An Approximation Approach to Computationally Expensive Multiobjective Optimization Problems

Markus Hartikainen

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Ph. Lic. in mathematics (mathematical analysis)
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- Interested in optimization, scientific computing, statistics, pedagogy
Area: multiobjective optimization
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Advisors: professor Kaisa Miettinen and visiting professor Margaret M. Wiecek
My Dissertation

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- A collection of articles

Two conference proceedings articles (that are to be refereed) submitted. Other one of these has been published in the report series of the department [Hartikainen et al., 2010].

One journal manuscript almost ready (to be submitted in 1−2 weeks). Two more papers planned (and I also have something on paper already).

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A Multiobjective optimization problem (MOP)

minimize \( f_1(x), \ldots, f_k(x) \)

so that \( x \in S \)
Objective functions

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\text{minimize} & \quad (f_1(x), \ldots, f_k(x)) \\
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\end{align*}
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- Decision variable
A Multiobjective optimization problem (MOP)

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- Decision variable

- Decision space

The set \(f(S)\), where \(f(x) = (f_1(x), \ldots, f_k(x))\) for all \(x \in S\), is called the outcome space. A vector \(z \in f(S)\) is called an outcome. An outcome \(z_1\) is said to dominate another outcome \(z_2\) if \(z_1^i \leq z_2^i\) for all \(i = 1, \ldots, k\) and \(z_1^j \leq z_2^j\) for some \(j = 1, \ldots, k\).

A decision variable \(x \in S\) is called a Pareto optimal (PO) solution if there does not exist an outcome \(z \in f(S)\) that dominates \(f(x)\) and the outcome \(f(x)\) given by a PO solution is called a PO outcome.
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- Calculation of objective functions is computationally expensive.
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Interactive methods fail because the decision maker (DM) gets anxious while waiting for new solutions to be computed according to his/her preferences.

Our approach:

1. Construct an approximation of the Pareto front (PF) based on a small set of PO outcomes.
2. Use interactive multiobjective optimization (MO) methods (see e.g., [Miettinen, 1999]) for finding the preferred outcome on the approximation.
3. Find the PO solution that is closest to the preferred point by means of an achievement scalarizing function [Wierzbicki, 1986].
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Inherent Nondominance

A Pareto front approximation $\Rightarrow$ inherent nondominance

Definition

A set $A \subset \mathbb{R}^k$ is called inherently nondominated (IND) if there does not exist $z^1, z^2 \in A$ so that $z^1$ dominates $z^2$. An IND set $A$ is called an IND PF approximation based on $P$, if $P \subset \mathbb{R}^k$ is a finite set of PO outcomes and $P \subset A$. 

See [Hartikainen et al., 2010] avoids misleading the DM in what is attainable and what not. Also the (actual) PF is by definition an IND PF approximation based on any set of PO outcomes.
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Constructing Inherently Nondominated Complexes: Basic Definitions

A collection of polytopes is a complex if
1. every face of a polytope in the collection is also in the collection and
2. an intersection of two polytopes in the collection is a face of each of them.

A complex $D$ is a Delaunay triangulation of a finite set $P \subset \mathbb{R}^k$ if $\bigcup_{K \in D} K = \text{conv}(P)$ and the complex only contains "polytopes with neighbouring vertices" (see [Edelsbrunner, 1987]).
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$\bigcup_{K \in \mathcal{D}} K = \text{conv}(P)$ and the complex only contains ”polytopes with neighbouring vertices” (see [Edelsbrunner, 1987]).
A complex $\mathcal{K}$ is said to be IND if the set $\bigcup_{K \in \mathcal{K}} K$ is IND.
Constructing an Inherently Nondominated Complex

- A complex $\mathcal{K}$ is said to be IND if the set $\bigcup_{K \in \mathcal{K}} K$ is IND.
- A polytope $K^1$ is said to dominate another polytope $K^2$ if there exists $s^1 \in K^1$ and $s^2 \in K^2$ so that $s^1$ dominates $s^2$. 

Theorem: A complex $\mathcal{K}$ is IND if and only if there does not exist $K^1, K^2 \in \mathcal{K}$ so that $K^1$ dominates $K^2$.

We aim to construct a complex that is an IND PF approximation. The aim is thus to construct such a complex $\mathcal{K}$ that $\{p\} \in \mathcal{K}$ for all $p \in P$ and there does not exist $K^1, K^2 \in \mathcal{K}$ so that $K^1$ dominates $K^2$.

With minor extra conventions, this can be seen as a shape reconstruction problem as defined in [Edelsbrunner, 1998].
Constructing an Inherently Nondominated Complex

- A complex $\mathcal{C}$ is said to be IND if the set $\bigcup_{K \in \mathcal{C}} K$ is IND.
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**Theorem:** A polytope $K^1$ dominates another polytope $K^2$ if and only if either

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   is less than 0 or
2. Optimal value in problem (1) is 0 and the optimal value in optimization problem
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We want to remove polytopes from the Delaunay triangulation so that there are no polytopes in the resulting collection of polytopes (which can be shown to be a complex) that dominate each other.

In a paper that we aim to submit "any day now" we propose a set of rules for doing the above.
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Properties of the Constructed IND PF Approximation

- The constructed approximation $\mathcal{K}$ is a complex.
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- The constructed approximation is maximal in a way that for all $K \in \mathcal{D} \setminus \mathcal{K}$ the collection of polytopes $\mathcal{K} \cup \{K\}$ is not IND.
- Some error estimates for the approximation can be given: For a given point $s$ in $\bigcup_{K \in \mathcal{K}} K$ we can estimate $d(s) \in \mathbb{R}^k$ so that
  1. there exists an outcome $z \in f(S)$ that is at least as good as $s + d(s)$ in all objectives and
  2. there does not exist an outcome $z \in f(S)$ that dominates $s - d(s)$. 

A preferred point on the approximation can be searched with interactive multiobjective optimization methods, because the approximation is IND and the approximation can be parametrized.
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A Delaunay triangulation of $m$ points in $k$ dimensions contains at most $O(m^\lceil k/2 \rceil)$ polytopes. This means that in order to find out all the dominations between polytopes, one may have to solve $O(m^k)$ optimization problems as described above. For this reason the first research question is to reduce the computational effort.
Further Research

- A Delaunay triangulation of $m$ points in $k$ dimensions contains at most $O(m^\lceil k/2 \rceil)$ polytopes. This means that in order to find out all the dominations between polytopes, one may have to solve $O(m^k)$ optimization problems as described above. For this reason the first research question is to reduce the computational effort.

- The next step is to develop better methods for finding the preferred point on the approximation.
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Any further applications are welcome. We are ready to try our approach with real world problems.
References


Thank You!

For further details or if you have ideas please contact Markus Hartikainen (markus.e.hartikainen@jyu.fi)