat{Q}^{\psi}$ and $\widehat{R}^{\psi}$ are superharmonic and (c) or (d) hold, one of them is thus a barrier at x_0. □

8. Balayage and polar sets

In this section we characterize polar sets, in particular in terms of balayage. See Kinnunen–Shanmugalingam [33] for earlier results on polar sets on metric spaces, and Chapter 10 in Heinonen–Kilpeläinen–Martio [24] for the weighted $\mathbf{R}^n$ case.

Definition 8.1. A set $E \subset \Omega$ is *polar*, if there exists a superharmonic function in Ω such that $u = \infty$ in E.

Theorem 8.2. *Assume that Ω is bounded and that $E \subset \Omega$. Then the following are equivalent*:

(a) *E is polar*;
(b) *there is a nonnegative superharmonic function u on Ω such that $u = \infty$ on E*;
(c) *there is a nonnegative superharmonic $u \in N^{1,p}(\Omega)$ such that $u = \infty$ on E*;
(d) *E is of capacity zero*;
(e) *$\widehat{R}_E^\psi \equiv 0$ for all functions ψ*;
(f) *$\widehat{R}_E^\psi \equiv 0$ for some function $\psi > 0$*;
(g) *$\widehat{Q}_E^\psi \equiv 0$ for all functions ψ*;
(h) *$\widehat{Q}_E^\psi \equiv 0$ for some function $\psi > 0$*;
(i) *there is a function $u \in N^{1,p}(\Omega)$ such that $u = \infty$ on E.*

The implication (a) $\Rightarrow$ (d) was obtained in Kinnunen–Shanmugalingam [33], Proposition 2.2. They also showed the converse implication under the assumption that E is relatively closed, see Theorem 3.4 in [33].

Note that the equivalence (a) $\Leftrightarrow$ (d) implies that a countable union of polar sets is polar. This is trivial in the linear case, as a countable sum of superhamonic functions is superharmonic (if not too large), but more difficult to show in the nonlinear theory.

Proof. (e) $\Rightarrow$ (f) $\Rightarrow$ (h), (c) $\Rightarrow$ (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (i) These implications are trivial.

(h) $\Rightarrow$ (d) By Theorem 4.4, $0 = \widehat{Q}_E^\psi \geq \psi > 0$ q.e. in E. Hence E must have capacity zero.

(d) $\Rightarrow$ (c) By Corollary 1.3 in Björn–Björn–Shanmugalingam [11], C_p is an outer capacity, i.e. there exists, for $j = 1, 2, \dots$, an open set $G_j \supset E$ with $C_p(G_j) < 2^{-jp}$ and thus a nonnegative $\varphi_j \in N^{1,p}(X)$ such that $\|\varphi_j\|_{N^{1,p}(X)} < 2^{-j}$ and $\varphi_j \geq \chi_{G_j}$. Let $\varphi = \sum_{j=1}^\infty \varphi_j$ and let w be the $\operatorname{ess\,lim\,inf}$-regularized solution of the $\mathcal{K}_{\varphi,\varphi}$-obstacle problem. Then $w \in N^{1,p}(\Omega)$ is a nonnegative superharmonic function and $w = \infty$ in E.

(b) $\Rightarrow$ (f) Let G be a component of Ω. As u is superharmonic, there is $x \in G$ such that $u(x) < \infty$. Moreover $\varepsilon u \in \Phi_E^1$ for every $\varepsilon > 0$, and hence $R_E^1(x) \leq \varepsilon u(x)$. Letting $\varepsilon \to 0$ shows that $\widehat{R}_E^1(x) \leq R_E^1(x) \leq 0$. By the strong minimum principle $\widehat{R}_E^1 \equiv 0$ in G. As G was an arbitrary component we have that $\widehat{R}_E^1 \equiv 0$ in Ω.

(a) $\Rightarrow$ (d) Let u be a superharmonic function such that $u = \infty$ on E, $\Omega_j = \{x \in \Omega : \operatorname{dist}(x, X \setminus \Omega) > 1/j\}$ and $E_j = E \cap \Omega_j$, $j = 1, 2, \dots$. As u is lower semicontinuous it is bounded from below on $\overline{\Omega}_j$. Hence it follows from the already proved implication (b) $\Rightarrow$ (d) (applied to $u - \inf_{\Omega_j} u$ and Ω_j) that $C_p(E_j) = 0$. By countable subadditivity, $E = \bigcup_{j=1}^\infty E_j$ is of zero capacity.

(d) $\Rightarrow$ (g) By definition $\Psi_E^\psi = \Psi^0$. Hence $\widehat{Q}_E^\psi = \widehat{Q}^0 \equiv 0$.

(g) $\Rightarrow$ (e) This follows from Theorem 4.10.

(i) $\Rightarrow$ (d) We have that $u/k \geq 1$ on E for all $k > 0$. Thus

$$C_p(E) \leq \left\| \frac{u}{k} \right\|_{N^{1,p}(\Omega)}^p = \frac{\|u\|_{N^{1,p}(\Omega)}^p}{k^p} \to 0, \quad \text{as } k \to \infty. \qquad \square$$

9. The case $\Omega = X$ bounded

Assume now that X is bounded and that Ω is an open subset such that $C_p(X \setminus \Omega) = 0$. In this case we first should observe that Definition 3.1 of (super)minimizers does not need any modifications. On the other hand, in the case when $\Omega = X$ we need to modify (iii) in Definition 3.3 of superharmonic functions, the reason being that if $\Omega = X$ then $V := X \Subset \Omega$, but $H_X v$ is not well-defined (one can talk about solutions of the corresponding obstacle problem, but there will be no uniqueness). The natural way is to also require that $C_p(X \setminus V) > 0$ in (iii), as done in A. Björn [2]. Note that this condition is automatically fulfilled when $V \Subset \Omega$ apart from in the case when $\Omega = X$ is bounded. In fact this modification, or something equivalent, is the only possible way if we want the restriction of a superharmonic function to be superharmonic. It follows (see below) that the only superharmonic functions on all of a bounded X are the constant functions.

Let now ψ be a bounded function on Ω. One can define Q^ψ and R^ψ (and their regularizations) as before. Let us first look at $\varphi \in \Psi^\psi$. As ψ is bounded, φ is bounded from below. By Theorem 6.3 in A. Björn [4], φ has a superharmonic extension $\widetilde{\varphi}$ to all of X. As $\widetilde{\varphi}$ is lower semicontinuous on the compact set X it attains its minimum. By the minimum principle, $\widetilde{\varphi}$ is constant in the component of X containing the minimum point, but as X is connected it follows that $\widetilde{\varphi}$ is constant, and thus also φ is constant.

From this we see that $Q^\psi = \widehat{Q}^\psi \equiv \operatorname{ess\,sup}_\Omega \psi$. Similarly $R^\psi = \widehat{R}^\psi \equiv \sup_\Omega \psi$. The theory of balayage in this case does not become very interesting. Let us however point out the following example.

Example 9.1. If there is a point $x \in X$ such that $C_p(\{x\}) = 0$, then $\widehat{Q}^\psi(X) \equiv 0$ and $\widehat{R}^\psi(X) \equiv 1$, where $\psi = \chi_{\{x\}}$.

The situation is also similar if $X = \mathbf{R}^2$ and $p = 2$ as in this case any superharmonic function bounded from below on X has an extension to a superharmonic function on the Riemann sphere and is thus constant.

10. Open problems

Our results in the previous sections leave the following problems open. They all are also open in the Euclidean setting.

Let us assume that Ω is bounded and that $C_p(X \setminus \Omega) > 0$.

Open problem 10.1. Is it true that $\widehat{R}^\psi = \widehat{Q}^\psi$ for all functions ψ?

In the linear case this is well known: Let $E = \{x \in \Omega : \widehat{Q}^\psi(x) < \psi(x)\}$ and $\varphi = \infty\chi_E$. As $C_p(E) = 0$, we know by Theorem 4.10 that $\widehat{R}^\varphi = \widehat{Q}^\varphi = 0$. Take $u \in \Psi^\psi$ and $v \in \Phi^\varphi$, then $u + v \in \Phi^\psi$ and thus $R^\psi \le Q^\psi + R^\varphi$ and $\widehat{R}^\psi \le \widehat{Q}^\psi + \widehat{R}^\varphi = \widehat{Q}^\psi$. The converse inequality is trivial.

In the nonlinear case this is more subtle. However, as we pointed out before Proposition 4.11 it would be enough to have an R-version of Proposition 4.11 to obtain a positive answer to Problem 10.1. A positive answer to the following problem would also give a positive answer to Problem 10.1 at least for all bounded ψ.

Open problem 10.2. If ψ is a superminimizer is it then true that $\widehat{R}^\psi = \psi^*$?

As ψ^* is superharmonic and $\psi^* = \psi$ q.e. it follows from Proposition 4.7 that $\widehat{Q}^\psi = \psi^*$. It follows that Open problem 10.2 has a positive answer in the linear case.

The following are some further obvious questions.

Open problem 10.3. Can the condition $\widehat{Q}^\psi \in N^{1,p}_{\rm loc}(\Omega)$ be removed from the statements of either or both of Theorems 6.5 and 6.7?

Open problem 10.4. Can the condition that ψ is bounded in $\Omega \setminus U$ be omitted from Theorem 7.5?

Open problem 10.5. Is the R-version of any of Theorems 4.10, 5.8, 6.5, 6.7, 6.8 or Proposition 5.6 true?

Open problem 10.6. Is the Q-version of Theorem 7.1 true?

In the linear case it is clear that Open problems 10.5 and 10.6 have positive answers.

In many of our results we assume that Ω is bounded. This is often due to usage of obstacle problems or boundary regularity for which the theory so far has been developed mainly just for bounded sets (at least on metric spaces). We do not know when this boundedness assumption is essential.

References

1. Armitage, D. H. and Gardiner, S. J., *Classical Potential Theory,* Springer, London, 2001.
2. Björn, A., Characterizations of p-superharmonic functions on metric spaces, *Studia Math.* **169** (2005), 45–62.
3. Björn, A., A weak Kellogg property for quasiminimizers, *Comment. Math. Helv.* **81** (2006), 809–825.
4. Björn, A., Removable singularities for bounded p-harmonic and quasi(super)-harmonic functions on metric spaces, *Ann. Acad. Sci. Fenn. Math.* **31** (2006), 71–95.
5. Björn, A. and Björn, J., Boundary regularity for p-harmonic functions and solutions of the obstacle problem, *J. Math. Soc. Japan* **58** (2006), 1211–1232.
6. Björn, A. and Björn, J., Approximations by regular sets and Wiener solutions in metric spaces, *Comment. Math. Univ. Carolin* **48** (2007), 343–355.
7. Björn, A. and Björn, J., *Nonlinear Potential Theory in Metric Spaces*, In preparation.
8. Björn, A., Björn, J. and Parviainen, M., Lebesgue points and the fundamental convergence theorem for superharmonic functions on metric spaces, to appear in *Rev. Mat. Iberoamericana.*
9. Björn, A., Björn, J. and Shanmugalingam, N., The Dirichlet problem for p-harmonic functions on metric spaces, *J. Reine Angew. Math.* **556** (2003), 173–203.
10. Björn, A., Björn, J. and Shanmugalingam, N., The Perron method for p-harmonic functions, *J. Differential Equations* **195** (2003), 398–429.
11. Björn, A., Björn, J. and Shanmugalingam, N., Quasicontinuity of Newton–Sobolev functions and density of Lipschitz functions on metric spaces, *Houston J. Math.* **34** (2008), 1197–1211.
12. Björn, A. and Martio, O., Pasting lemmas and characterizations of boundary regularity for quasiminimizers, to appear in *Results Math.*
13. Björn, J., Boundary continuity for quasiminimizers on metric spaces, *Illinois J. Math.* **46** (2002), 383–403.
14. Björn, J., MacManus, P. and Shanmugalingam, N., Fat sets and pointwise boundary estimates for p-harmonic functions in metric spaces, *J. Anal. Math.* **85** (2001), 339–369.

15. Capogna, L. and Garofalo, N., Boundary behavior of nonnegative solutions of subelliptic equations in NTA domains for Carnot–Carathéodory metrics, *J. Fourier Anal. Appl.* **4** (1998), 403–432.
16. Cheeger, J., Differentiability of Lipschitz functions on metric spaces, *Geom. Funct. Anal.* **9** (1999), 428–517.
17. Doob, J. L., *Classical Potential Theory and its Probabilistic Counterpart,* Springer, New York, 1984.
18. Farnana, Z., The double obstacle problem on metric spaces, to appear in *Ann. Acad. Sci. Fenn. Math.*
19. Hajłasz, P., Sobolev spaces on an arbitrary metric space, *Potential Anal.* **5** (1995), 403–415.
20. Hajłasz, P. and Koskela, P., *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (2000).
21. Hansen, W. and Hueber, H., The Dirichlet problem for sub-Laplacians on nilpotent Lie groups—geometric criteria for regularity, *Math. Ann.* **276** (1987), 537–547.
22. Heinonen, J. and Kilpeläinen, T., Polar sets for supersolutions of degenerate elliptic equations, *Math. Scand.* **63** (1988), 136–150.
23. Heinonen, J. and Kilpeläinen, T., On the Wiener criterion and quasilinear obstacle problems, *Trans. Amer. Math. Soc.* **310** (1988), 239–255.
24. Heinonen, J., Kilpeläinen, T. and Martio, O., *Nonlinear Potential Theory of Degenerate Elliptic Equations,* 2nd ed., Dover, Mineola, NY, 2006.
25. Heinonen, J. and Koskela, P., Quasiconformal maps in metric spaces with controlled geometry, *Acta Math.* **181** (1998), 1–61.
26. Keith, S. and Zhong, X., The Poincaré inequality is an open ended condition, *Ann. of Math.* **167** (2008), 575–599.
27. Kilpeläinen, T., Kinnunen, J. and Martio, O., Sobolev spaces with zero boundary values on metric spaces, *Potential Anal.* **12** (2000), 233–247.
28. Kinnunen, J. and Martio, O., The Sobolev capacity on metric spaces, *Ann. Acad. Sci. Fenn. Math.* **21** (1996), 367–382.
29. Kinnunen, J. and Martio, O., Choquet property for the Sobolev capacity in metric spaces, in *Proceedings on Analysis and Geometry* (Novosibirsk, Akademgorodok, 1999), pp. 285–290, Sobolev Institute Press, Novosibirsk, 2000.
30. Kinnunen, J. and Martio, O., Nonlinear potential theory on metric spaces, *Illinois Math. J.* **46** (2002), 857–883.
31. Kinnunen, J. and Martio, O., Potential theory of quasiminimizers, *Ann. Acad. Sci. Fenn. Math.* **28** (2003), 459–490.
32. Kinnunen, J. and Shanmugalingam, N., Regularity of quasi-minimizers on metric spaces, *Manuscripta Math.* **105** (2001), 401–423.
33. Kinnunen, J. and Shanmugalingam, N., Polar sets on metric spaces, *Trans. Amer. Math. Soc.* **358** (2006), 11–37.
34. Koskela, P. and MacManus, P., Quasiconformal mappings and Sobolev spaces, *Studia Math.* **131** (1998), 1–17.
35. Mäkäläinen, T., Removable sets for Hölder continuous p-harmonic functions on metric measure spaces, to appear in *Ann. Acad. Sci. Fenn. Math.*
36. Martio, O., Balayage in metric measure spaces, *Handwritten notes*, 2001.
37. Shanmugalingam, N., Newtonian spaces: An extension of Sobolev spaces to metric measure spaces, *Rev. Mat. Iberoamericana* **16** (2000), 243–279.
38. Shanmugalingam, N., Harmonic functions on metric spaces, *Illinois J. Math.* **45** (2001), 1021–1050.
39. Shanmugalingam, N., Some convergence results for p-harmonic functions on metric measure spaces, *Proc. London Math. Soc.* **87** (2003), 226–246.