INSIGHTS INTO STUDENTS’ ALGEBRAIC REASONING

John Francisco                          Markus Hähköniemi
Rutgers University, USA               University of Jyväskylä, Finland

This research examines the mathematical activity of a group of eight-grade students who participated in open-ended mathematical investigations of quadratic functions. In particular, the study traces the students’ mathematical reasoning and notation as they work on determining rules of quadratic functions. The study is also an outgrowth of a 3-Year NSF-funded longitudinal research on the development of mathematical ideas and ways of reasoning involving students from an urban minority school district in New Jersey, USA, before they experience formal instruction. The results provide insights into the students’ building of meaningful and powerful algebraic ideas and ways of reasoning in the context of the Guess My Rule approach.

INTRODUCTION

This paper describes patterns in the algebraic reasoning of three eight-grade students as they engaged in open-ended mathematical investigations of quadratic functions in the Guess My Rule approach (Alston & Davis, 1996). Typically, researchers make up a rule or equation for a function and ask the students to guess it. Then, they play a game, in which the students provide input values and the researchers return output values according to the rule. A table of ordered pairs is built and students use it to guess the rule. The students can construct the table from a problem situation or get it ready-made from the researchers. Boxes and triangles can also be used instead of the traditional $x$ and $y$ to denote input and output values, respectively. Students can also graph from the tables or rules.

Three research questions guided the present study: (1) what ideas or conceptions did the students build about functions, (2) how do they represent them, and (3) what connections did the students make between representations of functions? The study is an outgrowth of the Informal Mathematical Learning project (IML), an after-school 3-Year NSF-funded longitudinal study (Award REC-0309062) with support from the Rutgers University MetroMath Center for Learning. The project investigated the development of mathematical ideas and ways of reasoning in middle-grade students [6th-8th grades] in problem-solving investigations, involving challenging open-ended tasks in different mathematical domains, which include combinatorics, probability and algebra. In total, the IML project involved approximately fifty students from an urban minority community in New Jersey, USA. This study reports on the mathematical activity of three students in a 3-month algebra strand implemented in the last year of the project. The results provide insights into the students’ building of meaningful ideas and ways of reasoning about functions.
THEORETICAL FRAMEWORK

There is a substantial amount of literature on students’ ability to engage in algebraic reasoning at an early stage (Bellisio & Maher, 1998; Carraher & Earnest, 2003; Schliemann et al., 2003). The studies reflect particular views to Algebra. Davis (1985) distinguishes between two opposing views. In one, learning algebra involves the mere acquisition of rules and expertise in the manipulation of symbols. In the other, learning algebra involves the building, from experiences, of algebraic ideas and ways of reasoning about algebraic concepts such as a variable, function, and a graph. In the process, the students develop a mathematical language and notation, which help them describe their mathematical activity. The idea is to help students build meaningful and durable knowledge. Sfard and Linchevsky (1994) differentiate between the notions of symbols as unknown fixed numbers and symbols as variables, which correspond to algebra of a fixed value and functional algebra, respectively. They argue that algebraic objects may be conceived as series of operations (operational conception) or static entities (structural conception) and students should be allowed to engage in operational algebra before structural algebra. Confrey and Smith (1994) identify two approaches in the treatment of functions. The correspondence approach starts with the building of a rule of correspondence between x-values and y-values, usually as an equation of the form \( y = f(x) \). The covariational approach starts with a problem situation and the students construct a table of \((x, y)\) pairs, by first filling in \(x\)-values, which increase by 1, and then adding \(y\)-values through some operation constructed in the problem situation. They claim that covariational approach is “more powerful,” as it enhances reasoning about rate-of-change.

The Guess My Rule approach is consistent with the covariational approach regarding the emphasis placed on the construction of tables of ordered pairs and problem situations. It enhances the students’ building of the notion of a variable and a language to express their mathematical reasoning (Davis & Alston, 1996). The approach also enhances the students’ ability to make connections between different representations of functions (Davis & Maher, 1996). In a study using the Guess My Rule approach with seventh grade students, Bellisio and Maher (1998) reported students’ successful engagement in algebraic reasoning before the formal study of algebra. They also reported a movement whereby students first verbalized an idea before they attempted to write it in some symbolic form. Similarly, Stacey and MacGregor (1997) reported cases, where students predicted the value of \(y\), usually for large values of \(x\), but could not describe the relationship between \(x\) and \(y\), or write it in algebraic symbols. This study aims at deepening our understanding of the use of the Guess My Rule in promoting students’ algebraic reasoning. The results are consistent with Bellisio and Maher’s movement from verbalization to rule writing, Sfard and Linchevsky’s (1994) transition from operational to structural algebra, and provide insights into students’ development of meaningful and durable reasoning.
METHODOLOGY

This study relies on videotapes of two consecutive 1.5-hour problem-solving sessions with three eight-grade students, in which they worked on determining rules that fit tables of functions $Y = (X - 1)^2$ and $Y = (X + 1)^2$ within the Guess My Rule approach. The students are referred to as Chris, Ian and Jerel and had previously played the Guess My Rule games with functions of the type $Y = AX + B$ and $Y = X^2 + A$. The data is part of an extensive database of the IML project housed at the Robert Davis Institute for Learning, at Rutgers University, in New Jersey USA. Data analysis built on six video-related treatment procedures (Powell, Francisco & Maher, 2003). These included, (1) watching all videotapes of the sessions to have a sense of the content as a whole, (2) partitioning the data into significant episodes, (3) describing the episodes, (4) characterizing the significance of the episodes, (5) transcribing the episodes and (6) engaging in a structural analysis across the episodes to identify emerging themes about the students’ algebraic reasoning. Due to space limitation, this paper focuses only the mathematical activity and insights from four episodes.

RESULTS

This section describes the four episodes that illustrate the students’ reasoning. The insights and their significance are presented as conclusions in the in the next section.

Episode 1: Symmetry and a recursive pattern

When working on the rule $Y = (X - 1)^2$, Ian noticed a symmetry pattern in $y$-values, which helped him add the point (-3, 16). He had noticed that the numbers 1, 4, and 9 were below and above 0. Since an extra 16 was below 0, Ian guessed that a 16 had also to be above 0 and added the point [Fig. 1]:

Ian: I just noticed something. Look. Look, look, look. 9, 4, 1, 0, 1, 4, 9, 16 [Points at the Y-values column in the table]. It will be 16 up here and a negative three [Adds point (-3, 16) at the top of the table].

Figure 1: Ian notices symmetry in the table.

Ian then focused on changes in $y$-values in the table. He computed finite differences between $y$-values ($y_n - y_{n-1}$) and noticed another pattern. He plotted the ordered pairs and noticed the same pattern in the graph. The pattern was a recursive rule, whereby,
starting at zero, y-values increased by 1, 3, 5, and 7 up and down the y-column in the table and the Y-axis in the graph [Fig. 2]:

Ian: [Pointing in the graph] One. One, two, three. One, two, three, four, five. I was right. All that is right there. It’s odd, that’s it. Look. [Inaudible] One, [Pause] one. One, two, three. One, two, three, four, five. It’s right there.

Figure 2: Ian notices a recursive pattern in the table and graph.

The recursive pattern was more powerful than the symmetry, as Ian could add more points.

**Episode 2: Family of functions**

Ian continued to work on determining the rule for the function \( Y = (X - 1)^2 \). He wrote “5×5 – 9 = 16” and “4×4 – 7 = 9” and claimed to have the rule:

Ian: I got it but. I just got it. Look, four times four minus seven equals nine [writes 4×4 – 7 = 9]. Look. Then if you do the next three times three minus five [writes 3×3 – 5 = 4] I got the freaking answer, but it’s not freaking coming up. [Pause] Look. I got it. It’s right there! [Adds 2×2 – 3 = 1]. I just don’t know what the rule is. It’s x times x minus odd number.

Ian had noticed that he could change x-values into y-values by multiplying x by x and subtracting an odd number, which changed by two. This is the family of functions \( X \times X – (2X – 1) = Y \), where \( 2X – 1 \) is the odd number that changes systematically [Fig. 3].

Figure 3: Ian’s family of functions
The excerpt suggests that Ian may not have expected the rule to consist of a family of functions.

**Episode 3: Identity rules**

When working on the rule \( Y = (x + 1)^2 \), at some point Chris claimed to have a “rule.” Chris had come up with an identity expression [Fig. 4]. The other students rejected it because Chris had subtracted and added the same quantity:

Ian: [Checking the rule] I still don’t know how you did that. ‘Cause you are subtracting and then putting it back [sic adding]. You’re a cheater. It’s like you ain’t [sic aren’t] adding nothing [sic anything]. You just put zero. You’re a cheater.

Jerel: Oh yeah. He’s just subtracting and then putting it back. That’s cheating.

Chris: I am smart.

Ian even referred to the students’ past experience to question how the rule was written. He suggested that it did not look like the ones they used to write:

Ian: Can you subtract \( y \) from \( x \)? Is that possible? Can you subtract \( y \) from \( x \)? Cause before we couldn’t do that. We could never do that before. We could always put \( x \) plus \( y \). We could never put \( y \) minus \( x \).

The students subsequently dropped Chris’ “rule”. Moments later, Ian proposed another “rule”. However, it was another identity expression [Fig. 4]. Chris immediately pointed out that it had a similar “mistake” as his “rule”. He claimed that Ian had used multiplication and division, when he had used addition and subtraction, of the same quantity:

Chris: [Looks at Ian’s rule] It is the same think I am doing. You are just dividing and multiplying. It’s like this adding [Inaudible]. It is like him multiplying and dividing. It’s like me [adding and] subtracting, right?

Jerel: Yeah, you’re cheating.

The students dropped Ian’s “rule” and continued the search for another rule.

**Episode 4: A composite functional representation**

The students worked on determining the rule \( Y = (x + 1)^2 \), for long periods of time, without making much progress. They kept looking for rules of the type \( Y = X^2 + A \). This prompted an intervention from the researchers in which the students revisited the function \( Y = X^2 \) and re-played the Guess My Rule game for the function \( Y = (x + 1)^2 \). They stopped considering algebraic expressions with A terms and, for the first time, they also started adding or subtracting numbers to \( x \) before multiplying it by itself. Eventually, Jerel was able to verbalize the rule:
Jerel: I got it. I got it before all of you [smiles in joy]. It is \( x \) plus one, and then you multiply, I mean, then, you time the sum. I mean then you get the sum, then you time the sum. Ah, then time the sum, then you time the sum by the sum.

The students agreed to write down the rule. Chris’ used a notation that included the word “sum.” Jerel called it “the new X” and noted as "\( nx \)" [Fig. 5):

Chris: I get it. I get it. It’s \( x \) plus one [pause] \( x \) plus one times \( x \) equal \( y \).

Jerel: No. Equal the new \( x \). Times that \( x \) equals \( y \).

Chris: All right. I get it [writes his equation and show it to others].

Jerel: Yeah. It’s the same. But, it’s the same. The new \( x \) plus [sic times] \( x \) [sic \( x+1 \)] equals \( y \) [the students writes their equation].

Figure 5: Chris’ rule [left] and Jerel’s rule [right].

The students used pair \((-2,1)\) to explain their rule. They first computed the sum \((i)\) \(-2+1=1\), then the product \((ii)\) \((-1)\times(-1)=1\). Interestingly, they argued that the “-1” in the product was not the same as “-2+1” in the sum because, according to Jerel, in the product, “They would not know that that I added first.” So, the students had come up with a composite functional representation as \( x+1=z \) and \( z\times z = y \). In particular, \( z \) equals Jerel’s “new \( x \)” and Chris’ “sum”. For a relatively long period of time, the students used this representation. However when the researchers suggested writing the “new \( x \)” using parentheses as \((x+1)\), the students were able to re-write their rule as \((x+1)(x+1)=y\).

CONCLUSIONS

There is evidence that the students focused extensively on how \( x \) and \( y \)-values [co]varied with each other up and down the table and the graph of functions. In episode 1, they attended to the distribution of \( y \)-values in the table and graph using finite differences to compute changes in \( y \)-values. In episode 2, they explored relationships between \( x \)-values and \( y \)-values in the table and came up with a family of functions. This is consistent with Confrey and Smith’s (1994) claim regarding the advantages of the covariational approach. In episodes 1 and 2, the students came up with symmetry, a recursive pattern and a family of functions. In episode 1, it the students were able to make connections between a table and a graphical representation of the same function.

In episode 4, the students first verbalized a rule for the function before they were able to write it in symbolic form. This is consistent with Bellisio and Maher’s (1998)
findings about a movement in students’ algebraic thinking from verbalizing an idea to writing it down in some symbolic form. In the same episode, the students verbalized their rule using a term such as “the new \( x \)” and initially wrote down their rule with non-formal notation such as “\( nx \)”. This is consistent with the claim regarding the advantages of the Guess My Rule approach in helping students develop a personally meaningful language to express their mathematical activity and reasoning.

The students’ rules of the type \( Y = AX + B \) and \( Y = X^2 + A \) and their rule in episode 4 seemed to be rules for operations that should be done to the numbers in left column of the table to get the numbers in the right column. Their language in verbalizing their rule included expressions such as “It is \( x \) plus one”, “then you multiply”, “then you get the sum”, and “then you time the sum by the sum”, which support the claim that the students were thinking operationally. The students were also initially reluctant to accept that “\( z \)” [a product] be replaced by \( x+1 \) [a process]. Jerel’s claim that “They would not know that I added first”, further indicates that the distinction was based on their perceived roles as either product or process. Therefore, the students seemed to be thinking of algebraic expressions as series of commands or processes. Sfard and Linchevsky (1994) call this operational thinking. This may be one reason why the students had such difficulties to come up with a formal rule and notation of it in episode 4: if \( x+1 \) is an operation to be executed, then how could this operation be multiplied by another operation. The students solved this difficulty by inventing a composite functional representation, which allowed them to create an operational rule. This is consistent with the claim that operational reasoning is natural in students and precedes a structural approach. Sfard and Linchevsky (1994) argue that to avoid functional algebra becoming a mere application of arbitrary operations without meaning, students should build an operational basis for the structural algebra by learning first algebra of a fixed value (unknown). This study further suggests that students can build the operational basis in functional algebra.

Students’ difficulties in moving from their composite functional representation to the formula \((x+1)(x+1) = y\) are consistent with the challenge that Sfard and Linchevsky’s (1994) claim students to experience when trying to make a transition from operational algebra to structural algebra. They claim that it requires the building of a “dual outlook” or “process-product duality” interpretation of algebra formulae, which involves being able to consider algebraic expression as representing both process and product, as opposed to either processes or products, which must be kept separate from each other, often on different sides of the equal sign. Sfard and Linchevsky (1994) add that, in the absence of the dual outlook, “The equality symbol looses the basic notion of an equivalence predicate: it stops being symmetrical or transitive (p. 104).” They also warn that the transition from a purely operational to a dual process-product outlook is not “a smooth movement” and the reification process may require a “quantum leap”. So, it is inconclusive whether the students succeed in building a
[durable] dual outlook in rewriting their rule as \((x+1)(x+1) = y\) after the introduction of parentheses notation.

Finally, the students engaged in an interesting discussion in episode 2, regarding whether the identity expressions that they came up with were acceptable functional rules. This means that the search for a rule may also involve a parallel debate on what counts as an acceptable rule. These issues would be regarded elsewhere as being at a cognitive and epistemic level of reasoning, respectively (Kitchener, 1983). In particular, this shows that students are likely to naturally engage in a mathematical activity that emphasizes sense making and justification of ideas.

References


