Reliable Modeling: History, Achievements, Trends, Open Problems

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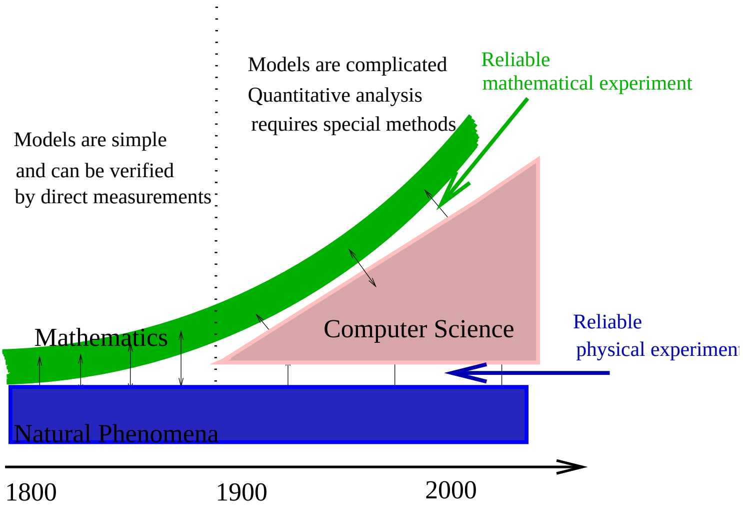
What is Reliable Mathematical Modeling?
Why it is indeed important?
Where is its place in scientific/technological process?


Models are complicated
Quantitative analysis
requires special methods
Reliable
mathematical experiment
Models are simple
and can be verified
by direct measurements
Mathematics
Computer Science
Reliable
physical experiment
Natural Phenomena
1800
1900
2000

1. XVI-XIX:
Creation of an adequate language able to describe real life models (calculus, theory of differential equations, etc.).

XIX: Mathematical Models <===> Differential Equations

2.**Beginning of XX:**
First attempts of quantitative analysis of PDE. However, without computers these attempts are confined to very special examples.

3. 50'-70':

A Priori Conception:

$\boxed{\text{Existence}} \Rightarrow \boxed{\text{Regularity}} \Rightarrow \boxed{\text{Approximation}}$

$\mathcal{A}u = f$, $\mathcal{A} : V \to V^*$:
$V_h \subset V$, $\dim V_h = N \sim 1/h < \infty$,
u_h: $\mathcal{A}_h u_h = f_h \; u_h \to u$

$$\|u - u_h\|_V \leq Ch^k, \quad C > 0, k > 0.$$

A priori error analysis is based on rather serious assumptions:

- **(a)** u **has an extra regularity;**
- **(b) All $\mathbf{V_h}$ are also regular (in some sense);**
- **(c)** u_h **is the exact solution of the corresponding finite dimensional problem.**

For external (nonconforming) approximations the stability condition must be appended.

A priori methods are focused on QUALITATIVE properties of solutions and on principal ability to approximate them using an infinitely powerful computer with unlimited memory.

4. 80': A new conception: "Adaptive Modeling"

A "good"approximation of the exact solution can be constructed only on the basis of consequent improvement adaptation of spaces V_h.

Adaptive methods have generated a serious interest to a posteriori error indicators:

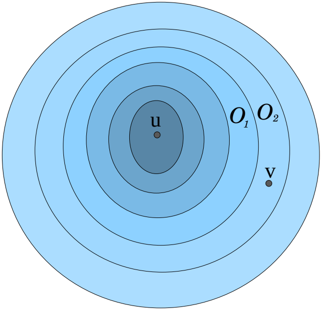
I. Babuska, W. Rheinboldt, T. Oden, R. Nochetto,
R. Rannacher, C. Johnson, E. Suli, P. Hansbo, C. Carstensen,
W. Wendland, R. Verfurth, M. Ainsworth,...

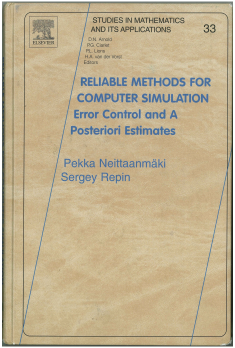
5. mid 90s: "Fully Reliable Modelling"

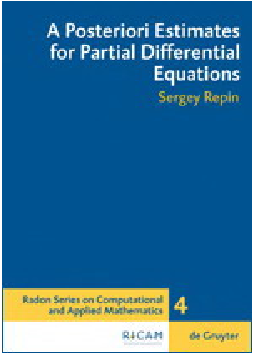
Quantitative analysis of mathematical models requires solution of a special mathematical problem, which together with existence and regularity creates a necessary component in analysis of a mathematical problem.

At that time new mathematical tools (functional a posteriori estimates) has been constructed. These estimates allow to really measure the accuracy of approximate solutions and evaluate modeling errors.

Let u be the exact solution and $\mathcal{O}_k$ is a system of neighborhoods (e.g., balls in the corresponding energy space). For any admissible function v, we need to find two arbitrary close neighborhoods $\mathcal{O}_1 \subset \mathcal{O}_2$ such that $v \in \mathcal{O}_2$ но $v \notin \mathcal{O}_1$.







A consequent exposition of the theory can be found in:
Elsevier 2004 Walter deGruyter 2008