Multiple Criteria Optimization: Some Introductory Topics

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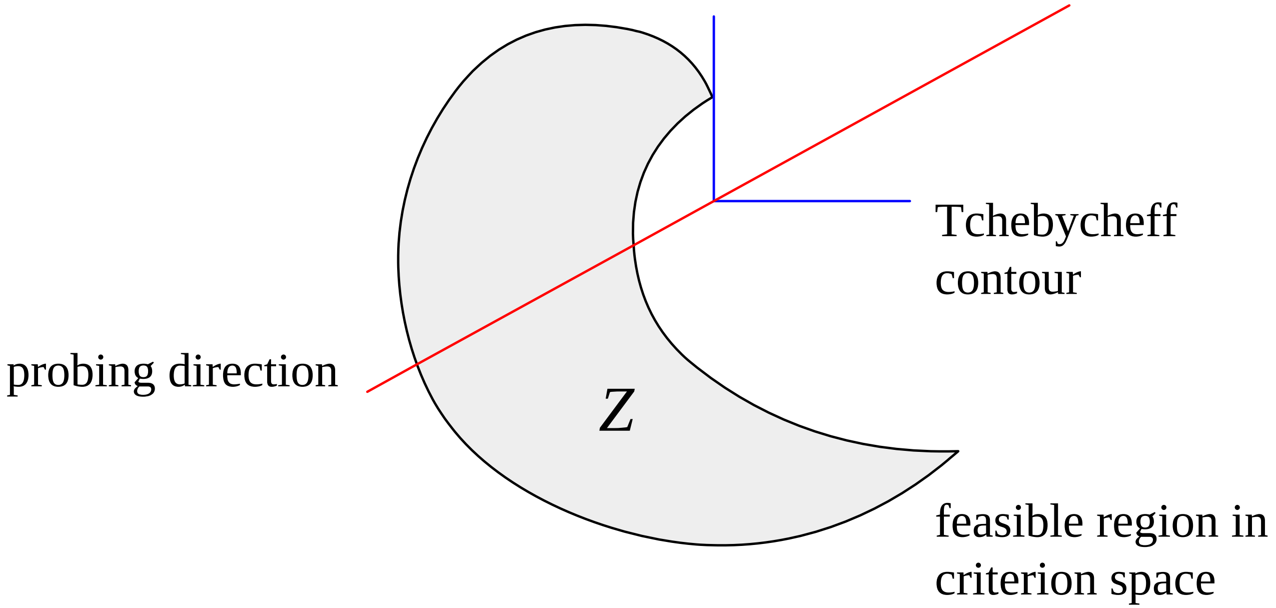
$$\max\{\ f_1(\mathrm{x}) = z_1\ \}$$
$$\vdots$$
$$\max\{\ f_k(\mathrm{x}) = z_k\ \}$$
$$s.t. \qquad \mathrm{x} \in S$$



Production planning

min { cost }
min { fuel consumption }
min { production in a given geographical area }

River basin management

achieve { BOD standards }
min { nitrate standards }
min { pollution removal costs }
achieve { municipal water demands }
min { groundwater pumping }

Oil refining

min { cost }
min { imported crude }
min { environmental pollution }
min { deviations from demand slate }

Sausage blending

min { cost }

max { protein }

min { fat }

min { deviations from moisture target }

Portfolio selection in finance

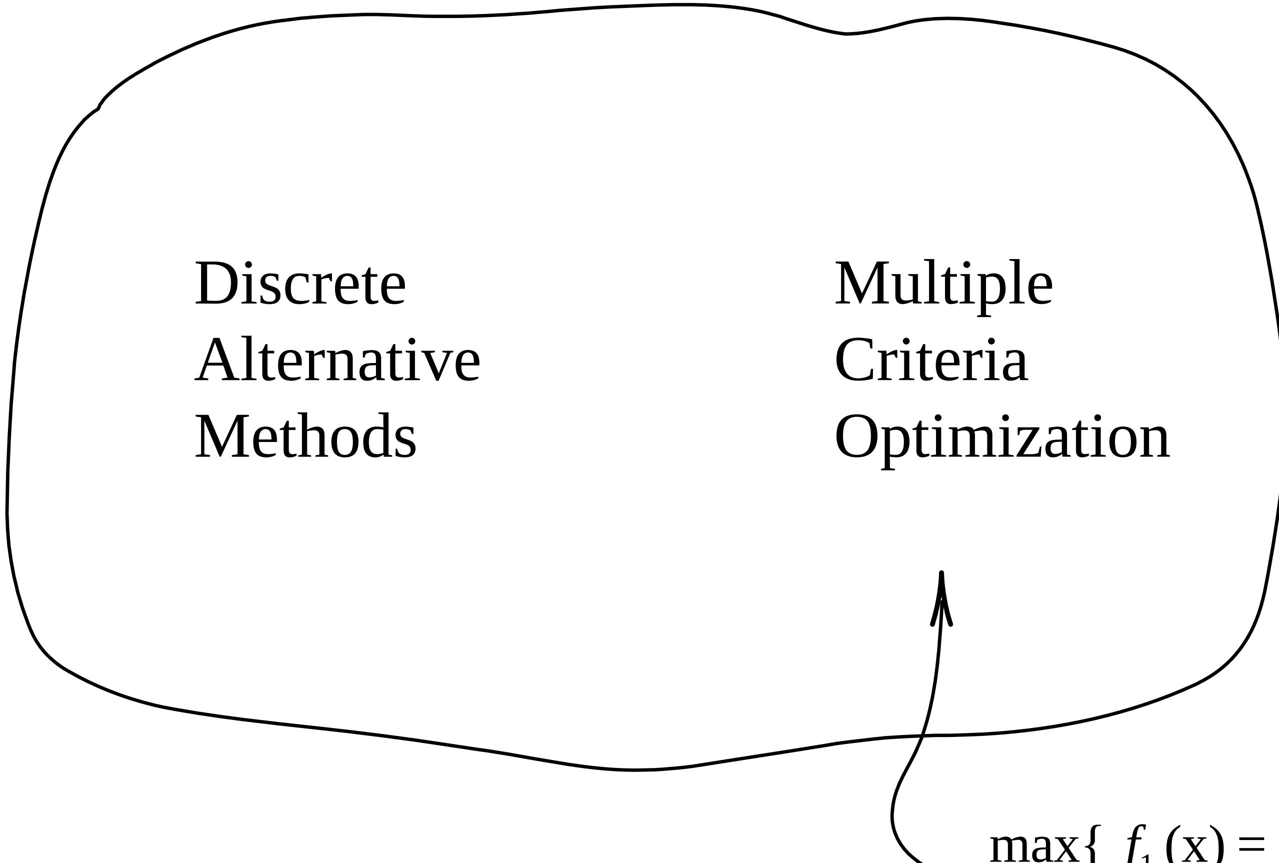
min { variance }

max { expected return }

max { dividends }

max { liquidity }

max { social responsibility }



$$\begin{aligned}
&\max\{\ f_1(\mathrm{x}) = z_1\ \}\\
&\quad\vdots\\
&\max\{\ f_k(\mathrm{x}) = z_k\ \}\\
&s.t. \qquad \mathrm{x} \in S
\end{aligned}$$

1. **Decision Space vs. Criterion Space**
2. **Contenders for Optimality**
3. **Criterion and Semi-Positive Polar Cones**
4. **Graphical Detection of the Efficient Set**
5. **Graphical Detection of the Nondominated Set**
6. **Nondominated Set Detection with Min and Max Objectives**
7. **Image/Inverse Image Relationship and Collapsing**
8. **Unsupported Nondominated Criterion Vectors**

In the general case, we write

$$\begin{aligned}&\max\{\ f_1(\mathrm{x}) = z_1\ \}\\ &\quad\vdots\\ &\max\{\ f_k(\mathrm{x}) = z_k\ \}\\ &s.t. \qquad \mathrm{x} \in S\end{aligned}$$

But if all objectives and constraints are linear, we write

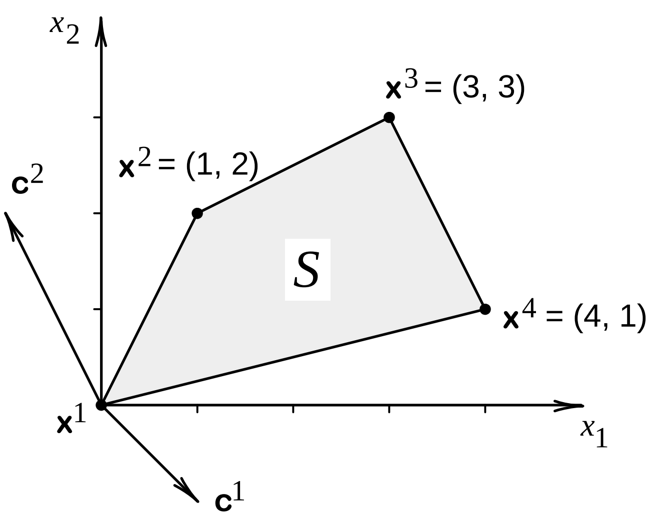
$$\begin{aligned}&\max\{\ \mathrm{c}^1\mathrm{x} = z_1\ \}\\ &\quad\vdots\\ &\max\{\ \mathrm{c}^k\mathrm{x} = z_k\ \}\\ &s.t. \qquad \mathrm{x} \in S\end{aligned}$$

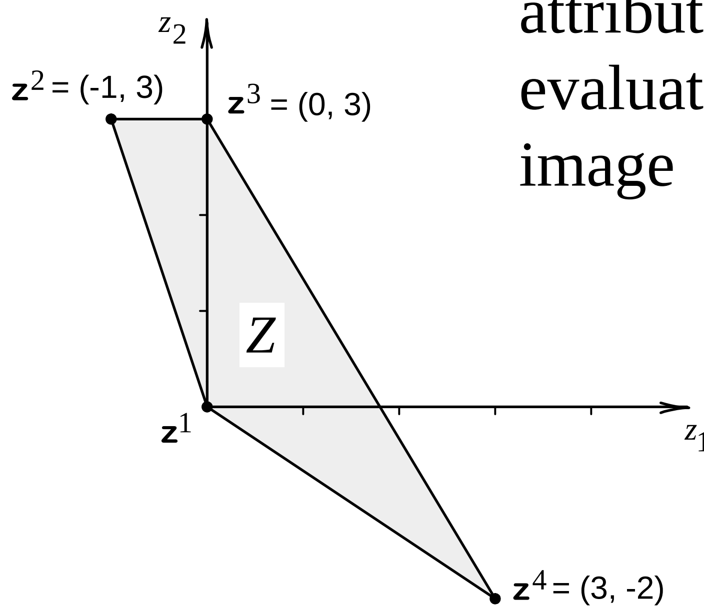
in which case we have a multiple objective linear program (MOLP).

1. Decision Space vs. Criterion Space

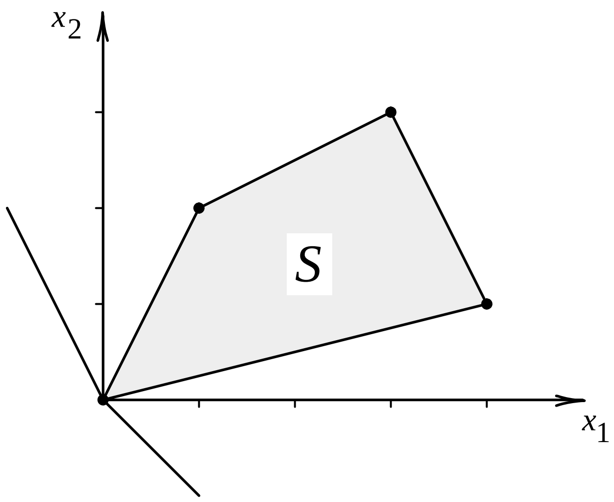
$$\max\{\ \ x_1 - x_2 = z_1\}$$
$$\max\{-x_1 + 2x_2 = z_2\}$$
$$s.t. \qquad \mathrm{x} \in S$$

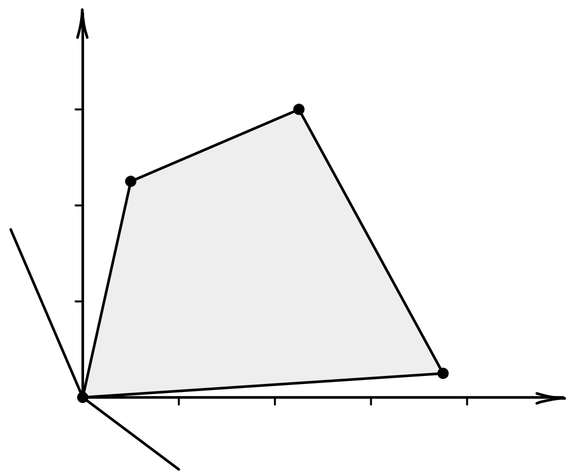
criterion
objective
outcome
attribute
evaluation
image

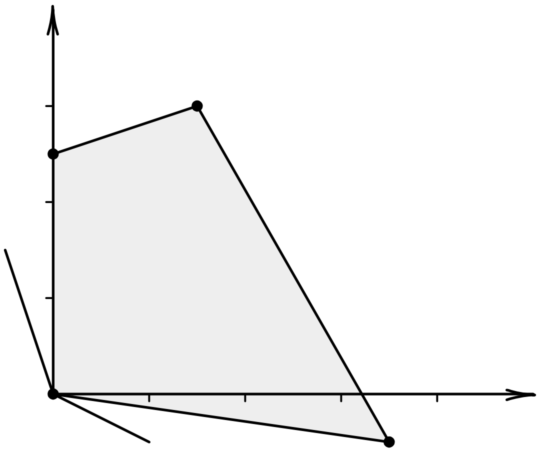


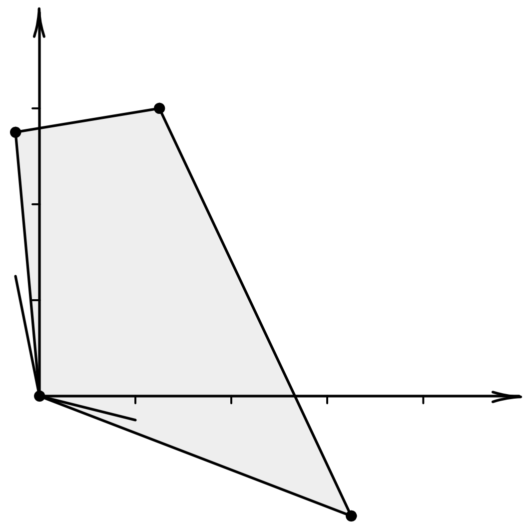


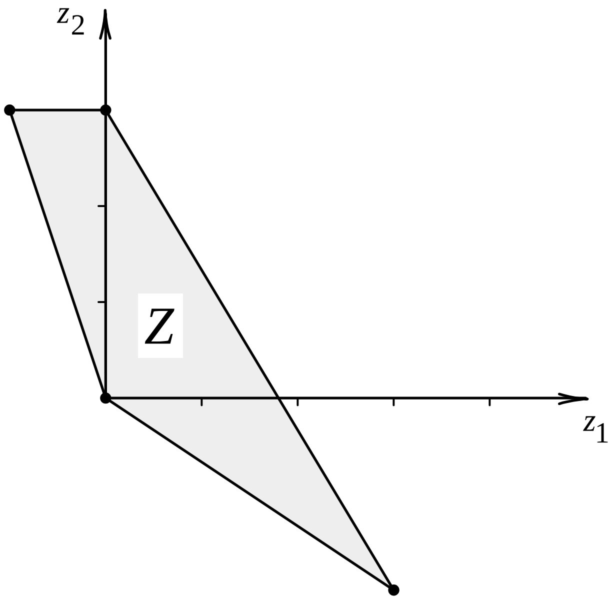
Morphing of S into Z as we change coordinate system










z_2
Z
z_1

2. Contenders for Optimality

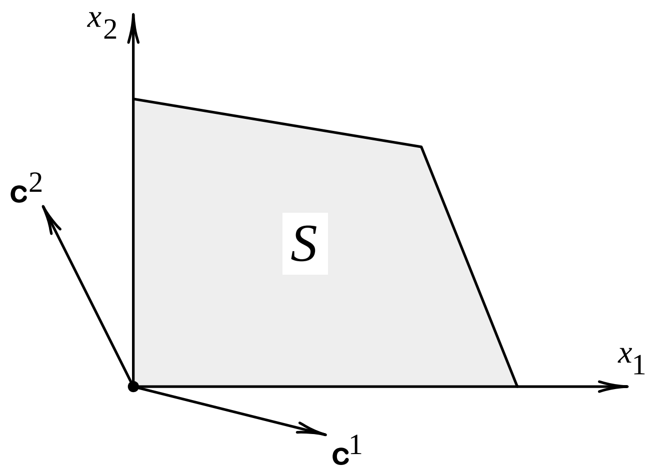
Points (criterion vectors) in criterion space are either nondominated or dominated.

Their points in decision space are either efficient or inefficient.

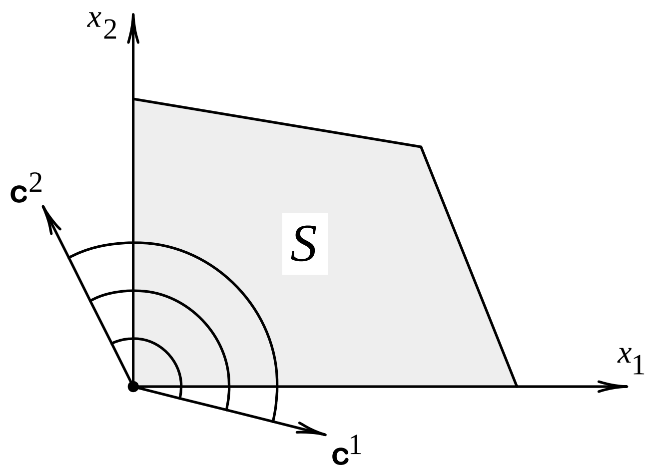
We are interested in nondominated criterion vectors and their efficient points because only they are contenders for optimality.

3. Criterion and Nonnegative Polar Cones

<u>Criterion cone</u> -- convex cone generated by the gradients of the objective functions.



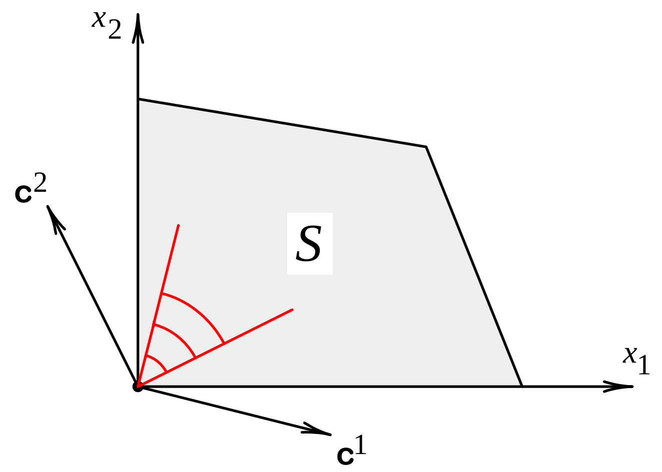
The larger the criterion cone (i.e., the more conflict there is in the problem), the bigger the efficient set.



Nonnegative polar of the criterion cone -- set of vectors that make 90° or less angle with all objective function gradients. In the case of an MOLP, given by

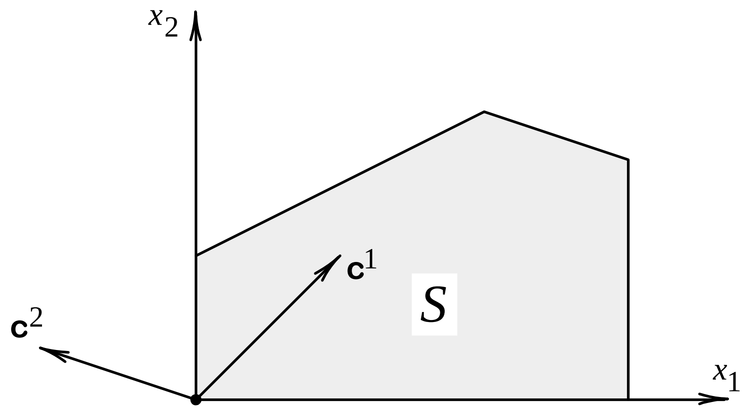
$$\{ \mathrm{y} \in R^n : \; \mathrm{c}^i \mathrm{y} \geq 0, \; i = 1, \ldots, k \}$$

Contains all points that dominate its vertex.

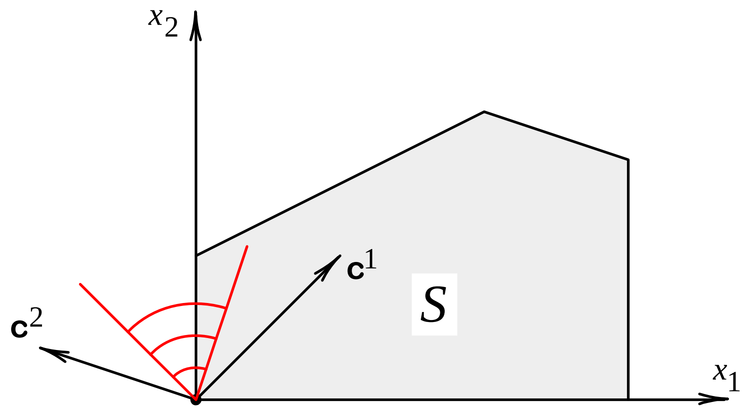


4. Graphical Detection of the Efficient Set

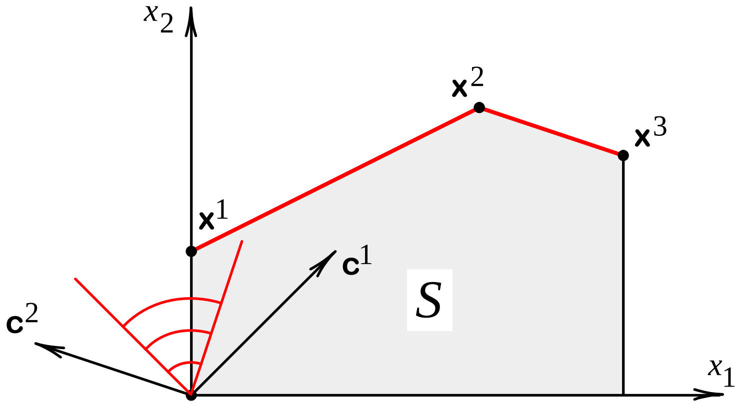
Example 1



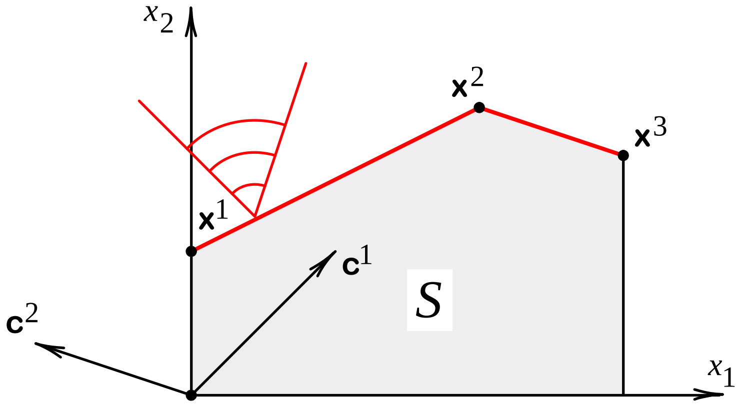
observe the criterion cone.



form nonnegative polar cone

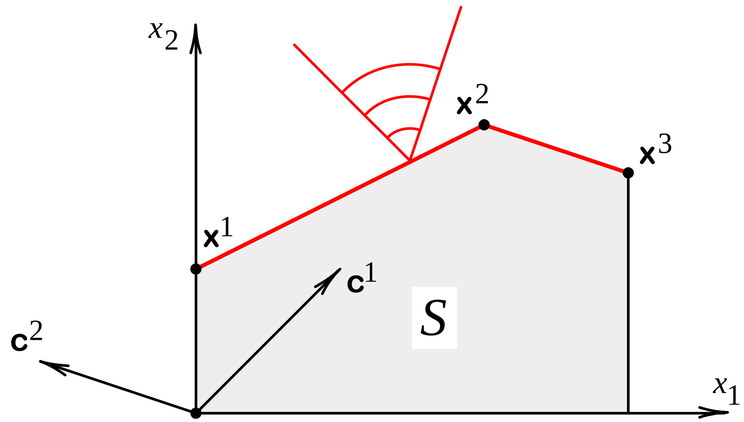


move it around



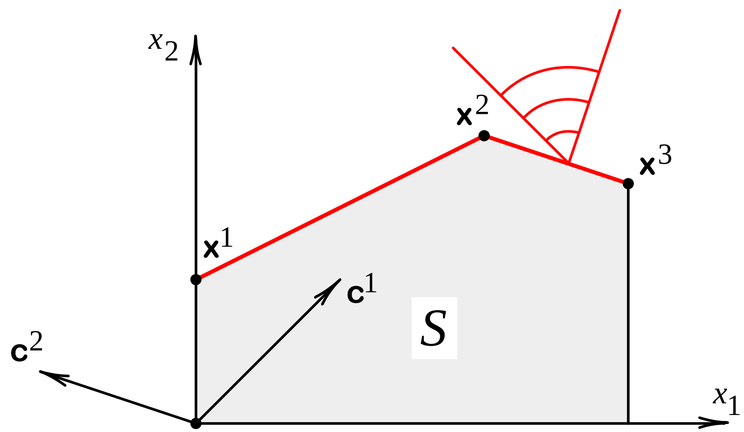
efficient set $\equiv E = \gamma[x^1, x^2] \cup \gamma[x^2, x^3]$

set of efficient extreme points $\equiv E_x = \{x^1, x^2, x^3\}$



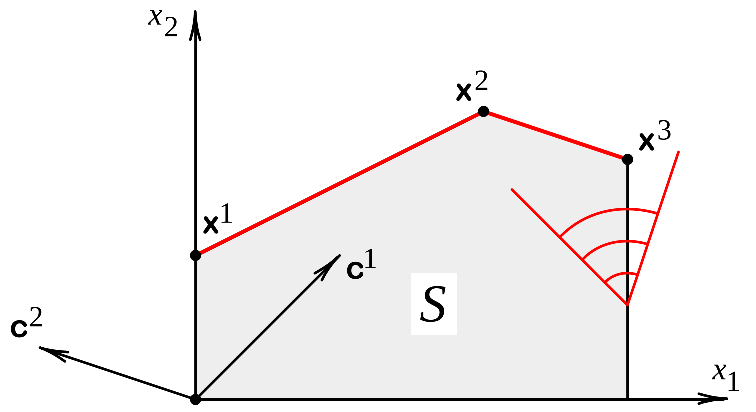
$$\text{efficient set} \equiv E = \gamma[x^1, x^2] \cup \gamma[x^2, x^3]$$

$$\text{set of efficient extreme points} \equiv E_x = \{x^1, x^2, x^3\}$$



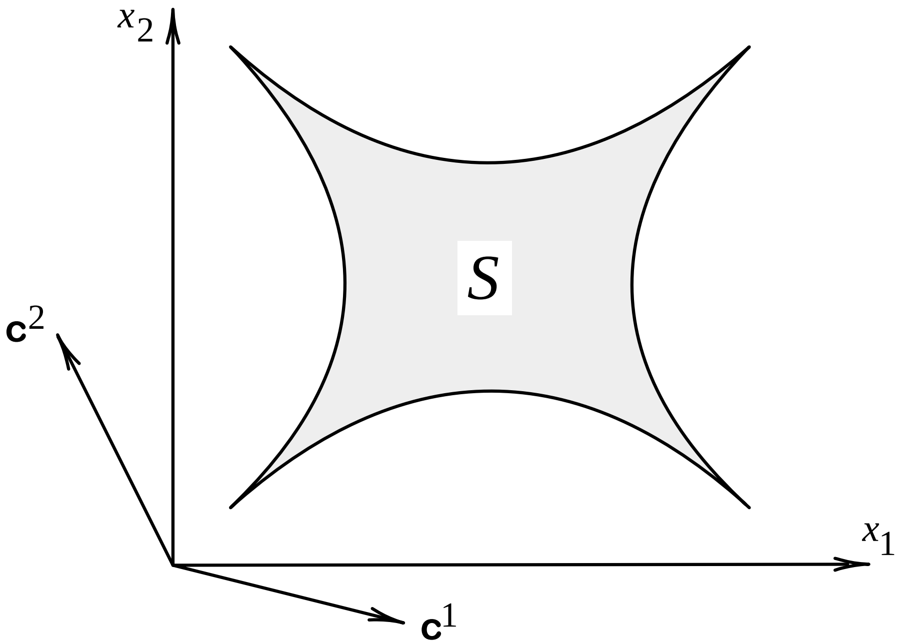
$$\text{efficient set} \equiv E = \gamma[x^1, x^2] \cup \gamma[x^2, x^3]$$

$$\text{set of efficient extreme points} \equiv E_x = \{x^1, x^2, x^3\}$$

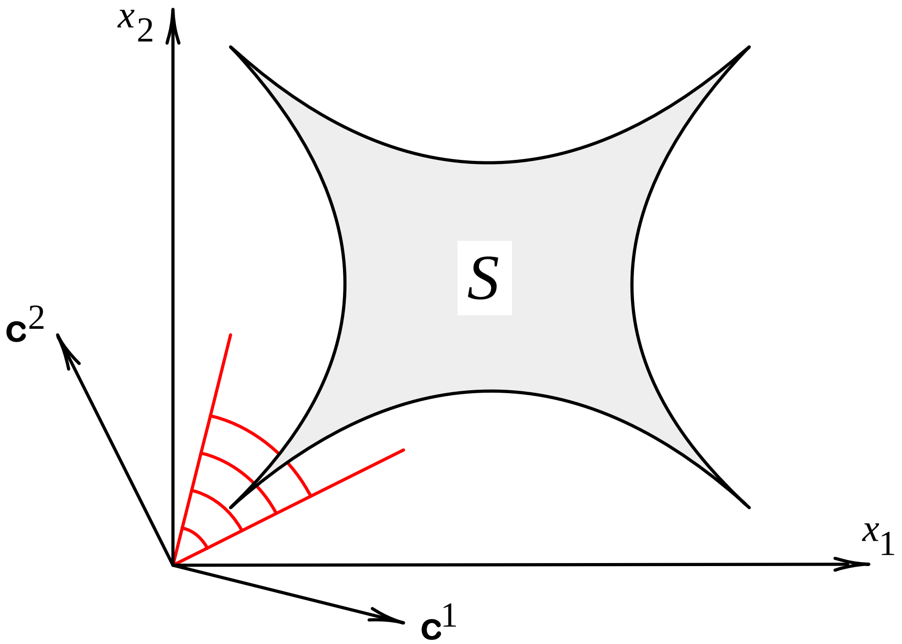


Only when there is no intersection at other than the vertex of the cone

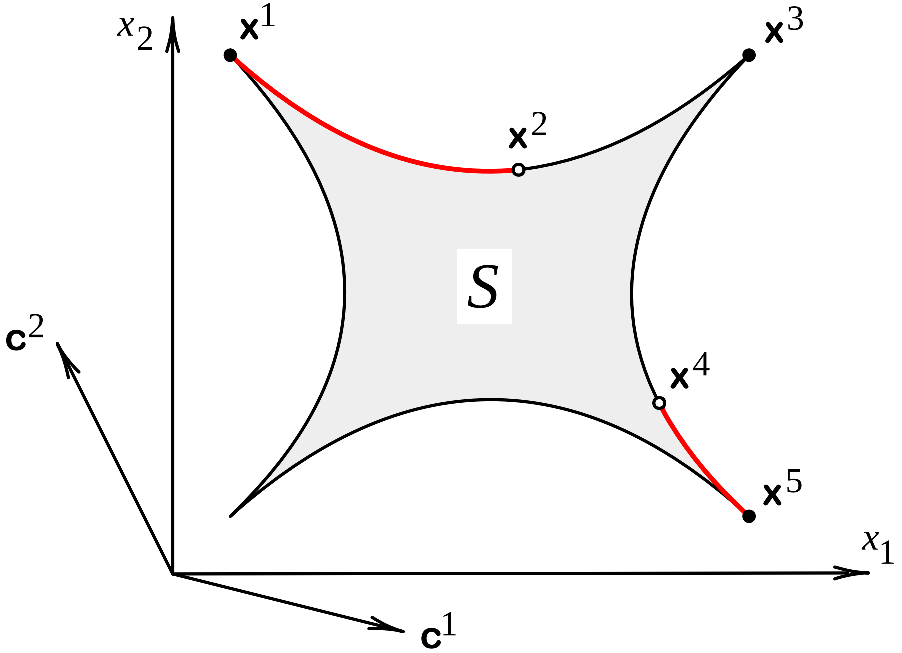
Example 2



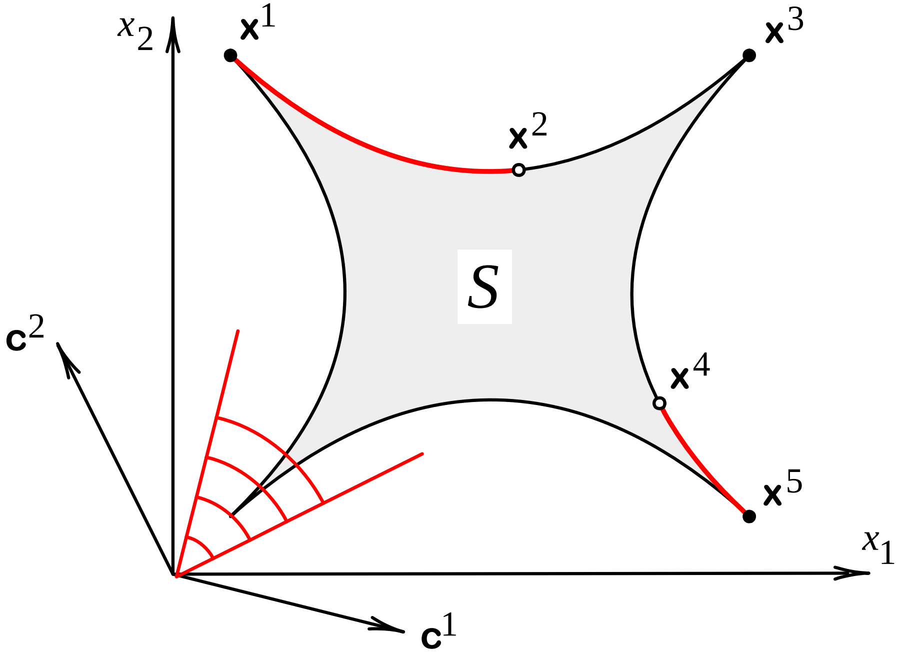
observe criterion cone



form nonnegative polar cone



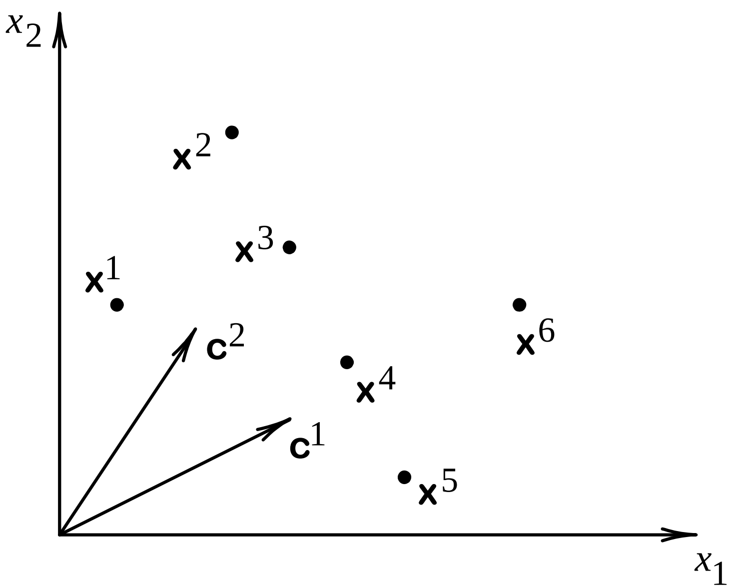
$$E = \partial[x^1, x^2) \cup \{x^3\} \cup \partial(x^4, x^5]$$



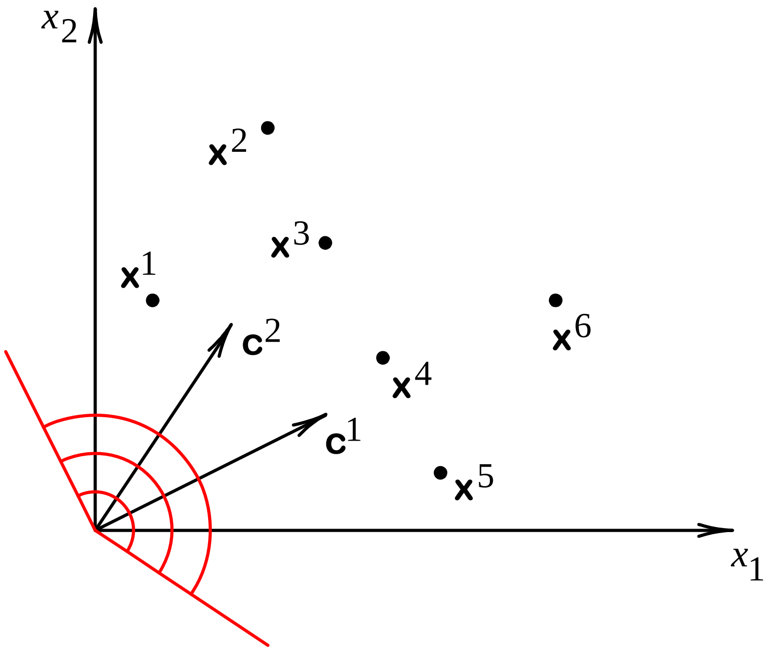
$$E = \partial[x^1, x^2) \cup \{x^3\} \cup \partial(x^4, x^5]$$

(observe that x^2 and x^4 are not efficient)

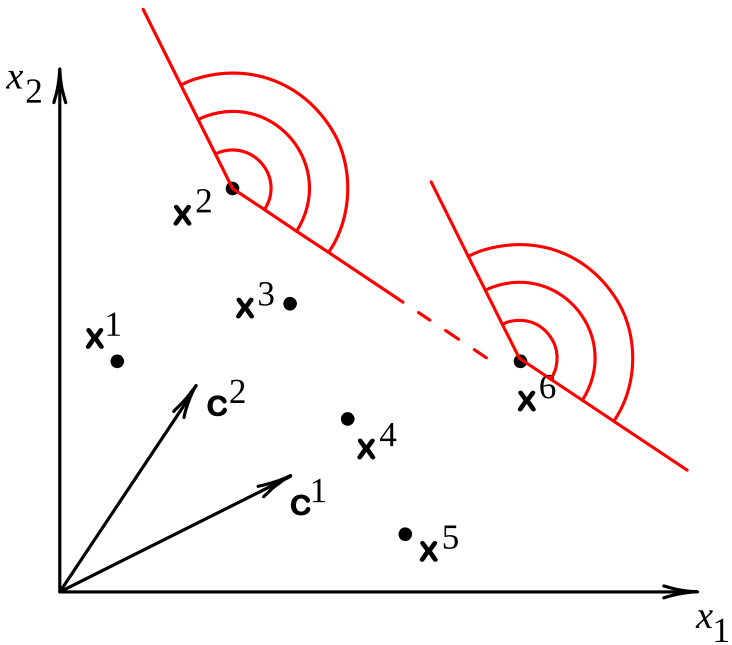
Example 3



Note small size of criterion cone and that S consists of only 6 points.



Small criterion cone results in a large nonnegative polar cone.
(this makes it harder for points to be efficient).

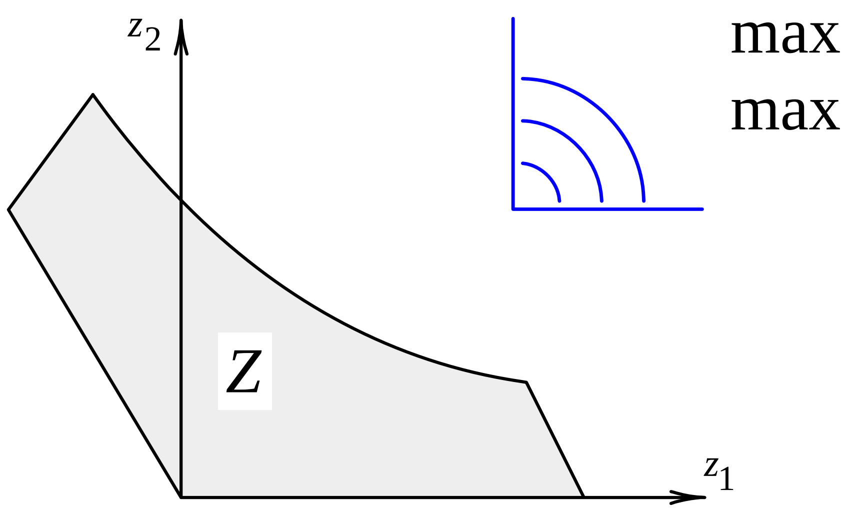


Moving nonnegative polar cone around.

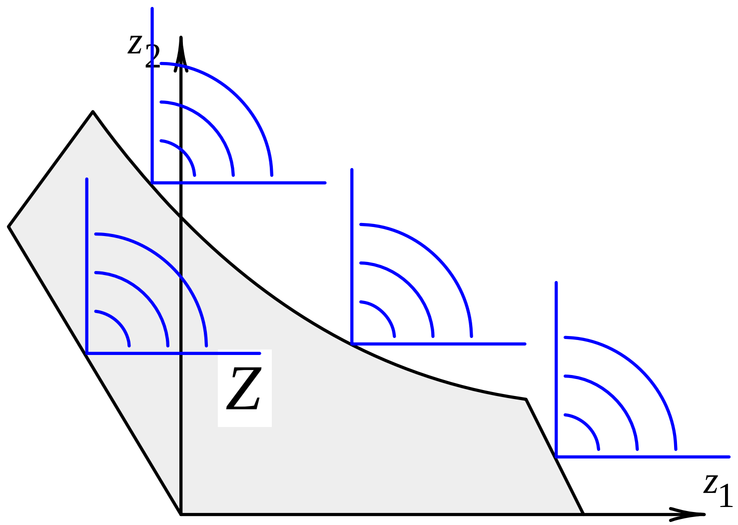
$$E = \{x^6\}$$

5. Graphical Detection of the Nondominated Set

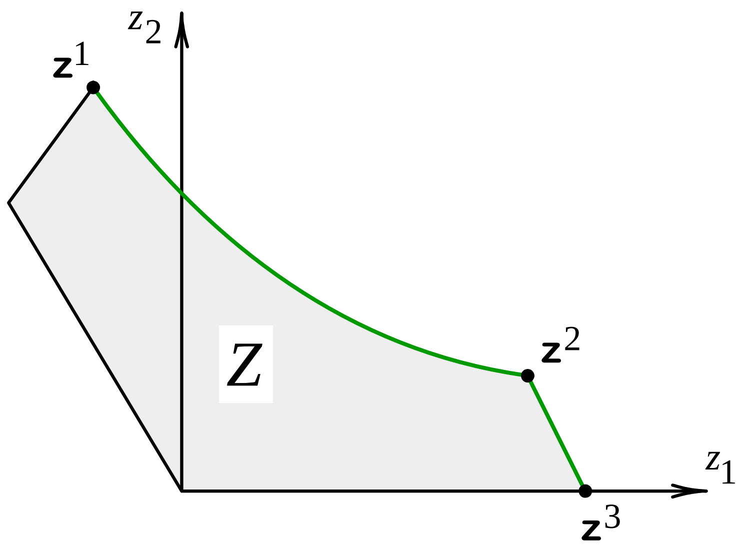
To determine if a criterion vector in *Z* is nondominated, translate nonnegative orthant in R^k to the point.



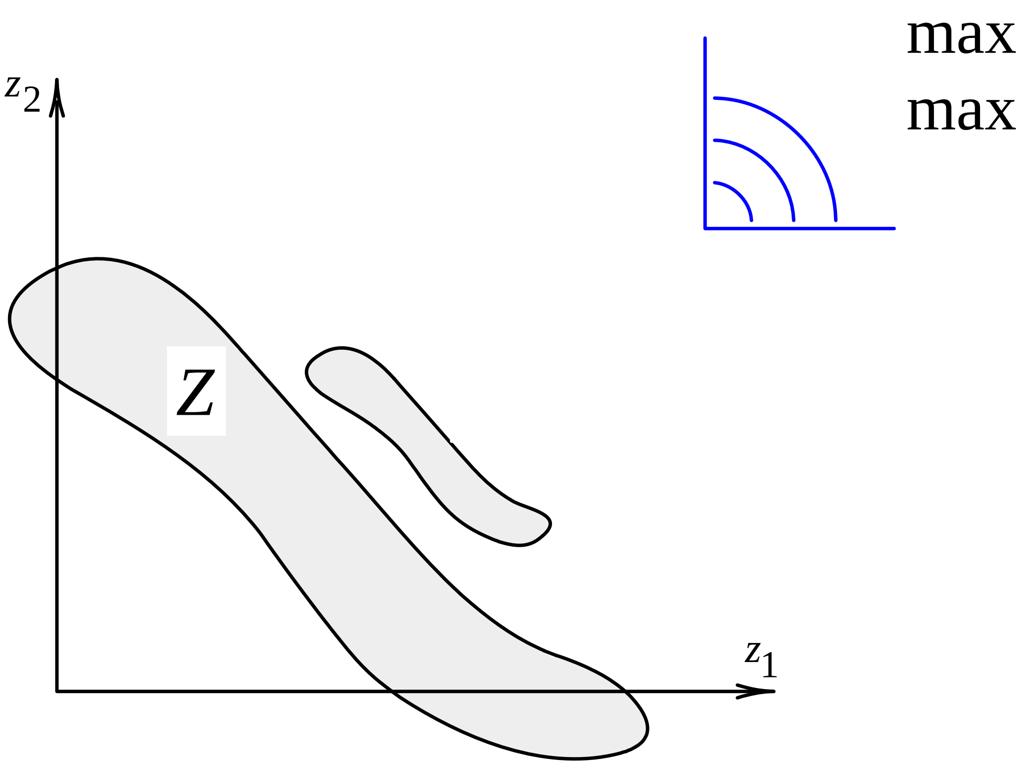
move nonnegative orthant around



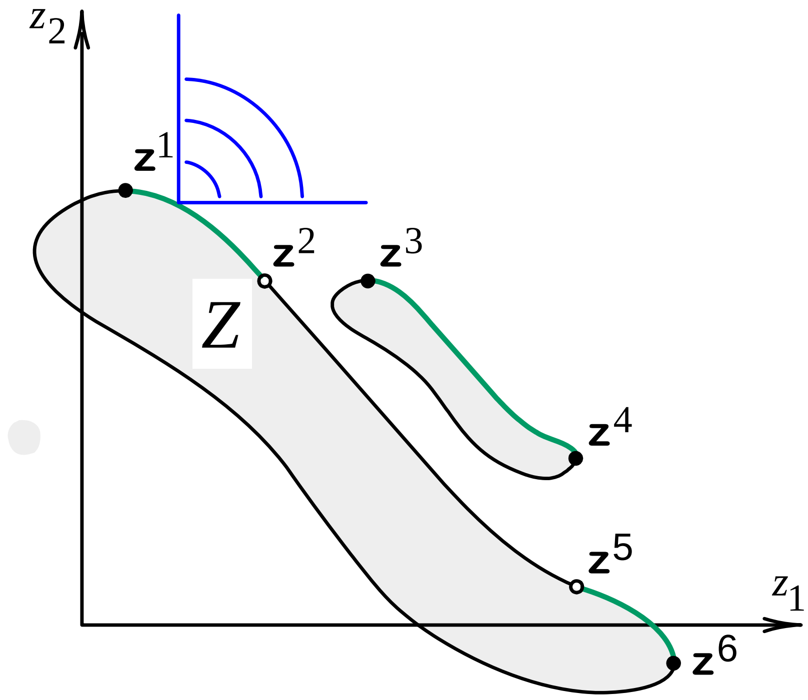
try to identify the entire nondominated set



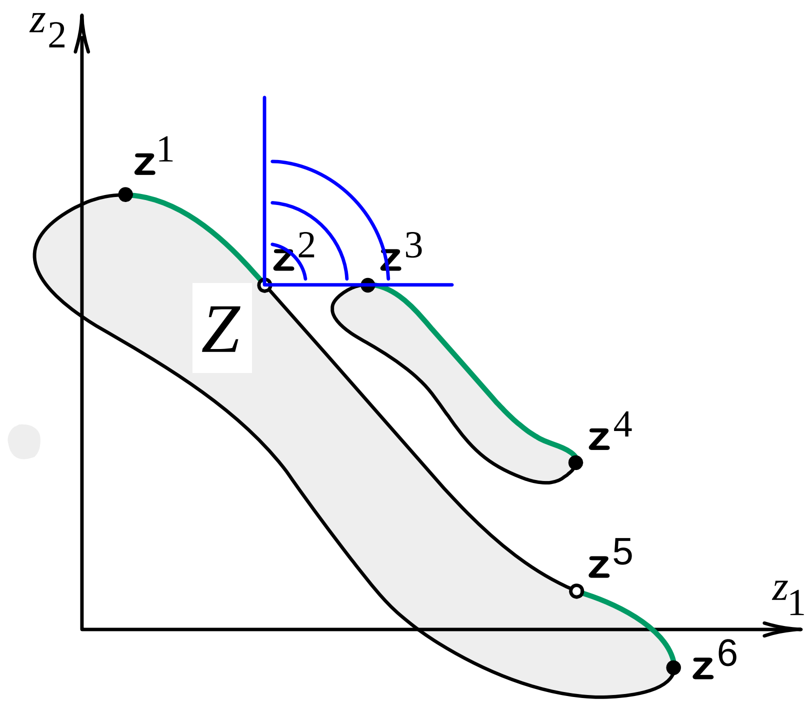
nondominated set $\equiv N = \partial[z^1, z^2] \cup \gamma[z^2, z^3]$



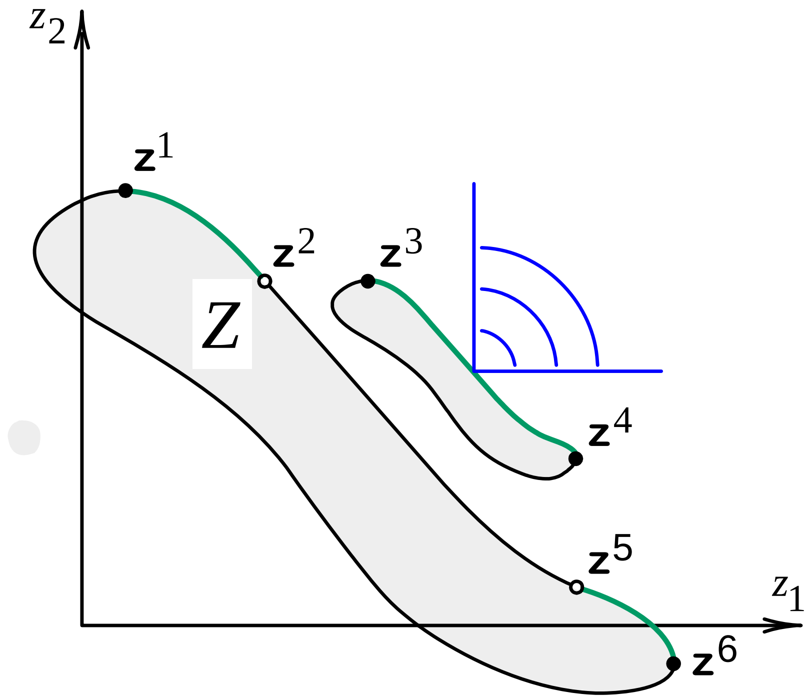
Now, move nonnegative orthant around.



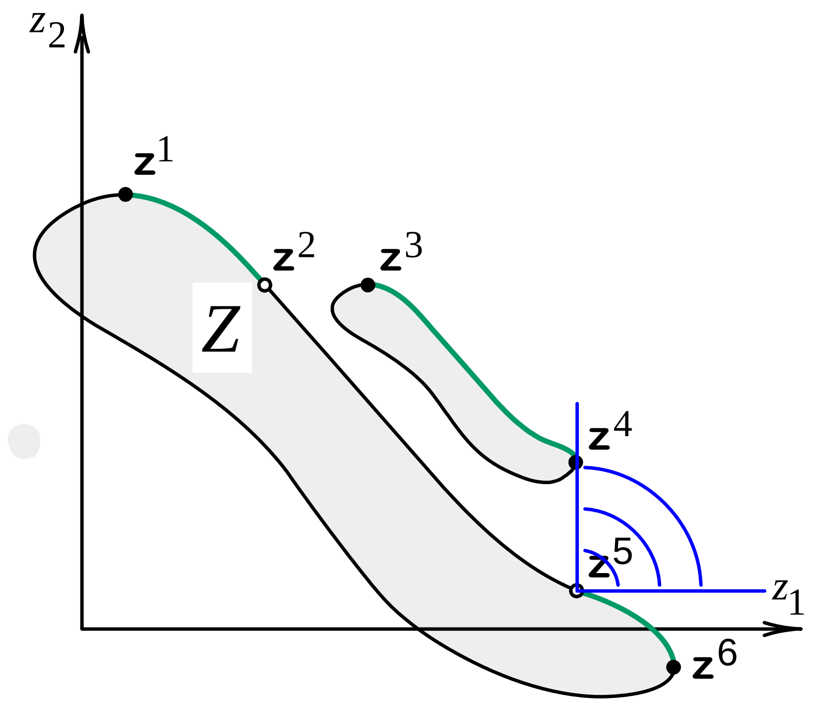
z_2
$\mathbf{z}^1$
$\mathbf{z}^2$
$\mathbf{z}^3$
Z
$\mathbf{z}^4$
$\mathbf{z}^5$
z_1
$\mathbf{z}^6$

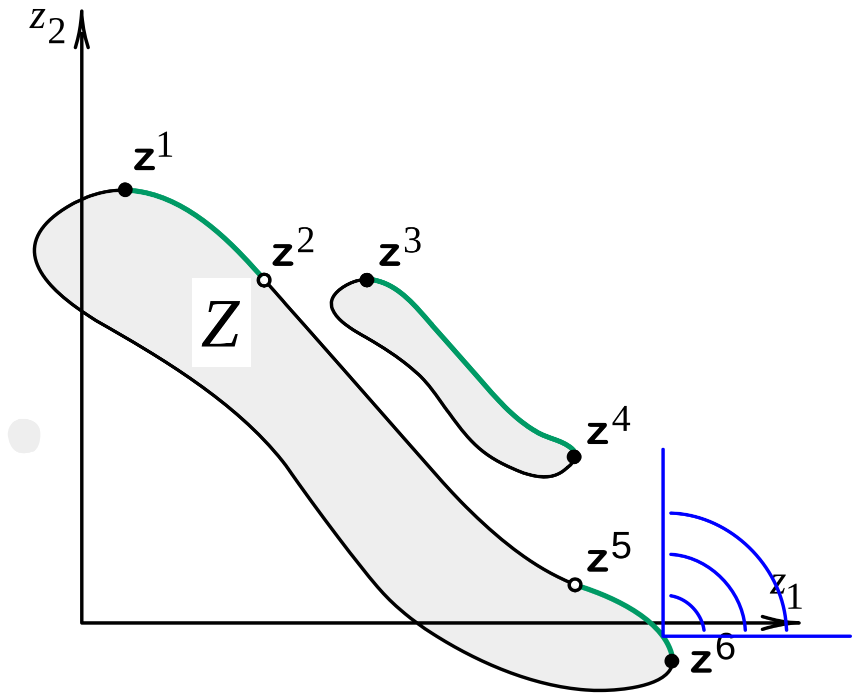


z_2
$\mathbf{z}^1$
$\mathbf{z}^2$
$\mathbf{z}^3$
Z
$\mathbf{z}^4$
$\mathbf{z}^5$
z_1
$\mathbf{z}^6$



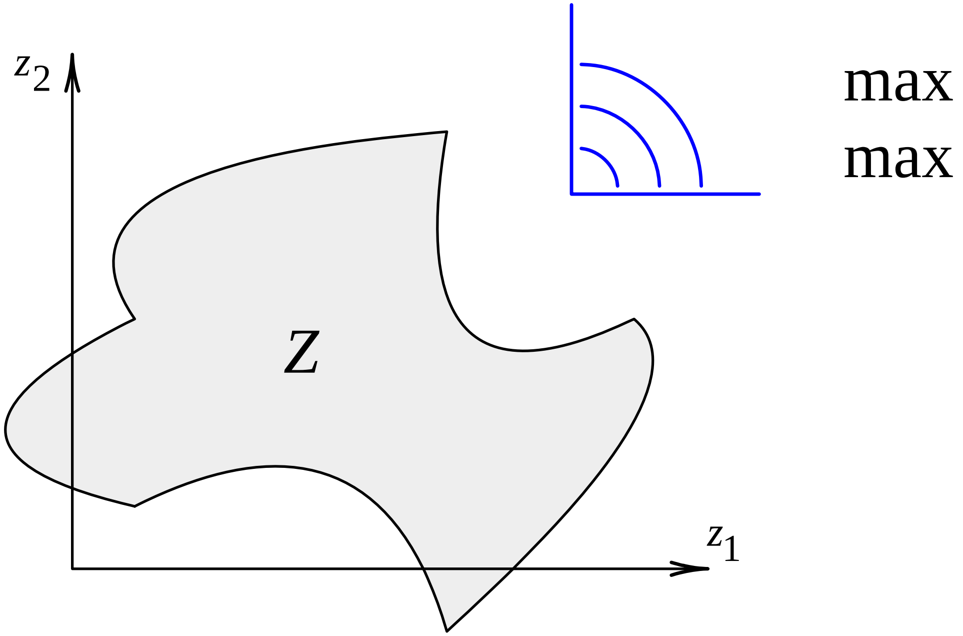
z_2
$\mathbf{z}^1$
$\mathbf{z}^2$
$\mathbf{z}^3$
Z
$\mathbf{z}^4$
$\mathbf{z}^5$
z_1
$\mathbf{z}^6$

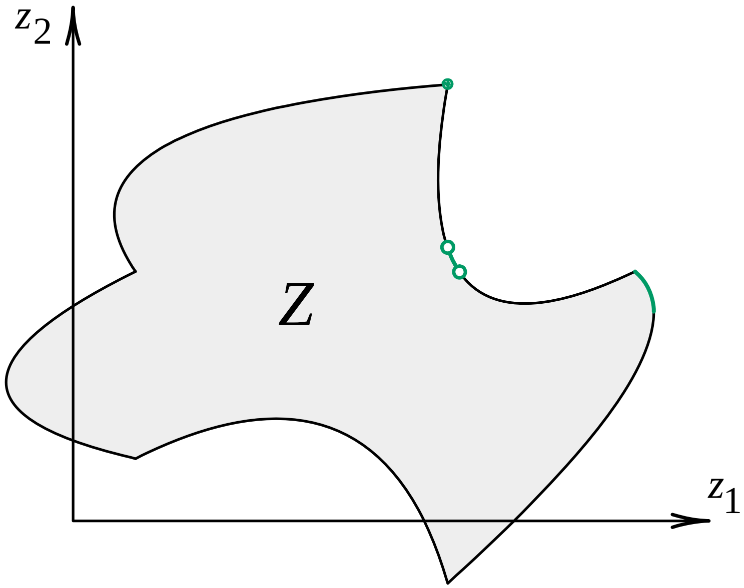

z_2
$\mathbf{z}^1$
$\mathbf{z}^2$
$\mathbf{z}^3$
Z
$\mathbf{z}^4$
$\mathbf{z}^5$
z_1
$\mathbf{z}^6$



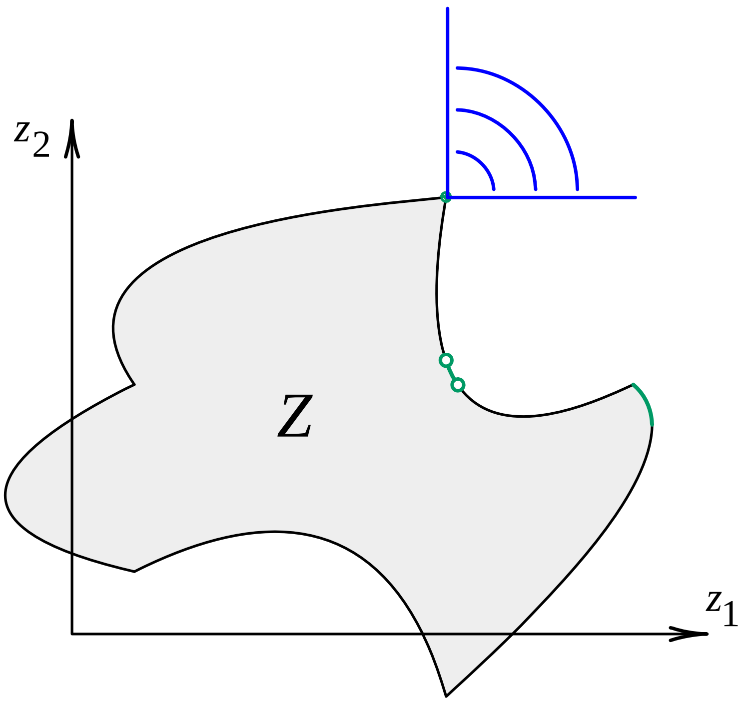
$$N = \partial[z^1, z^2) \cup \partial[z^3, z^4] \cup \partial(z^5, z^6]$$

6. Nondominated Set Detection with Min and Max Objectives

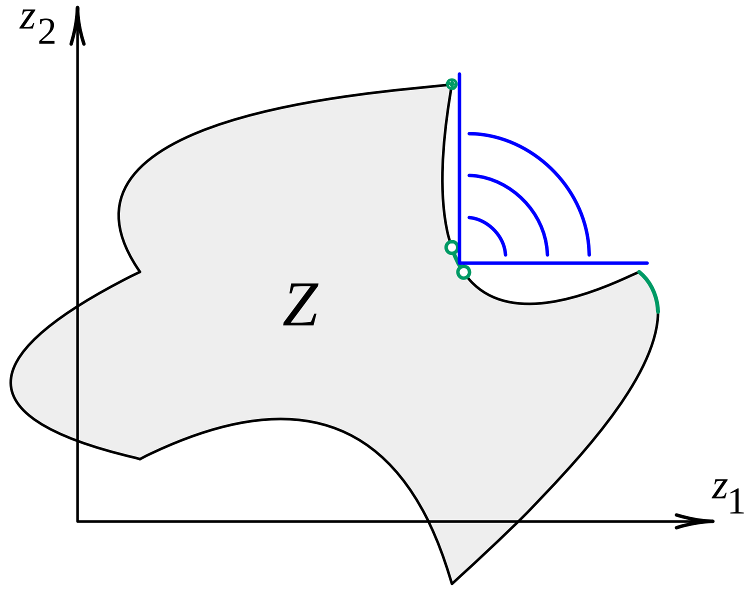




max
max

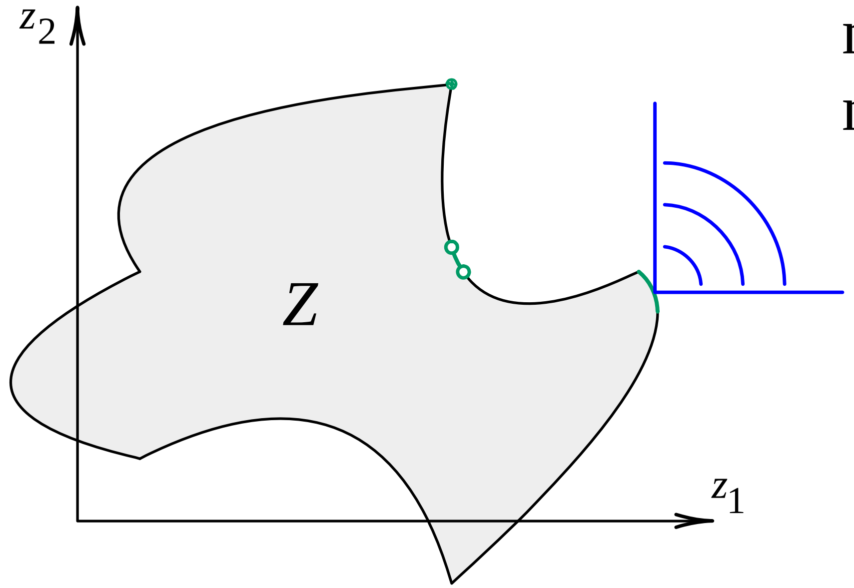


max
max

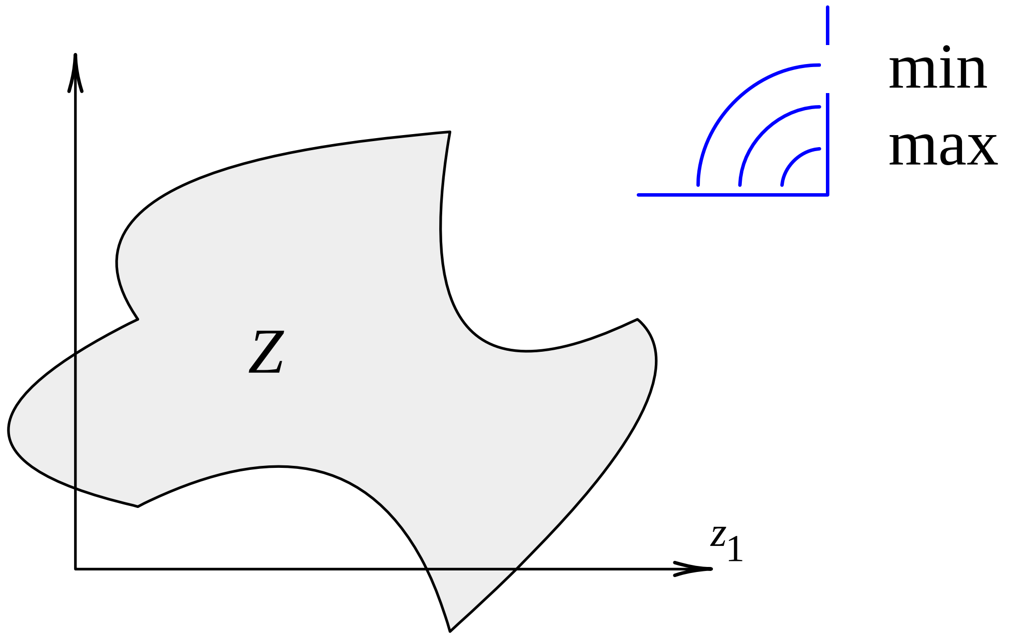


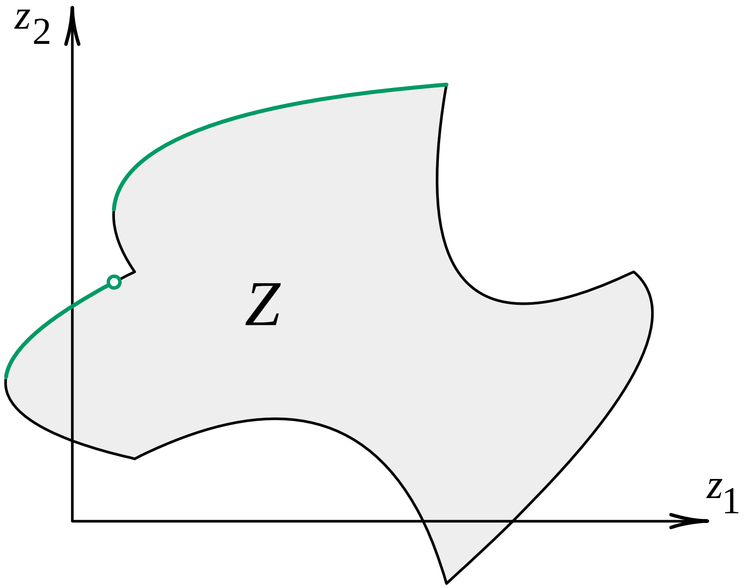
max

max

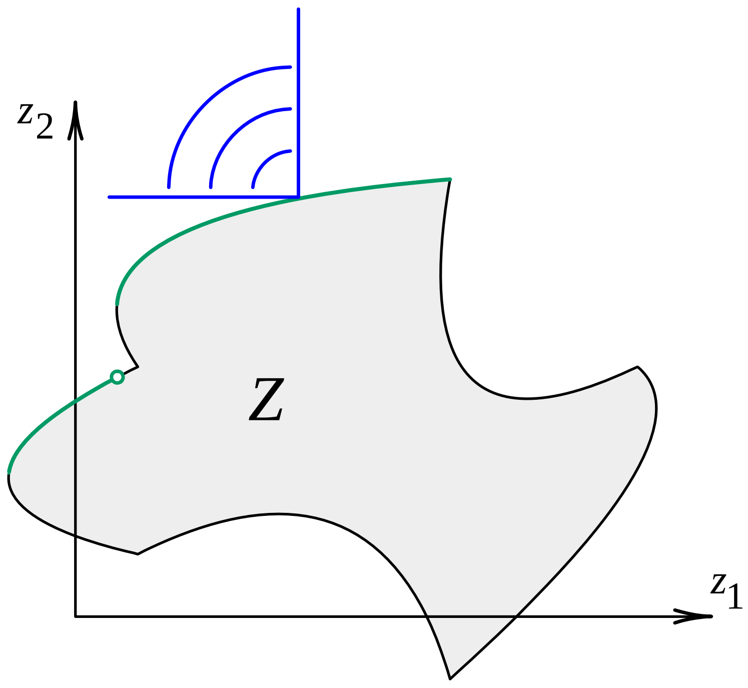


max
max


min
max
Z
z_1

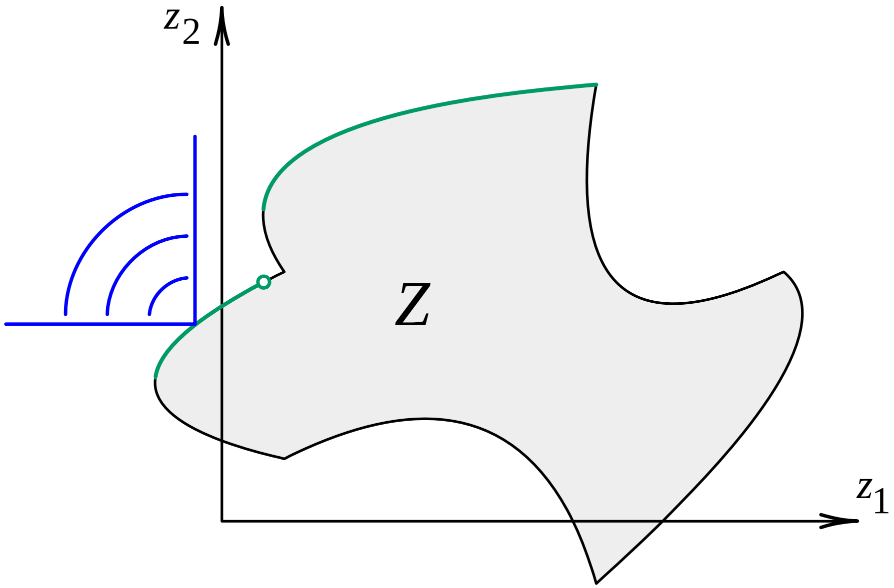


min
max



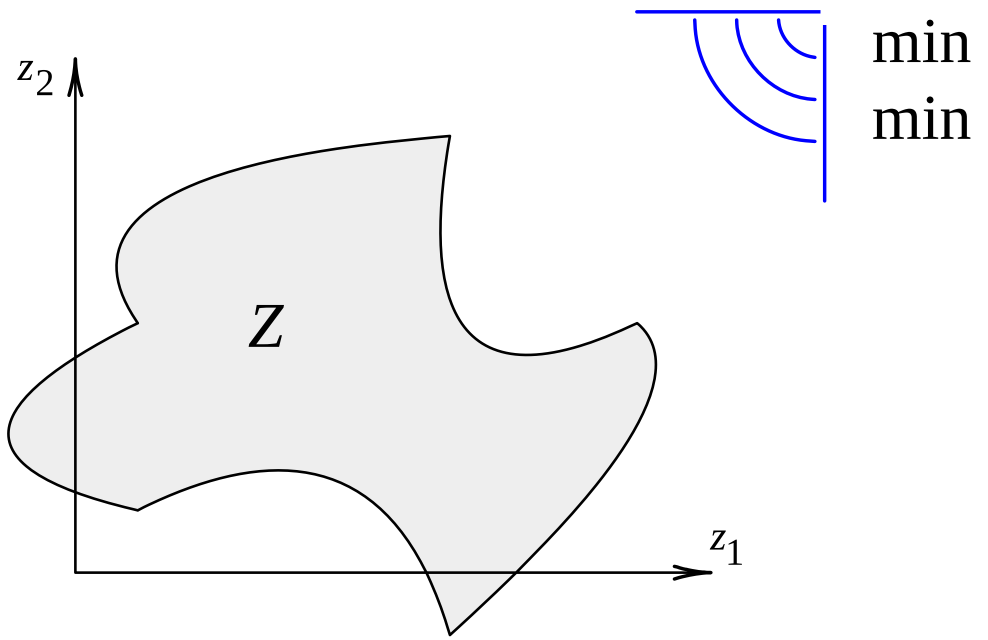
min

max

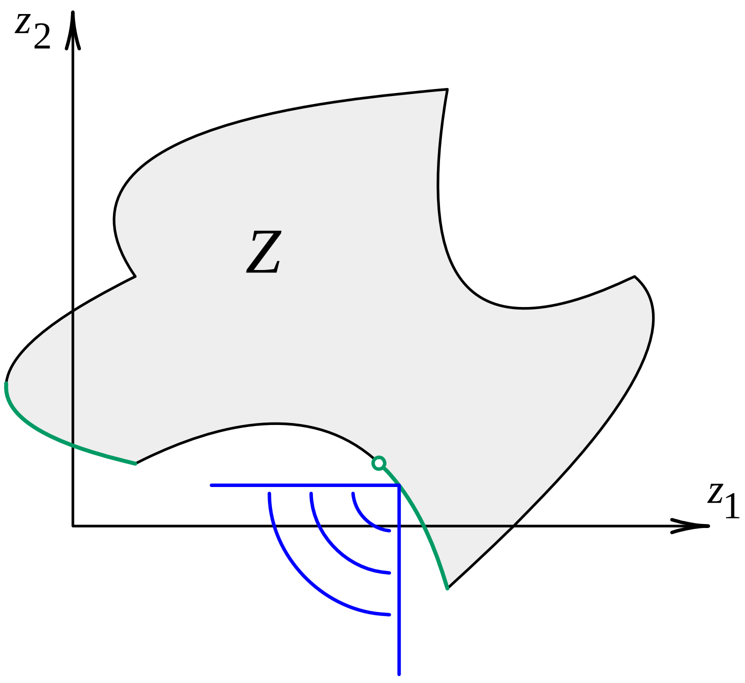


min

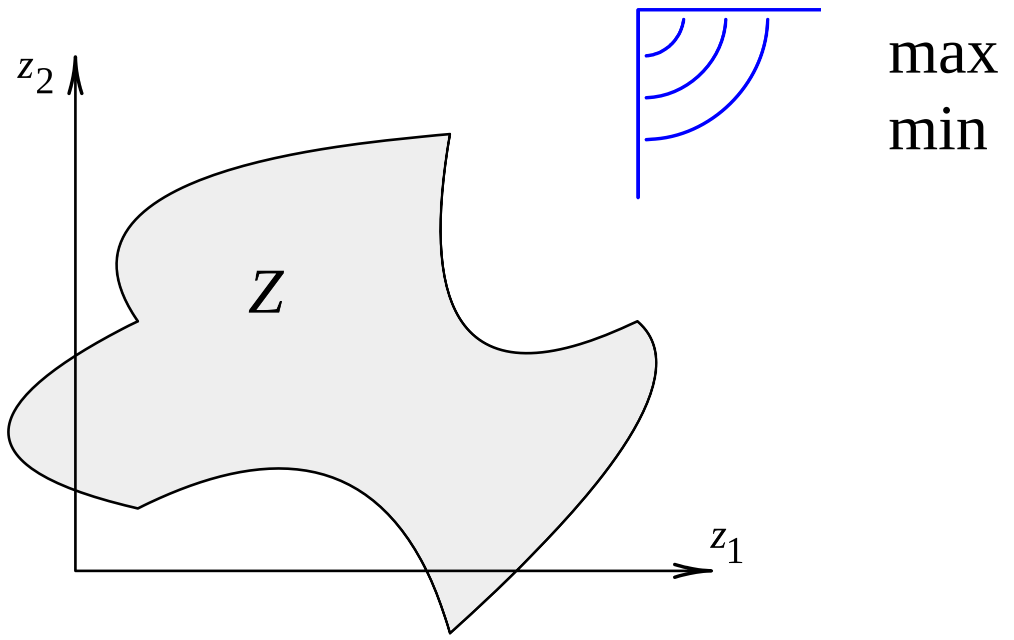
max

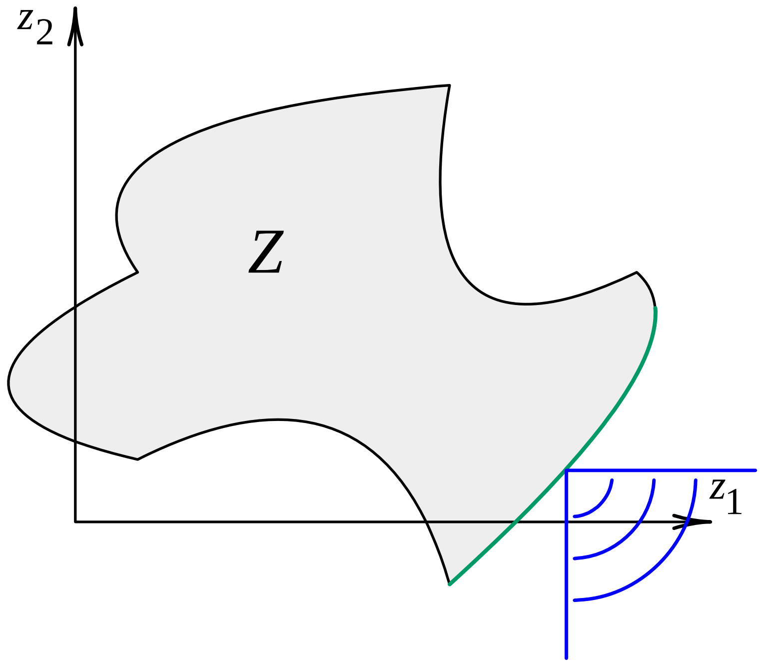


min
min
z_2
Z
z_1



min
min


z_2
z_1
Z
max
min



max
min

Let $\bar{z} \in Z$. Then $\bar{z}$ is *nondominated* if and only if there does not exist another $z \in Z$ such that $z_i \geq \bar{z}_i$ for all i and $z_j > \bar{z}_j$ for at least one j. Otherwise, $\bar{z}$ is *dominated*.

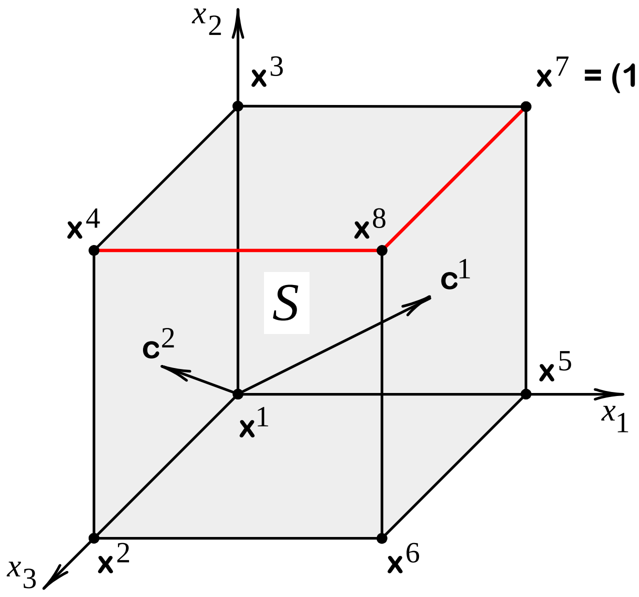
Let $\bar{x} \in S$. Then $\bar{x}$ is *efficient* if and only its criterion vector $\bar{z}$ is nondominated. Otherwise, $\bar{x}$ is *inefficient*.

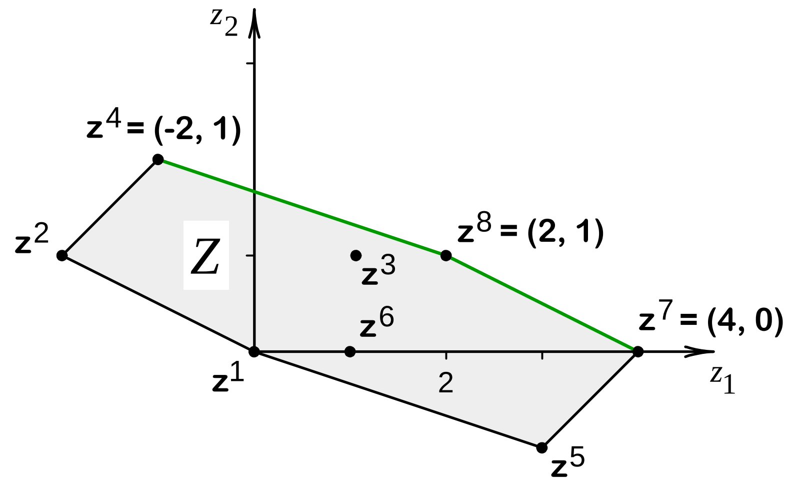
In other words,

image of an efficient point is a nondominated criterion vector
inverse image of a nondominated criterion vector is an eff point

7. Image/Inverse Image Relationship and Collapsing

$$\max\{3x_1 + x_2 - 2x_3 = z_1\}$$
$$\max\{-x_1 + x_2 + x_3 = z_2\}$$
$$s.t. \quad \mathrm{x} \in S = \text{unit cube}$$





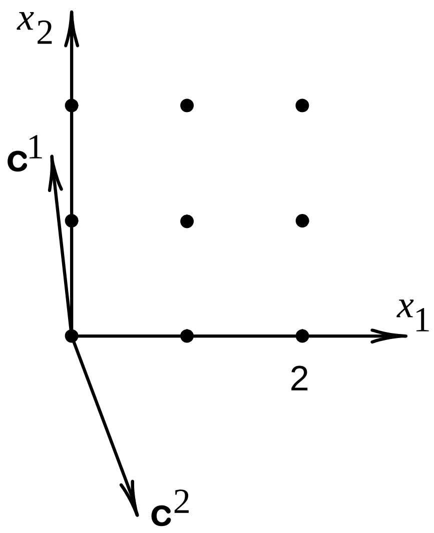
dimensionality of S is n, but dimensionality of Z is k.

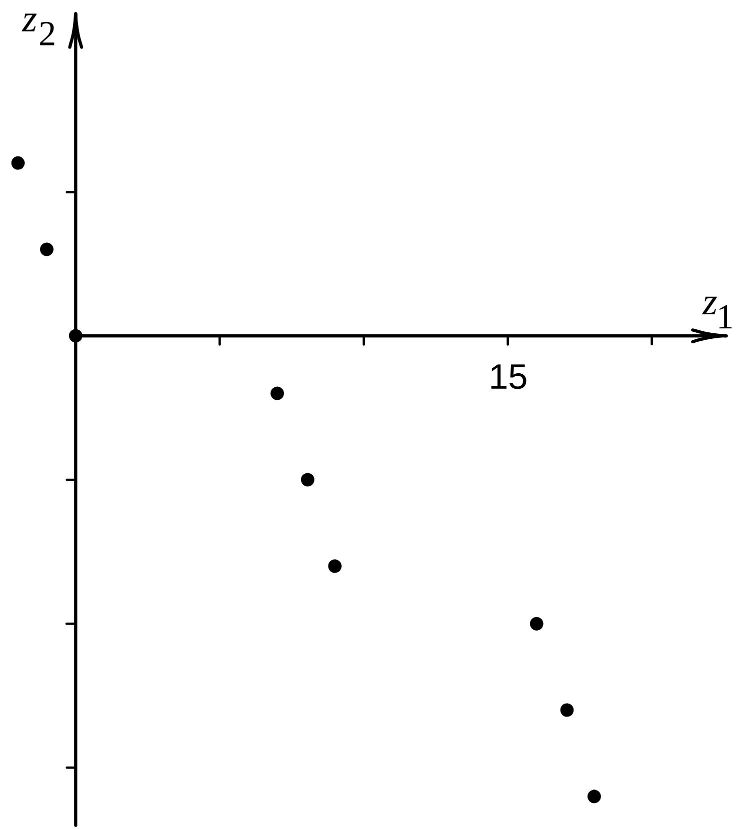
8. Unsupported Nondominated Criterion Vectors

A nondominated criterion vector is supported or unsupported.

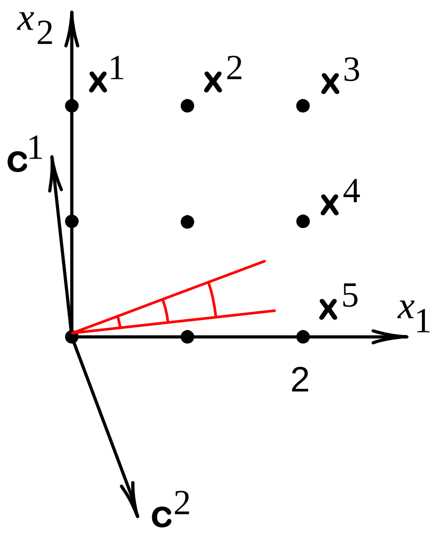
Unsupported if dominated by a convex combination of other feasible criterion vectors.

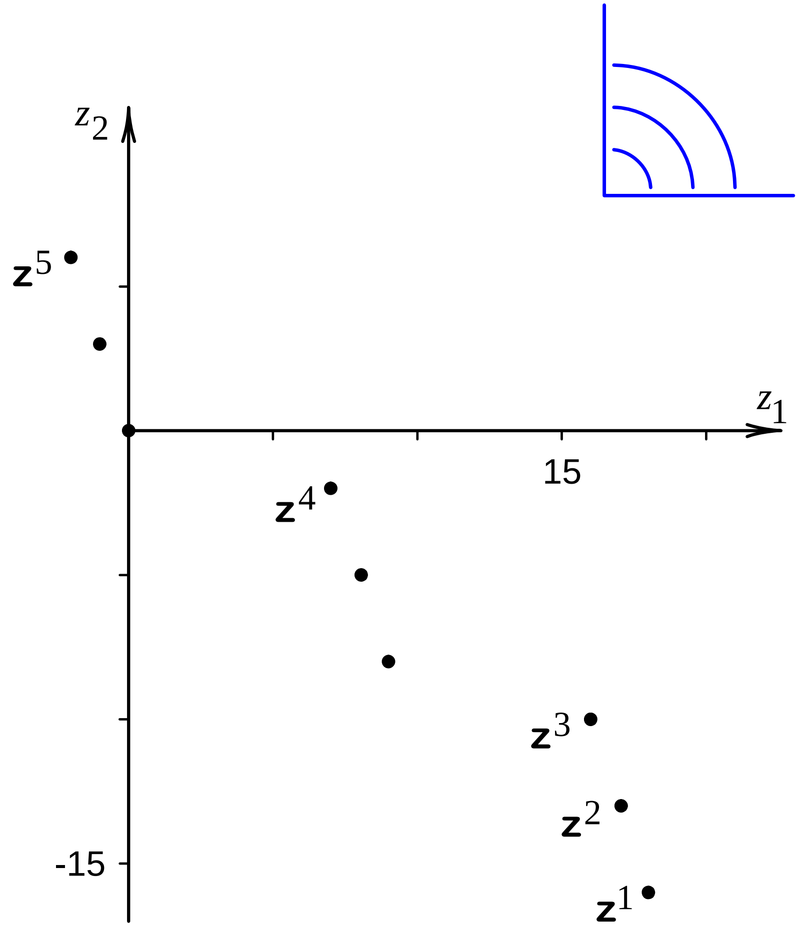
Unsupported nondominated criterion vectors are typically hard to compute.





$$\max\{-x_1 + 9x_2 = z_1\}$$
$$\max\{\ 3x_1 - 8x_2 = z_2\}$$
$$s.t. \qquad \mathrm{x} \in S$$





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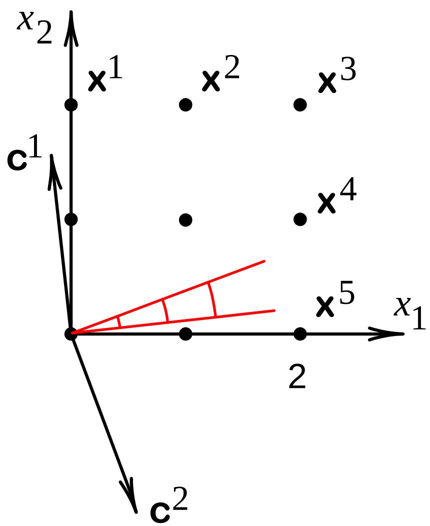
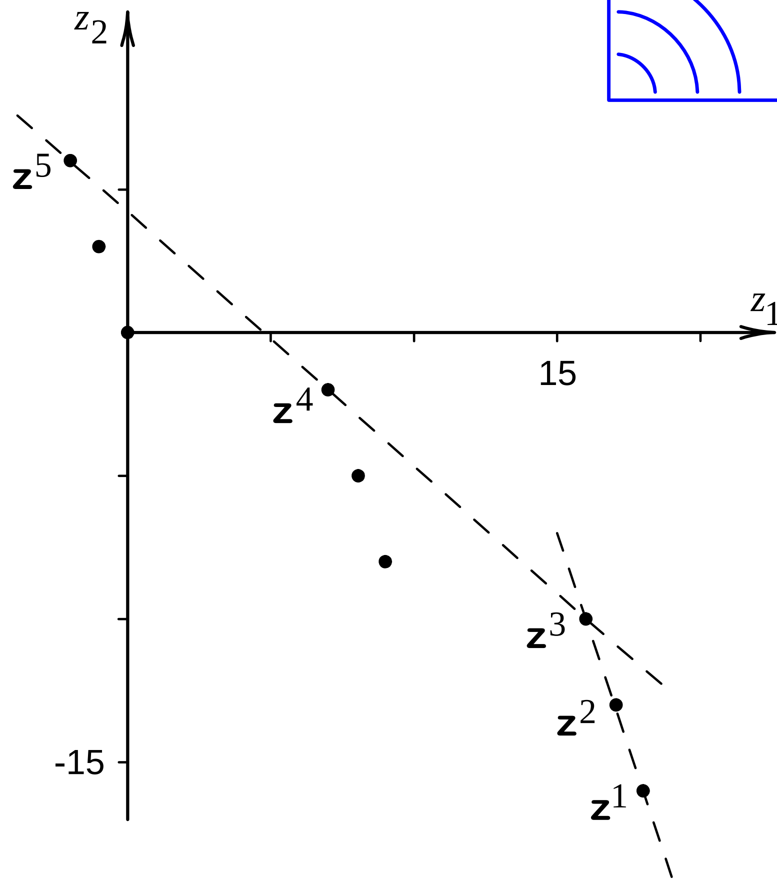
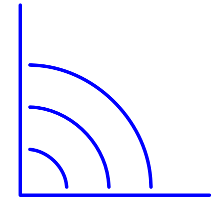
$$\max\{\ 3x_1 - 8x_2 = z_2\}$$

$$s.t. \qquad \mathrm{x} \in S$$







$$\max\{-x_1+9x_2 = z_1\}$$

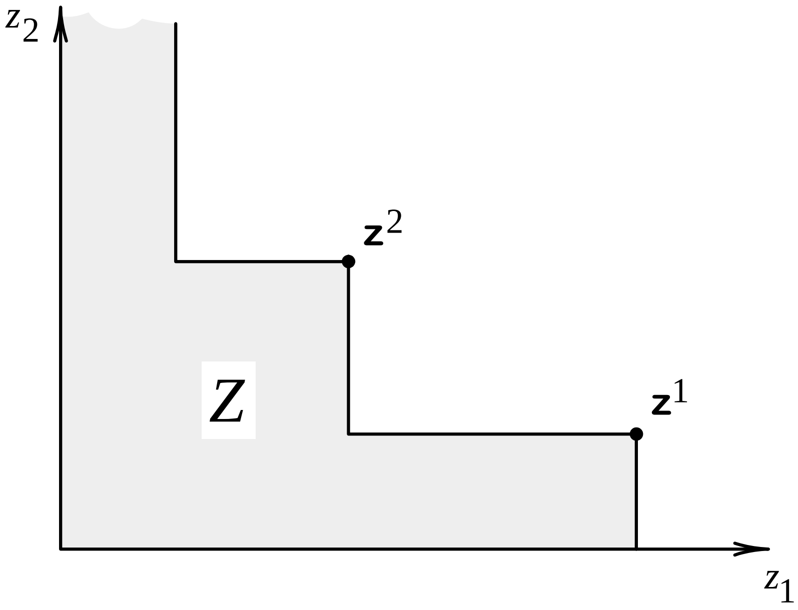
$$\max\{\ 3x_1-8x_2 = z_2\}$$

$$s.t. \qquad \mathrm{x} \in S$$

$$\mathrm{N} = Z$$

$$\mathrm{N}^{\mathrm{supp}} = \{\mathrm{z}^1, \mathrm{z}^2, \mathrm{z}^3, \mathrm{z}^4, \mathrm{z}^5\}$$

$$\mathrm{N}^{\mathrm{unsupp}} = Z - \mathrm{N}^{\mathrm{supp}}$$



$$N^{supp} = \{ z^1 \}$$

$$N^{unsupp} = \{ z^2 \}$$

Multiple Criteria Optimization: An Introduction (Continued)

Ralph E. Steuer
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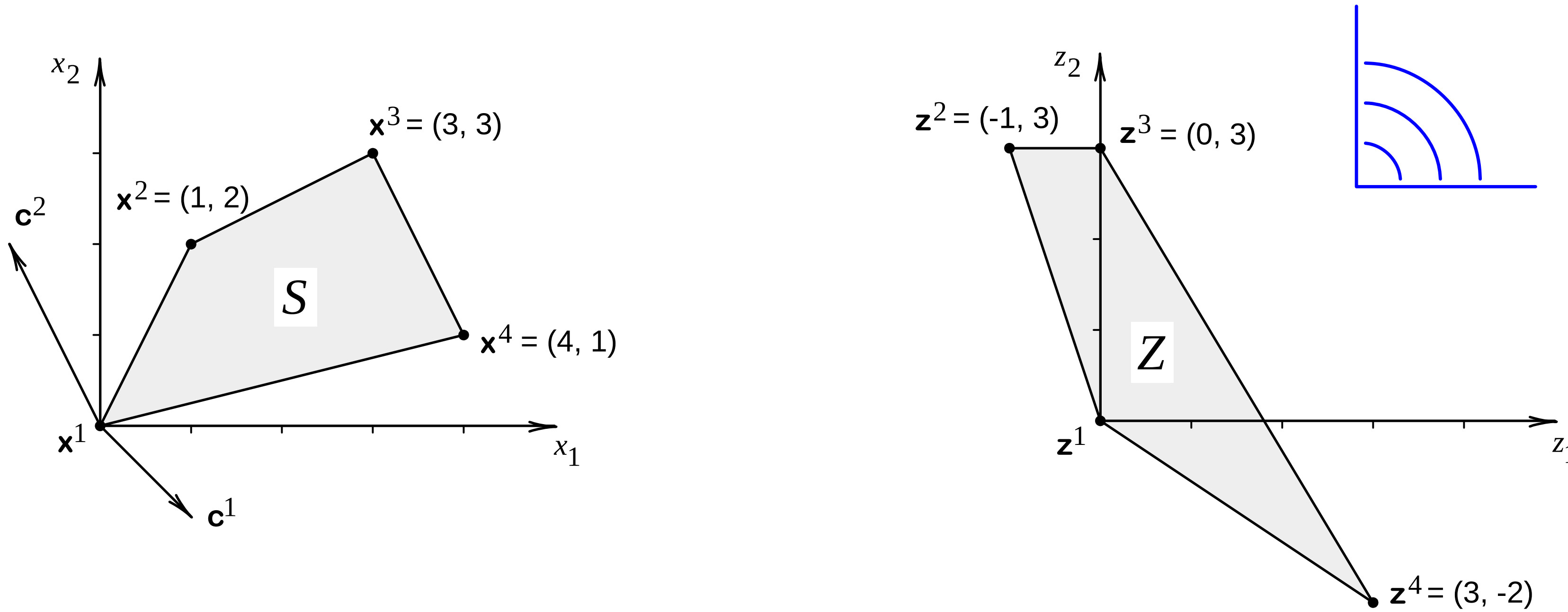
Recall

9. **Ideal way?**
10. **Contours, Upper Level Sets and Quasiconcavity**
11. **More-Is-Always-Better-Than-Less vs. Quasiconcavity**
12. **ADBASE**
13. **Size of the Nondominated Set**
14. **Criterion Value Ranges over Nondominated Set**
15. **Nadir Criterion Values**
16. **Payoff Tables**
17. **Filtering**
18. **Stamp/Coin Example**
19. **Weighted-Sums Method**
20. **e-Constraint Method**

Recall

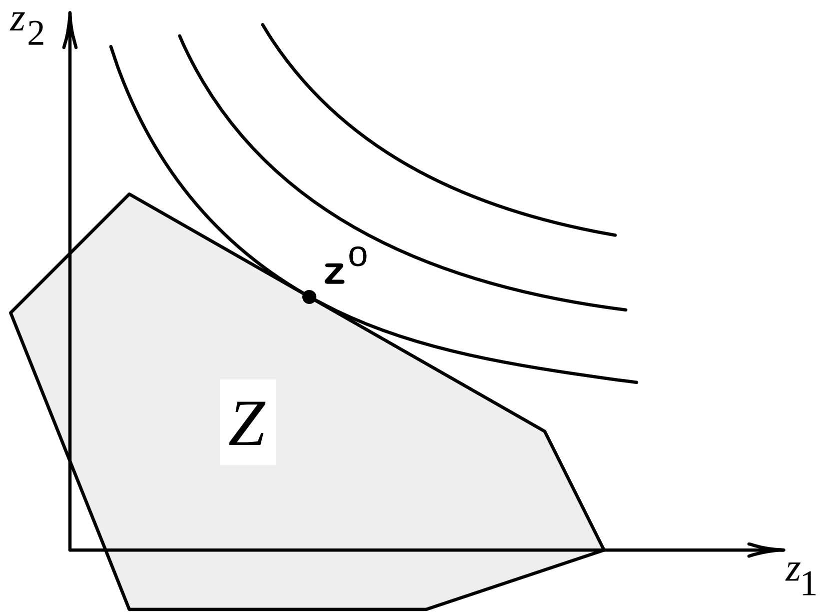
$$\max\{\ \ x_1 - x_2 = z_1\}$$
$$\max\{-x_1 + 2x_2 = z_2\}$$
$$s.t. \qquad \mathrm{x} \in S$$



9. Ideal Way?

Assess a decision maker's utility function $U : R^k \to R$ and solve

$$\max\{U(z_1, \ldots, z_k)\}$$
$$s.t. \quad f_i(\mathrm{x}) = z_i \quad i = 1, \ldots, k$$
$$\mathrm{x} \in S$$

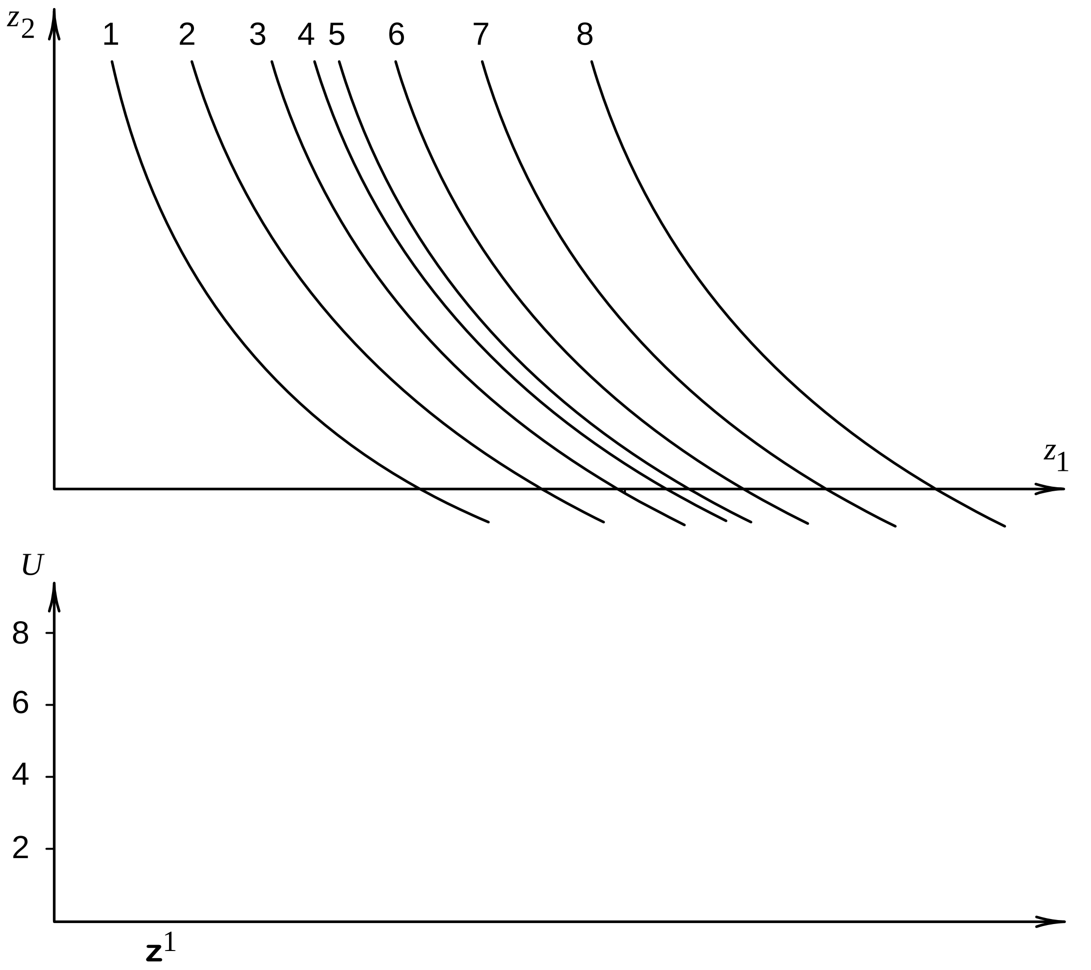


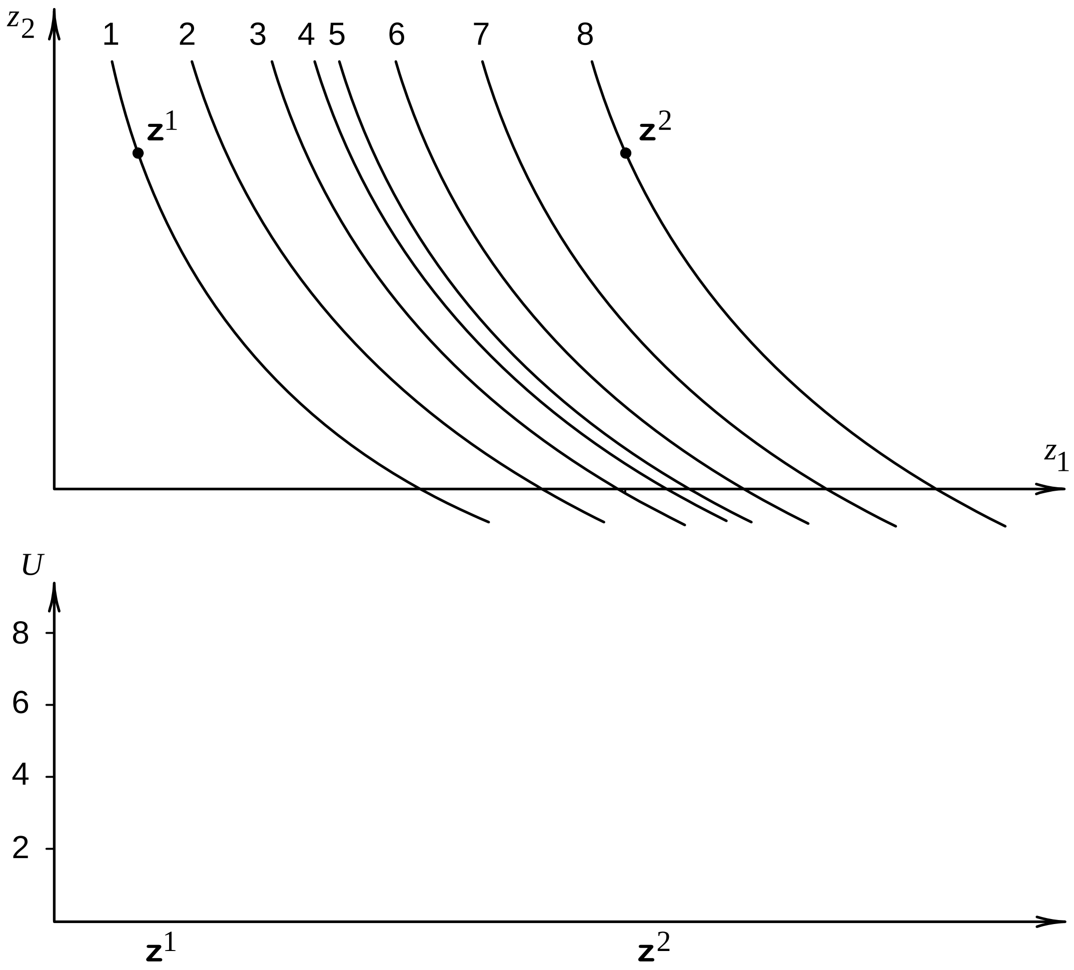
Maybe not good for four reasons.

1. Difficulty in assessing U
2. U is almost certainly nonlinear
3. Generates only one solution
4. Does not allow for learning

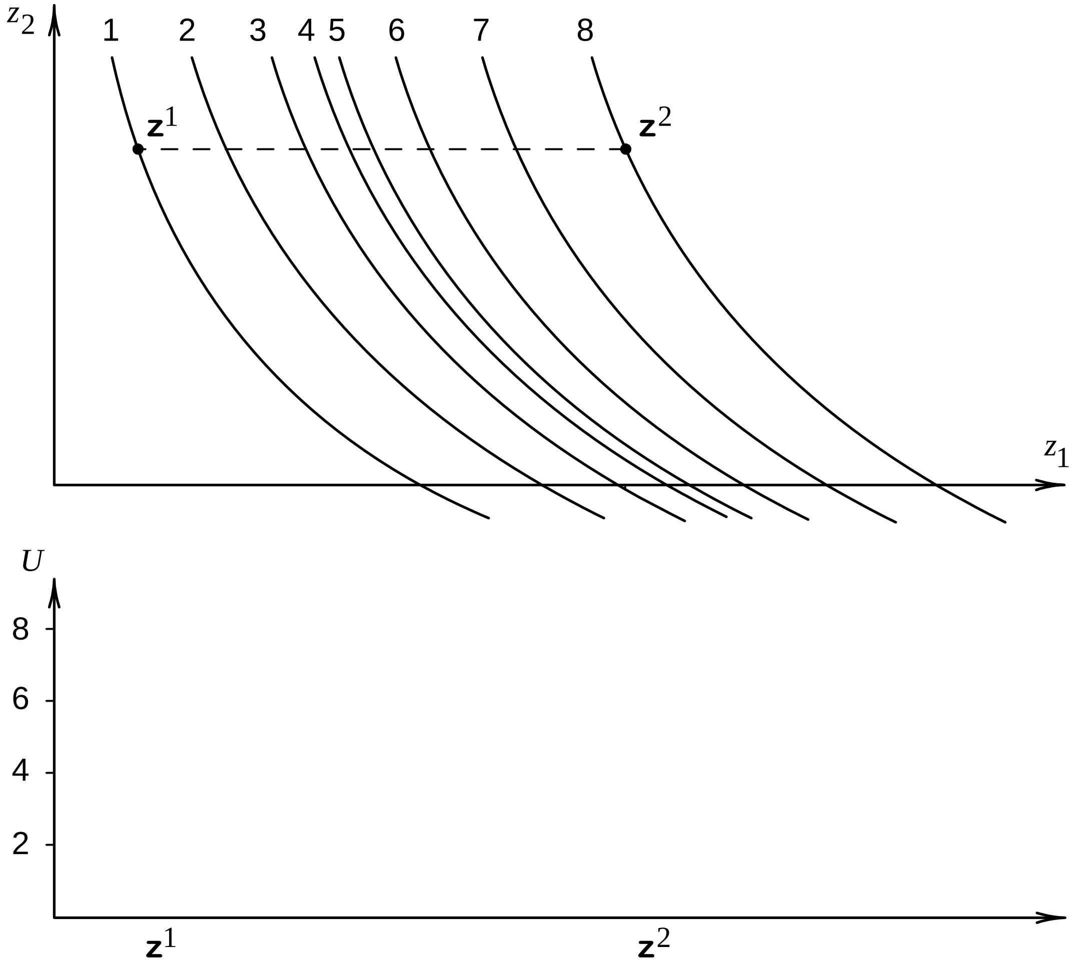
10. Contours, Upper Level Sets, and Quasiconcavity

A U is quasiconcave if all upper level sets are convex.

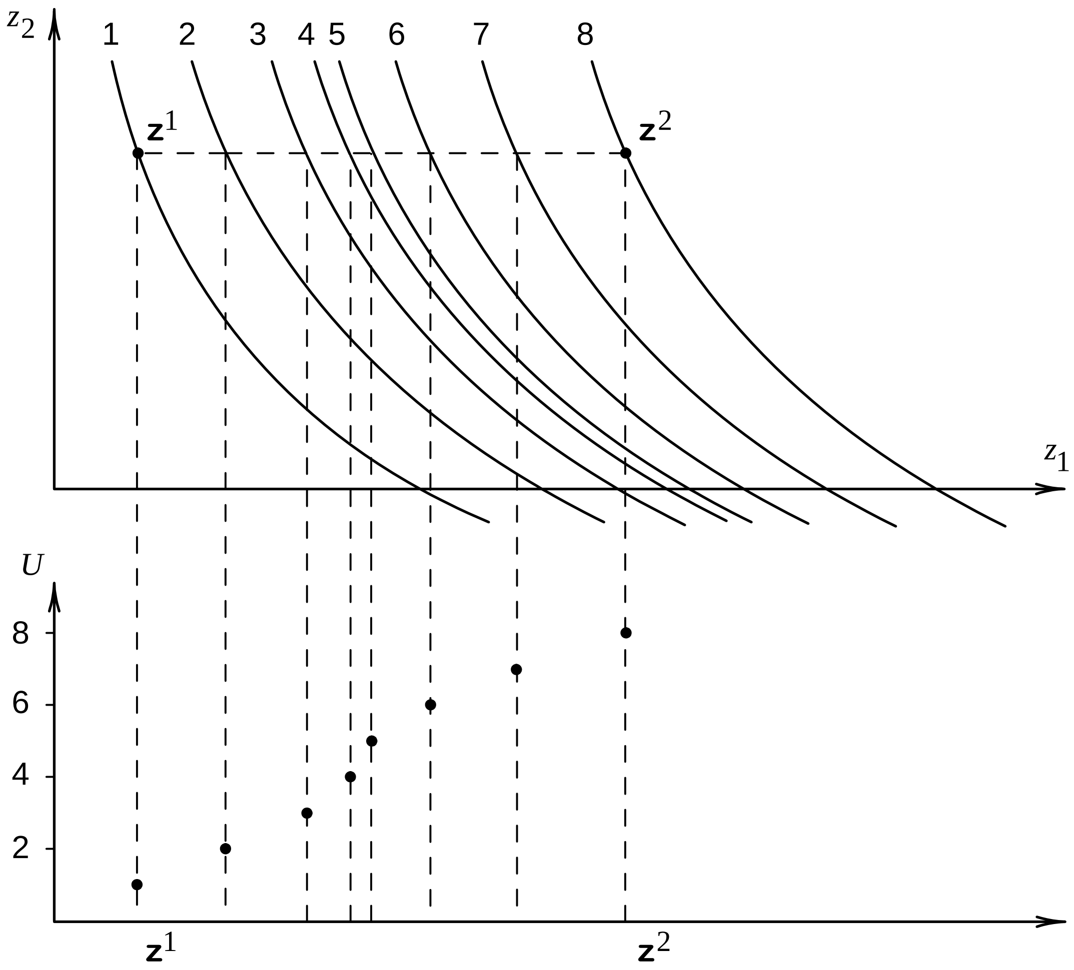




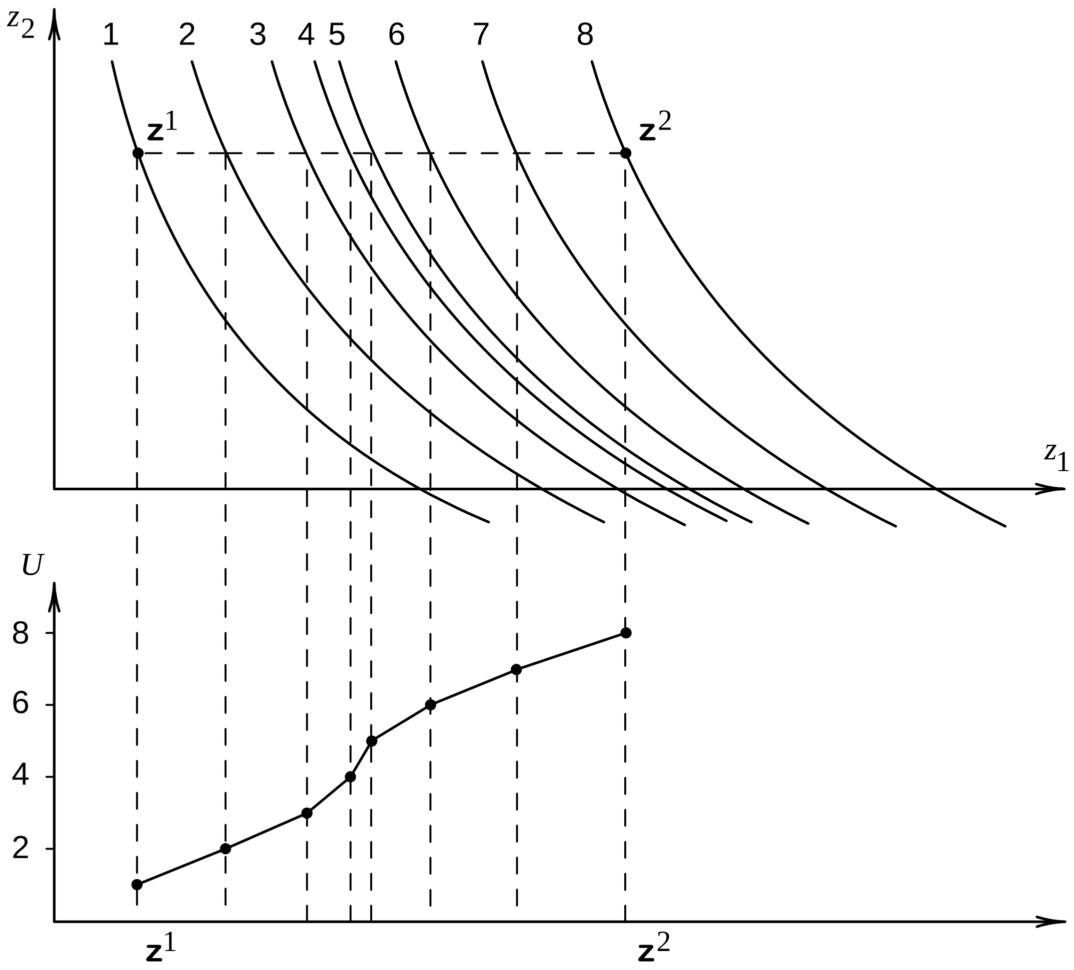
z_2
1 2 3 4 5 6 7 8
$\mathbf{z}^1$
$\mathbf{z}^2$
z_1
U
8
6
4
2
$\mathbf{z}^1$
$\mathbf{z}^2$



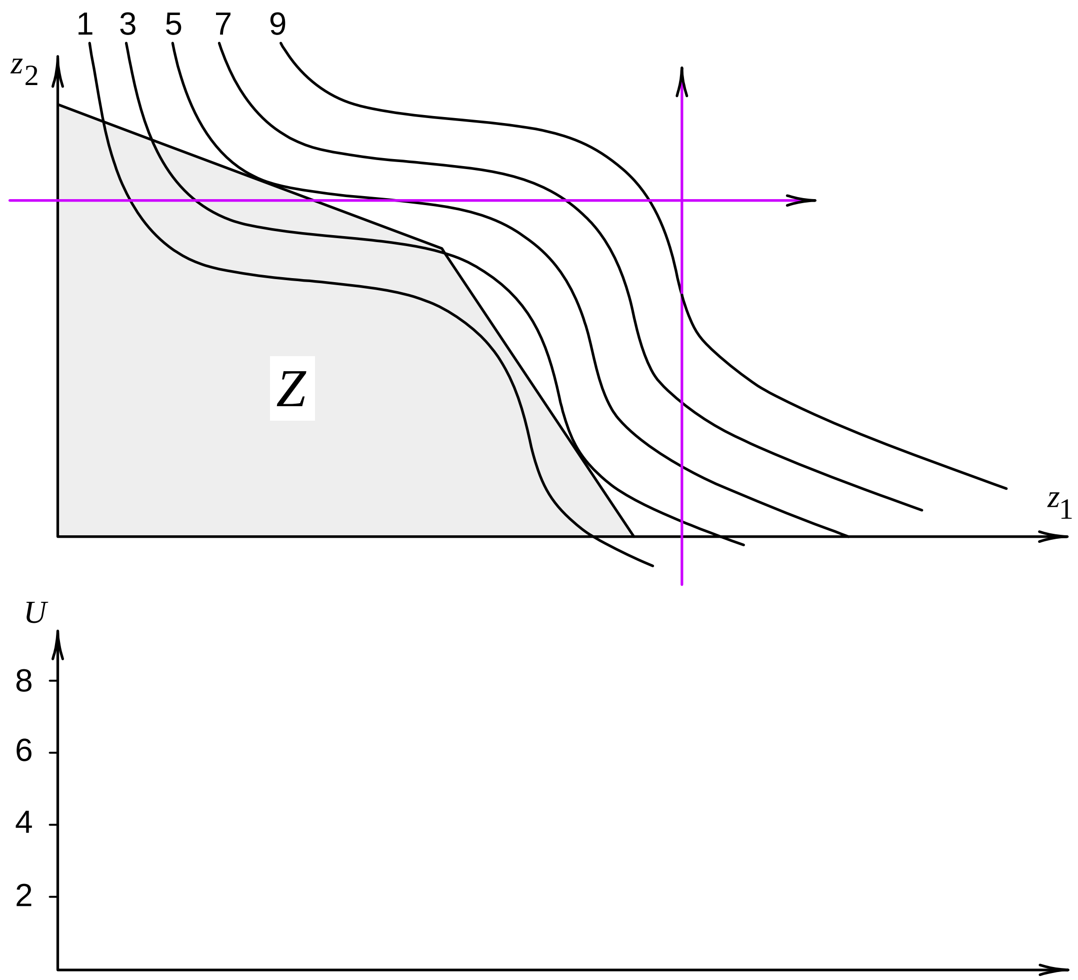
z_2
1
2
3
4
5
6
7
8
$\mathbf{z}^1$
$\mathbf{z}^2$
z_1
U
8
6
4
2
$\mathbf{z}^1$
$\mathbf{z}^2$


z_2
1
2
3
4
5
6
7
8
$\mathbf{z}^1$
$\mathbf{z}^2$
z_1
U
8
6
4
2
$\mathbf{z}^1$
$\mathbf{z}^2$

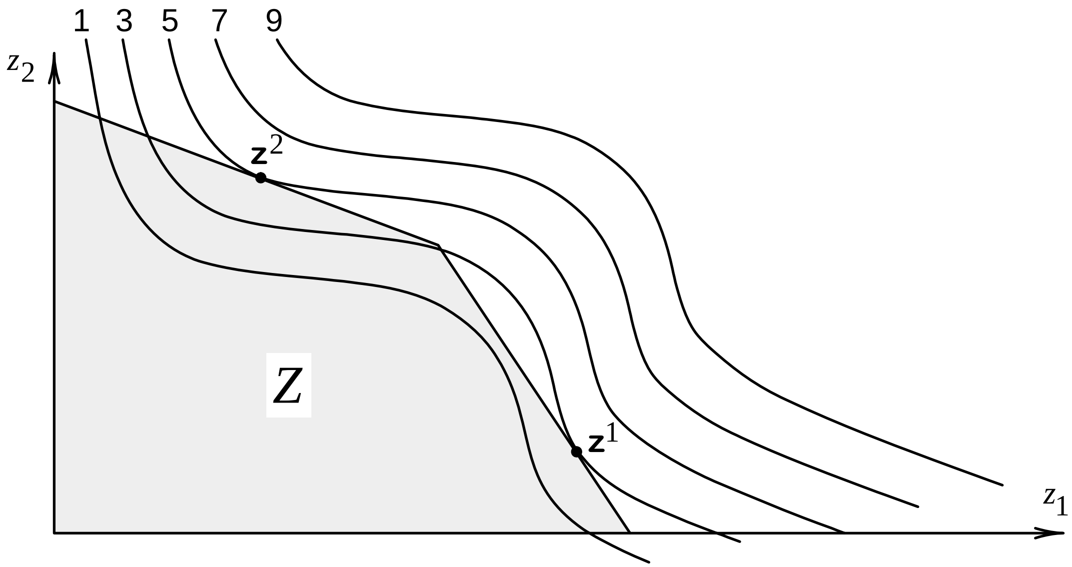
Quasiconcave functions have at most one top.



11. More-Is-Always-Better-Than-Less (i.e, Coordinate-wise increasing) vs. Quasiconcavity

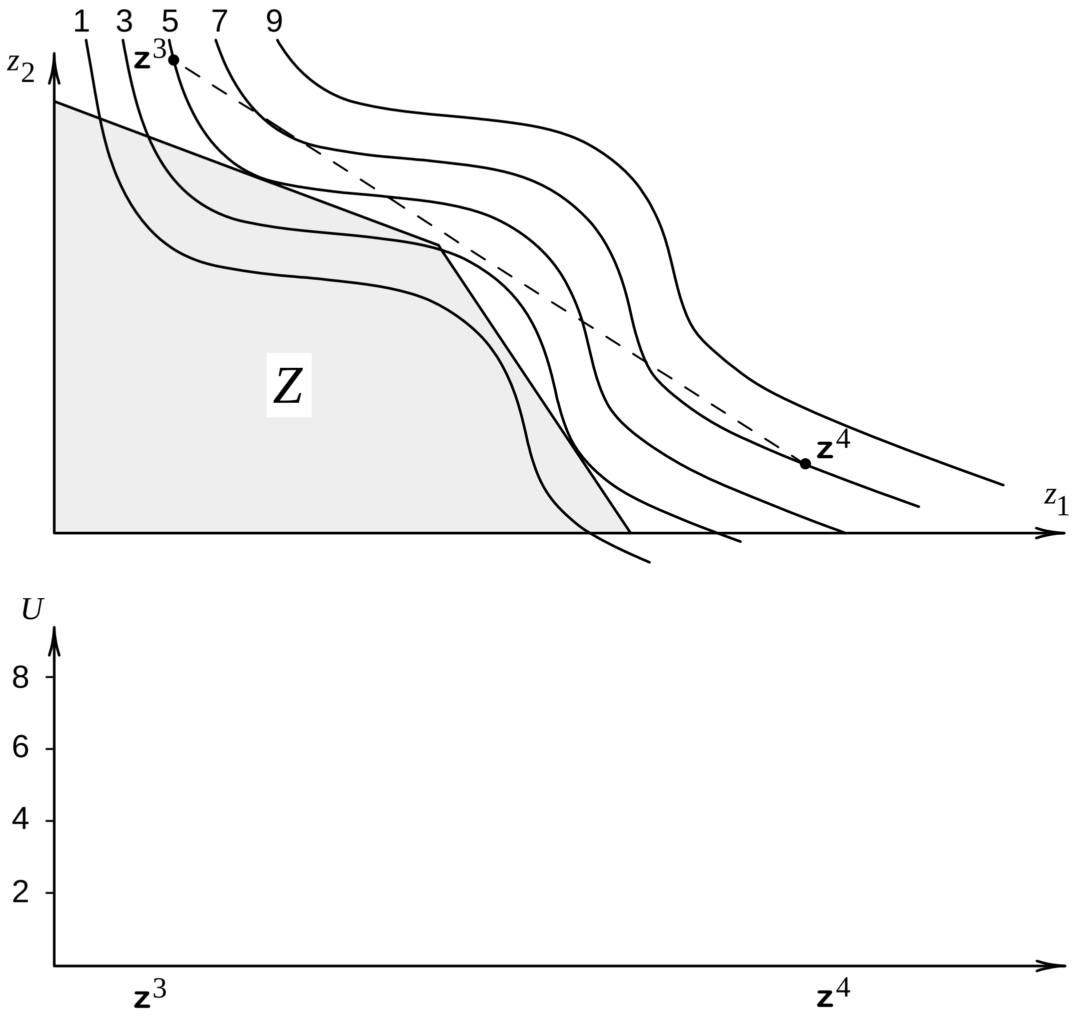


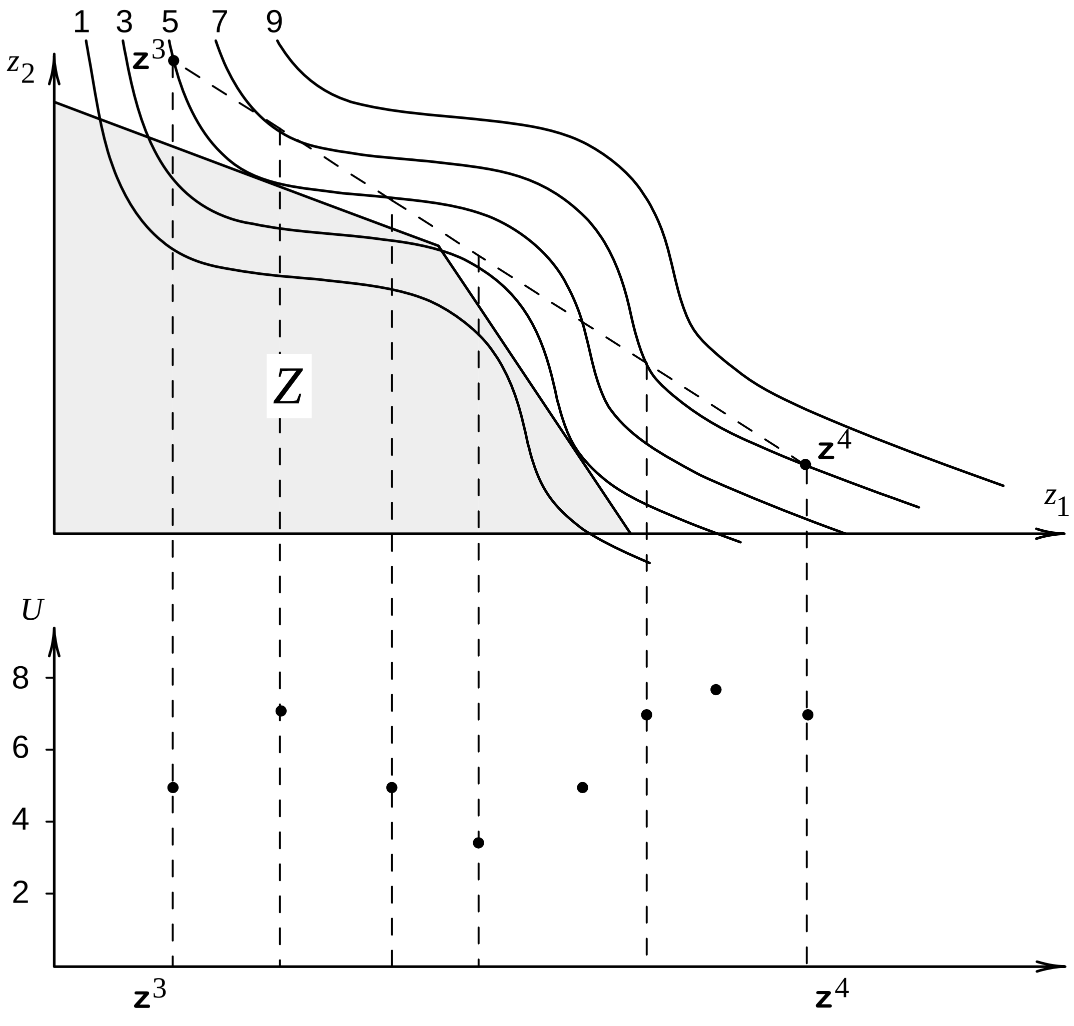
More-is-always-better-than-less does not imply that all local optima are global optima

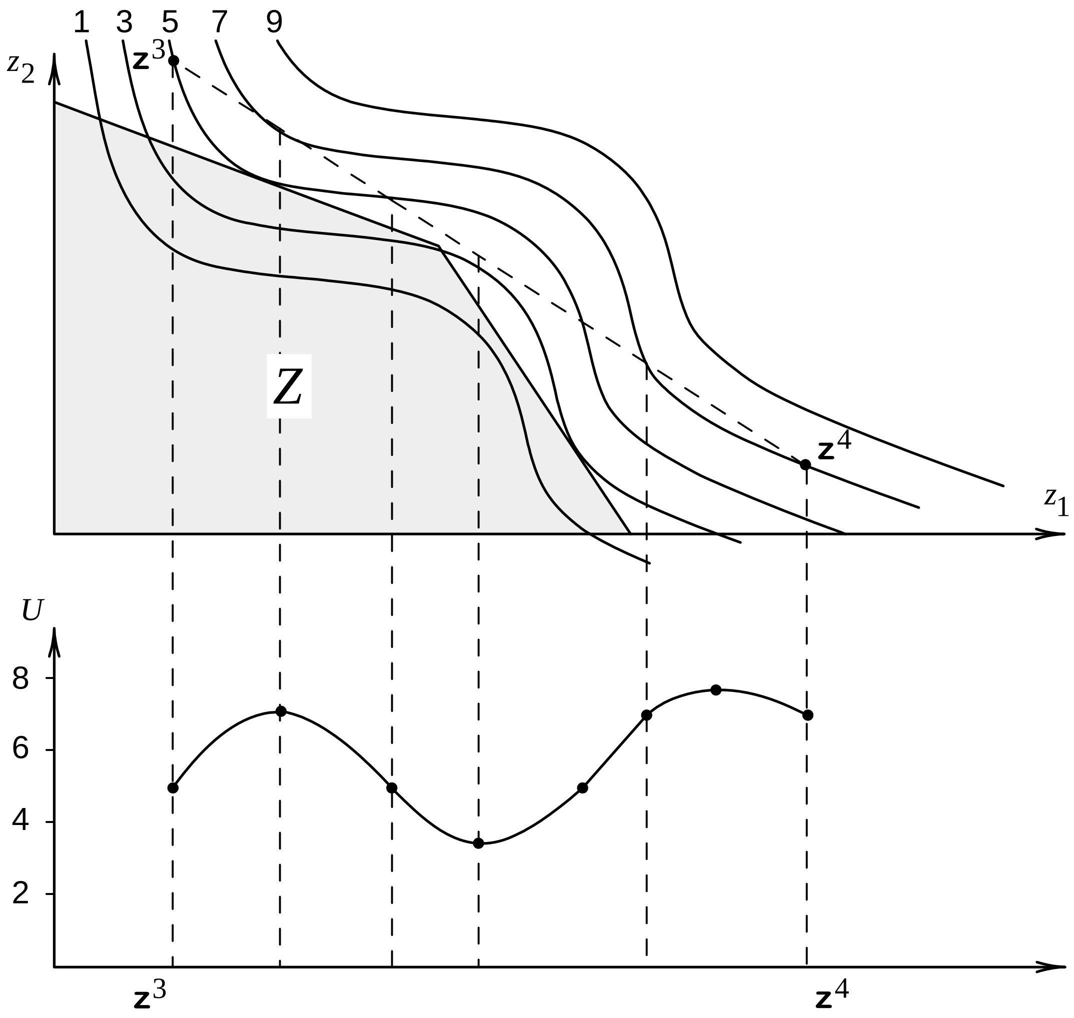


$\mathbf{z}^1$ is a local optimum, but $\mathbf{z}^2$ is the global optimum.

More-is-always-better-than-less does not imply quasiconcavity




1
3
5
7
9
z_2
$\mathbf{z}^3$
Z
$\mathbf{z}^4$
z_1
U
8
6
4
2
$\mathbf{z}^3$
$\mathbf{z}^4$


1
3
5
7
9
z_2
$\mathbf{z}^3$
Z
$\mathbf{z}^4$
z_1
U
8
6
4
2
$\mathbf{z}^3$
$\mathbf{z}^4$

Assuming that U is coordinate-wise increasing:

- Nondominated set N -- set of all potentially optimal criterion vectors.
- Efficient set E -- set of all potentially optimal solutions.

12. ADBASE

In an MOLP, of course, efficient set is a portion of the surface of *S*, and nondominated set is a portion of the surface of *Z*.

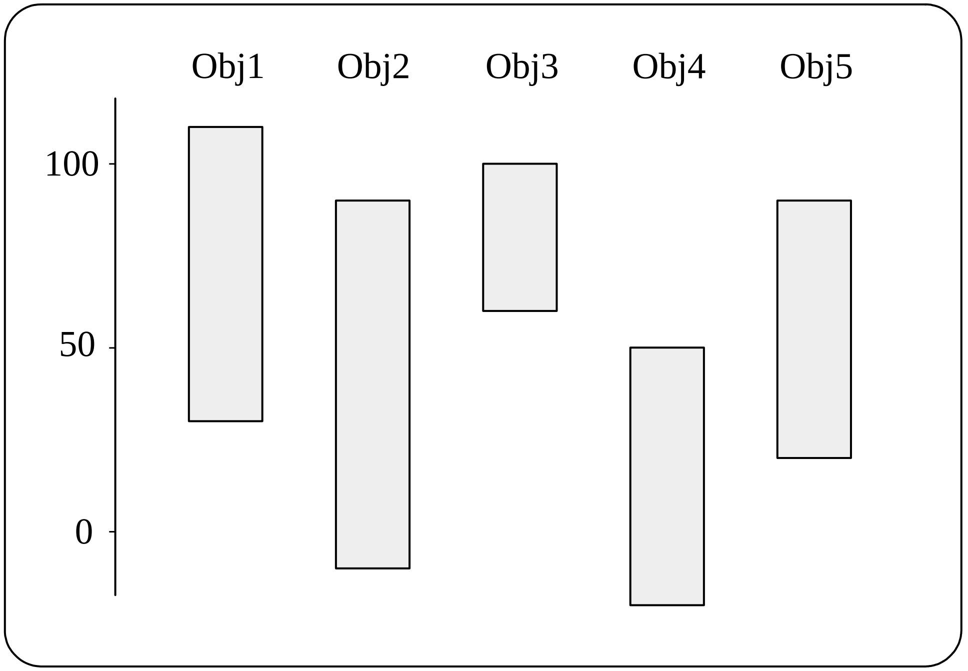
ADBASE is for MOLPs. It computes all of the extreme points of *S* that efficient, and hence all of the vertices of *Z* that are nondominated in an MOLP.

13. Size of the Efficient and Nondominated Sets

MOLP problem size	ave efficient extreme pts
3 x 100 x 150	13,415
3 x 250 x 375	285,693
4 x 50 x 75	19,921
5 x 35 x 45	15,484
5 x 60 x 90	414,418

14. Criterion Value Ranges over the Nondominated Set

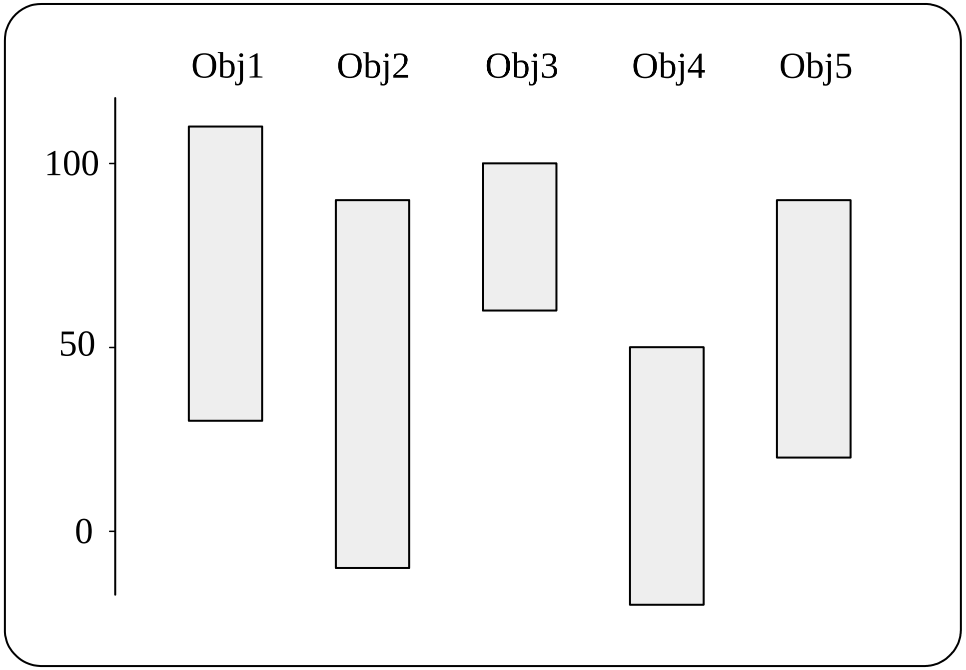
If know nondominated set ahead of time, can “warm up” decision maker with following information.



The lower bounds on the ranges are called nadir criterion values.

15. Nadir Criterion Values

If don't know nondominated set ahead of time, true nadir criterion vector can be difficult to obtain.

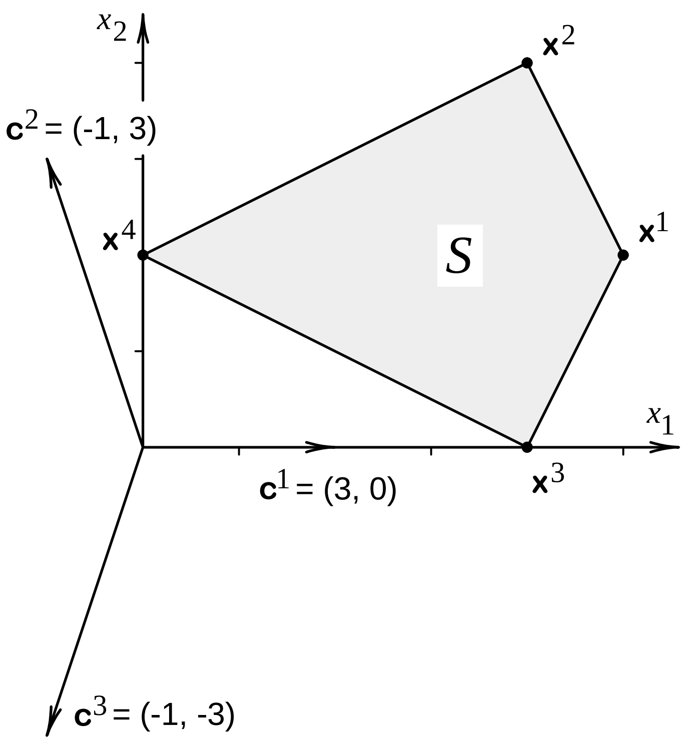


$$\mathbf{z}^{\max} = (110,\ 90, 100,\ 50,\ 40)$$

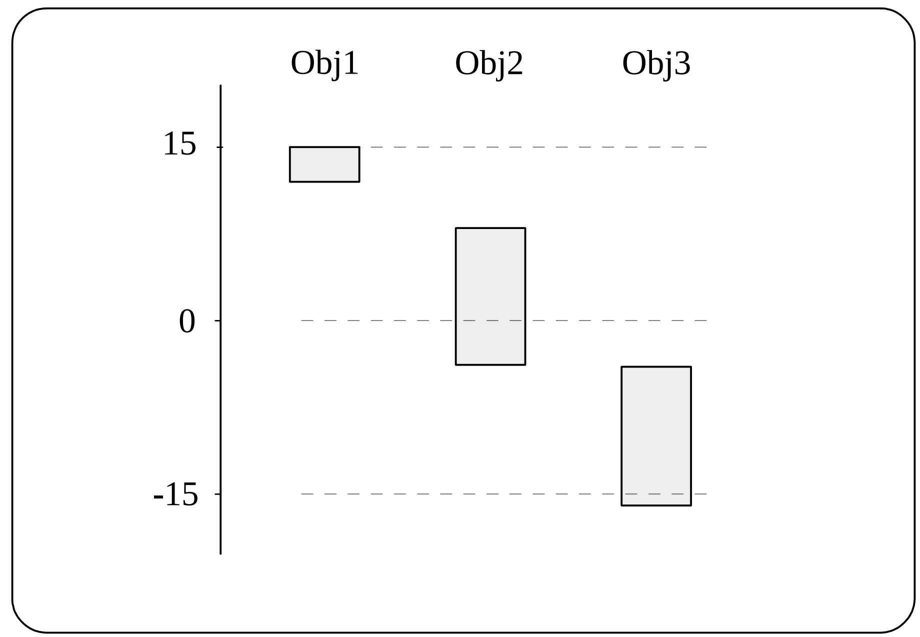
$$\text{estimated } \mathbf{z}^{\text{nad}} = (30, -10,\ 60, -20,\ 20)$$

16. Payoff Table

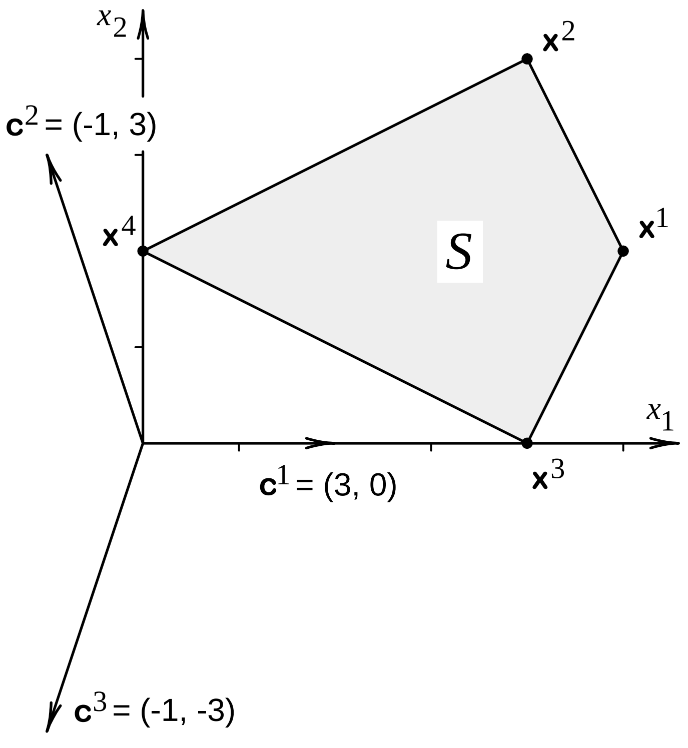
Obtained by individually maximizing each objective over S.
But minimum column values often over-estimate nadir values.



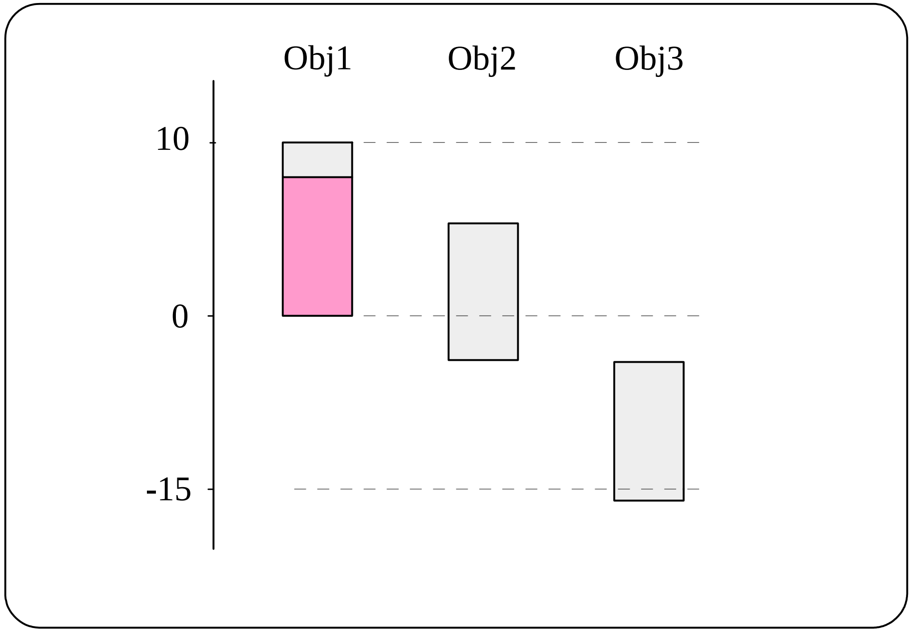
$\mathbf{z}^1$	15	1	-11
$\mathbf{z}^2$	12	8	-16
$\mathbf{z}^3$	12	-4	-4



In this problem $E = S$. True nadir value for Obj1 is 0 not 12.



$\mathbf{z}^1$	15	1	-11
$\mathbf{z}^2$	12	8	-16
$\mathbf{z}^3$	12	-4	-4

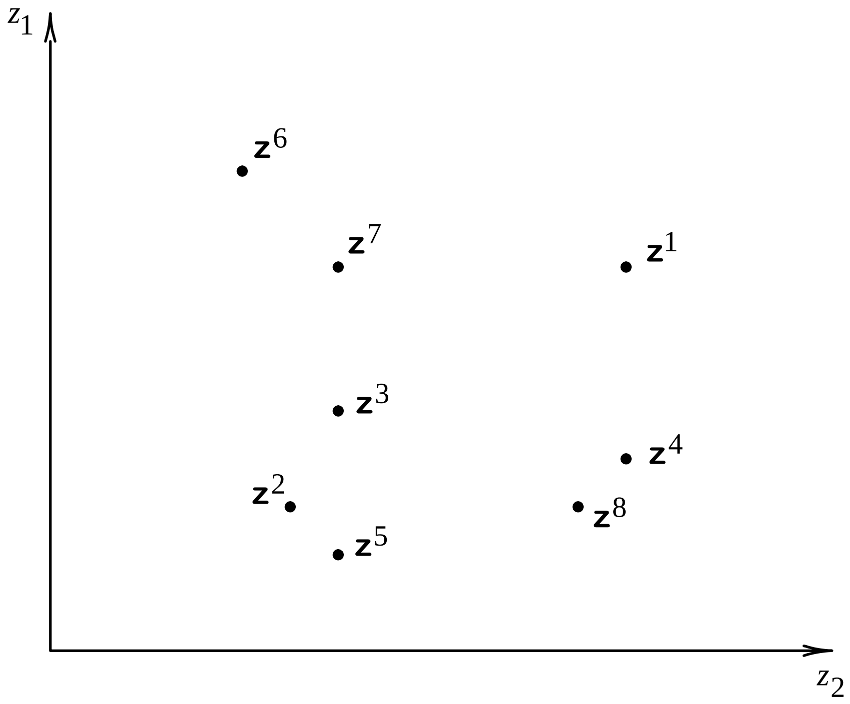


The larger the problem, the greater the likelihood that the payoff table column minimum values will be wrong.

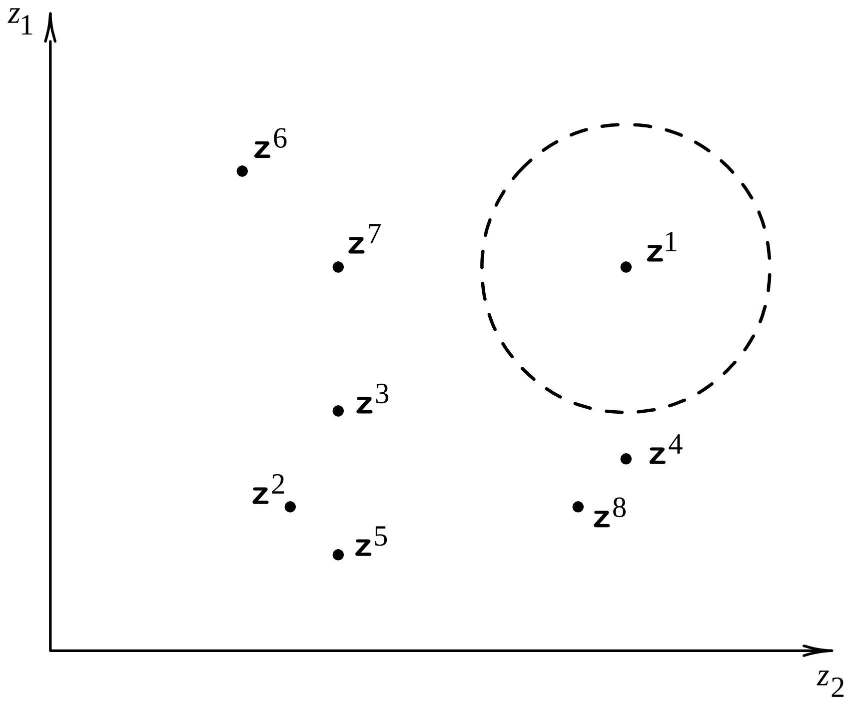
After about 5 x 20 x 30, most will be wrong.

17. Filtering

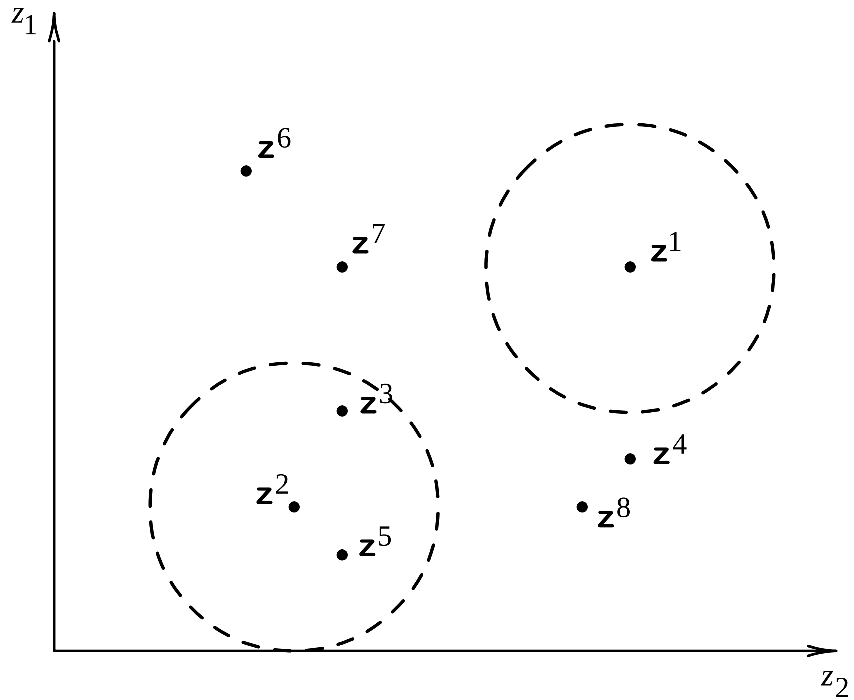
Reducing 8 vectors down to a dispersed subset of size 5



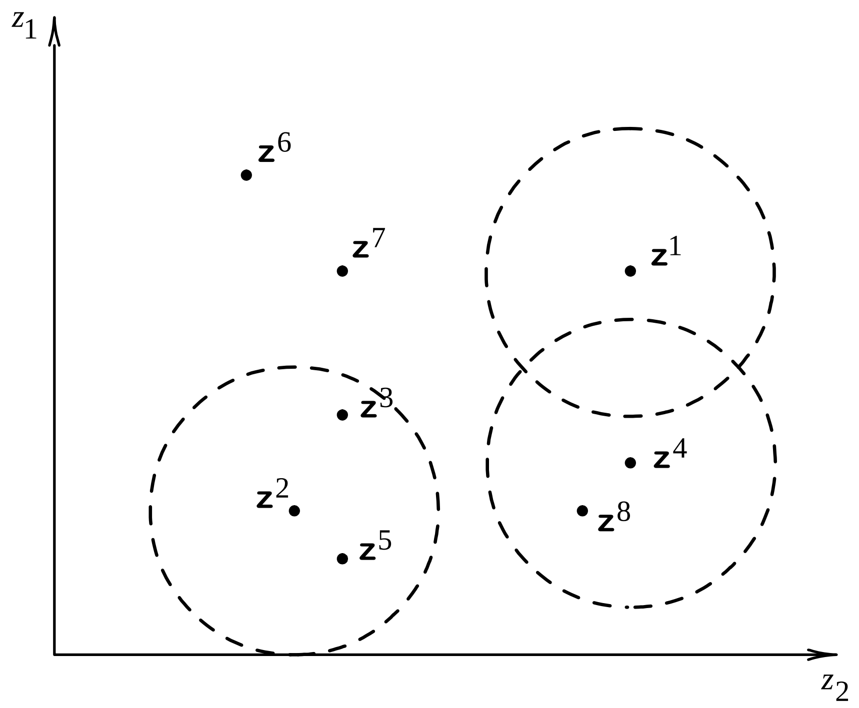
First point $\mathbf{z}^1$ always retained by filter.



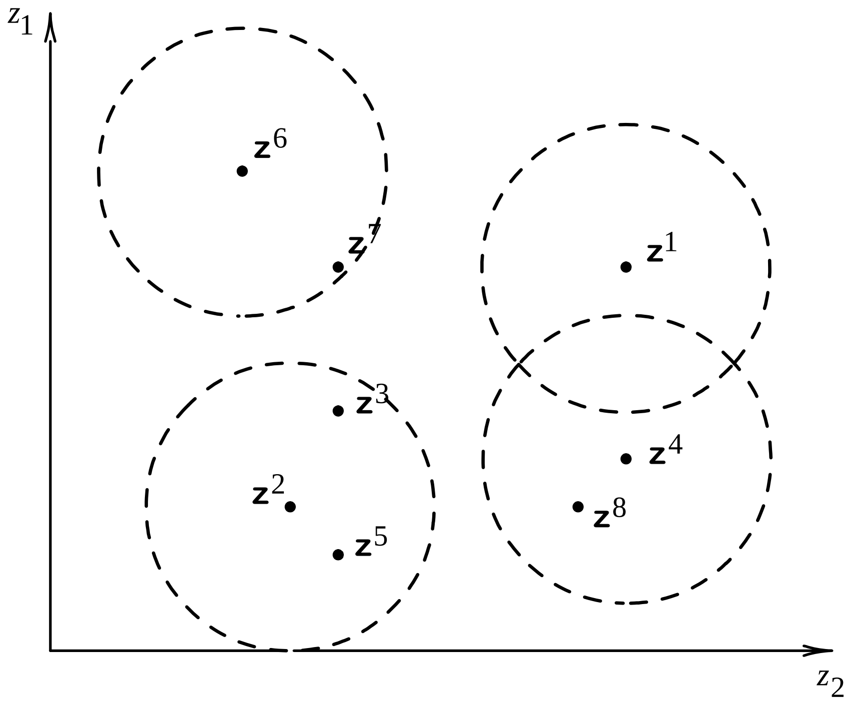
$\mathbf{z}^2$ retained by filter, but $\mathbf{z}^3$ and $\mathbf{z}^5$ discarded.



$\mathbf{z}^4$ retained by filter, but $\mathbf{z}^8$ discarded.

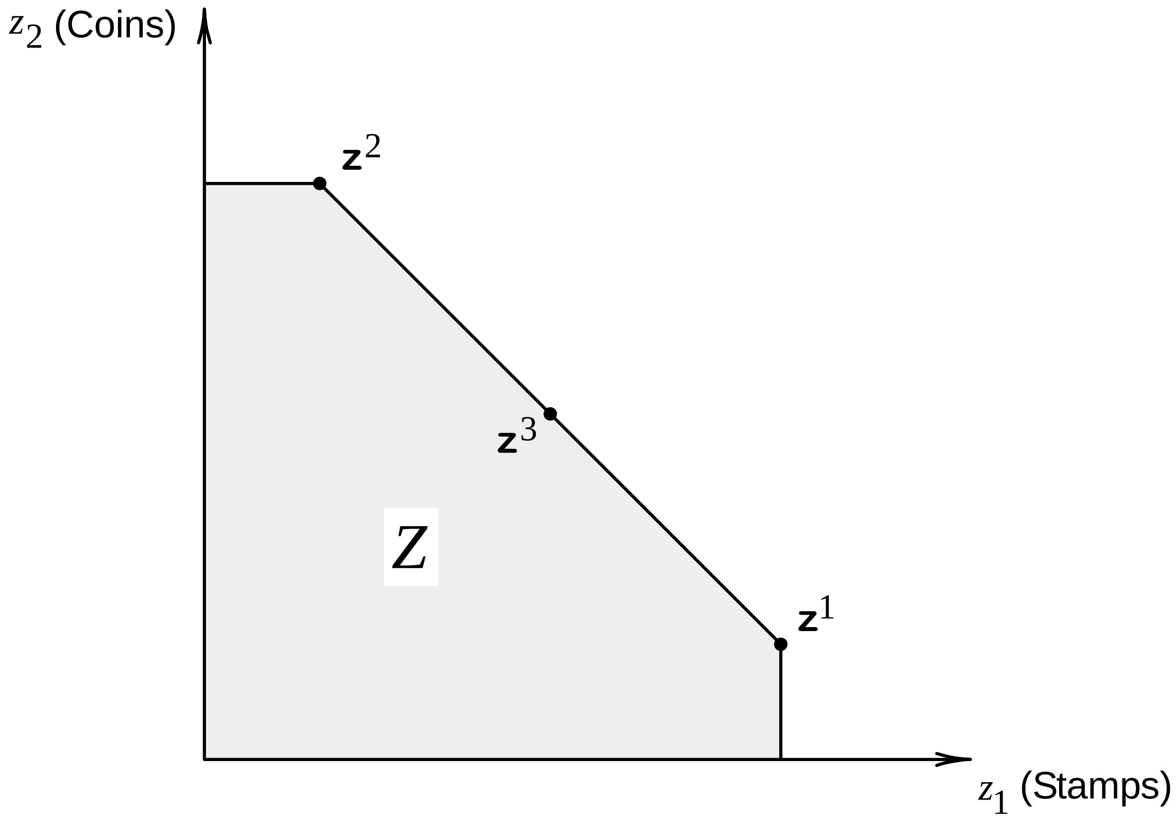


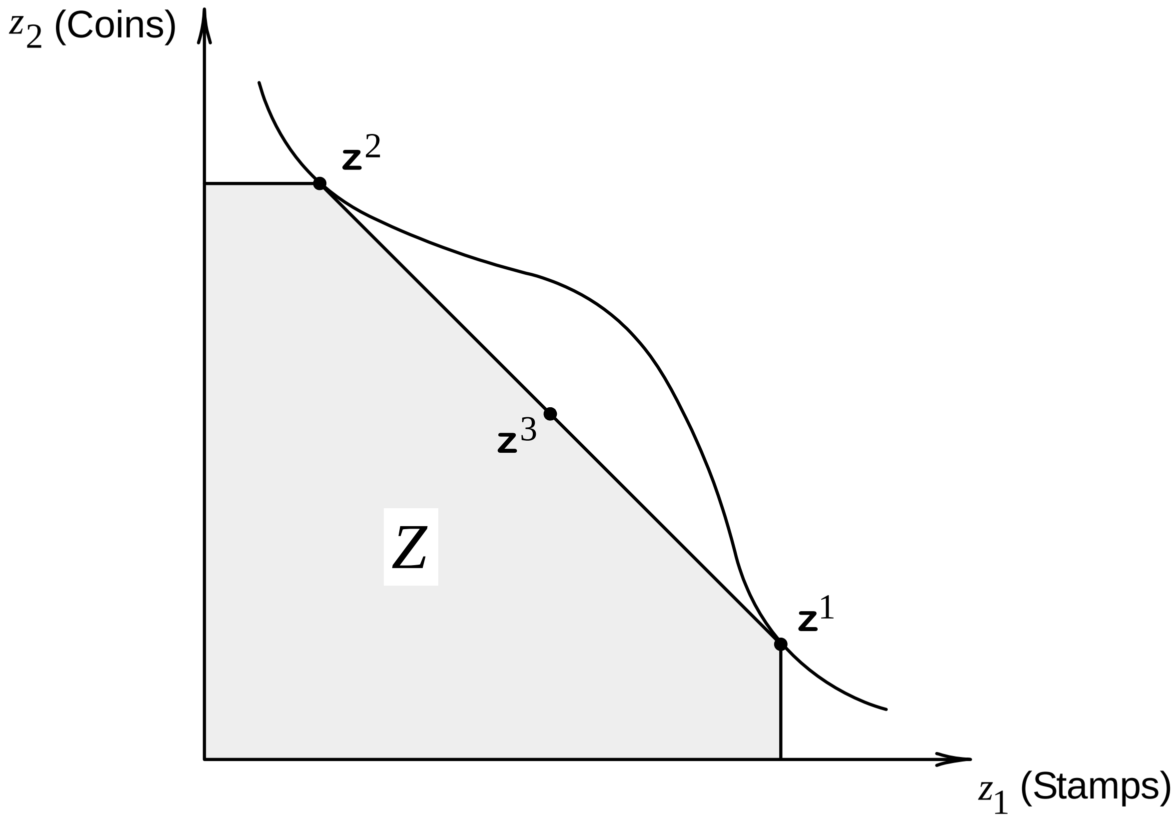
$\mathbf{z}^6$ retained by filter, but $\mathbf{z}^7$ discarded.



Wanted 5 but got 4. Reduce neighborhood, then do again. After a number of iterations, will converge to desired size.

18. Stamp/Coin Example




z_2 (Coins)
$\mathbf{z}^2$
$\mathbf{z}^3$
Z
$\mathbf{z}^1$
z_1 (Stamps)

19. Weighted-Sums Method

$$
\begin{aligned}
&\max\{\, c^1 x = z_1 \,\} \\
&\quad \vdots \\
&\max\{\, c^k x = z_k \,\} \\
&s.t. \qquad x \in S
\end{aligned}
\qquad\qquad
\begin{aligned}
&\max\{\, \lambda^T C x \,\} \\
&s.t. \quad x \in S
\end{aligned}
$$

But how to pick the weights because they are a function of

1. decision-maker's preferences.
2. scale in which the objectives are measured (e.g., cubic feet versus board feet of lumber).
3. shape of the feasible region

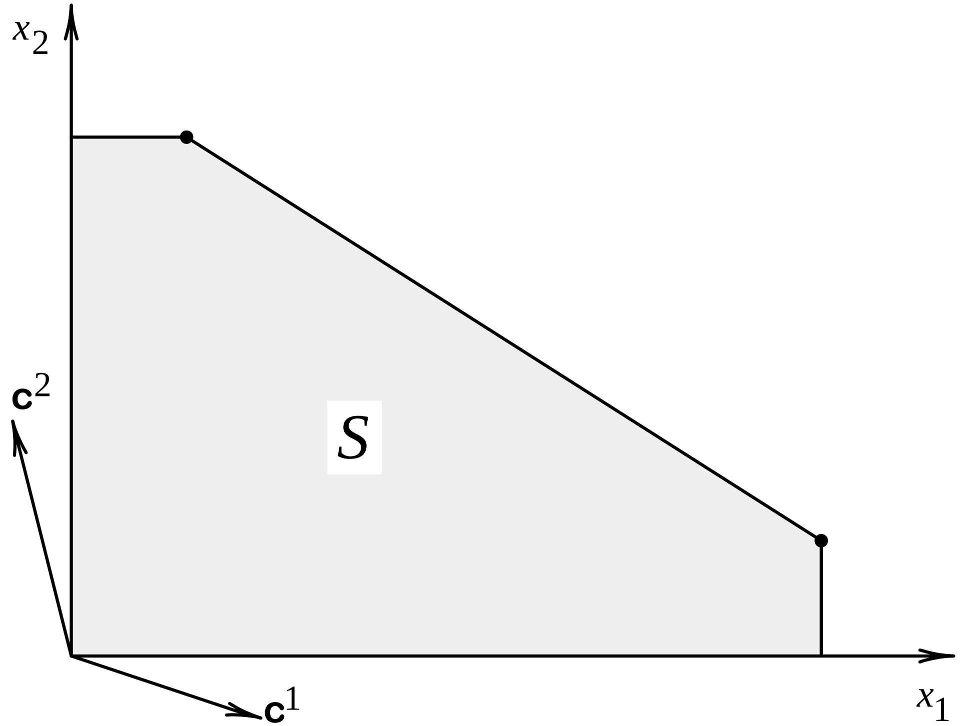
May also get flip-flopping behavior.

Purpose of weighted-sums approach is to obtain information from the DM to create a λ-vector that causes "composite gradient" $\lambda^T C$ in the weighted-sums program

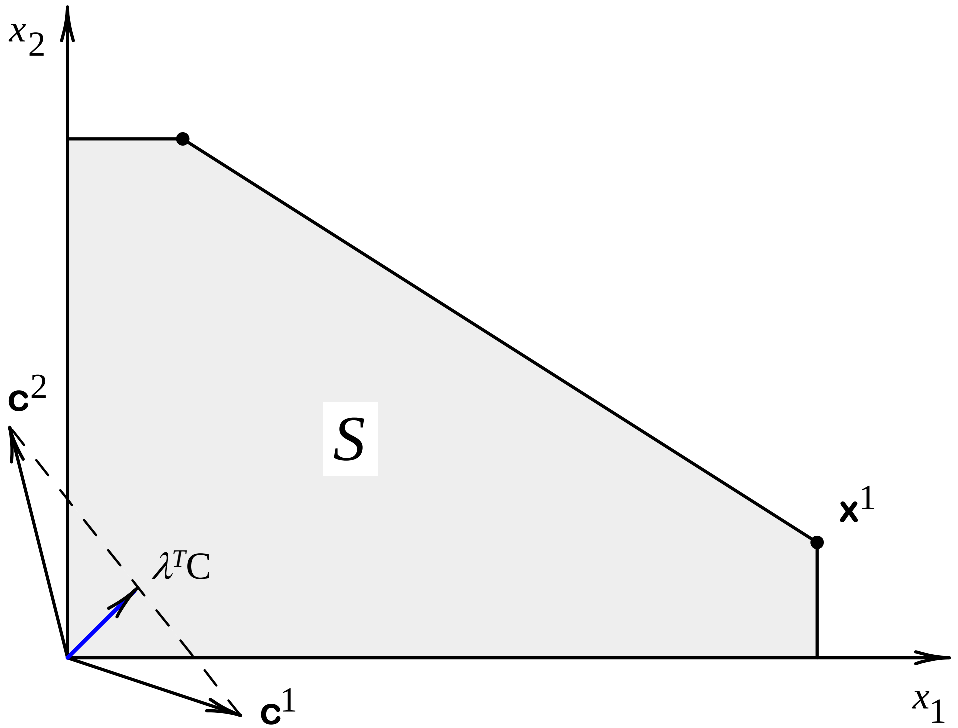
$$\max\{\lambda^T Cx\}$$
$$s.t. \quad x \in S$$

to point in the same direction as the utility function gradient.

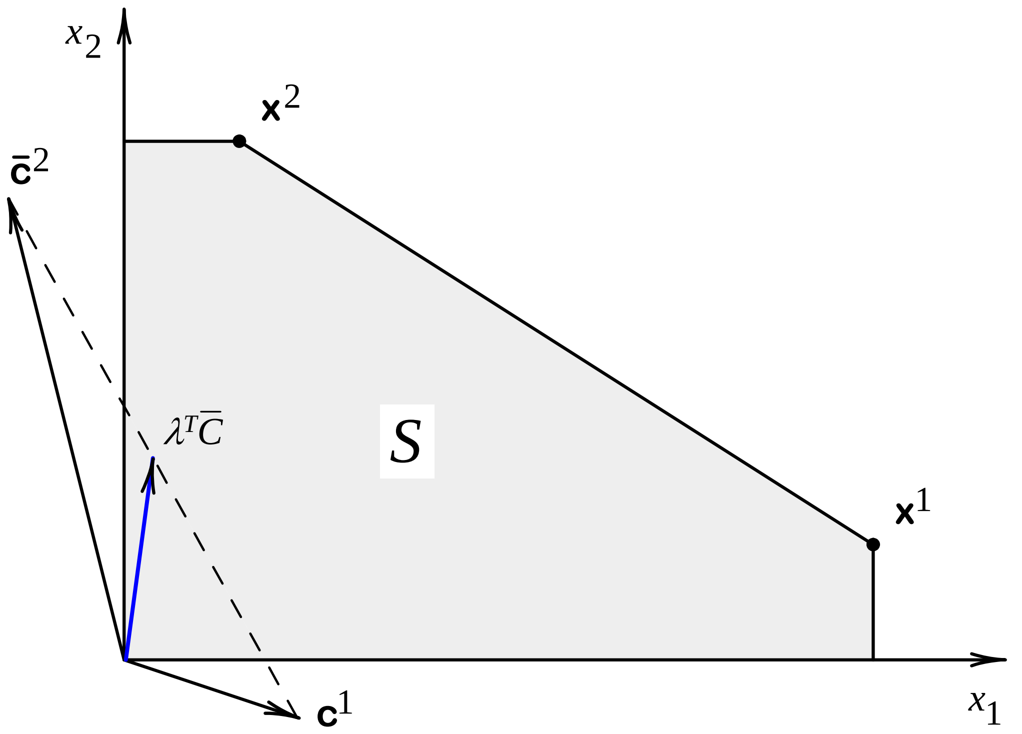
2.



Boss says to go with 50/50 weights.

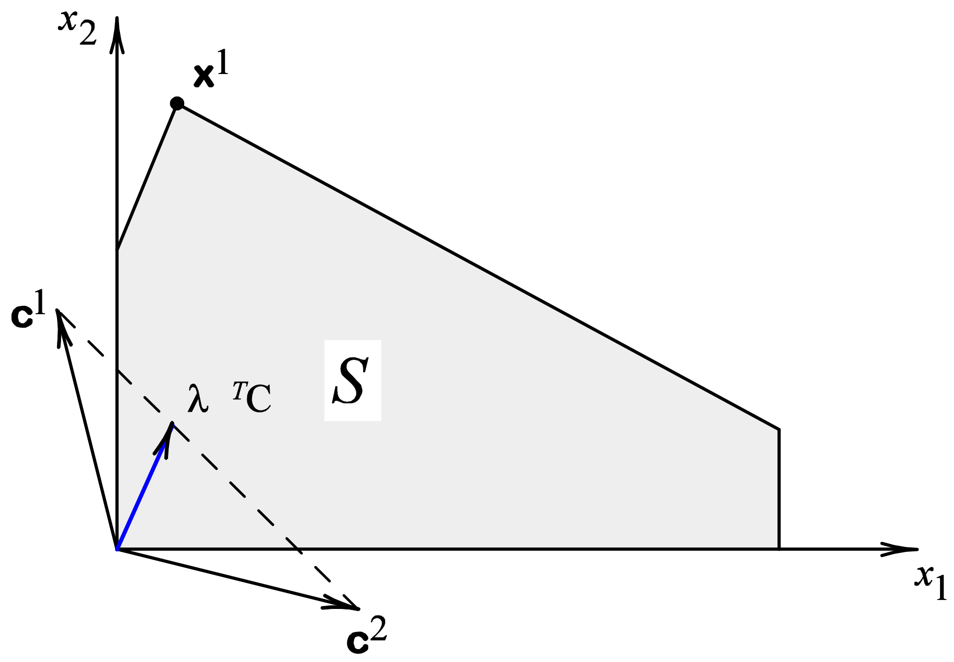


Boss likes resulting solution and is proud his 50/50 weights. Then asks that second objective be changed from cubic feet to board feet of timber production.

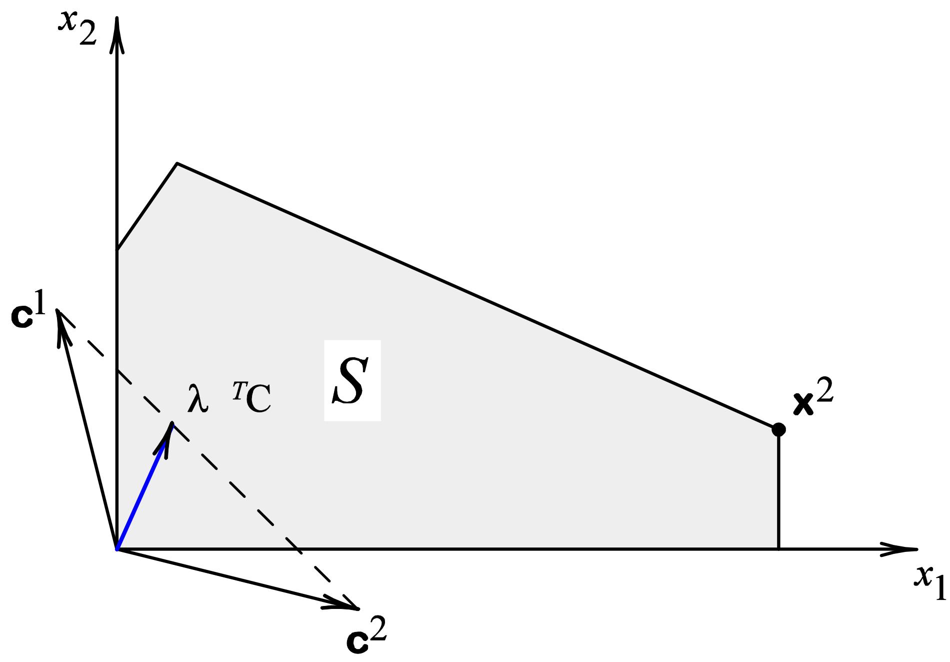


With 50/50 weights, this causes composite gradient to point in a different direction. Get a completely different solution.

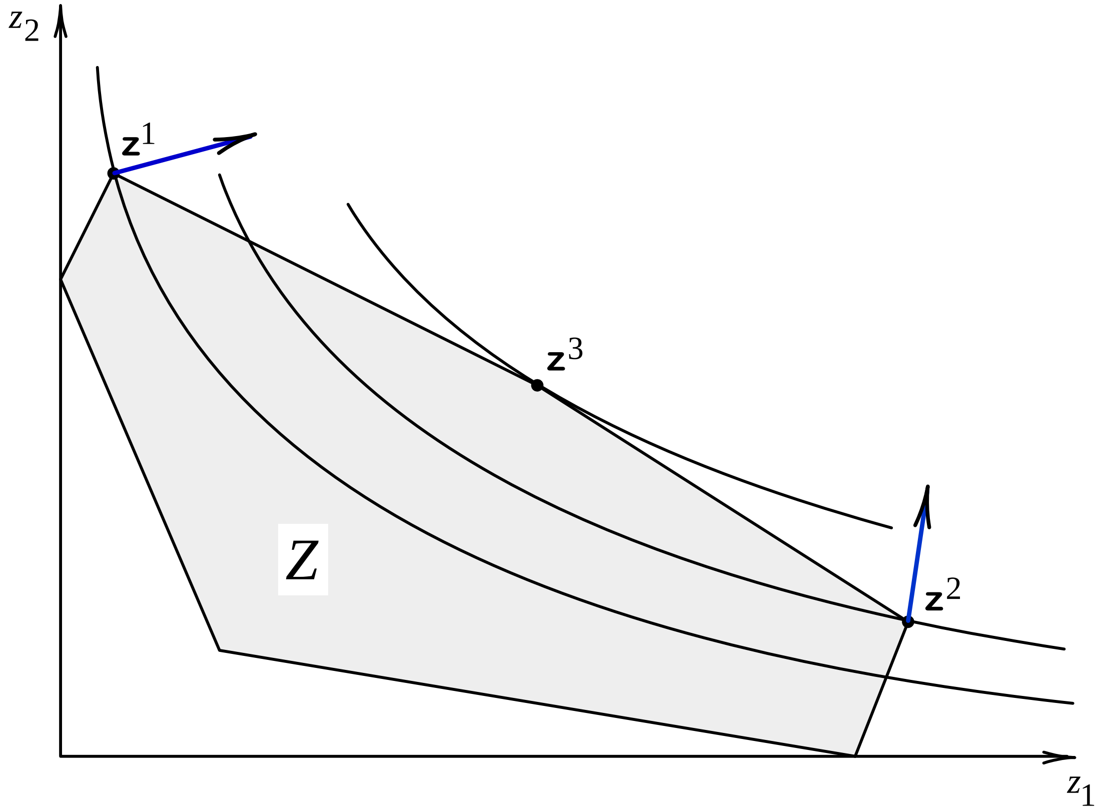
3.



Boss says use 60/40 weights.



Then a constraint needs to be changed slightly.
Get a completely different solution.



Utility function is quasiconcave. Assuming we get perfect information from DM, weighted-sums method will iterate forever!

20. *e*-Constraint Method

$$\begin{aligned} &\max\{\, c^1 x = z_1 \,\} \\ &\quad \vdots \\ &\max\{\, c^k x = z_k \,\} \\ &s.t. \qquad x \in S \end{aligned}$$

$$\begin{aligned} &\max\{\, c^j x = z_j \,\} \\ &s.t. \quad c^i x \geq e_i \quad i \neq j \\ &\qquad\quad x \in S \end{aligned}$$

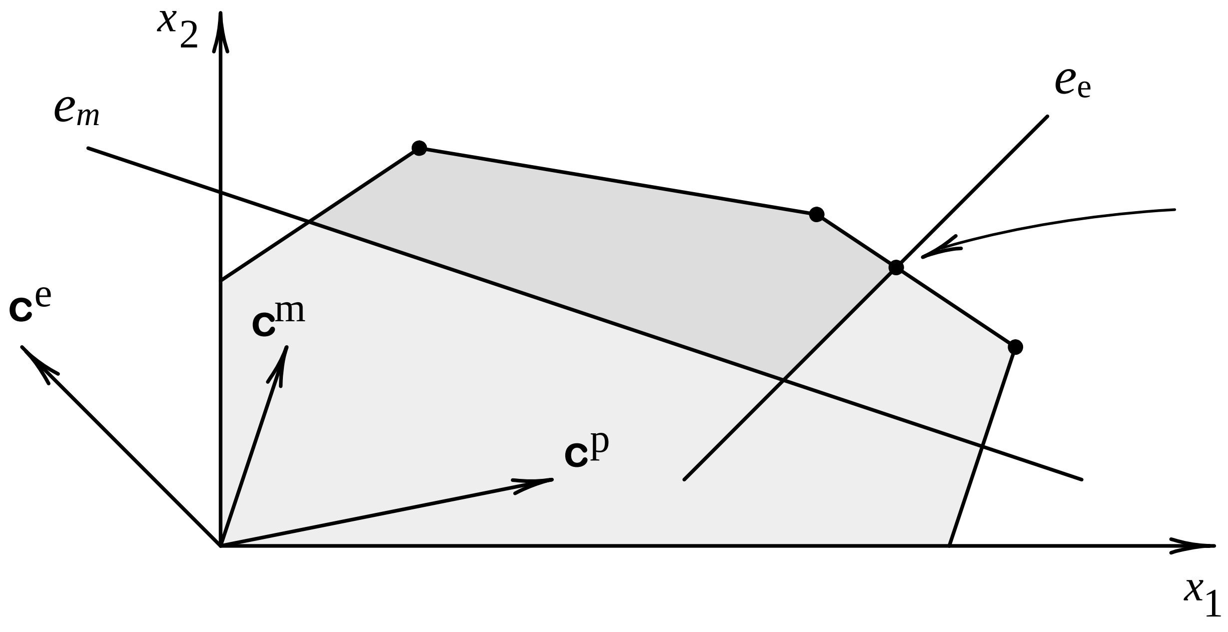
Basically trial-and-error

$$\max\{\, c^p x = z_p \,\}$$

$$s.t. \quad c^m x \geq e_m$$

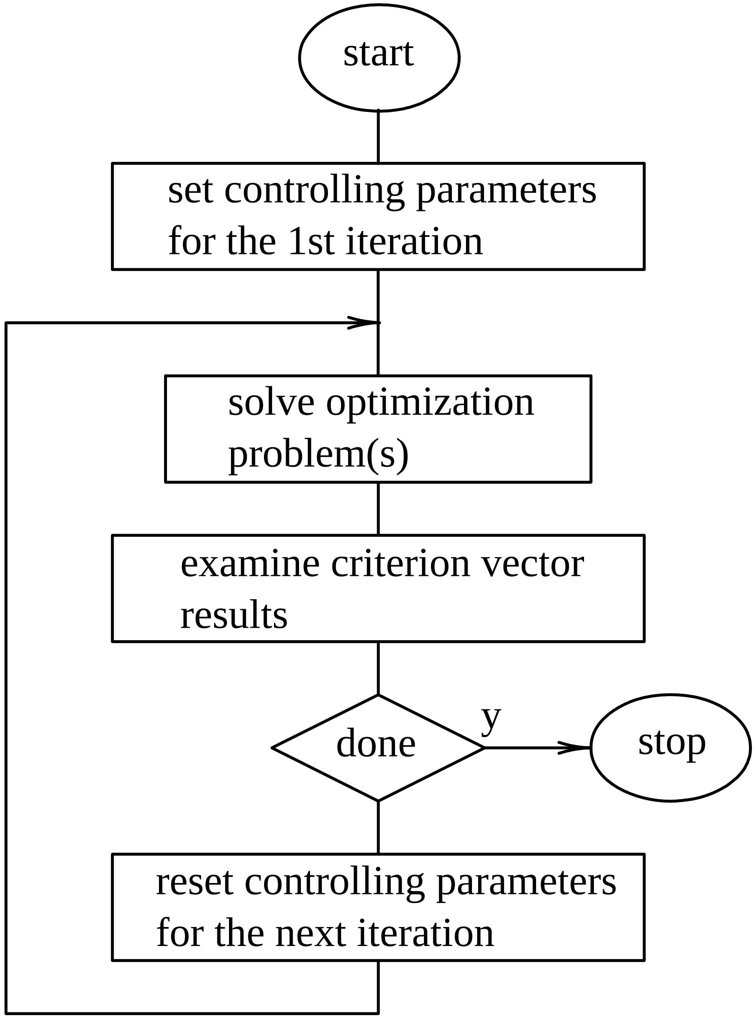
$$c^e x \geq e_e$$

$$x \in S$$



22. **Overall Interactive Algorithmic Structure**
23. **Vector-Maximum/Filtering**
24. **Goal Programming**
25. **Lp-Metrics**
26. **Weighted Lp-Metrics**
27. **Reference Criterion Vector**
28. **Wierzbicki's Aspiration Criterion Vector Method**
29. **Lexicographic Tchebycheff Sampling Program**
30. **Tchebycheff Procedure (overview)**
31. **Tchebycheff Procedure (in more detail)**
32. **Tchebycheff Vertex λ-Vector**
33. **How to Compute Dispersed Probing Rays**
34. **Projected Line Search Method**
35. **List of Interactive Procedures**

22. Overall Interactive Algorithmic Structure



Controlling Parameters:

- weighting vector
- e_i RHS values
- aspiration vector
- others

23. Vector-Maximum/Filtering

- Let number of solutions shown be 8, convergence rate be 1/6.
- Solve an MOLP for, say 66,000, nondominated extreme points.
- Filter to obtain the 8 most different among the 66,000.
    - Decision maker selects $\mathbf{z}^{(1)}$, the most preferred of the 8.
- Filter to obtain the 8 most different among the 11,000 closest to $\mathbf{z}^{(1)}$.
    - Decision maker selects $\mathbf{z}^{(2)}$, the most preferred of the new 8.
- Filter to obtain the 8 most different among the 1,833 closets to $\mathbf{z}^{(2)}$.
    - Decision maker selects $\mathbf{z}^{(3)}$, the most preferred of the new 8.
- And so forth.

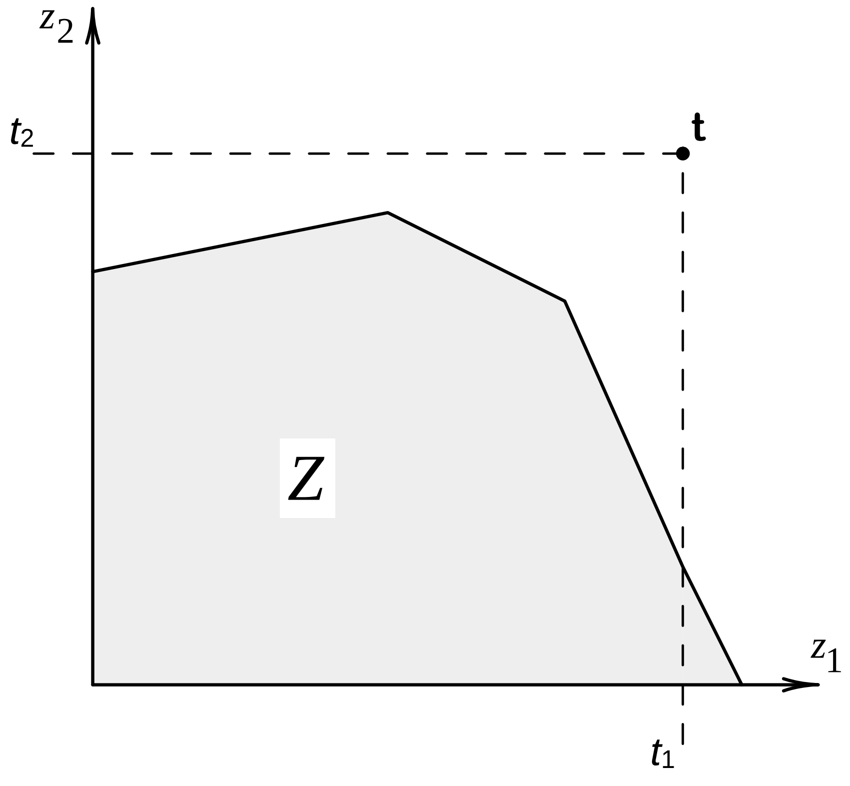
24. Goal Programming

$$\begin{aligned} \max\{\, c^1 x = z_1 \,\} \\ \text{achieve}\{\, c^2 x = z_2 \,\} \\ \min\{\, c^3 x = z_3 \,\} \\ s.t. \quad x \in S \end{aligned}$$

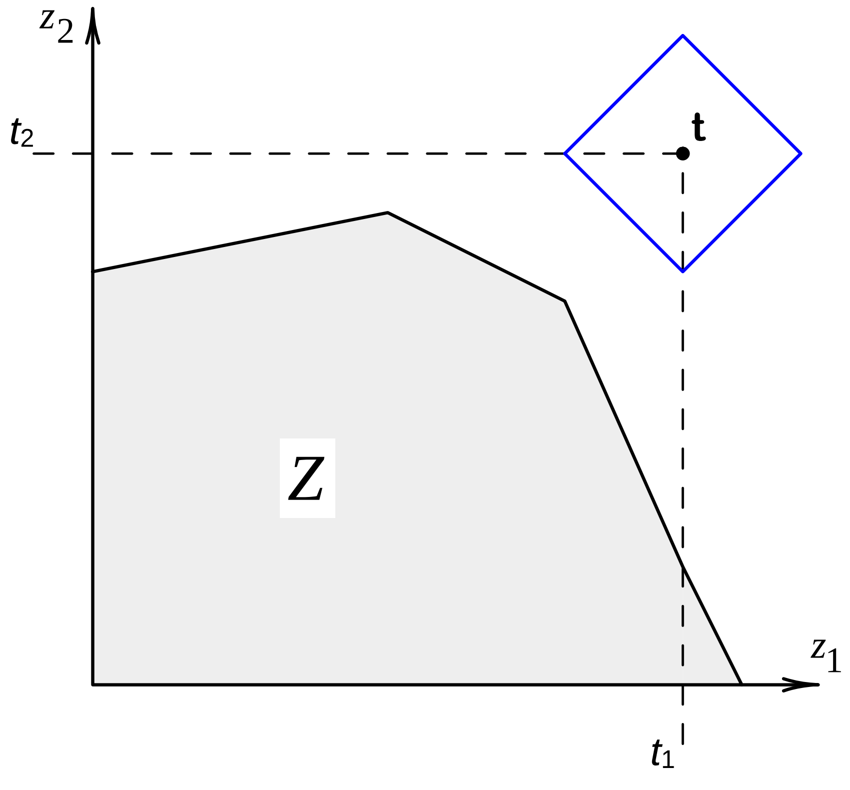
$$\begin{aligned} \min\{\, w_1^- d_1^- + w_2^- d_2^- + w_2^+ d_2^+ + w_3^+ d_3^+ \,\} \\ s.t. \quad c^1 x + d_1^- \geq t_1 \\ c^2 x + d_2^- - d_2^+ = t_2 \\ c^3 x + d_3^+ \leq t_3 \\ x \in S \\ \text{all } d_i^-, d_i^+ \geq 0 \end{aligned}$$

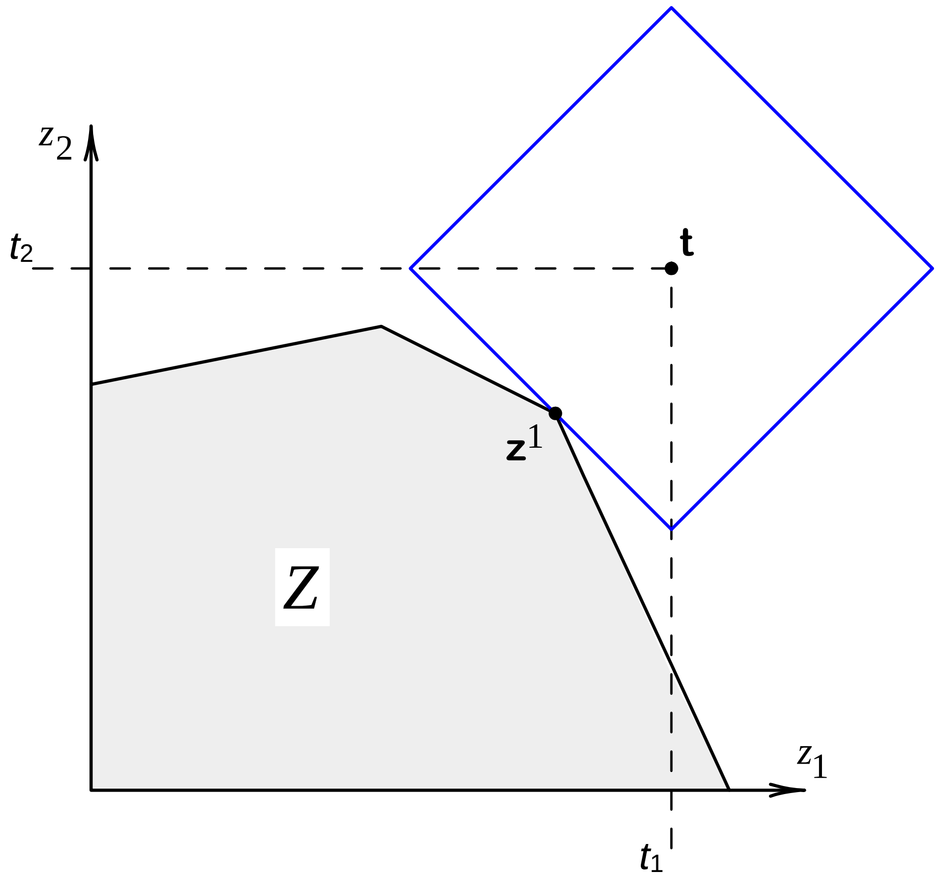
Must choose a target vector and then select deviational variable weights.

Goal programming uses weighted L_1-metric.

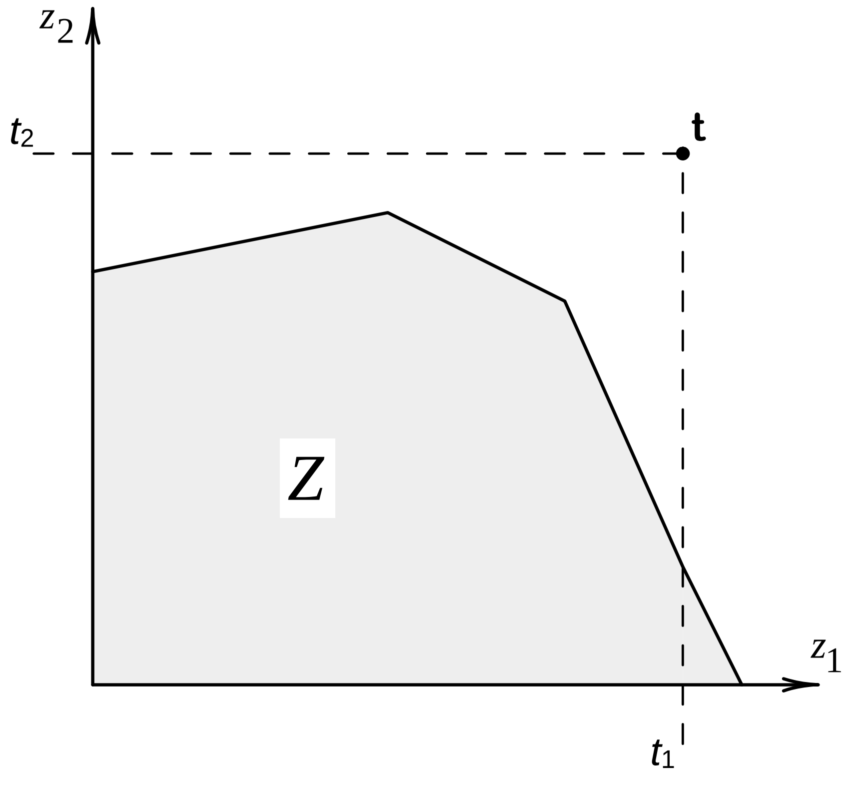


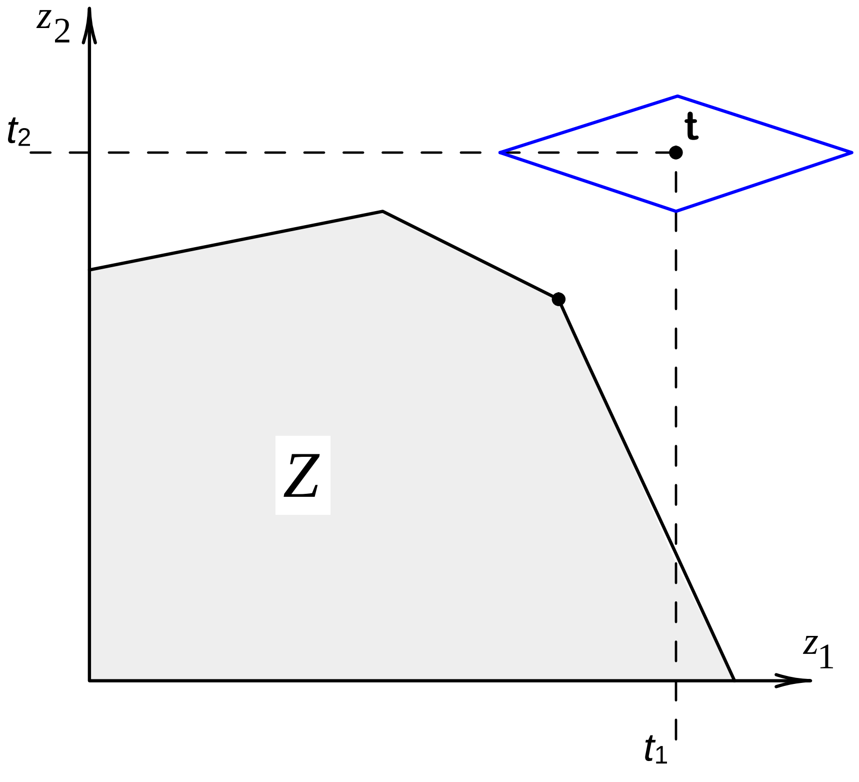
z_2
t_2
t
Z
z_1
t_1


z_2
t_2
t
Z
z_1
t_1

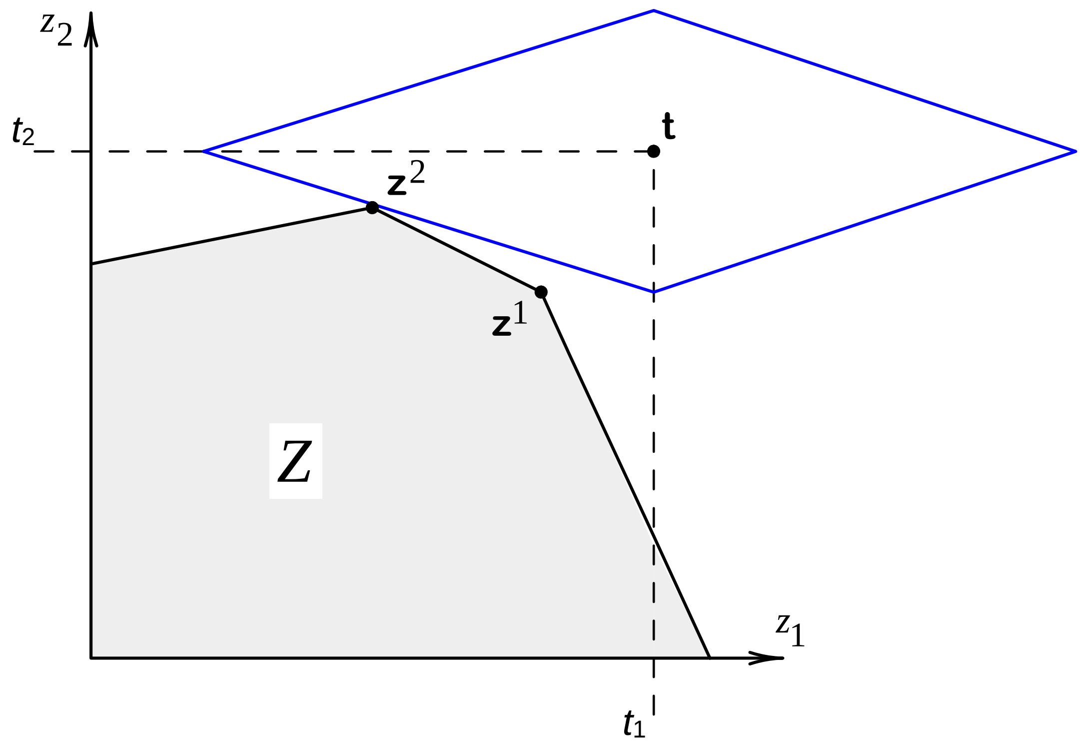


z_2
t_2
$\mathbf{t}$
$\mathbf{z}^1$
Z
z_1
t_1


z_2
t_2
t
Z
z_1
t_1



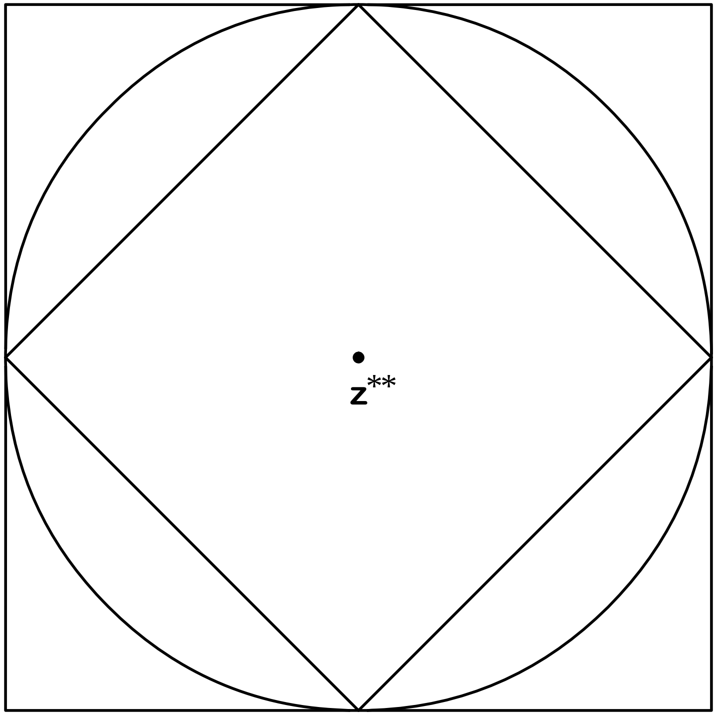
z_2
t_2
t
Z
z_1
t_1



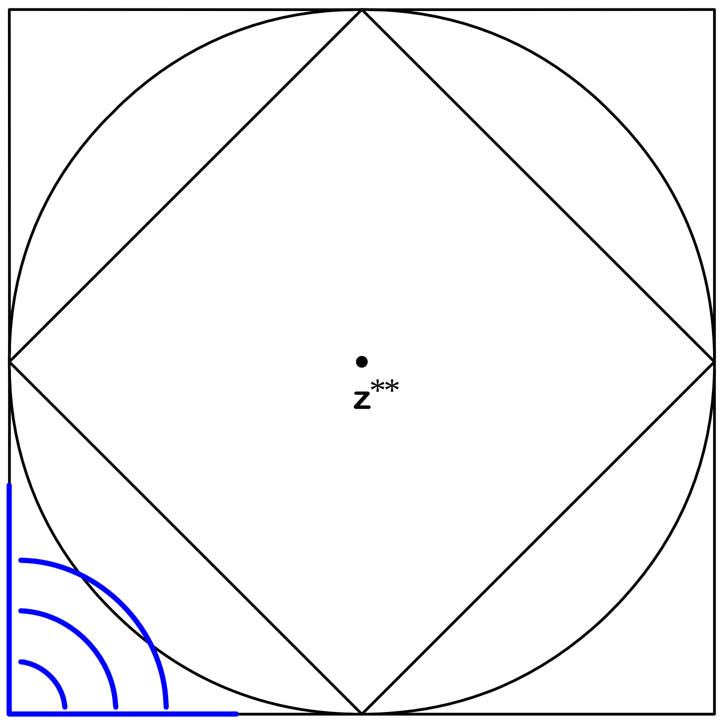
z_2
t_2
$\mathbf{t}$
$\mathbf{z}^2$
$\mathbf{z}^1$
Z
z_1
t_1

25. Lp-Metrics

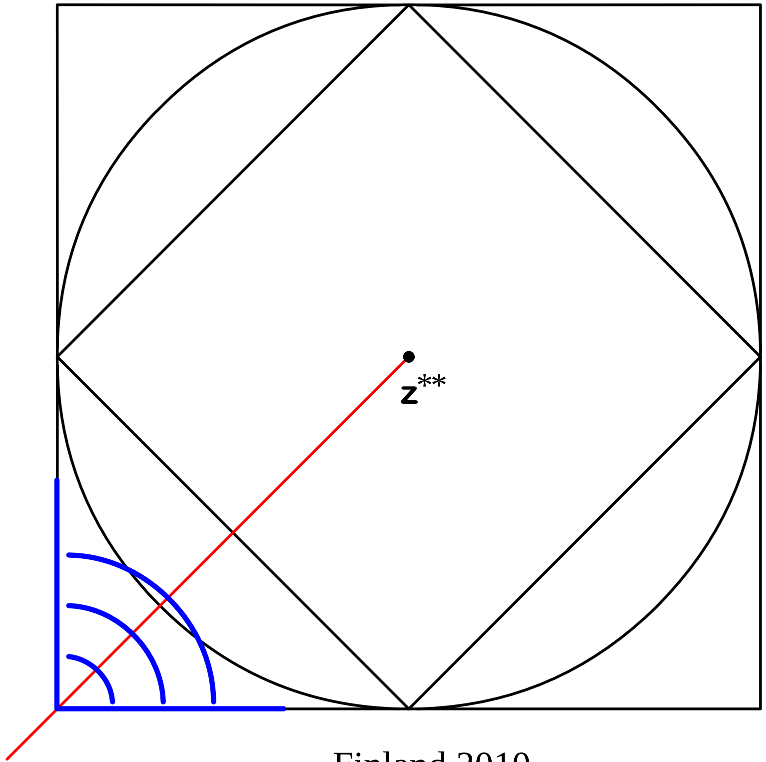
$$\left\| z - z^{**} \right\|_p = \begin{cases} \left[\sum_{i=1}^{k} \left| z_i - z_i^{**} \right|^p \right]^{\frac{1}{p}} & p = 1, 2, \ldots \\ \max\limits_{1 \le i \le k} \left\{ \left| z_i - z_i^{**} \right| \right\} & p = \infty \end{cases}$$



$$\left\| z - z^{**} \right\|_p = \begin{cases} \left[\sum_{i=1}^{k} \left| z_i - z_i^{**} \right|^p \right]^{\frac{1}{p}} & p = 1, 2, \ldots \\ \max\limits_{1 \le i \le k} \left\{ \left| z_i - z_i^{**} \right| \right\} & p = \infty \end{cases}$$

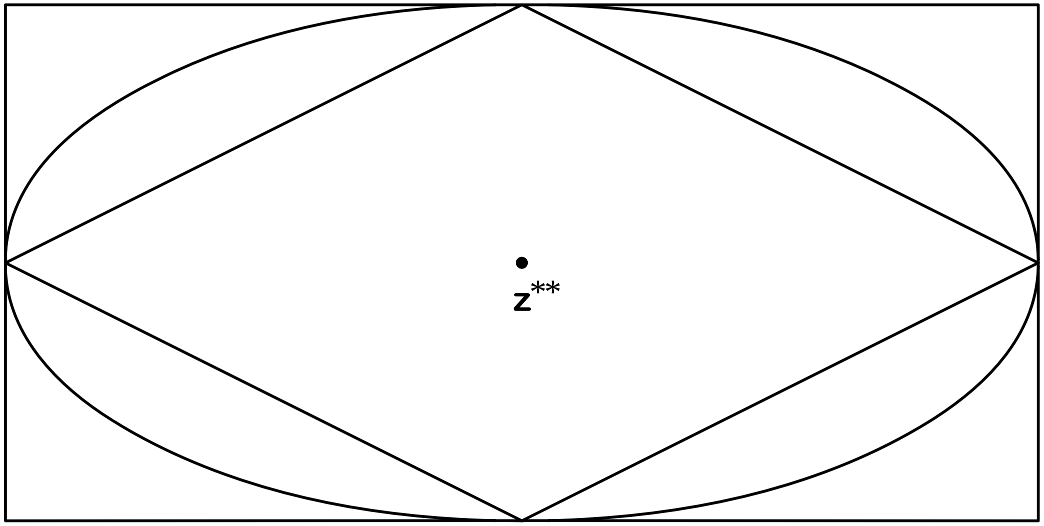


$$\left\| z - z^{**} \right\|_p = \begin{cases} \left[\sum_{i=1}^{k} \left| z_i - z_i^{**} \right|^p \right]^{\frac{1}{p}} & p = 1, 2, \ldots \\ \max\limits_{1 \le i \le k} \left\{ \left| z_i - z_i^{**} \right| \right\} & p = \infty \end{cases}$$

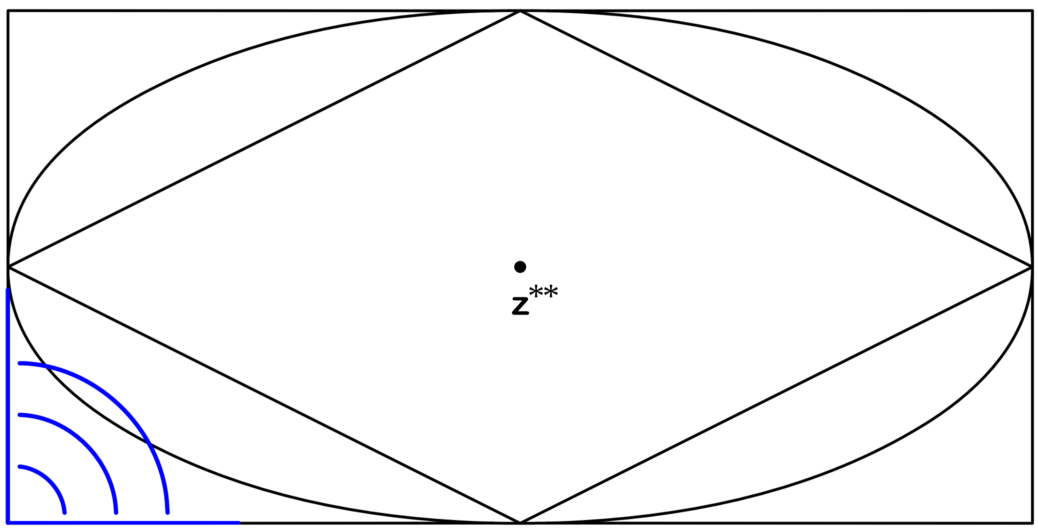


26. Weighted Lp-Metrics

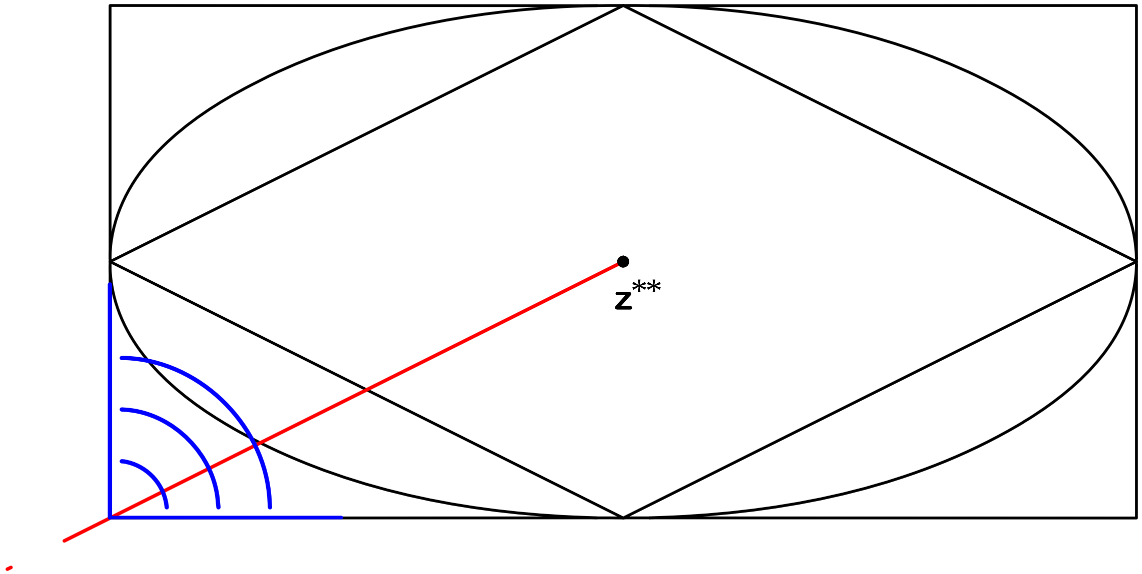
$$\left\| z - z^{**} \right\|_p^{\lambda} = \begin{cases} \left[\sum_{i=1}^{k} \left(\lambda_i \left| z_i - z_i^{**} \right|^p \right) \right]^{\frac{1}{p}} & p = 1, 2, \ldots \\ \max\limits_{1 \le i \le k} \left\{ \lambda_i \left| z_i - z_i^{**} \right| \right\} & p = \infty \end{cases}$$



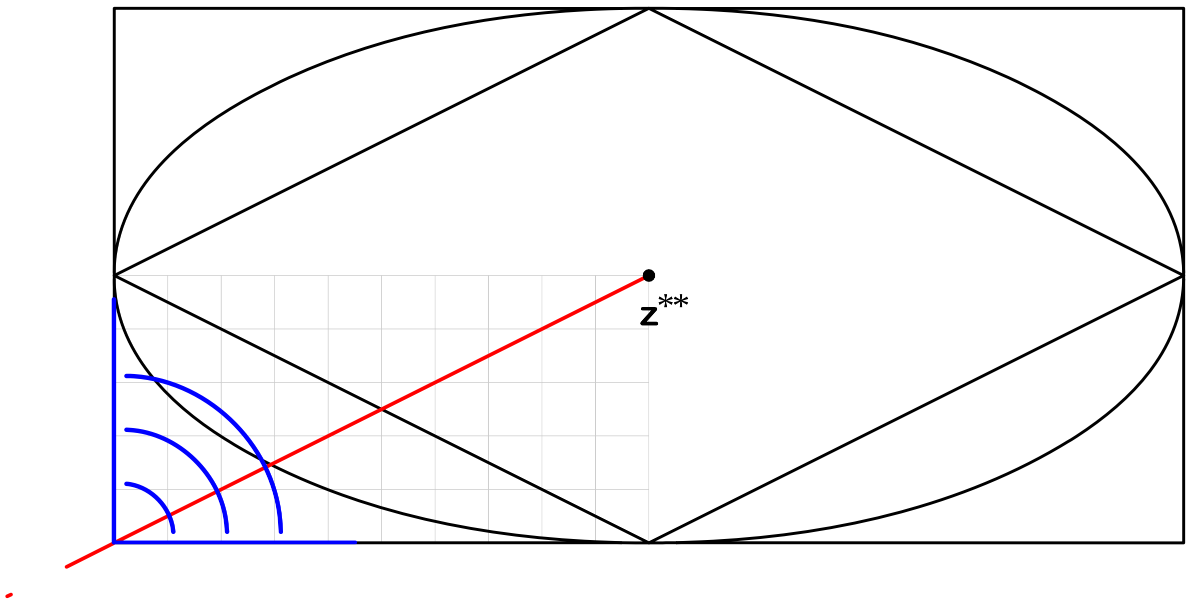
$$\left\| z - z^{**} \right\|_p^{\lambda} = \begin{cases} \left[\sum_{i=1}^{k} \left(\lambda_i \left| z_i - z_i^{**} \right|^p \right) \right]^{\frac{1}{p}} & p = 1, 2, \ldots \\ \max\limits_{1 \le i \le k} \left\{ \lambda_i \left| z_i - z_i^{**} \right| \right\} & p = \infty \end{cases}$$



$$\left\| z - z^{**} \right\|_p^{\lambda} = \begin{cases} \left[\sum_{i=1}^{k} \left(\lambda_i \left| z_i - z_i^{**} \right|^p \right) \right]^{\frac{1}{p}} & p = 1, 2, \ldots \\ \max\limits_{1 \le i \le k} \left\{ \lambda_i \left| z_i - z_i^{**} \right| \right\} & p = \infty \end{cases}$$

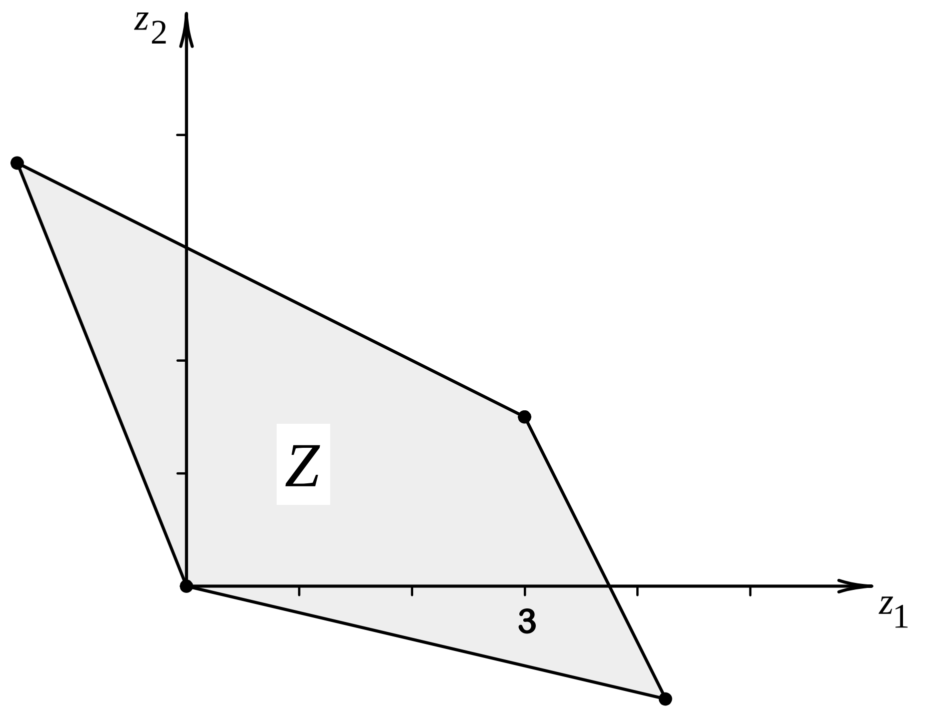


$$\left\| z - z^{**} \right\|_p^{\lambda} = \begin{cases} \left[\sum_{i=1}^{k} \left(\lambda_i \left| z_i - z_i^{**} \right|^p \right) \right]^{\frac{1}{p}} & p = 1, 2, \ldots \\ \max\limits_{1 \le i \le k} \left\{ \lambda_i \left| z_i - z_i^{**} \right| \right\} & p = \infty \end{cases}$$



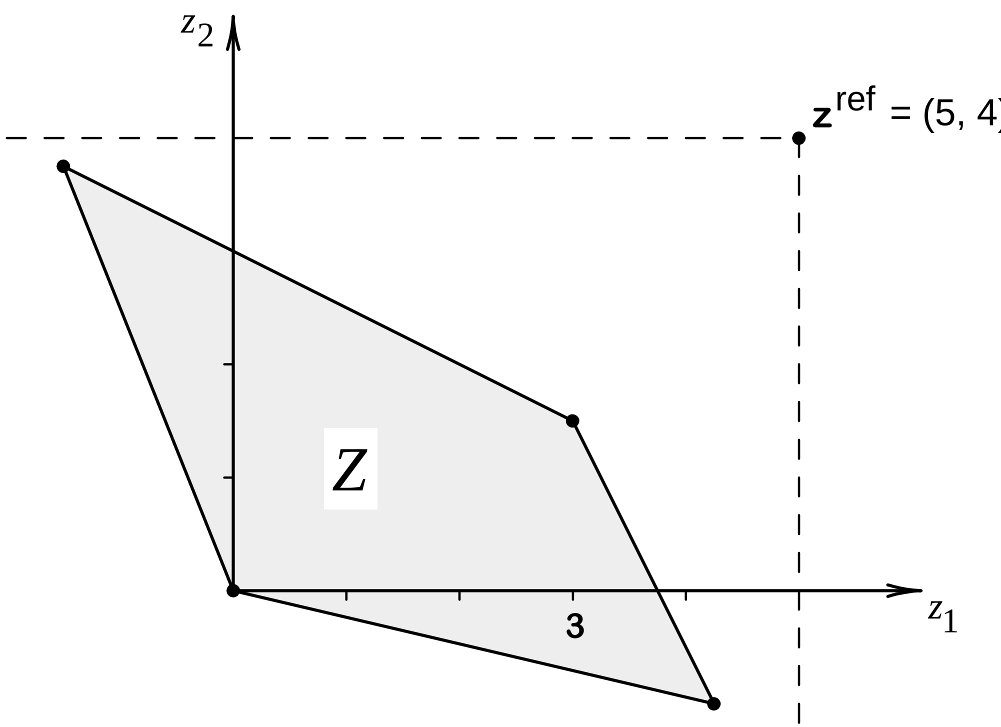
27. Reference Criterion Vector

Constructed so as to dominate every point in the nondominated set

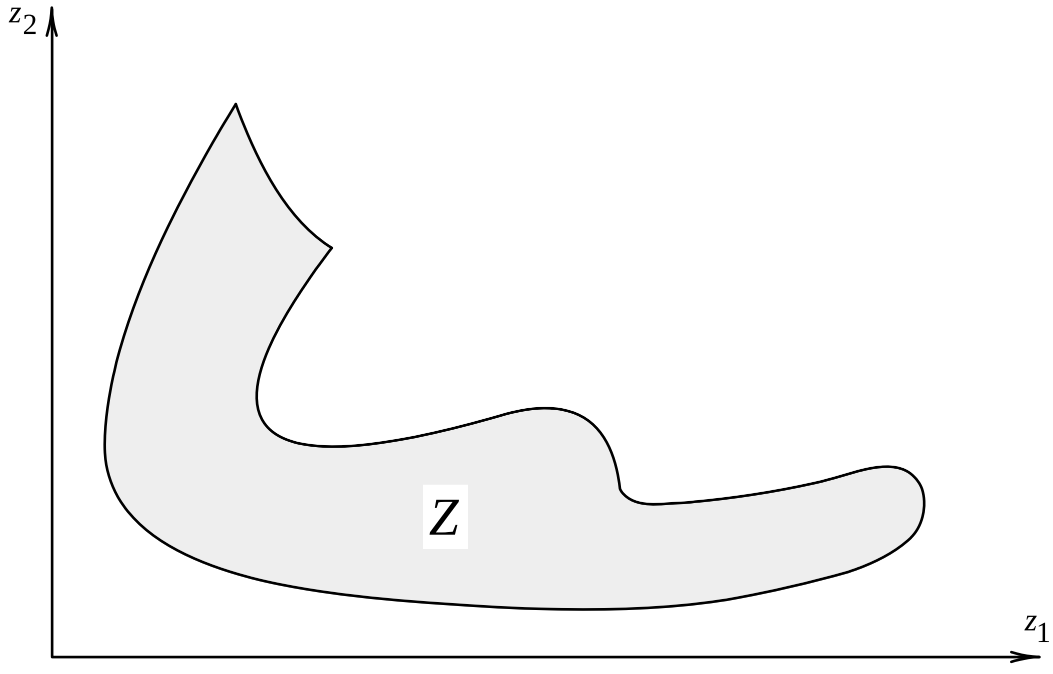


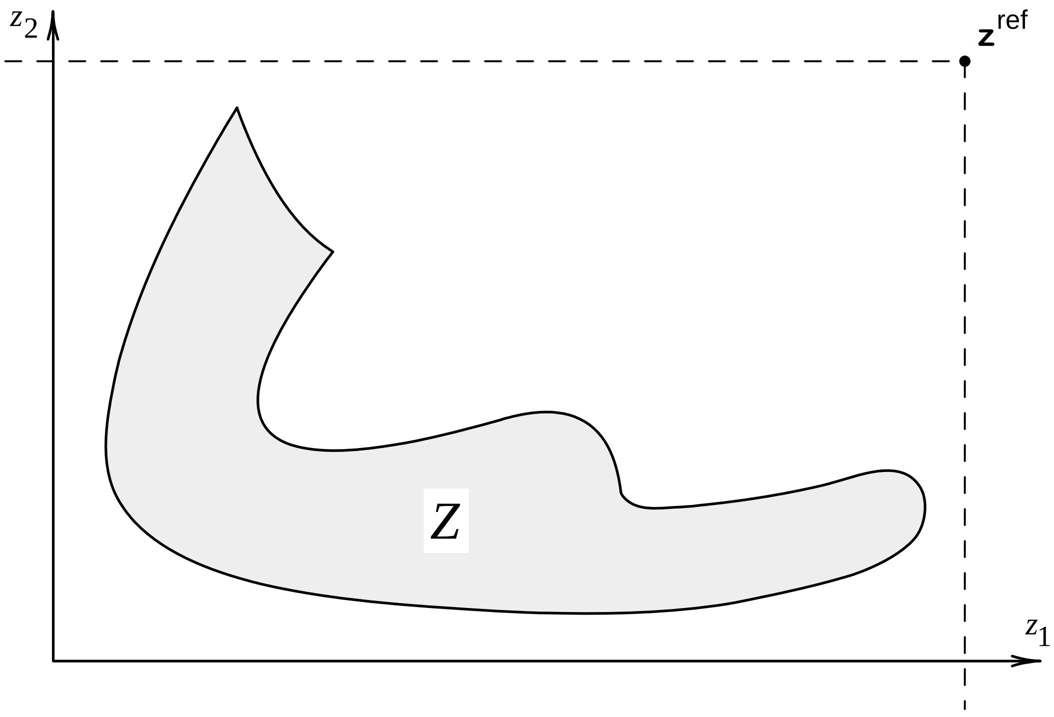
usually good enough to round to next largest integer

$$z_i^{\text{ref}} = z_i^{\max} + \varepsilon_i$$

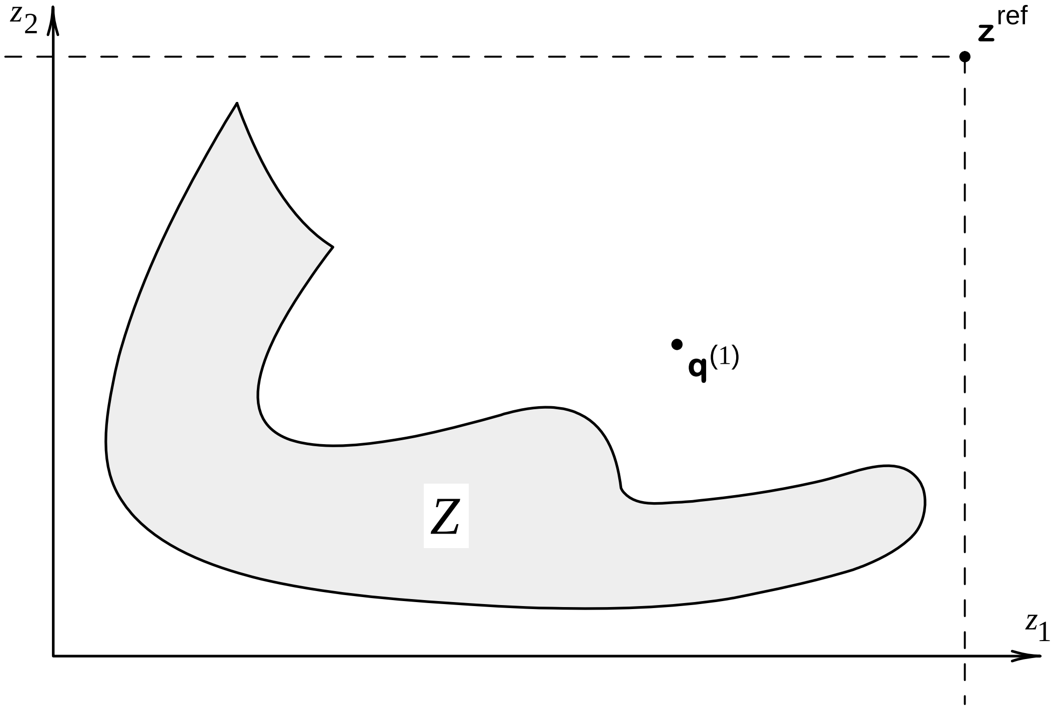


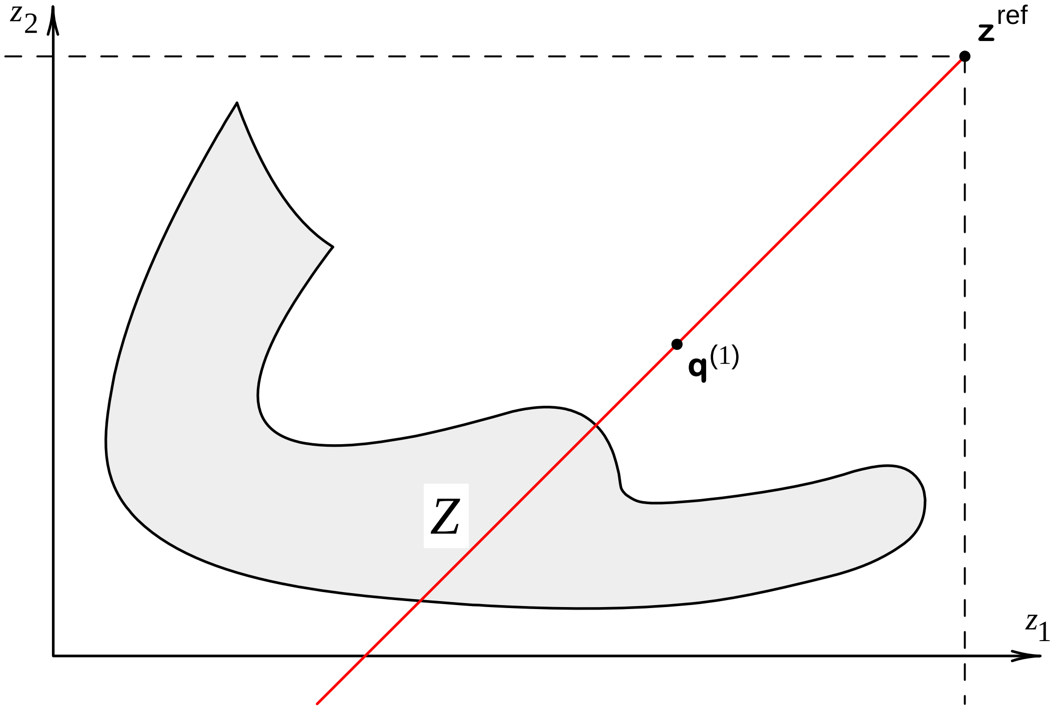
28. Wierzbicki's Reference Point Procedure

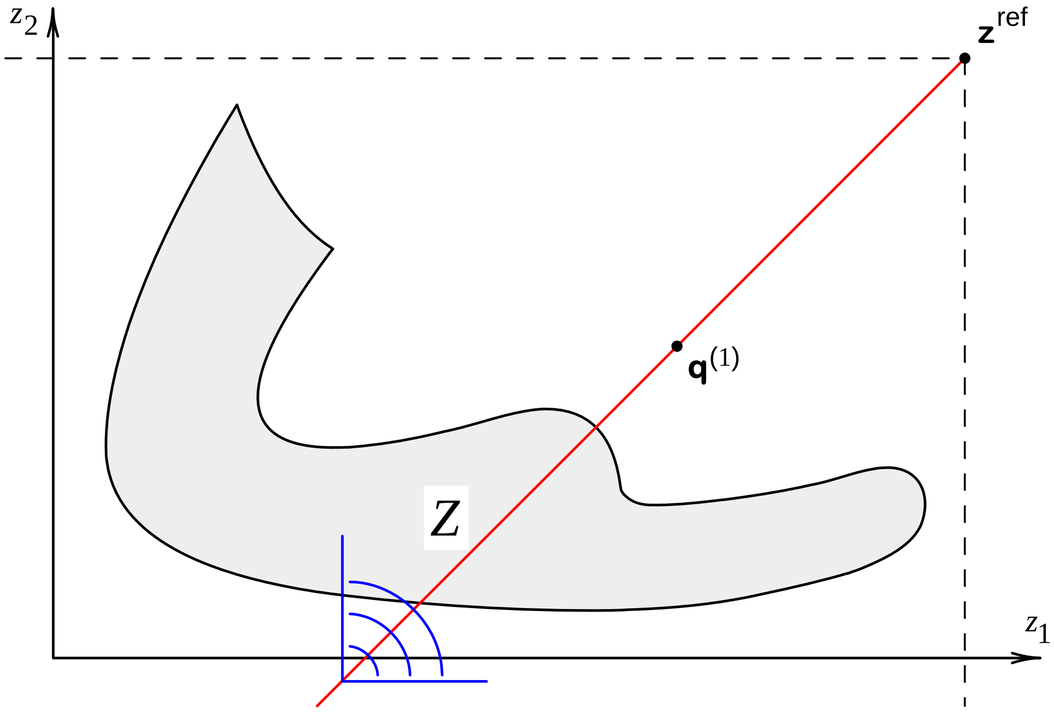



z_2
$\mathbf{z}^{\text{ref}}$
Z
z_1

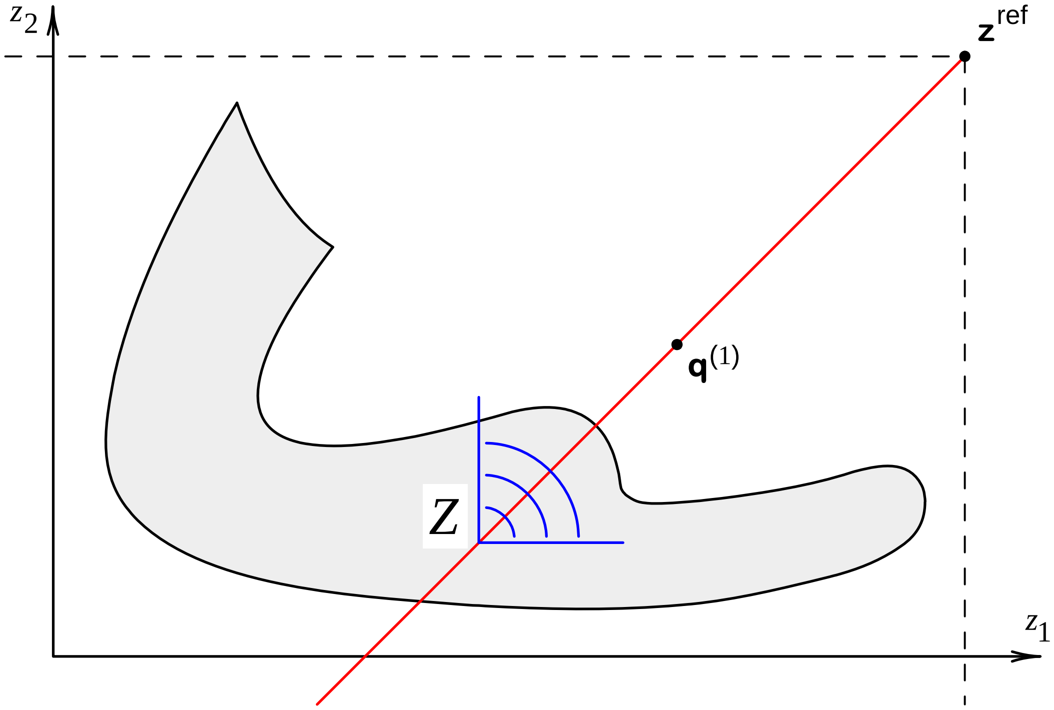
First iteration



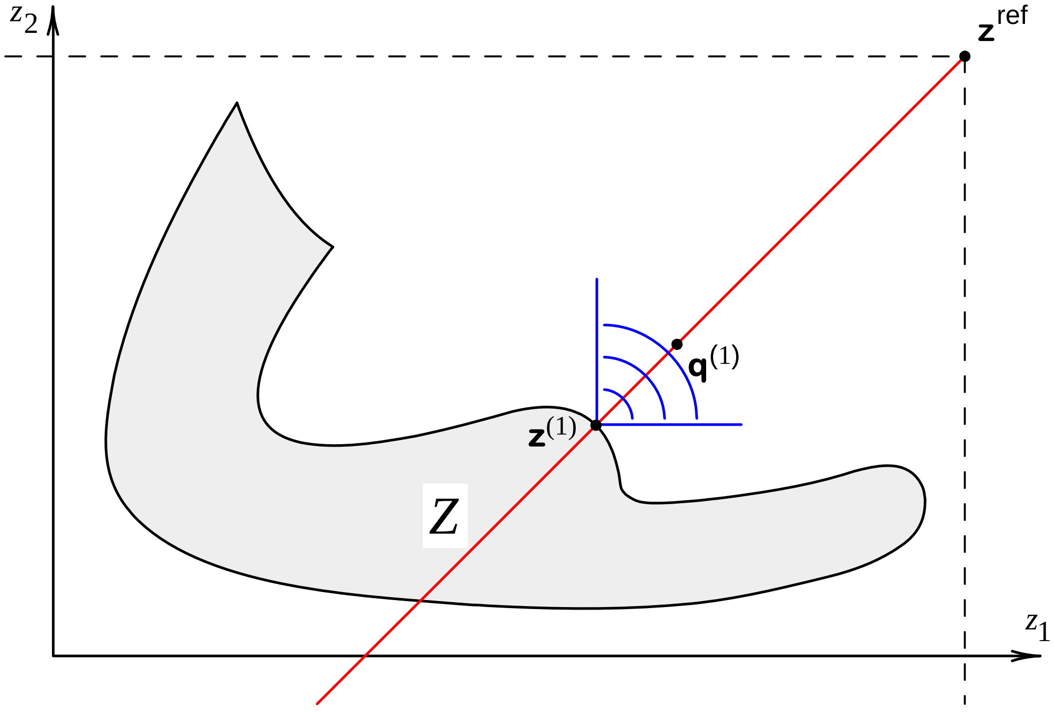

z2
z^ref
q^(1)
Z
z1



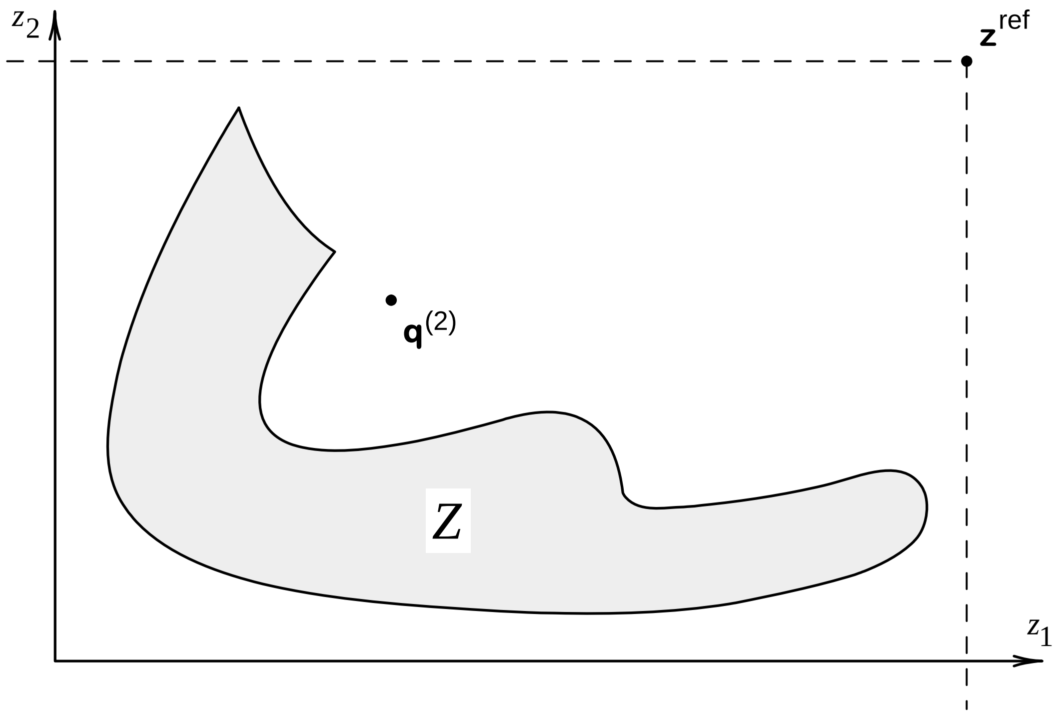
z_2
$\mathbf{z}^{\mathrm{ref}}$
$\mathbf{q}^{(1)}$
Z
z_1

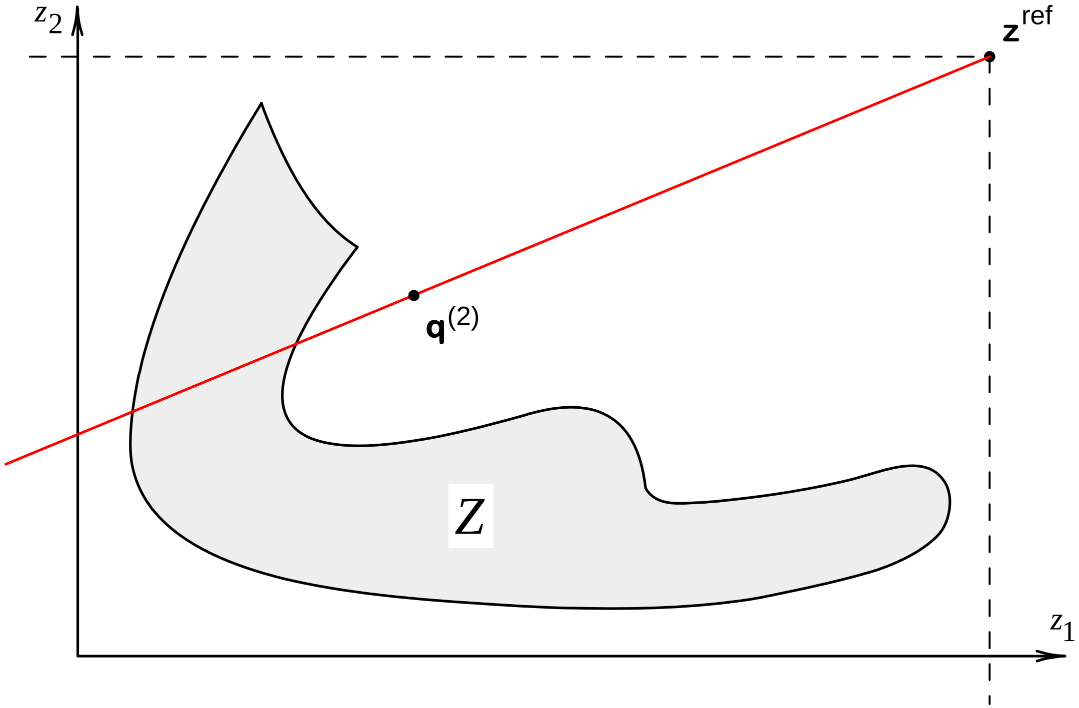


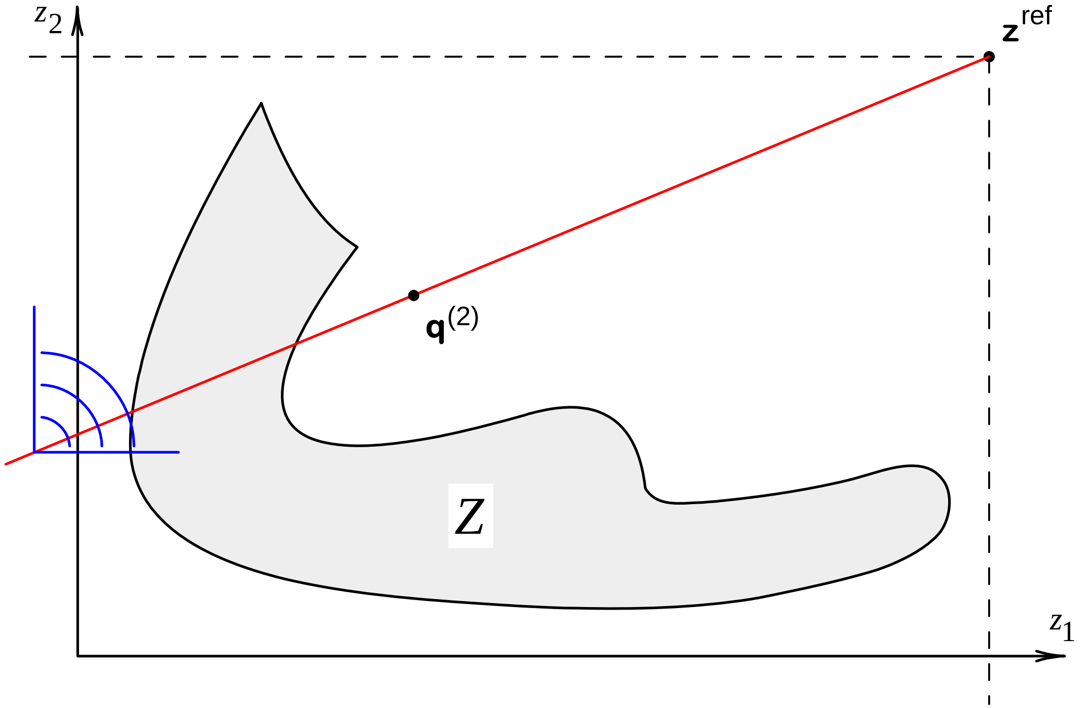
z_2
$\mathbf{z}^{\mathrm{ref}}$
$\mathbf{q}^{(1)}$
Z
z_1


z_2
$\mathbf{z}^{\text{ref}}$
$\mathbf{q}^{(1)}$
$\mathbf{z}^{(1)}$
Z
z_1

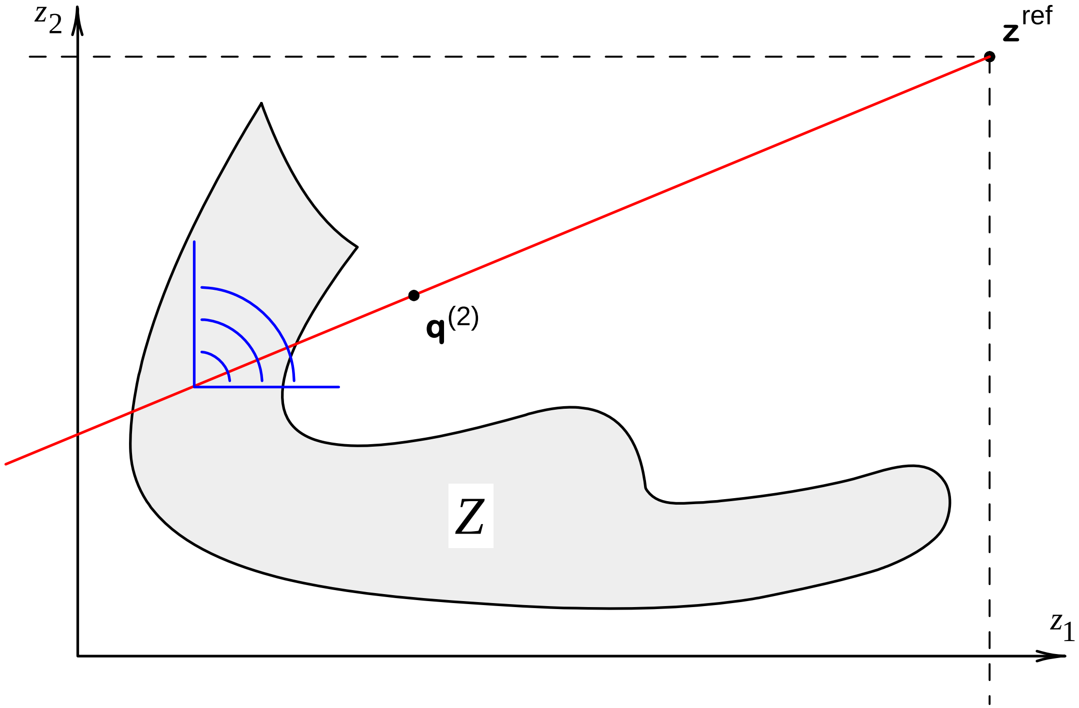
Second iteration



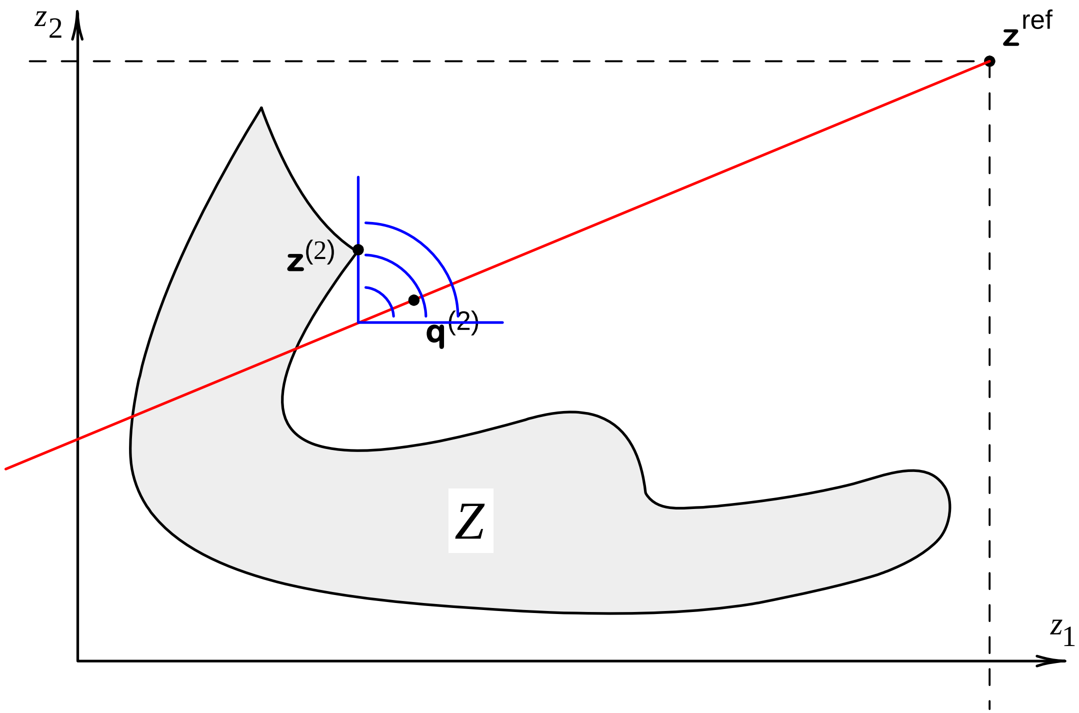

z_2
$\mathbf{z}^{\mathrm{ref}}$
$\mathbf{q}^{(2)}$
Z
z_1



z_2
$\mathbf{z}^{\text{ref}}$
$\mathbf{q}^{(2)}$
Z
z_1

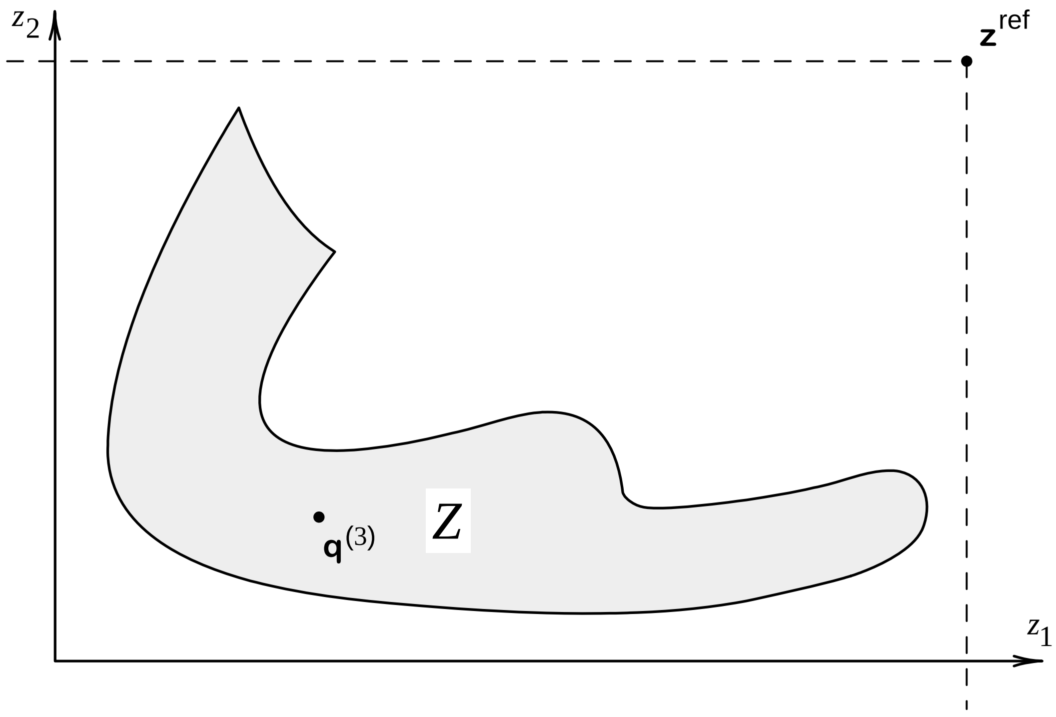


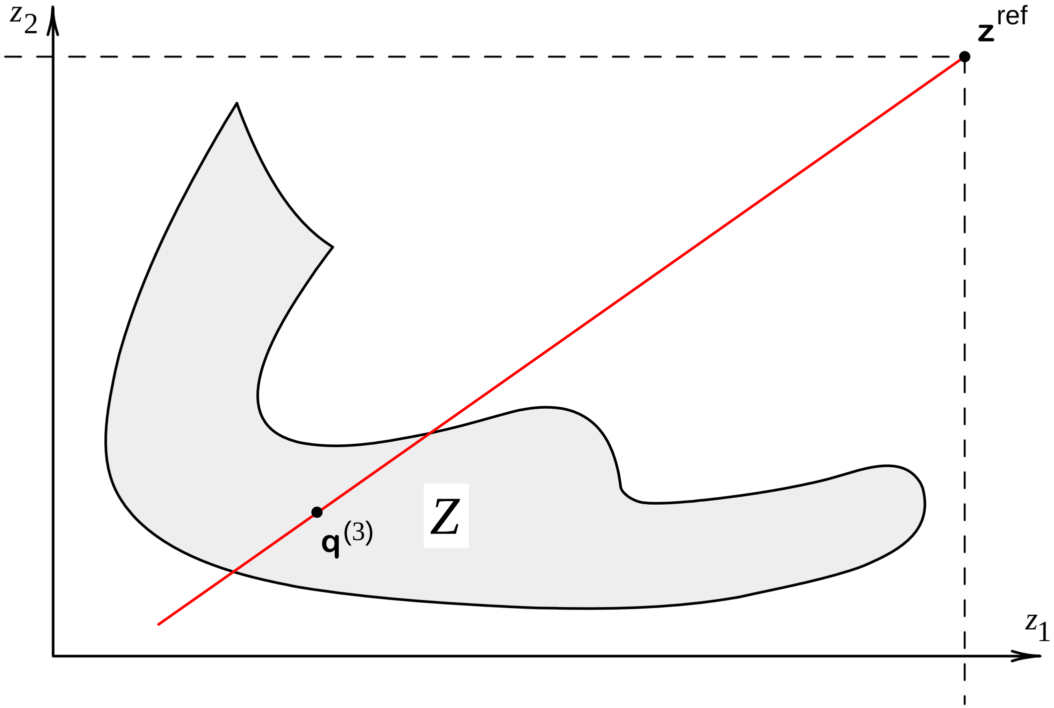
z_2
$\mathbf{z}^{\text{ref}}$
$\mathbf{q}^{(2)}$
Z
z_1



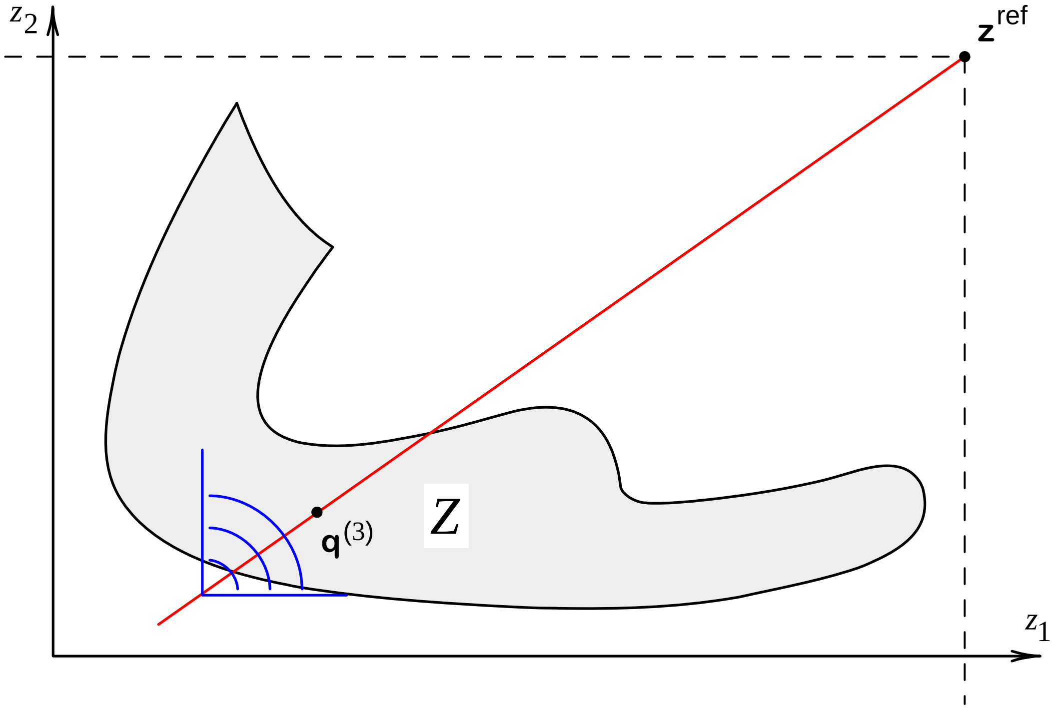
z_2
$\mathbf{z}^{\text{ref}}$
$\mathbf{z}^{(2)}$
$\mathbf{q}^{(2)}$
Z
z_1

Third iteration

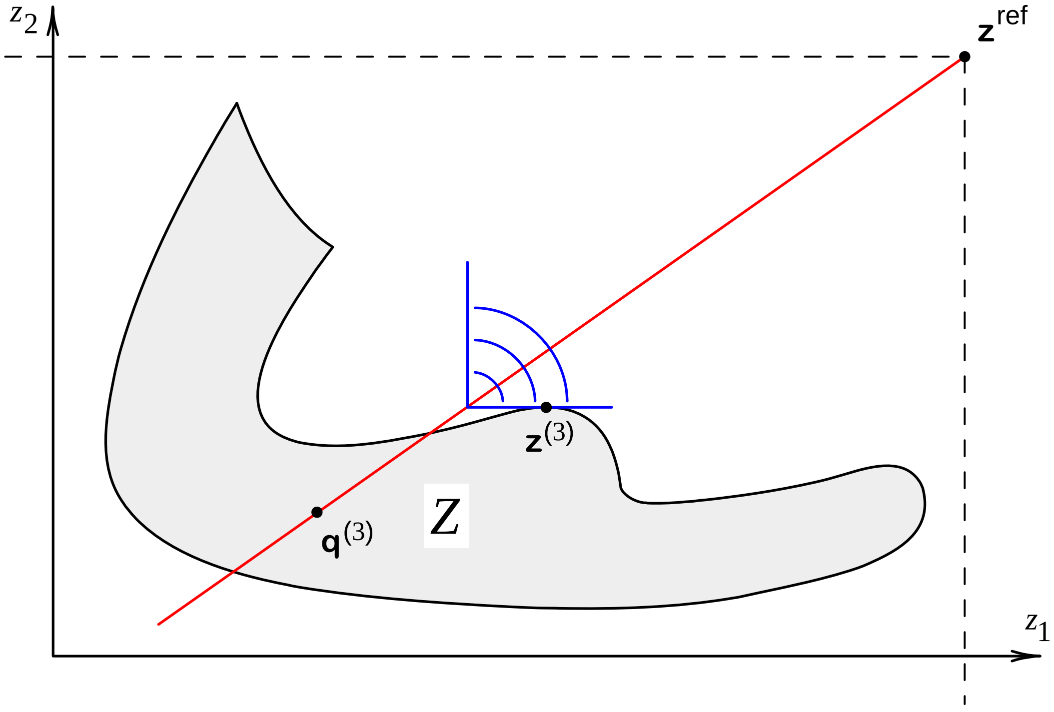




z_2
$\mathbf{z}^{\text{ref}}$
$\mathbf{q}^{(3)}$
Z
z_1



z_2
$\mathbf{z}^{\text{ref}}$
Z
$\mathbf{q}^{(3)}$
z_1



z_2
$\mathbf{z}^{\text{ref}}$
$\mathbf{z}^{(3)}$
$\mathbf{q}^{(3)}$
Z
z_1

29. Lexicographic Tchebycheff Sampling Program

Geometry carried out by lexicographic Tchebycheff sampling program

$$\text{lex min}\{\ \alpha,\quad -\sum_{i=1}^{k} z_i\ \}$$

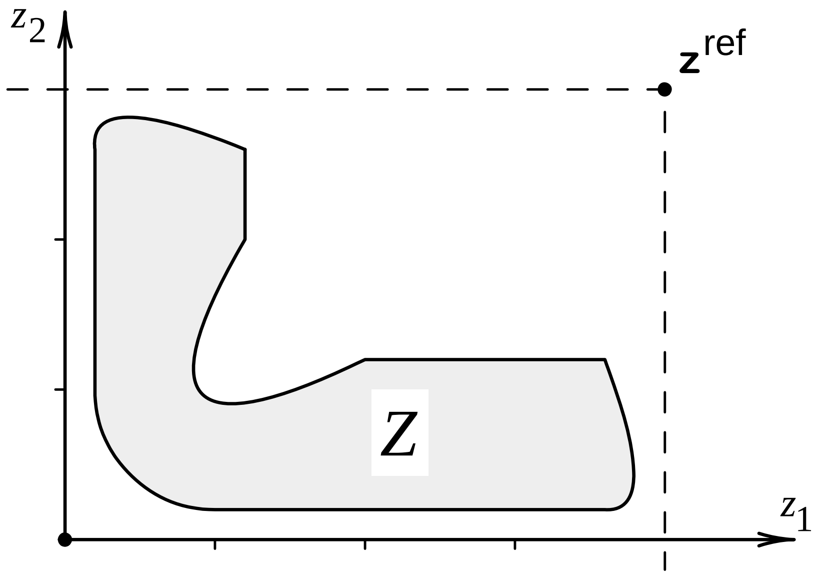
$$\begin{aligned} s.t. \qquad & \alpha \geq \lambda_i (z_i^{ref} - z_i) && i = 1,\ldots,k \\ & f_i(\mathrm{x}) = z_i && i = 1,\ldots,k \\ & \mathrm{x} \in S \end{aligned}$$

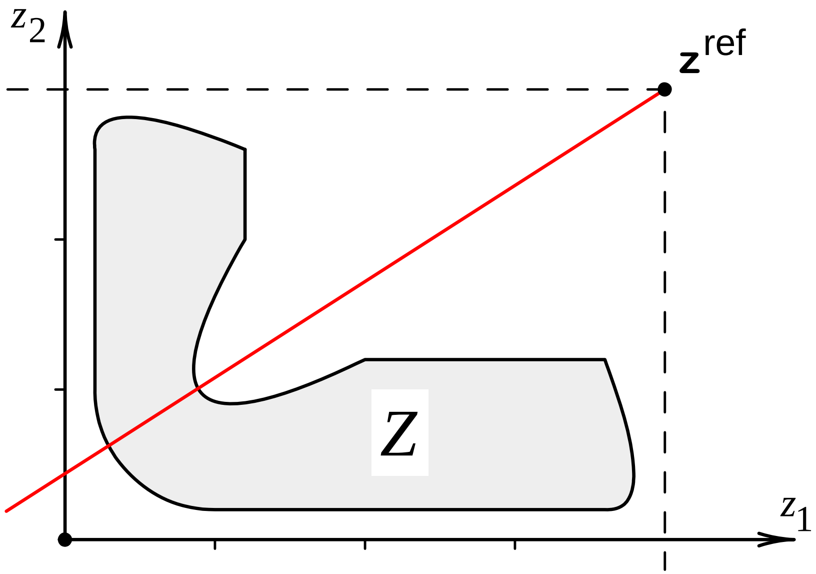
Minimizing α causes non-negative orthant contour to slide up the probing ray until it last touches the feasible region *Z*.

Perturbation term $-\sum_{i=1}^{k} z_i$ is there to break ties.

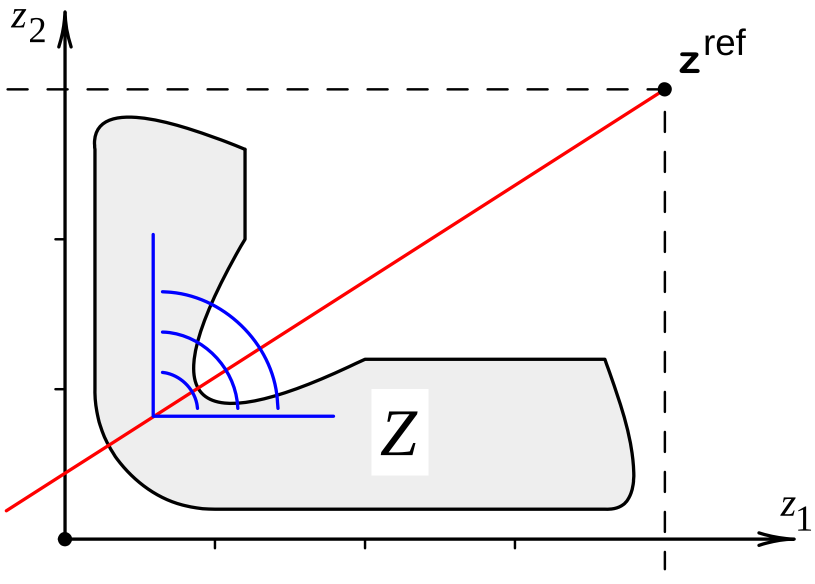
Direction of the probing ray emanating from z^{ref} is given by

$$-\left(\frac{1}{1/\lambda_i},\ldots,\frac{1}{1/\lambda_k}\right)$$


z_2
$\mathbf{z}^{\text{ref}}$
Z
z_1

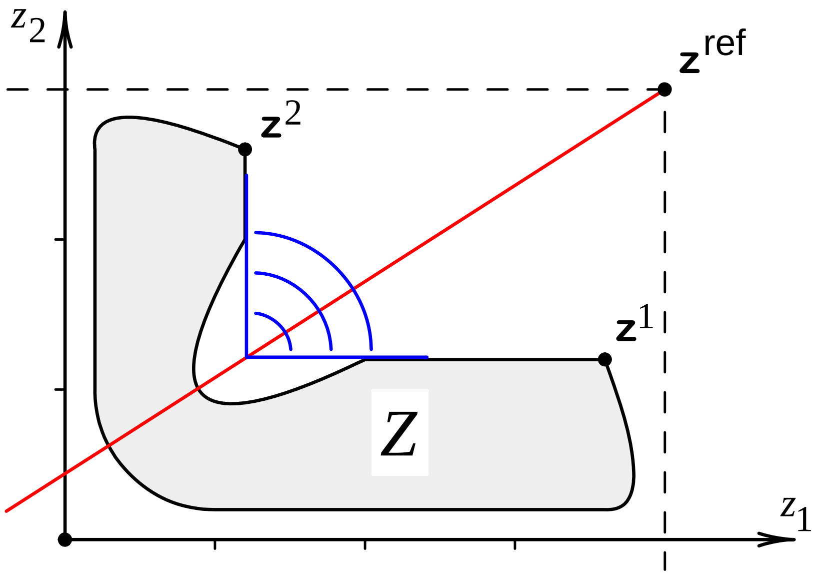


z_2
$\mathbf{z}^{\text{ref}}$
Z
z_1

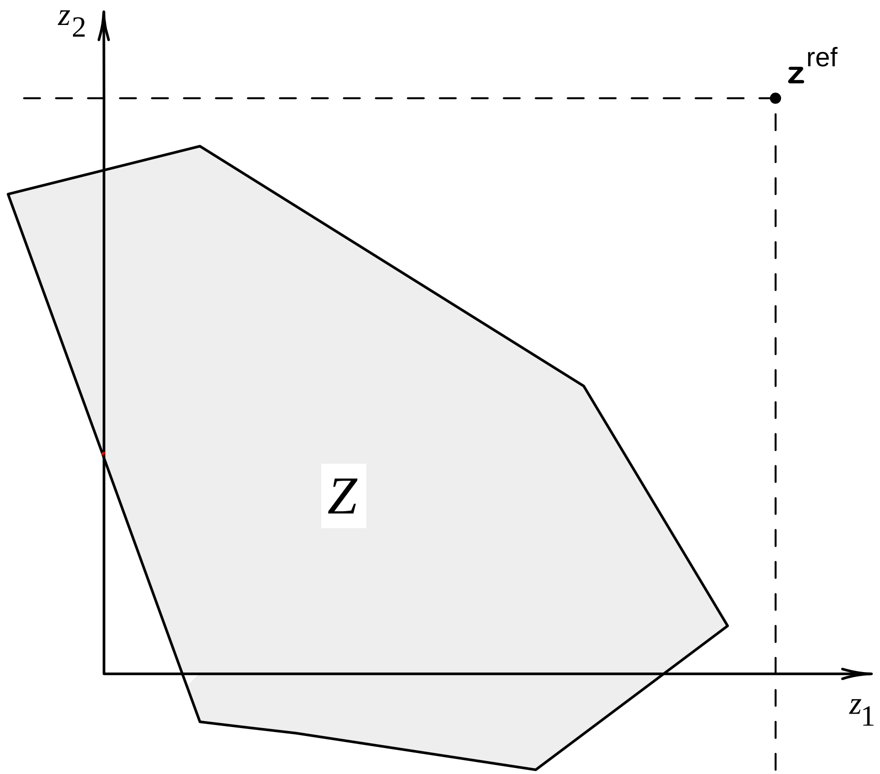


z_2
$\mathbf{z}^{\text{ref}}$
Z
z_1

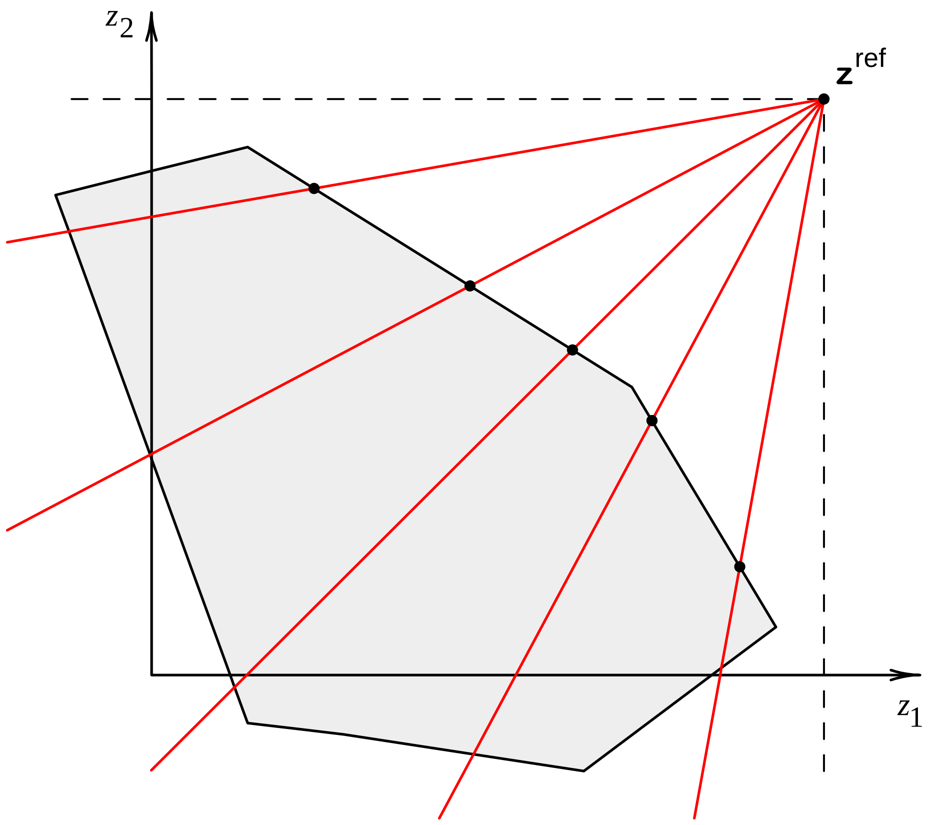
Two lexicographic minimum solutions, but both nondominated

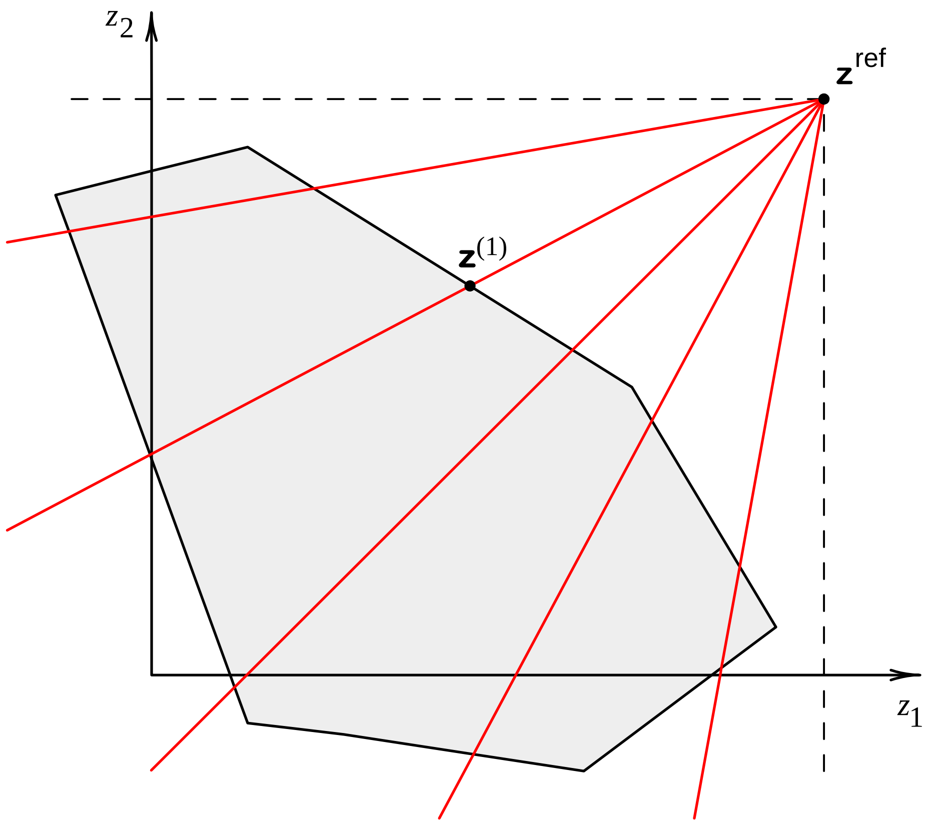


30. Tchebycheff Method (Overview)

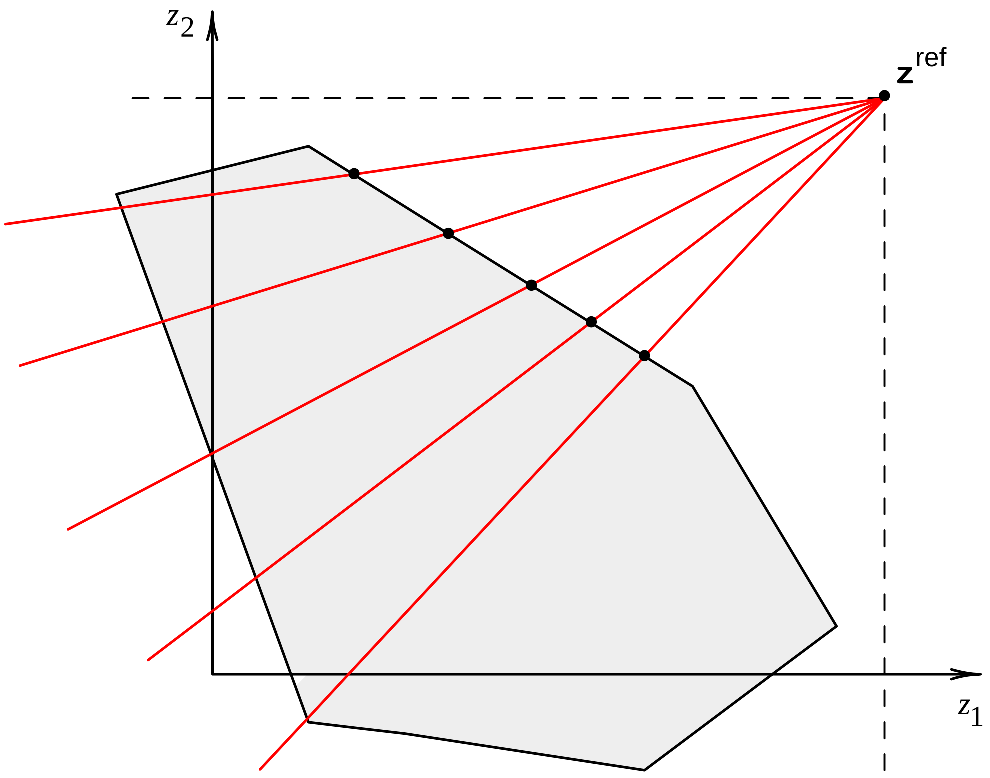


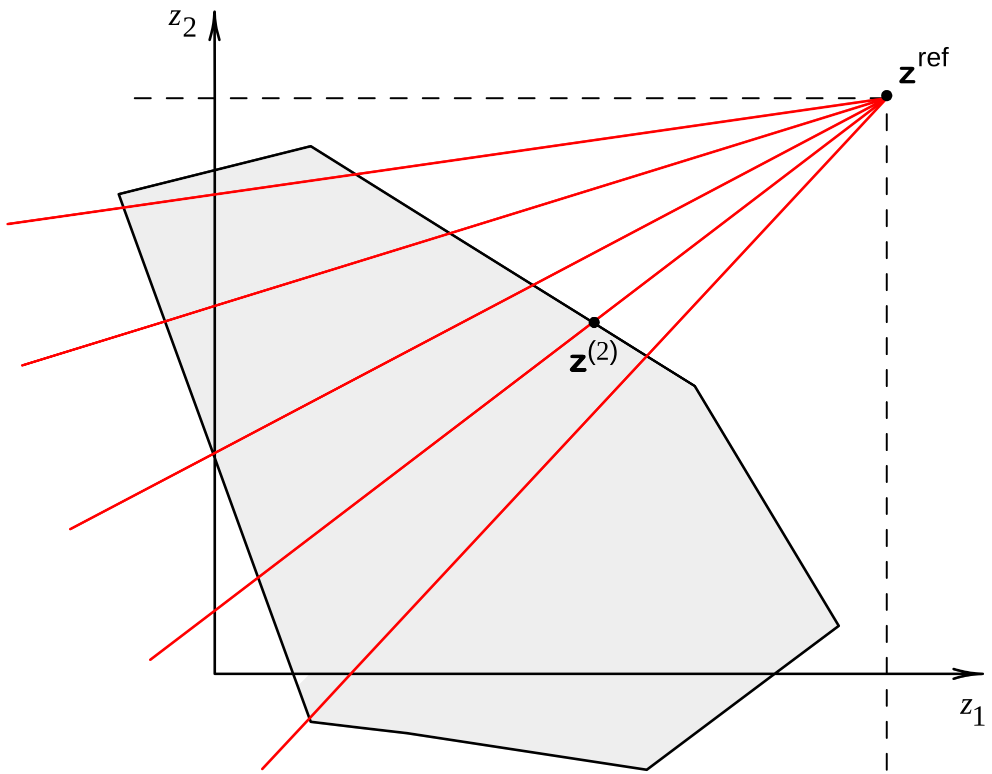
First iteration



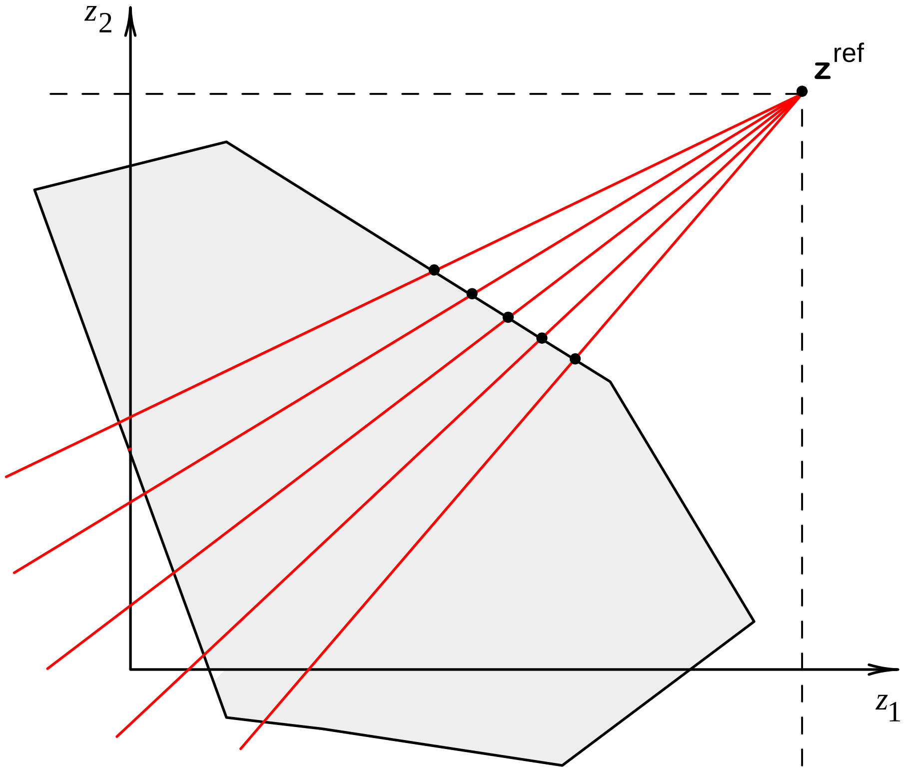

z_2
$\mathbf{z}^{\text{ref}}$
$\mathbf{z}^{(1)}$
z_1

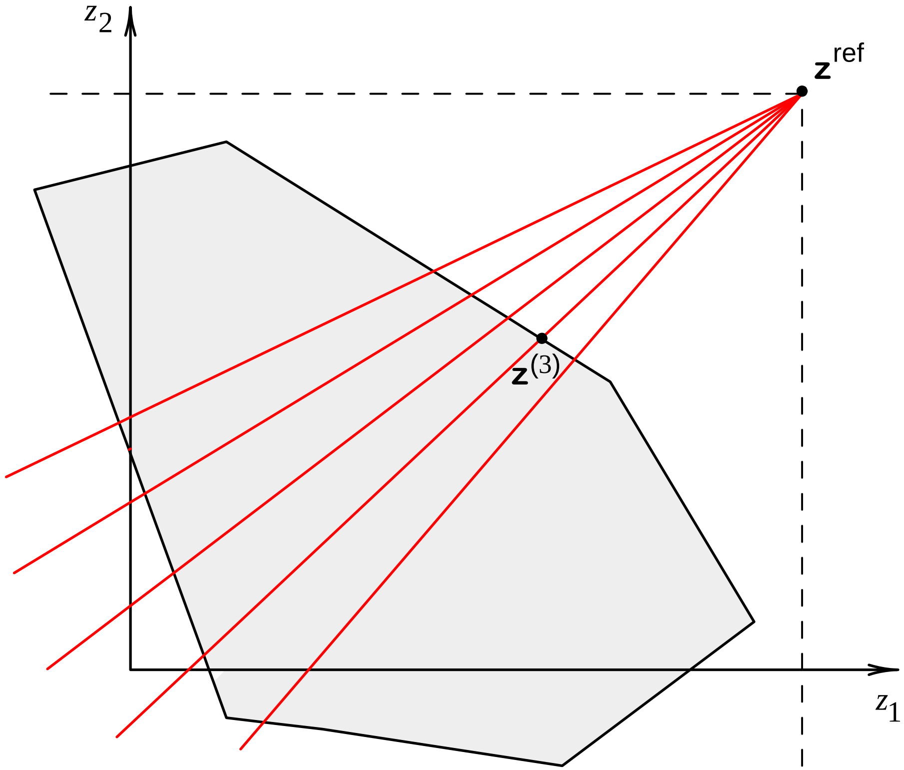
Second iteration



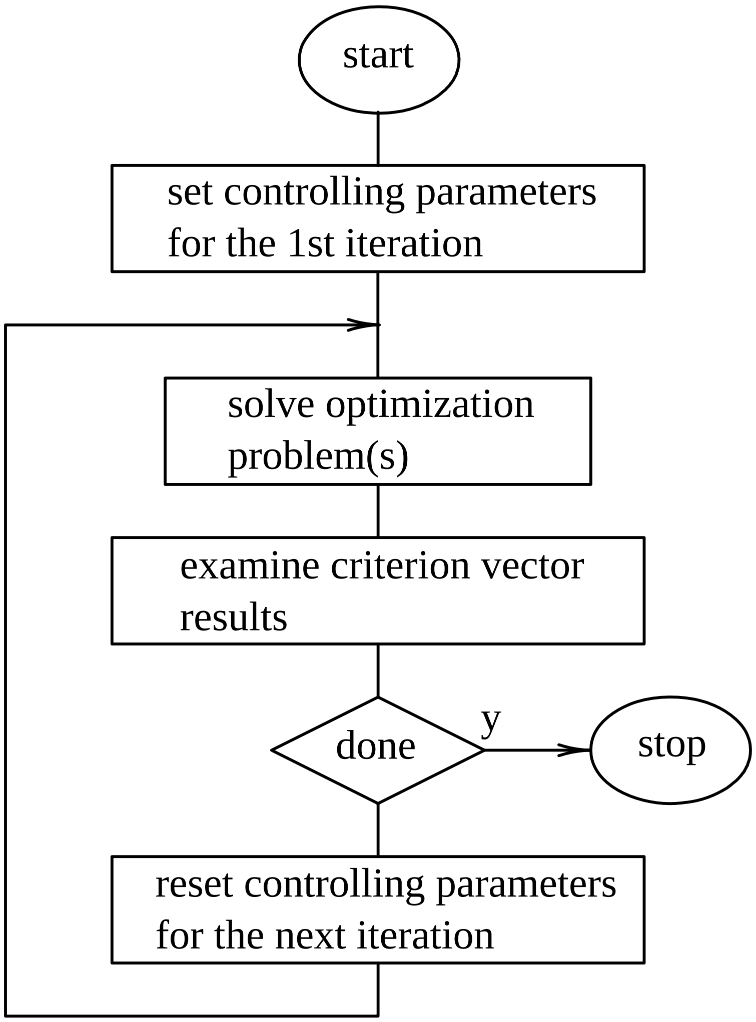

z_2
$\mathbf{z}^{\text{ref}}$
$\mathbf{z}^{(2)}$
z_1

Third iteration





z_2
$\mathbf{z}^{\text{ref}}$
$\mathbf{z}^{(3)}$
z_1



Controlling Parameters:

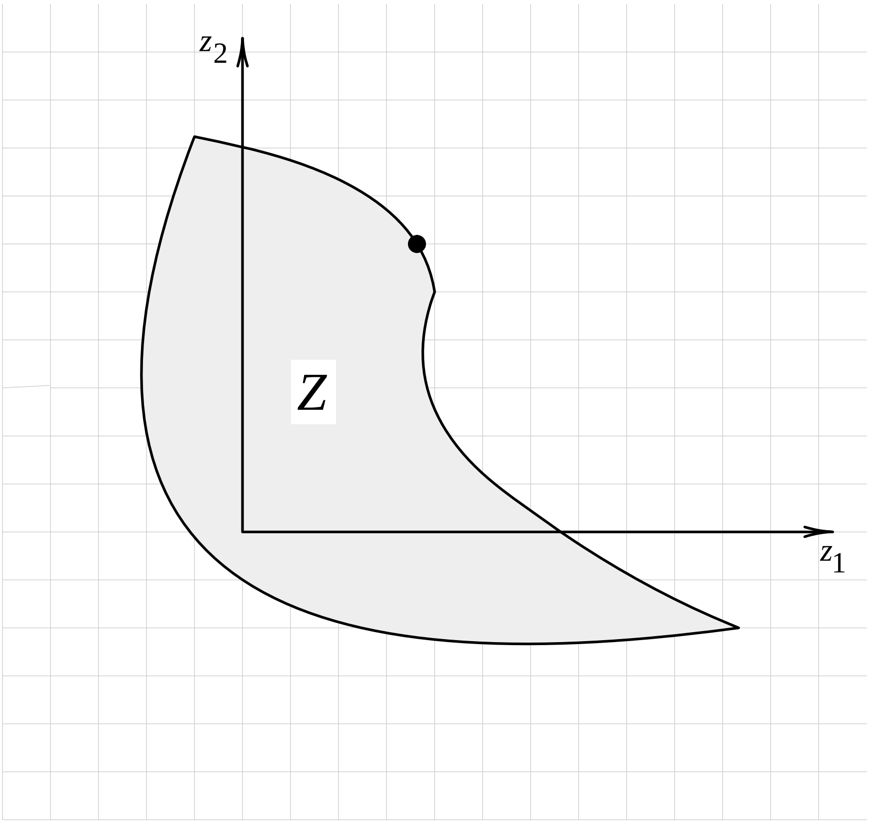
target vector, weights
$q^{(i)}$ aspiration vectors
λ_i multipliers

31. Tchebycheff Method (in more detail)

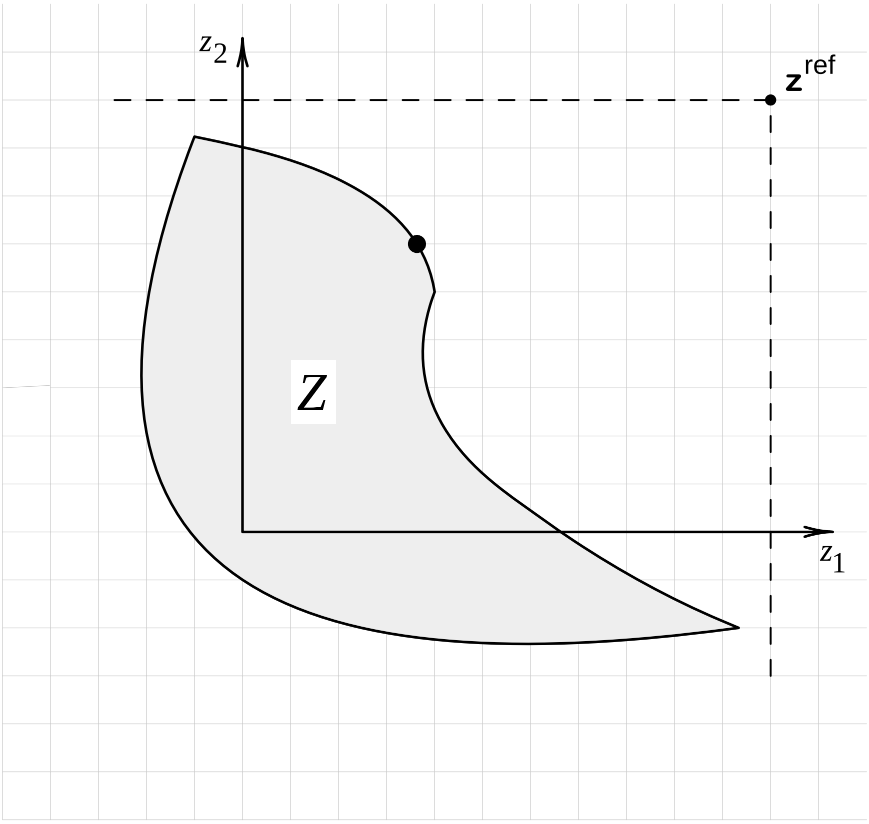
Let P = number of solutions to be presented to the DM at each iteration = 4

Let r = reduction factor = 0.5

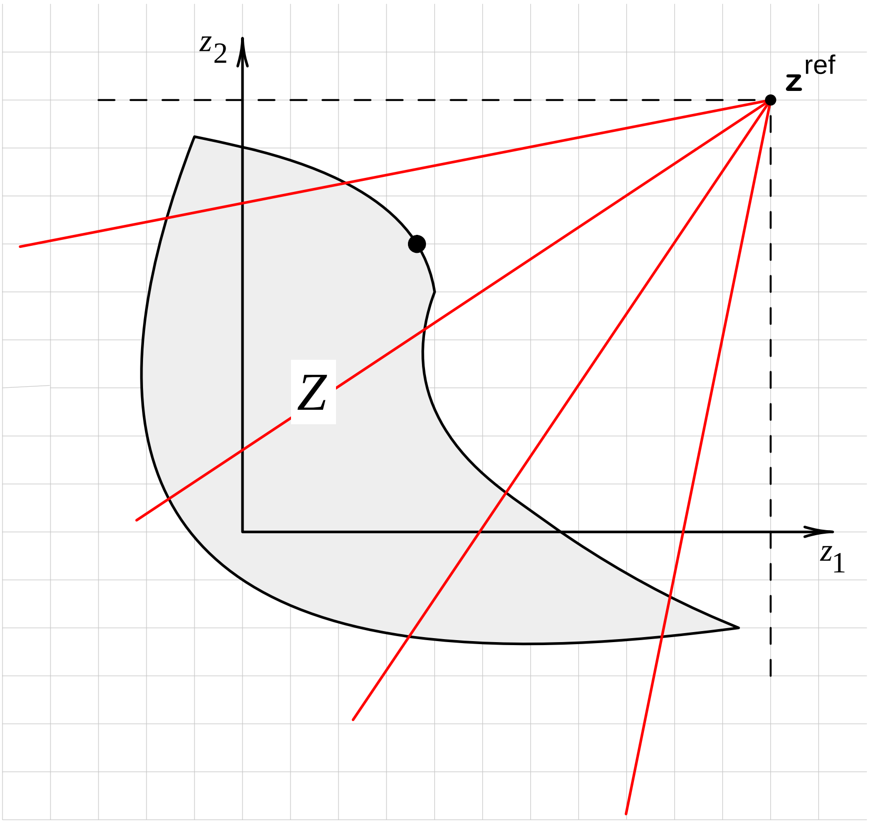
Let t = number of iterations = 4



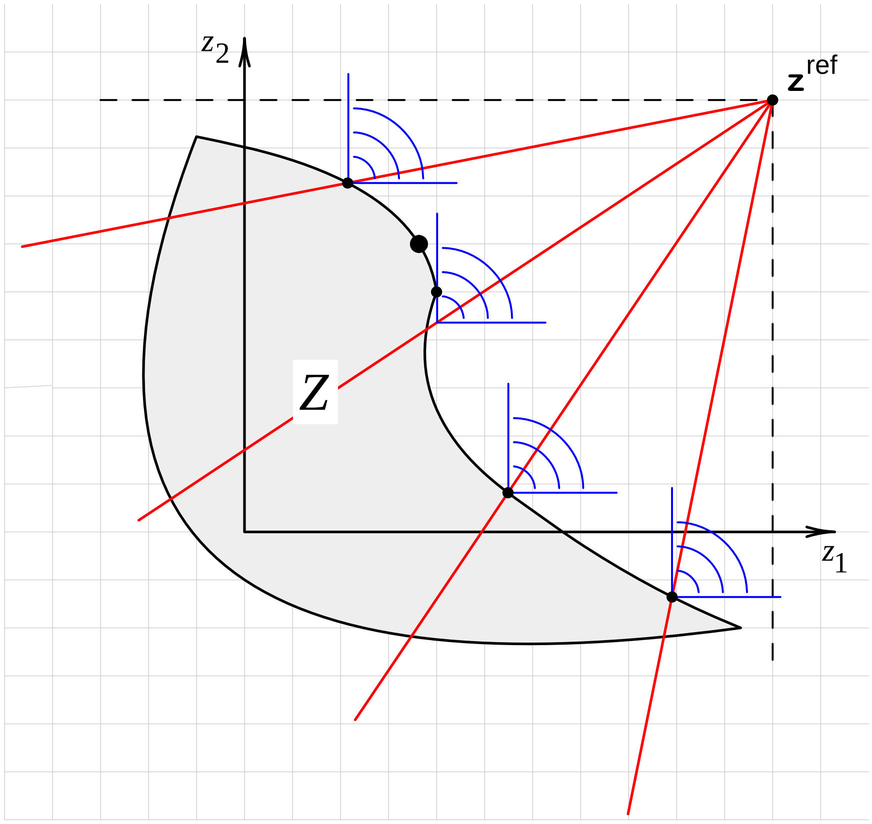
Now, form reference criterion vector z^{ref}.



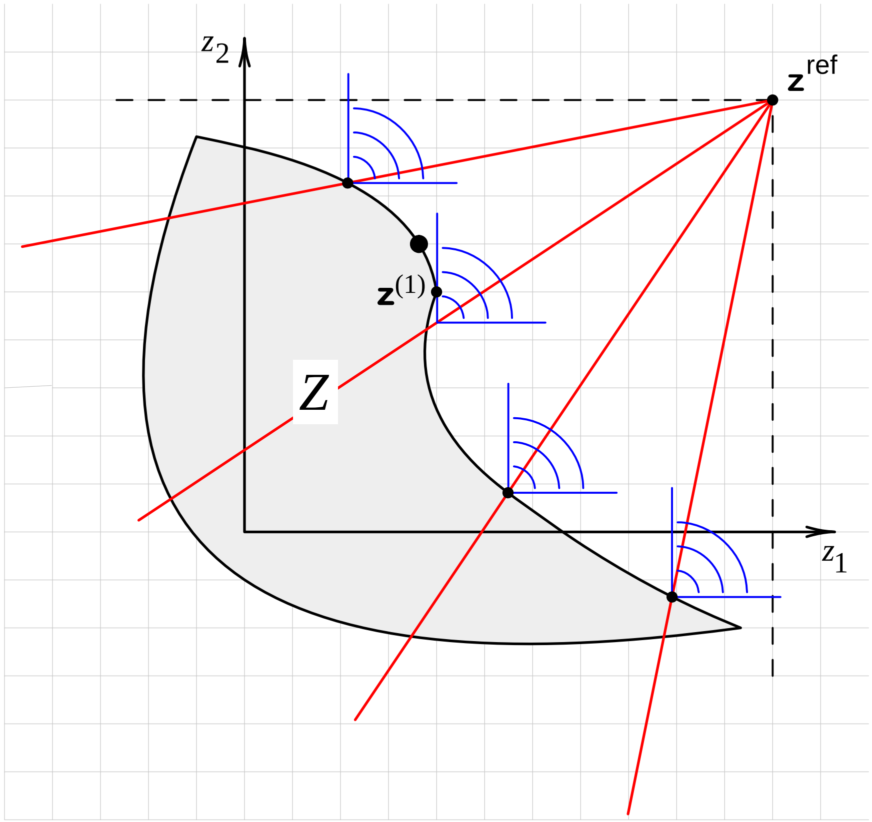
Now, form $\Lambda^{(1)}$ and obtain 4 dispersed λ-vectors from it.



Now, solve four lexicographic Tchebycheff sampling programs (one for each probing ray).

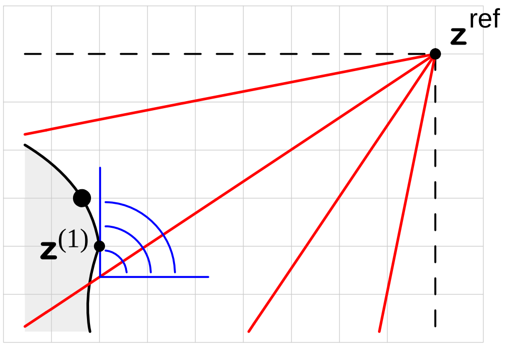


Now, select most preferred, designating it $z^{(1)}$.

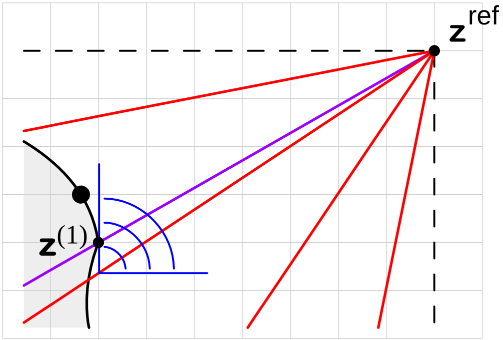


Now, form $\Lambda^{(2)}$ and obtain 4 dispersed λ-vectors from it.

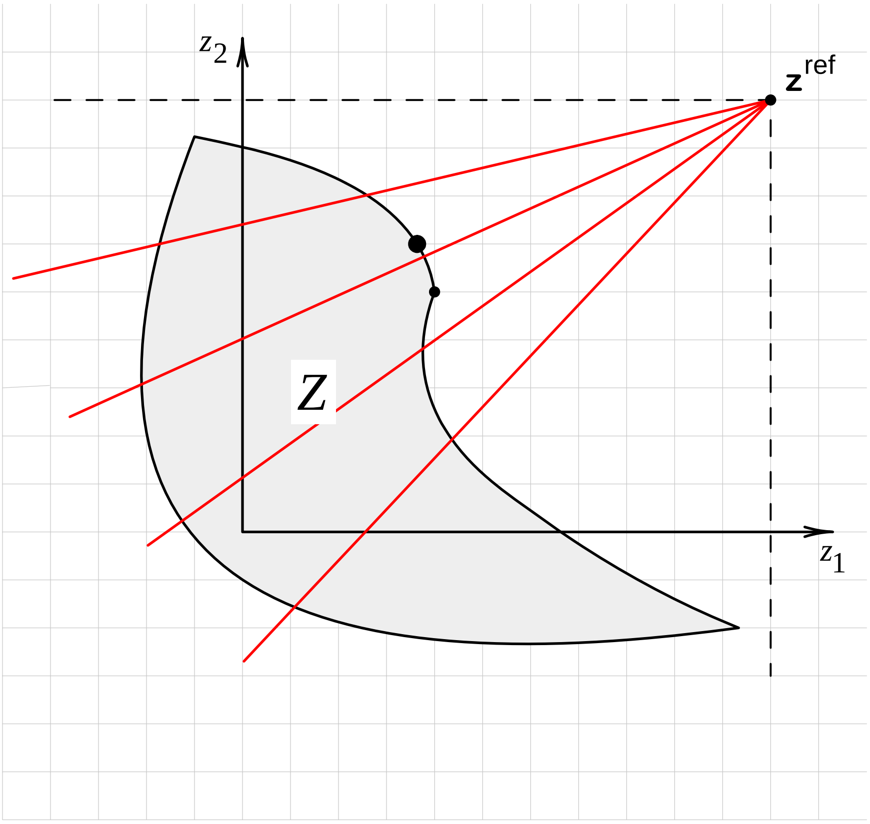
32. Tchebycheff Vertex λ-Vector



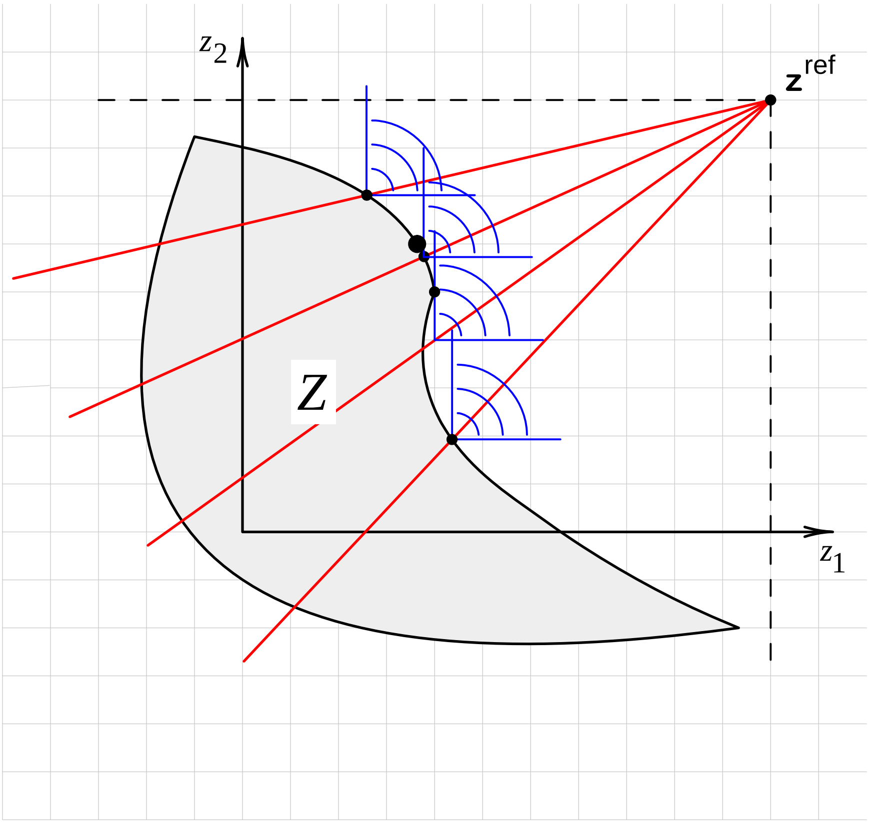
33. How to Compute Dispersed Probing Rays



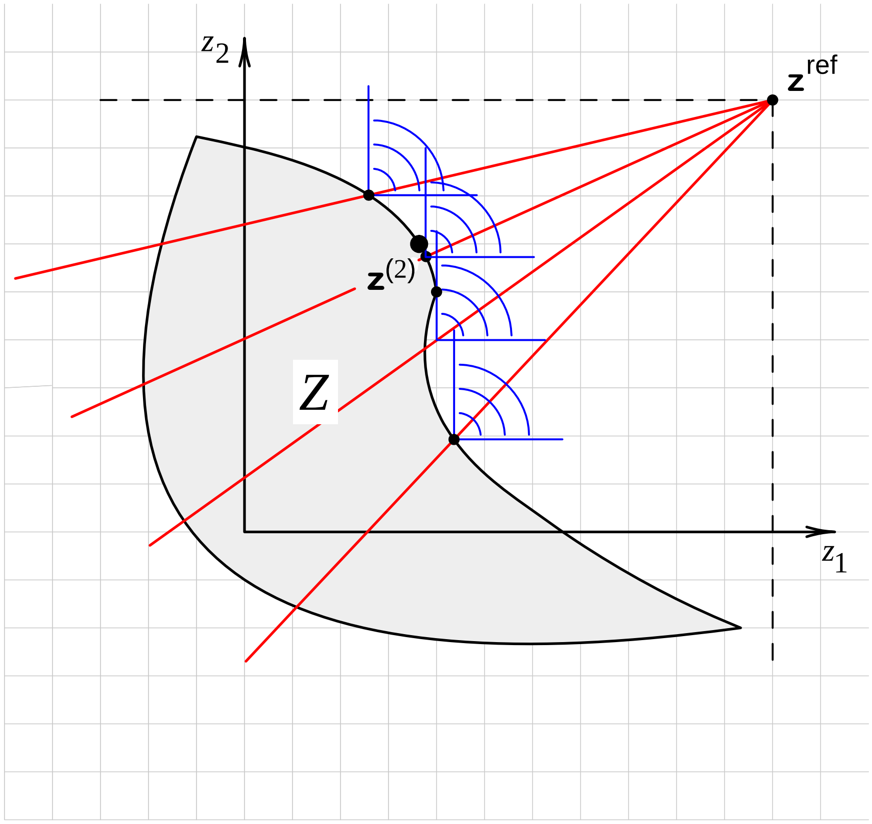
$$\lambda_i^{(1)} = \frac{1}{z_i^{ref} - z_i^{(1)}} \left[\sum_{j=1}^{k} \frac{1}{z_j^{ref} - z_j^{(1)}} \right]^{-1}$$



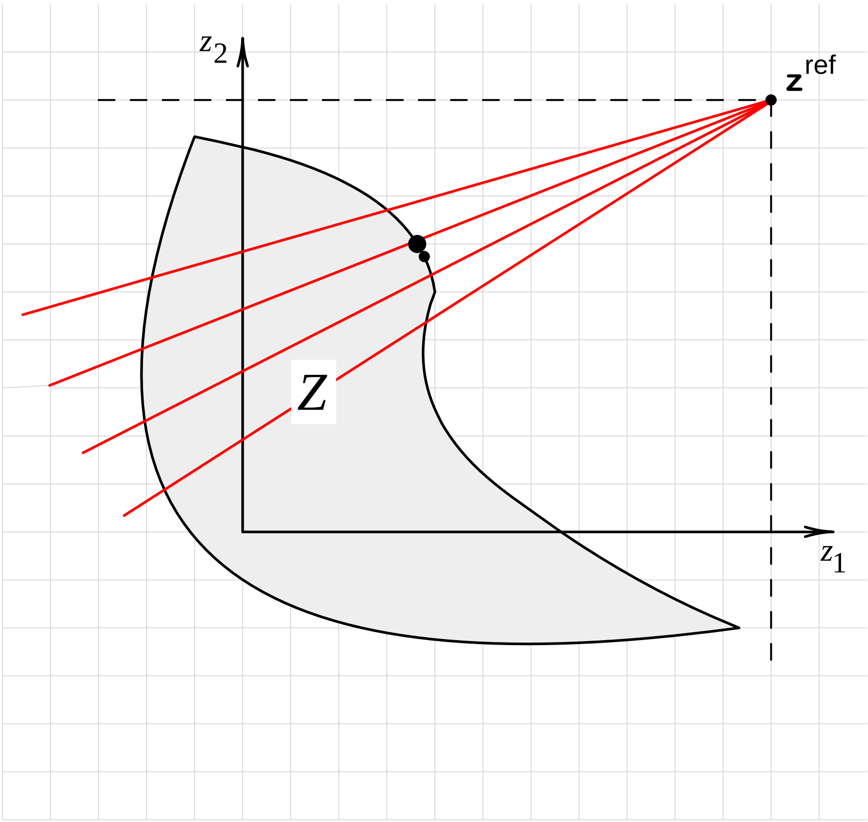
Now, solve four lexicographic Tchebycheff sampling programs.



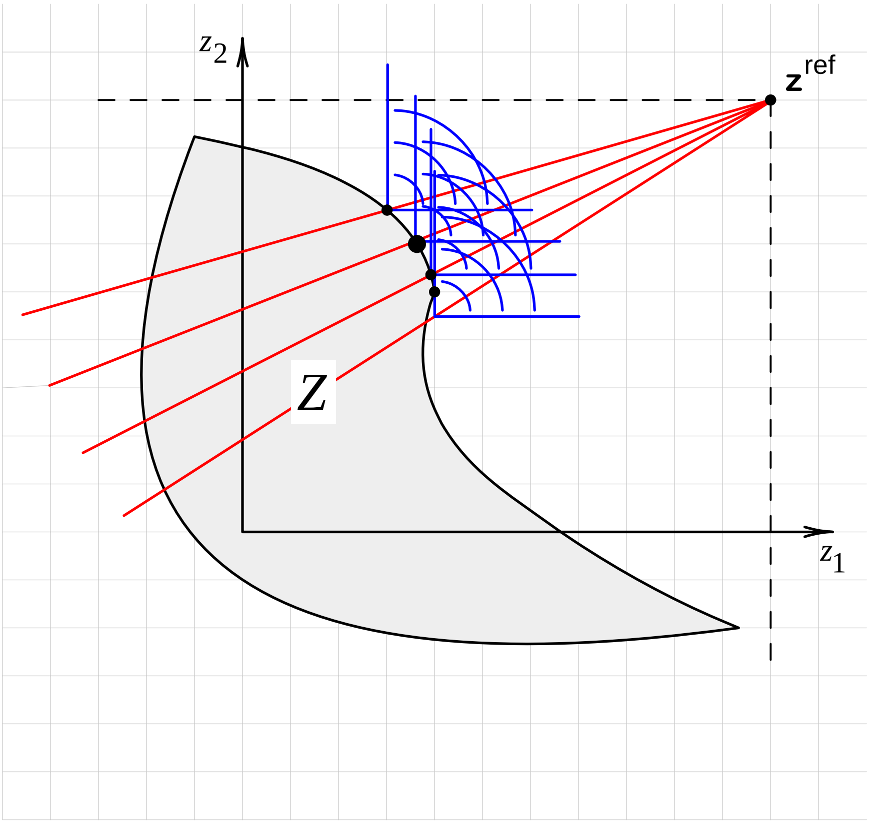
Now, select most preferred, designating it $z^{(2)}$.



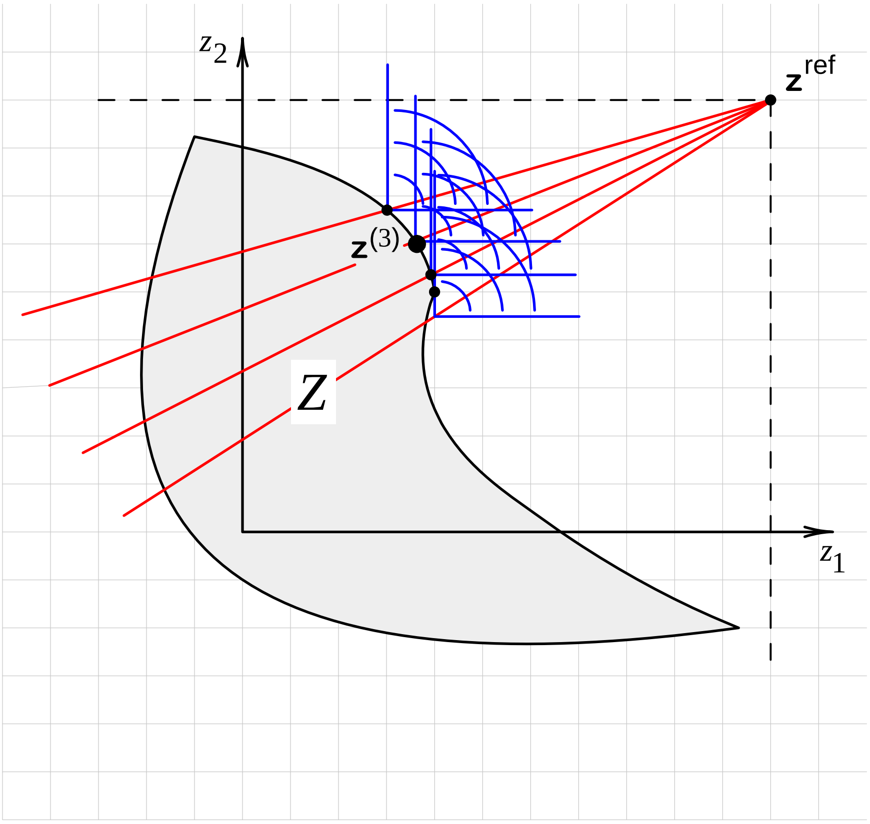
Now, form $\Lambda^{(3)}$ and obtain 4 dispersed λ-vectors from it.



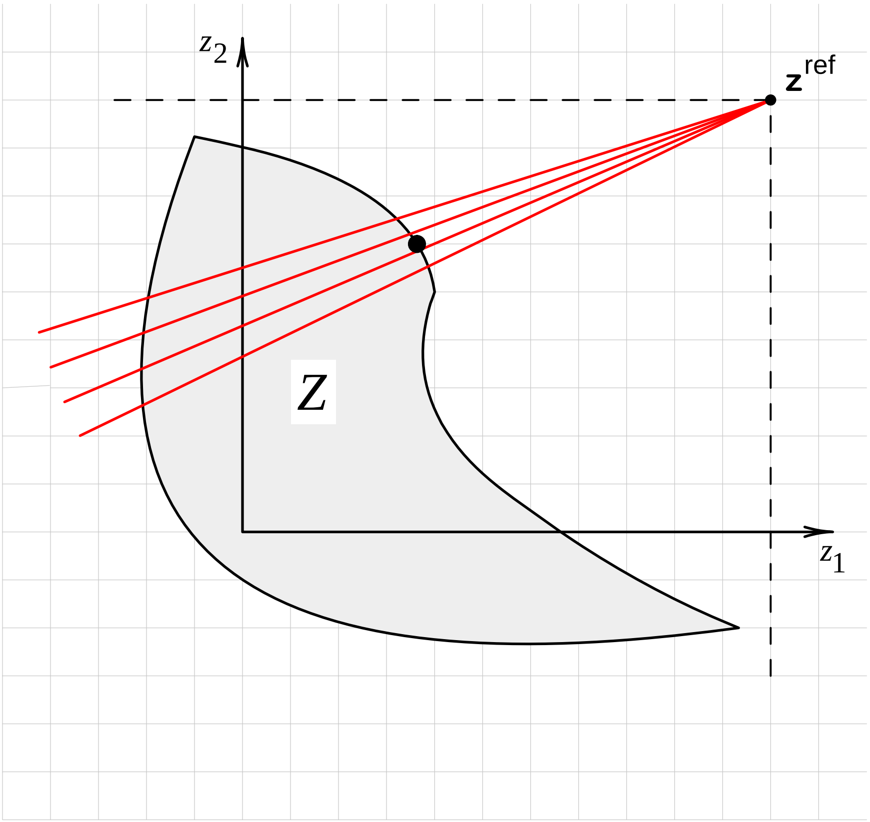
Now, solve four lexicographic Tchebycheff sampling programs.



Now, select most preferred, designating it $z^{(3)}$

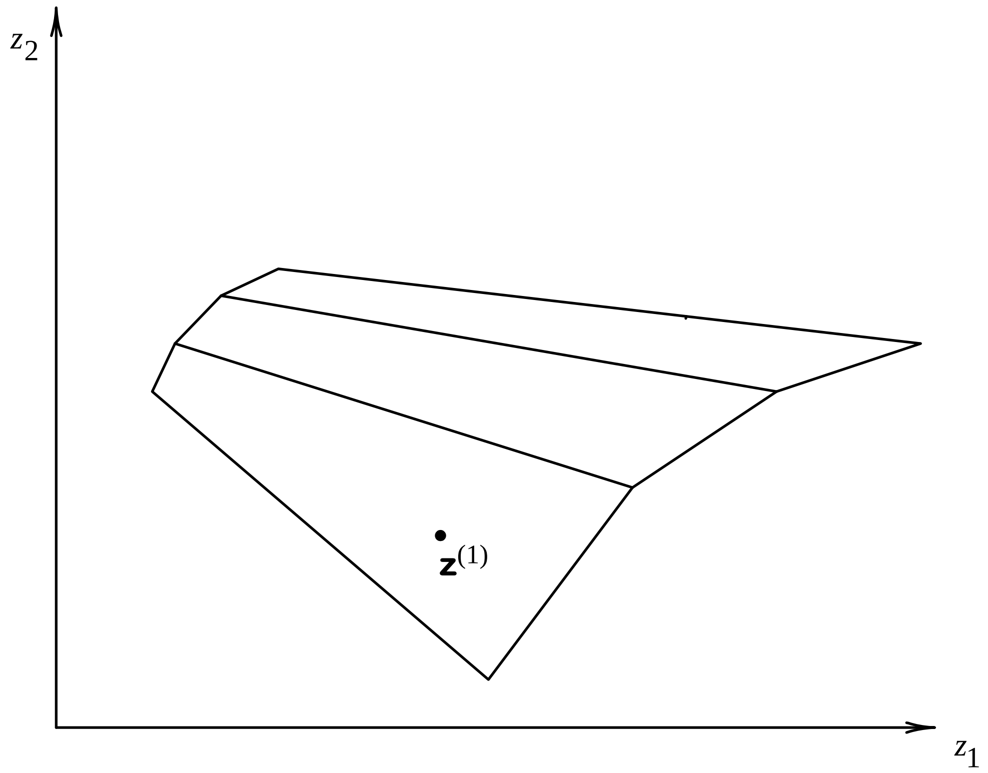


Now, form $\Lambda^{(4)}$ and obtain 4 dispersed λ-vectors from it.

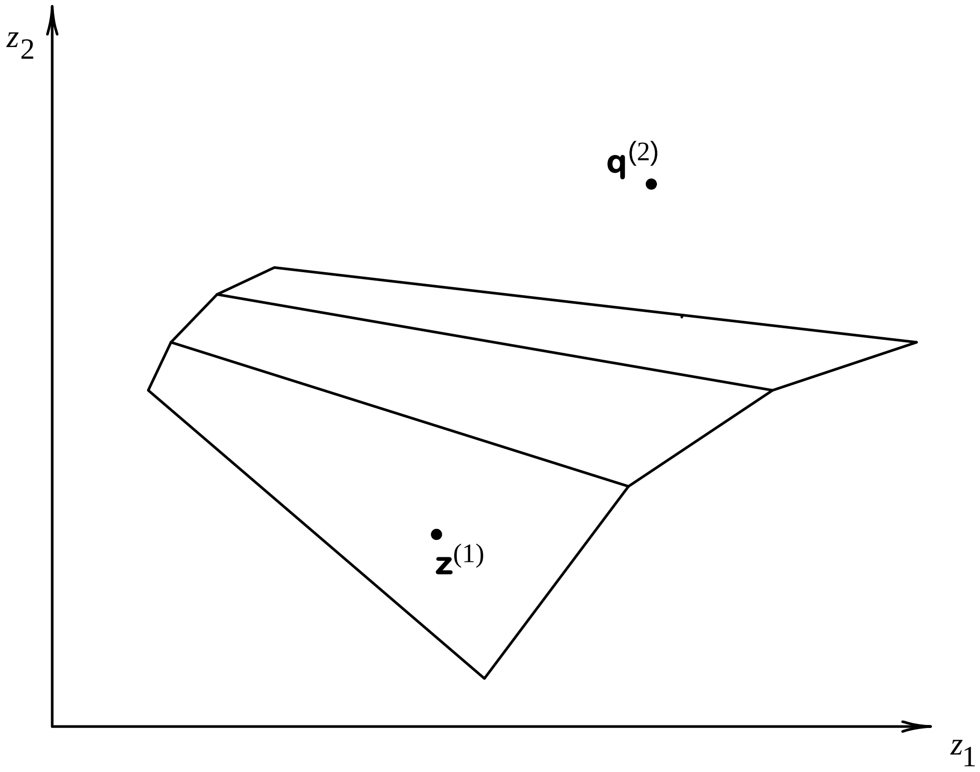


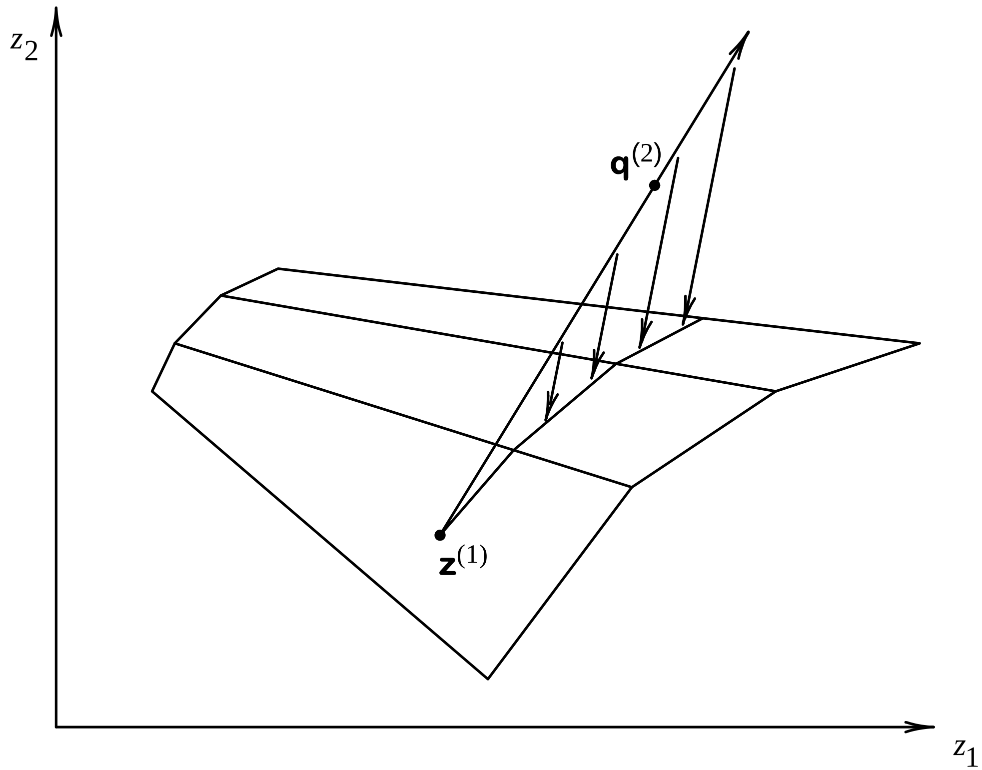
And so forth.

34. Projected Line Search Method

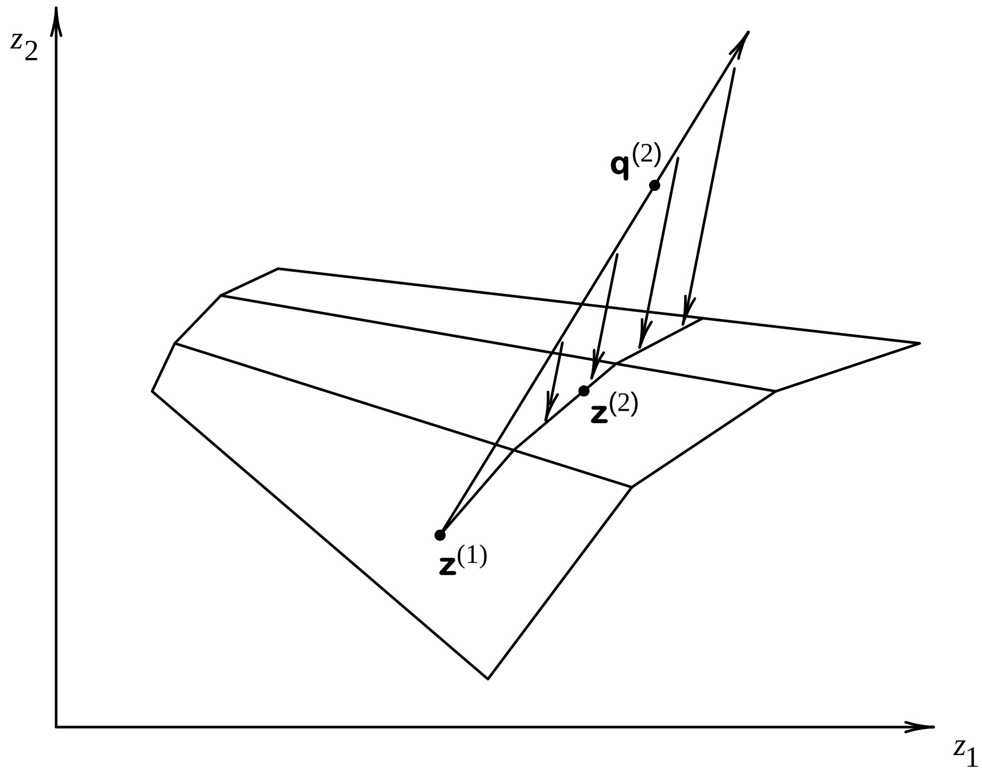


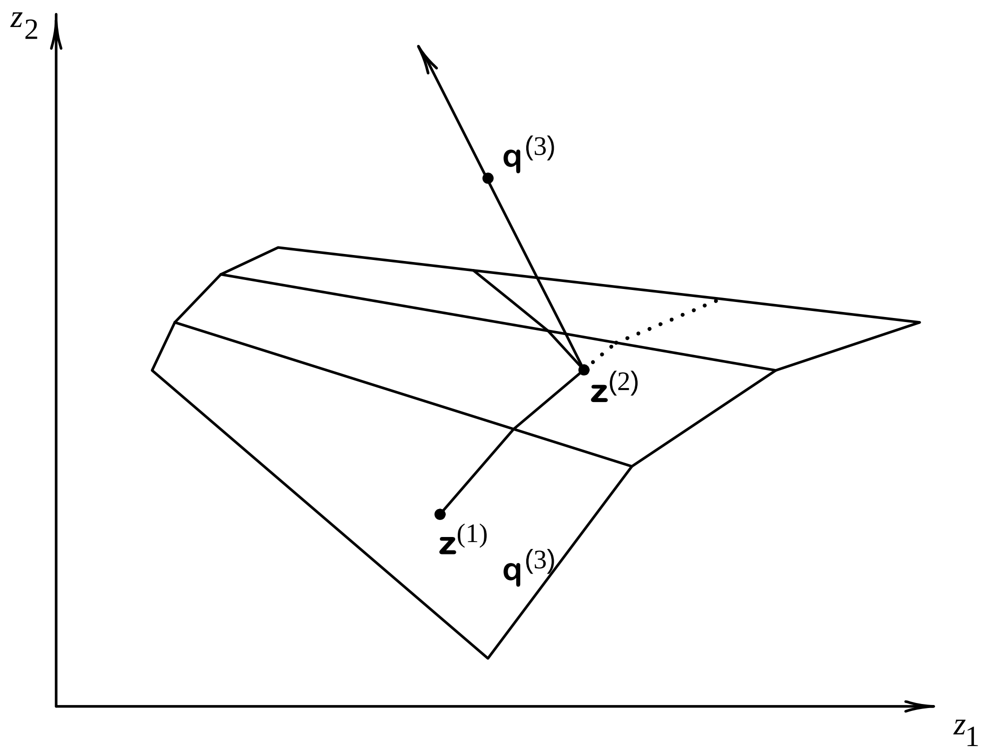
Like driving across surface of moon.

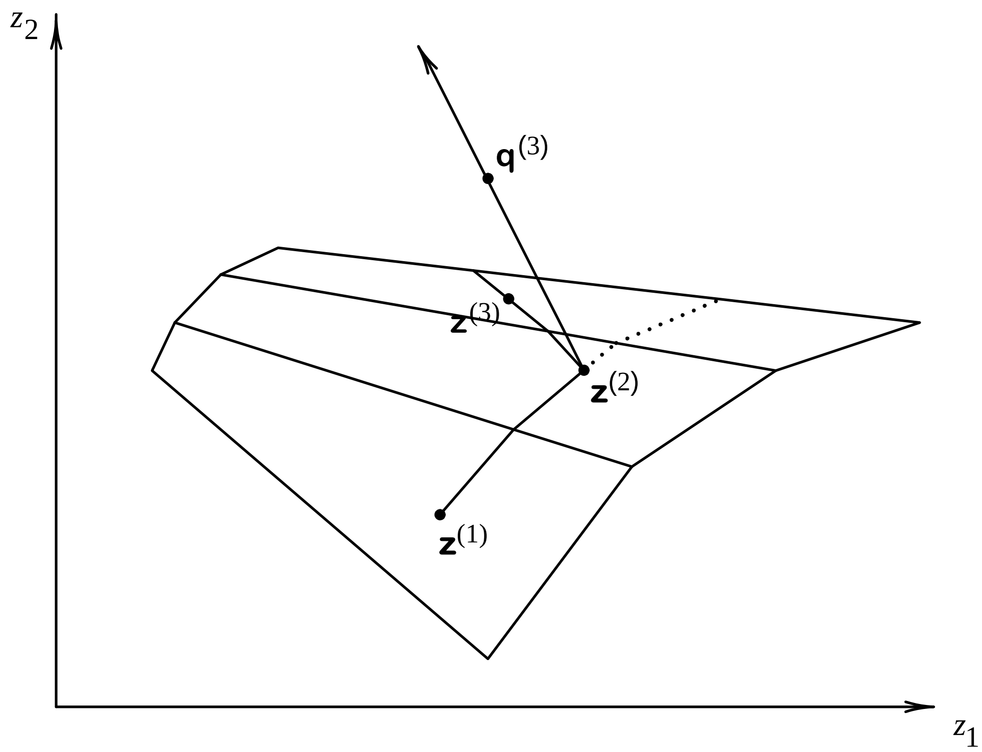

z_2
$\mathbf{q}^{(2)}$
$\mathbf{z}^{(1)}$
z_1

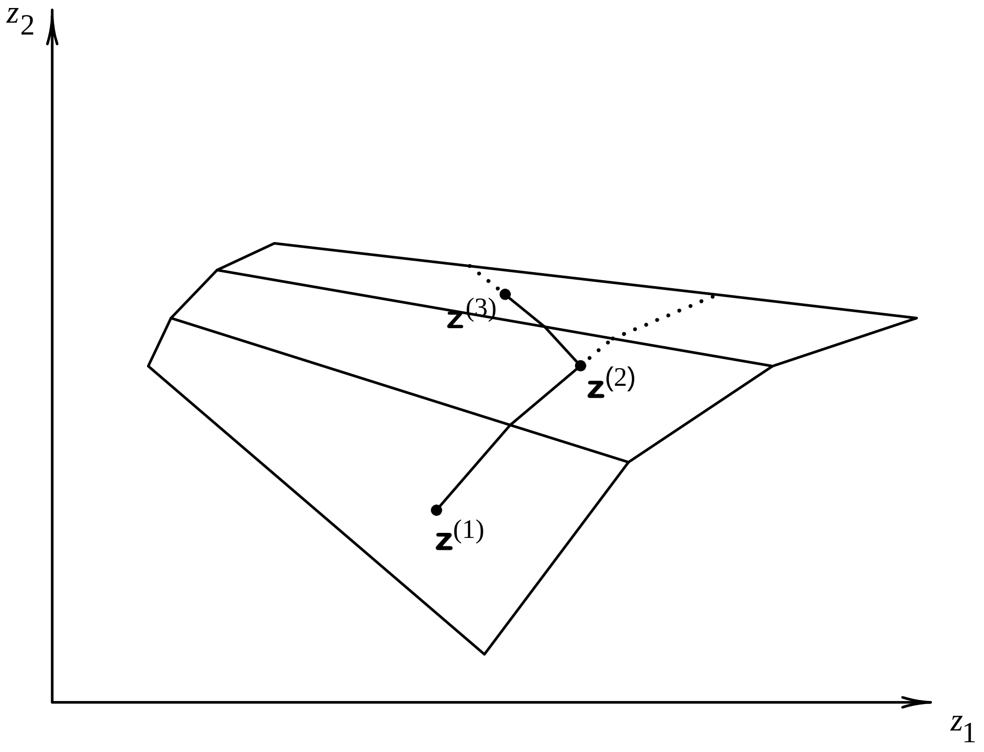


z2
q(2)
z(1)
z1


z_2
$\mathbf{q}^{(2)}$
$\mathbf{z}^{(2)}$
$\mathbf{z}^{(1)}$
z_1


z_2
$\mathbf{q}^{(3)}$
$\mathbf{z}^{(2)}$
$\mathbf{z}^{(1)}$
$\mathbf{q}^{(3)}$
z_1


z2
q(3)
z(3)
z(2)
z(1)
z1



Drive straight awhile, turn, drive straight awhile, turn, drive straight awhile, and so forth.

35. List of Interactive Procedures

1. Weighted-sums (traditional)
2. e-constraint method (traditional)
3. Goal programming (mostly US, 1960s)
4. STEM (France & Russia, 1971)
5. Geoffrion, Dyer, Feinberg procedure (US, 1972)
6. Vector-maximum/filtering (US, 1976)
7. Zionts-Wallenius Procedure (US & Finland, 1976)
8. Wierzbicki's reference point method (Poland, 1980)
9. Tchebycheff method (US & Canada, 1983)
10. Satisficing tradeoff method (Japan, 1984)
11. Pareto Race (Finland, 1986)
12. AIM (US & South Africa, 1995)
13. NIMBUS (Finland, 1998)

The End