EXHAUSTIONS OF CIRCLE DOMAINS

DIMITRIOS NTALAMPEKOS AND KAI RAJALA

Abstract. Koebe's conjecture asserts that every domain in the Riemann sphere is conformally equivalent to a circle domain. We prove that every domain Ω satisfying Koebe's conjecture admits an *exhaustion*, i.e., a sequence of interior approximations by finitely connected domains, so that the associated conformal maps onto finitely connected circle domains converge to a conformal map f from Ω onto a circle domain. Thus, if Koebe's conjecture is true, it can be proved by utilizing interior approximations of a domain. The main ingredient in the proof is the construction of *quasiround* exhaustions of a given circle domain Ω. In the case of such exhaustions, if $\partial\Omega$ has area zero then f is a Möbius transformation.

1. Introduction

A domain in the Riemann sphere $\hat{\mathbb{C}}$ is a *circle domain* if each connected component of its boundary is a point or a circle. A long-standing problem in complex analysis is Koebe's conjecture [Koe08], predicting that every domain in $\hat{\mathbb{C}}$ can be conformally mapped to a circle domain. Koebe himself established the conjecture for finitely connected domains, but it took over 70 years until the conjecture was established for countably connected domains by He–Schramm [HS93]; an alternative argument was provided by Schramm [Sch95]. The general case remains open. See also [Bon16, HS95, HK90, HvdM09, SV20] for other results related to Koebe's conjecture.

The proofs of the special cases of the conjecture follow the scheme of approximating the domain by finitely connected domains, uniformizing conformally these domains by circle domains using Koebe's theorem, and then passing to a limit. However, the limiting map will not always be a conformal map onto a circle domain, and a careful choice of the approximations is required. Thus, one of the difficulties in establishing the general case of Koebe's conjecture is that a universal approximation scheme that works for all domains is still to be found. Another subtle difficulty is that the uniformizing conformal map, if it exists, is not necessarily unique in the uncountably connected case. Therefore, one can say that there is no standard procedure that yields the desired conformal map in a unique way. For uniqueness results related to Koebe's conjecture see [HS94, NY20, Nta23b, You16].

The existence proofs by He–Schramm and Schramm apply approximation of a given domain from *outside* by a decreasing sequence of finitely connected

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domains together with Koebe's theorem to construct a sequence of conformal maps whose limit has circle domain image. Recently the second-named author [Raj] studied the uniformization problem by approximating a given domain from *inside* by exhaustions, i.e., increasing sequences of finitely connected subdomains, and gave an alternative proof of Koebe's conjecture in the countably connected case.

An *exhaustion* of a domain $\Omega \subset \hat{\mathbb{C}}$ is a sequence of domains $\Omega_j \subset \Omega$, $j \in \mathbb{N}$, each bounded by finitely many disjoint Jordan curves in Ω, such that

$$\Omega_j \subset \Omega_{j+1} \text{ for all } j \in \mathbb{N} \quad \text{and} \quad \Omega = \bigcup_{j\in\mathbb{N}} \Omega_j.$$

Given an exhaustion $(\Omega_j)_{j\in\mathbb{N}}$, we fix distinct points $a_1, a_2, a_3 \in \Omega_1$. By Koebe's theorem every Ω_j admits a unique conformal map $f_j : \Omega_j \to D_j$ onto a finitely connected circle domain D_j so that $f_j(a_k) = a_k$ for $k = 1, 2, 3$.

We say that a domain $\Omega \subset \hat{\mathbb{C}}$ satisfies Koebe's conjecture if there exists a conformal map f from Ω onto a circle domain.

THEOREM 1.1. *Let $\Omega \subset \hat{\mathbb{C}}$ be a domain that satisfies Koebe's conjecture. Then there are an exhaustion $(\Omega_j)_{j\in\mathbb{N}}$ of Ω and a circle domain $D \subset \hat{\mathbb{C}}$ so that $(f_j)_{j\in\mathbb{N}}$ converges locally uniformly in Ω to a conformal homeomorphism $f\colon \Omega \to D$.*

Thus, if Koebe's conjecture is true, then it can be proved by using exhaustions. The method of using exhaustions also appears in uniformizing a domain by horizontal slit domains. Namely, if Ω is any domain in $\mathbb{C}$ and $(\Omega_j)_{j\in\mathbb{N}}$ is *any* exhaustion of Ω, then the conformal maps f_j from Ω_j onto finitely connected slit domains, normalized appropriately, converge, after passing to a subsequence, to a conformal map from Ω onto a slit domain [Cou50, Theorem 2.1, p. 54].

In sharp contrast to that result, the conclusion of Theorem 1.1 is not true for *any* exhaustion, as was shown by the second-named author [Raj, Theorems 1.1 and 1.2]. Hence a careful choice of $(\Omega_j)_{j\in\mathbb{N}}$ is required. Theorem 1.1 is an immediate consequence of the next theorem.

THEOREM 1.2. *For every circle domain $\Omega \subset \hat{\mathbb{C}}$ there are an exhaustion $(\Omega_j)_{j\in\mathbb{N}}$ of Ω and a circle domain $D \subset \hat{\mathbb{C}}$ so that $(f_j)_{j\in\mathbb{N}}$ converges locally uniformly in Ω to a conformal homeomorphism $f\colon \Omega \to D$. In addition, if $\partial\Omega$ has area zero, then $D = \Omega$ and f is the identity map.*

We list the main steps for the proof of Theorem 1.2.

(1) Every circle domain Ω admits a *quasiround* exhaustion $(\Omega_j)_{j\in\mathbb{N}}$. That is, there exists $K \geqslant 1$ so that for each $j \in \mathbb{N}$, each complementary component p of Ω_j is K-quasiround in the sense that there are $a \in p$ and $r > 0$ such that

$$\overline{\mathbb{D}}(a, r) \subset p \subset \overline{\mathbb{D}}(a, Kr).$$

We prove the existence of quasiround exhaustions with $K = 43$ in Section 2. In Section 7 we show that one cannot take K arbitrarily close to 1.

(2) If $(\Omega_j)_{j\in\mathbb{N}}$ is a quasiround exhaustion of a circle domain Ω and $f_j\colon \Omega_j \to D_j$, $j \in \mathbb{N}$, is a conformal map provided by Koebe's theorem that fixes three distinct points of Ω_1, then, after passing to a subsequence, $(f_j)_{j\in\mathbb{N}}$ converges locally uniformly in Ω to a conformal map f from Ω onto a circle domain D. This statement is proved in Section 3 by applying estimates involving the *transboundary modulus* of curve families introduced by Schramm in [Sch95].

(3) If $\partial\Omega$ has area zero, then the map $g = f^{-1}$ satisfies a type of *transboundary upper gradient inequality*, as stated in Section 5.

(4) Any conformal map between circle domains that satisfies the transboundary upper gradient inequality of Section 5 is the restriction of a Möbius transformation of $\hat{\mathbb{C}}$; see Theorem 6.1. In the last two steps we apply methods that have recently been developed by the first-named author in [Nta23, Nta23b, Nta].

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2. Construction of quasiround exhaustions

Given a domain $G \subset \hat{\mathbb{C}}$, we denote by $\mathcal{C}(G)$ the collection of connected components of $\hat{\mathbb{C}} \setminus G$. Moreover, $\mathcal{C}(G) = \mathcal{C}_N(G) \cup \mathcal{C}_P(G)$, where elements of $\mathcal{C}_N(G)$ have positive diameter and elements of $\mathcal{C}_P(G)$ are singletons. If $(G_j)_{j\in\mathbb{N}}$ is an exhaustion of G, $\bar{p} \in \mathcal{C}(G)$, and $j \geqslant 1$, we denote by $p_j(\bar{p})$ the element of $\mathcal{C}(G_j)$ containing $\bar{p}$.

We denote $\hat{G} = \hat{\mathbb{C}}/\sim$, where

$$x \sim y \text{ if either } x = y \in G \text{ or } x, y \in p \text{ for some } p \in \mathcal{C}(G).$$

The corresponding quotient map is $\pi_G : \hat{\mathbb{C}} \to \hat{G}$. Identifying each $x \in G$ and $p \in \mathcal{C}(G)$ with $\pi_G(x)$ and $\pi_G(p)$, respectively, we have

$$\hat{G} = G \cup \mathcal{C}(G).$$

A homeomorphism $f : G \to G'$ has a homeomorphic extension $\hat{f} : \hat{G} \to \hat{G}'$; see [NY20, Section 3] for a detailed discussion. By Moore's theorem [Moo25], the quotient $\hat{G}$ is homeomorphic to $\hat{\mathbb{C}}$.

All distances below refer to the Euclidean metric of $\mathbb{C}$. In what follows, if $D = \mathbb{D}(z, r)$ is a disk and $\tau > 0$ then $\tau D = \mathbb{D}(z, \tau r)$. Moreover, $\mathbb{S}(z, r)$ is the boundary circle of $\mathbb{D}(z, r)$.

For $K \geqslant 1$ we say that a set $A \subset \mathbb{C}$ is *K-quasiround*, if there are $z_A \in \mathbb{C}$ and $r_A > 0$ such that

$$\overline{\mathbb{D}}(z_A, r_A) \subset A \subset \overline{\mathbb{D}}(z_A, K r_A).$$

Moreover, we say that a domain $G \subset \hat{\mathbb{C}}$ is K-quasiround if every $p \in \mathcal{C}_N(G)$ is K-quasiround, and that a sequence of domains $G_j \subset \hat{\mathbb{C}}$, $j \in \mathbb{N}$, is K-quasiround if every G_j is K-quasiround.

THEOREM 2.1. *Every circle domain $\Omega \subset \hat{\mathbb{C}}$ with $\infty \in \Omega$ has a 43-quasiround exhaustion $(\Omega_j)_{j\in\mathbb{N}}$.*

The rest of this section is devoted to the proof of Theorem 2.1. The collection $\mathcal{C}_N(\Omega)$ is empty or consists of finitely or countably many disks

$$D_k = \overline{\mathbb{D}}(z_k, r_k), \quad \text{where} \quad r_1 \geqslant r_2 \geqslant r_3 \cdots .$$

We will construct the required quasiround exhaustion of Ω inductively. Let Ω_0 be the complement of a large closed disk that contains $\hat{\mathbb{C}} \setminus \Omega$ in its interior. Assume that $j \geqslant 1$ and that we have defined a domain $\Omega_{j-1} \subset \overline{\Omega_{j-1}} \subset \Omega$ that is bounded by finitely many disjoint Jordan curves in Ω so that $\hat{\mathbb{C}} \setminus \overline{\Omega_{j-1}}$ is bounded. Theorem 2.1 follows if we find a 43-quasiround domain Ω_j that is bounded by finitely many disjoint Jordan curves in Ω and satisfies

$$\Omega_{j-1} \subset \Omega_j \subset \Omega \quad \text{and} \quad \hat{\mathbb{C}} \setminus \Omega_j \subset N_{1/j}(\hat{\mathbb{C}} \setminus \Omega), \tag{2.1}$$

where $N_{1/j}(A)$ is the open $1/j$-neighborhood of A. We define

$$\delta = \frac{\min\left\{\frac{1}{j}, \operatorname{dist}(\partial\Omega, \partial\Omega_{j-1})\right\}}{100} > 0 \tag{2.2}$$

and

$$\mathcal{C}_L(\Omega) = \{D_1, \dots, D_\alpha\} \subset \mathcal{C}_N(\Omega),$$

where α is the largest index for which $r_\alpha \geqslant \delta/4$.

Next, we denote

$$U = \hat{\mathbb{C}} \setminus \Big(\overline{\Omega_{j-1}} \cup \Big(\bigcup_{\beta=1}^{\alpha} (1 + 2\delta/r_\beta) D_\beta \Big)\Big).$$

Observe that the radius of the disk $(1+2\delta/r_\beta)D_\beta$ equals $r_\beta+2\delta$ and that each such disk is disjoint from $\overline{\Omega_{j-1}}$. By the $5r$-covering lemma [Hei01, Theorem 1.2], since U is bounded, there exists a finite collection $\mathcal{B}'$ of pairwise disjoint disks of radius δ centered at U, so that

$$U \subset \bigcup_{B \in \mathcal{B}'} 5B.$$

We let $\mathcal{U}' = \mathcal{C}_L(\Omega) \cup \mathcal{B}'$ and for $B \in \mathcal{U}'$ we define

$$V_B = \{z \in \hat{\mathbb{C}} \setminus \overline{\Omega_{j-1}} : \operatorname{dist}(z, B) < \operatorname{dist}(z, B') \text{ for every } B' \in \mathcal{U}',\ B' \neq B\}.$$

See Figure 1 for an illustration.

Lemma 2.2 (Properties of $\mathcal{U}'$)**.** *The following statements are true.*

(1) *The closed disks B, $B \in \mathcal{U}'$, are pairwise disjoint.*
(2) *The sets V_B, $B \in \mathcal{U}'$, are open and pairwise disjoint.*
(3) *For every $B \in \mathcal{U}'$,*

$$\{z \notin \overline{\Omega_{j-1}} \cup V_B : \operatorname{dist}(z, B) \leqslant \operatorname{dist}(z, B') \text{ for all } B' \in \mathcal{U}'\} = \partial V_B \setminus \overline{\Omega_{j-1}}.$$

(4) *For every $B \in \mathcal{U}'$, if $\overline{V_B} \cap \overline{\Omega_{j-1}} = \emptyset$, then V_B is star-like with respect to the center of B.*
(5) *For every $B \in \mathcal{U}'$, if $\overline{V_B} \cap \overline{\Omega_{j-1}} = \emptyset$, then V_B is a Jordan region.*
(6) $\hat{\mathbb{C}} \setminus \overline{\Omega_{j-1}} \subset \bigcup_{B \in \mathcal{U}'} \overline{V_B}$.

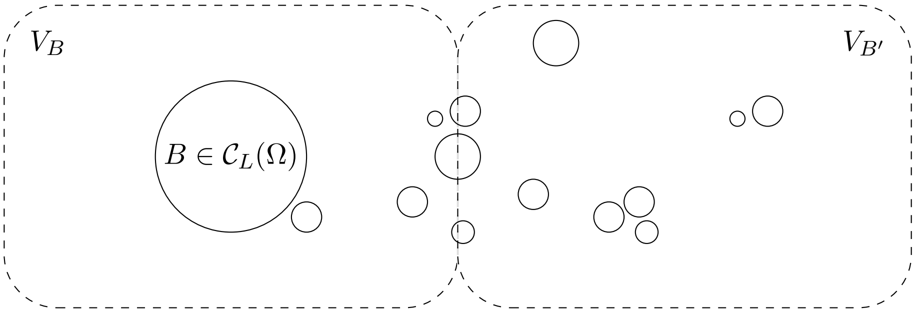

FIGURE 1. A region V_B corresponding to some $B \in \mathcal{C}_L(\Omega)$ and a region V_B' corresponding to some $B' \in \mathcal{B}'$. The disks are components of $\hat{\mathbb{C}} \setminus \Omega$; B' is not visible in the figure.

(7) *For each* $B = \overline{\mathbb{D}}(z_B, r_B) \in \mathcal{U}'$ *we have*

$$V_B \subset \overline{\mathbb{D}}(z_B, r_B + 4\delta) \subset \overline{\mathbb{D}}(z_B, 17r_B).$$

Moreover, if $\overline{V_B} \cap \overline{\Omega_{j-1}} = \emptyset$*, then* $\overline{\mathbb{D}}(z_B, r_B) \subset V_B$.

Proof. The disks of $\mathcal{B}'$ are pairwise disjoint and the disks of $\mathcal{C}_L(\Omega)$ are pairwise disjoint. Also, since each $B \in \mathcal{B}'$ is centered at U and has radius δ, its distance from all disks of $\mathcal{C}_L(\Omega)$ is at least δ. This shows that the collection $\mathcal{U}'$ is disjointed, as required in (1). Part (2) follows immediately from the definition of V_B.

Let $B = \overline{\mathbb{D}}(z_B, r_B) \in \mathcal{U}'$. For $\theta \in [0, 2\pi)$ and $t \geqslant 0$, let $w_t = z_B + te^{i\theta}$ and $r_t = \operatorname{dist}(w_t, B)$. Suppose that $w_t \notin \overline{\Omega_{j-1}} \cup V_B$, and $\operatorname{dist}(w_t, B) \leqslant \operatorname{dist}(w_t, B')$ for all $B' \in \mathcal{U}'$. Since $w_t \notin V_B$, there exists $B'' \in \mathcal{U}'$ with $B'' \neq B$ such that

$$r_t = \operatorname{dist}(w_t, B) = \operatorname{dist}(w_t, B'').$$

Since B and B'' are disjoint, we must have $r_t > 0$, so $t > r_B$. The disk $\mathbb{D}(w_t, r_t)$ is externally tangent to both B and B'', and it is disjoint from all $B' \in \mathcal{U}'$. For $s \in (r_B, t)$ the disk $\mathbb{D}(w_s, r_s)$ is a strict subset of $\mathbb{D}(w_t, r_t)$ and is tangent only to B and not to any $B' \in \mathcal{U}' \setminus \{B\}$. Hence, we have $\operatorname{dist}(w_s, B) < \operatorname{dist}(w_s, B')$ for all $B' \in \mathcal{U}' \setminus \{B\}$. Also, if s is sufficiently close to t, then $w_s \in \hat{\mathbb{C}} \setminus \overline{\Omega_{j-1}}$. This implies that $w_s \in V_B$ for all $s < t$ near t. Thus, $w_t \in \partial V_B$. Conversely, if $w_t \in \partial V_B \setminus \overline{\Omega_{j-1}}$, then by the definition of V_B we must have $\operatorname{dist}(z, B) \leqslant \operatorname{dist}(z, B')$ for all $B' \in \mathcal{U}'$. This proves (3). The proof also shows that if $w_t \in \partial V_B \setminus \overline{\Omega_{j-1}}$, then

$$\operatorname{dist}(w_s, B) < \operatorname{dist}(w_s, B') \text{ for } s \in [0, t) \text{ and } B' \in \mathcal{U} \setminus \{B\}. \tag{2.3}$$

Suppose that $\overline{V_B} \cap \overline{\Omega_{j-1}} = \emptyset$. If $w_t \in \partial V_B$, then by (2.3), we have $w_s \in V_B$ for all $s < t$ near t. If there exists $s \in [0, t)$ with $w_s \notin V_B$, then by (2.3) we must have $w_s \in \overline{\Omega_{j-1}}$. Hence, there exists $s' \in [s, t)$ with $w_{s'} \in \overline{V_B} \cap \overline{\Omega_{j-1}}$.

This is a contradiction. Therefore, the segment $\{w_s : 0 \leqslant s < t\}$ is contained in V_B and V_B is star-like with respect to z_B as claimed in (4).

For (6), let $z \in \hat{\mathbb{C}} \setminus \overline{\Omega_{j-1}}$ and consider $B \in \mathcal{U}'$ that minimizes $\operatorname{dist}(z, B')$ over all $B' \in \mathcal{U}'$. Then $\operatorname{dist}(z, B) \leqslant \operatorname{dist}(z, B')$ for all $B' \in \mathcal{U}'$. By (3), $z \in \overline{V_B}$.

Next, for (7), observe that if $z \in \hat{\mathbb{C}} \setminus \overline{\Omega_{j-1}}$, then either $z \in (1 + 2\delta/r_\beta)D_\beta$ for some $\beta \in \{1, \dots, \alpha\}$, in which case $\operatorname{dist}(z, D_\beta) < 2\delta$, or $z \in U$ so $z \in 5B = \overline{\mathbb{D}}(z_B, 5\delta)$ for some $B \in \mathcal{B}'$. In any case,

$$\text{if } z \in \hat{\mathbb{C}} \setminus \overline{\Omega_{j-1}}, \text{ then } \operatorname{dist}(z, B) < 4\delta \text{ for some } B \in \mathcal{U}'.$$

Now, let $B \in \mathcal{U}'$ and $z \in V_B$. Since $z \in \hat{\mathbb{C}} \setminus \overline{\Omega_{j-1}}$, by the above we have $\operatorname{dist}(z, B') < 4\delta$ for some $B' \in \mathcal{U}'$. The definition of V_B implies that $\operatorname{dist}(z, B) \leqslant \operatorname{dist}(z, B') < 4\delta$. This implies the first inclusion claimed in (7). Moreover, since the radius of every $B \in \mathcal{U}'$ is at least $\delta/4$, the second inclusion holds as well.

Suppose that $\overline{V_B} \cap \overline{\Omega_{j-1}} = \emptyset$, as the last part of (7). If $B \not\subset V_B$, since $z_B \in V_B$ by (4), we have $B \cap \partial V_B \neq \emptyset$. If $z \in B \cap \partial V_B$, we have $z \in \hat{\mathbb{C}} \setminus \overline{\Omega_{j-1}}$ and $\operatorname{dist}(z, B) = 0 < \operatorname{dist}(z, B')$ for all $B' \in \mathcal{U}' \setminus \{B\}$. This implies that $z \in V_B$, a contradiction. Therefore $B \subset V_B$, as desired.

For (5), observe that V_B is an open set that is simply connected by part (4). Since V_B is bounded by part (7), ∂V_B is a continuum. Since V_B is star-like and bounded, $\mathbb{C} \setminus \overline{V_B}$ is connected. Hence $\mathbb{C} \setminus \partial V_B$ has two connected components. If $z \in \partial V_B \subset \hat{\mathbb{C}} \setminus \overline{\Omega_{j-1}}$, by the definition of V_B there exists $B' \in \mathcal{U}' \setminus \{B\}$, such that

$$\operatorname{dist}(z, B) = \operatorname{dist}(z, B') \leqslant \operatorname{dist}(z, B'')$$

for all $B'' \in \mathcal{U}'$. By part (3), we have $z \in \partial V_{B'}$. By (4) there exists a line segment connecting z_B to z and whose interior is contained in V_B. By (2.3), there exists a line segment connecting z to a point of $V_{B'}$ and whose interior is contained in $V_{B'} \subset \mathbb{C} \setminus \overline{V_B}$. This shows that each point of the continuum ∂V_B is accessible from both complementary components of ∂V_B. The inverse of the Jordan curve theorem [Kur68, Theorem 61.II.12, p. 518] implies that ∂V_B is a Jordan curve. □

Next, we let

$$\mathcal{U} = \{B \in \mathcal{U}' : \overline{V_B} \cap N_{10\delta}(\Omega_{j-1}) = \emptyset\}.$$

Lemma 2.3 (Properties of $\mathcal{U}$)**.** *We have $\mathcal{C}_L(\Omega) \subset \mathcal{U}$ and*

$$N_{10\delta}(\hat{\mathbb{C}} \setminus \Omega) \subset \bigcup_{B \in \mathcal{U}} \overline{V_B}.$$

Proof. For each $B \in \mathcal{C}_L(\Omega)$ we have $V_B \subset N_{4\delta}(B)$ by Lemma 2.2 (7). Since $B \subset \hat{\mathbb{C}} \setminus \Omega$, we see that $V_B \subset N_{4\delta}(\hat{\mathbb{C}} \setminus \Omega)$. By the definition of δ in (2.2), we have

$$\operatorname{dist}(N_{10\delta}(\Omega_{j-1}), N_{10\delta}(\hat{\mathbb{C}} \setminus \Omega)) \geqslant 80\delta. \tag{2.4}$$

This shows that $\overline{V_B}$ is disjoint from $N_{10\delta}(\Omega_{j-1})$, so $B \in \mathcal{U}$ and $\mathcal{C}_L(\Omega) \subset \mathcal{U}$.

Next, the sets $\overline{V_B}$, $B \in \mathcal{U}'$, cover $\hat{\mathbb{C}} \setminus \overline{\Omega_{j-1}}$ by Lemma 2.2 (6). By (2.4), $N_{10\delta}(\hat{\mathbb{C}} \setminus \Omega)$ is also covered by $\overline{V_B}$, $B \in \mathcal{U}'$. By the first inclusion in Lemma 2.2 (7), if $B \in \mathcal{B}'$, then $\overline{V_B}$ has diameter at most 10δ, so it cannot intersect both sets in (2.4). Therefore, each point of $N_{10\delta}(\hat{\mathbb{C}} \setminus \Omega)$ is contained in a set $\overline{V_B}$ such that $B \in \mathcal{B}'$ and $\overline{V_B} \cap N_{10\delta}(\Omega_{j-1}) = \emptyset$ or $B \in \mathcal{C}_L(\Omega) \subset \mathcal{U}$. This completes the proof. □

We will need the following general topological lemma, which is a consequence of Moore's theorem.

Lemma 2.4 (Perturbation lemma). *Let $\varepsilon > 0$, $m \in \mathbb{N}$, and $A_1, \dots, A_m \subset \mathbb{C}$ be pairwise disjoint and non-separating continua with* $\operatorname{diam} A_i < \varepsilon$ *for each $i \in \{1, \dots, m\}$. Then for each closed nowhere dense set $E \subset \mathbb{C}$ there exists a homeomorphism $\phi\colon \mathbb{C} \to \mathbb{C}$ such that*

$$\phi(E) \cap \Big(\bigcup_{i=1}^{m} A_i\Big) = \emptyset \quad \textit{and} \quad \sup_{z \in \mathbb{C}} |\phi(z) - z| < \varepsilon.$$

Moreover, if $F \subset \mathbb{C} \setminus \bigcup_{i=1}^m A_i$ is a compact set with $E \cap F = \emptyset$, then we may also have $\phi(E) \cap F = \emptyset$.

Proof. By Moore's theorem [Dav86, Theorem 25.1], the decomposition of $\hat{\mathbb{C}}$ induced by the sets A_i, $i \in \{1, \dots, m\}$, and singleton points $z \in \hat{\mathbb{C}} \setminus \bigcup_{i=1}^m A_i$ is *strongly shrinkable.* In particular, this implies that there exists a set of points $\{p_1, \dots, p_m\} \subset \hat{\mathbb{C}}$ and a continuous and surjective map $\pi\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that π is the identity map outside a disk $\mathbb{D}(0, R)$ that contains $\bigcup_{i=1}^m A_i$, π is injective outside $\bigcup_{i=1}^m A_i$, and $\pi^{-1}(p_i) = A_i$, $i \in \{1, \dots, m\}$. Moreover, there exists a sequence of homeomorphisms $\pi_k\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $k \in \mathbb{N}$, that converge uniformly to π, and π_k is also the identity map outside $\mathbb{D}(0, R)$ for each $k \in \mathbb{N}$; see [Dav86, Section 5] for properties of strongly shrinkable decompositions.

By a compactness argument, there exists $\eta > 0$ such that if $A \subset \mathbb{C}$ is a compact set and $\operatorname{diam} A \leqslant \eta$, then $\operatorname{diam} \pi^{-1}(A) < \varepsilon$. By uniform convergence, there exists $k_1 \in \mathbb{N}$ such that if A is compact and $\operatorname{diam} A \leqslant \eta$, then we also have $\operatorname{diam} \pi_k^{-1}(A) < \varepsilon$ for all $k \geqslant k_1$.

Let $F \subset \mathbb{C} \setminus \bigcup_{i=1}^m A_i$ be a compact set that is disjoint from E. Then $\pi(E) \cap \pi(F) = \emptyset$, since π is injective in $\hat{\mathbb{C}} \setminus \bigcup_{i=1}^m A_i$. The set $\pi(E)$ is nowhere dense. For $i \in \{1, \dots, m\}$, let

$$S_i = \{z \in \overline{\mathbb{D}}(0, \eta) : p_i \in \pi(E) - z\} = \overline{\mathbb{D}}(0, \eta) \cap (\pi(E) - p_i).$$

The set $\bigcup_{i=1}^m S_i$ is nowhere dense. Hence, there exists $z_0 \in \overline{\mathbb{D}}(0, \eta)$, arbitrarily close to 0, such that the set $\pi(E) - z_0$ is disjoint from $\{p_1, \dots, p_m\}$. If

z_0 is sufficiently close to 0, then we may have that $\pi(E)-z_0$ is also disjoint from the compact set $\pi(F)$. By uniform convergence, there exists $k_2\in\mathbb{N}$ such that for $k\geqslant k_2$ the set $\pi_k(E)-z_0$ is disjoint from $\bigcup_{i=1}^m \pi_k(A_i)$ and from $\pi_k(F)$.

Fix $k\geqslant \max\{k_1,k_2\}$ and let $\psi(z)=z-z_0$ on $\hat{\mathbb{C}}$. We define $\phi=\pi_k^{-1}\circ\psi\circ\pi_k$, which is a homeomorphism of $\hat{\mathbb{C}}$ fixing ∞. By construction, $\phi(E)$ is disjoint from $\bigcup_{i=1}^m A_i$ and from F. For $z\in\mathbb{C}$ and $w=\pi_k(z)$, we have $|\psi(w)-w|=|z_0|\leqslant\eta$, so

$$|\phi(z)-z|\leqslant \operatorname{diam}\pi_k^{-1}(\{\psi(w),w\})<\varepsilon.$$

This completes the proof. ☐

By Lemma 2.3 we have $\mathcal{C}_L(\Omega)\subset\mathcal{U}$ and hence

$$\mathcal{U}=\mathcal{C}_L(\Omega)\cup\mathcal{B},\ \text{ for some } \mathcal{B}\subset\mathcal{B}'.$$

Let $J_0=\bigcup_{B\in\mathcal{U}}\partial V_B$. By Lemma 2.2 (5), J_0 is the union of finitely many Jordan curves. Moreover, by the last part of Lemma 2.2 (7),

$$J_0\cap B=\emptyset \quad \text{for all } B\in\mathcal{U}.$$

However, the set J_0 might intersect $\hat{\mathbb{C}}\setminus\Omega$. We wish to use Lemma 2.4 to deform slightly J_0 into a set $J_1\subset\Omega$ that consists of Jordan curves bounding quasiround regions. This is made precise in the following lemma.

Lemma 2.5. *There exists a collection q_B, $B\in\mathcal{U}$, of closed Jordan regions with the following properties.*

(1) *For each $B\in\mathcal{U}$ we have $\partial q_B\subset\Omega$.*
(2) *For each $B\in\mathcal{U}$ we have $q_B\subset\hat{\mathbb{C}}\setminus\overline{\Omega_{j-1}}$.*
(3) *The sets $\operatorname{int} q_B$, $B\in\mathcal{U}$, are pairwise disjoint.*
(4) *$\hat{\mathbb{C}}\setminus\Omega\subset\bigcup_{B\in\mathcal{U}} q_B$.*
(5) *For each $B=\overline{\mathbb{D}}(z_B,r_B)\in\mathcal{U}$ we have*

$$\overline{\mathbb{D}}(z_B,r_B/2)\subset q_B\subset\overline{\mathbb{D}}(z_B,r_B+5\delta)\subset\overline{\mathbb{D}}(z_B,21r_B).$$

(6) *For each $B\in\mathcal{U}$, if $q_B\setminus\Omega\neq\emptyset$, then $q_B\subset N_{1/j}(\hat{\mathbb{C}}\setminus\Omega)$.*

Proof. For $B\in\mathcal{C}_L(\Omega)$, by Lemma 2.2 (7) we have $B\subset V_B$. Since $B\subset\hat{\mathbb{C}}\setminus\Omega$, we may find a closed Jordan region $\widetilde{B}\subset V_B$ such that $\partial\widetilde{B}\subset\Omega$ and $B\subset\widetilde{B}$. Recall that all components of $\hat{\mathbb{C}}\setminus\Omega$ that are not in $\mathcal{C}_L(\Omega)$ have diameter less than $\delta/2$. Hence, all complementary components of the domain $\Omega\cup(\bigcup_{B\in\mathcal{C}_L(\Omega)}\widetilde{B})$ have diameter less than $\delta/2$. It follows that there exist pairwise disjoint closed Jordan regions

$$A_1,\dots,A_m\subset\hat{\mathbb{C}}\setminus\Big(\partial\Omega_{j-1}\cup\Big(\bigcup_{B\in\mathcal{C}_L(\Omega)}\widetilde{B}\Big)\Big)$$

of diameter less than $\delta/2$ such that $\partial A_i\subset\Omega$ for each $i\in\{1,\dots,m\}$, and

$$\hat{\mathbb{C}}\setminus\Omega\subset\Big(\bigcup_{B\in\mathcal{C}_L(\Omega)}\widetilde{B}\Big)\cup\Big(\bigcup_{i=1}^m A_i\Big).$$

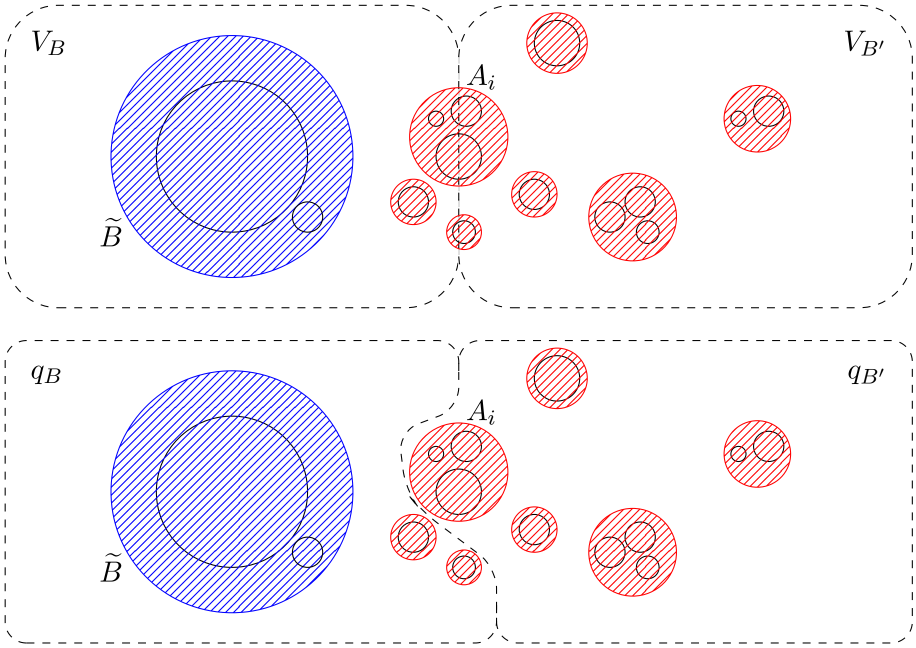

FIGURE 2. Top: The region $\widetilde{B}$ (blue) corresponding to $B \in \mathcal{C}_L(\Omega)$ and some of the regions A_i (red). Bottom: The regions q_B and $q_{B'}$.

See Figure 2 for an illustration of the regions $\widetilde{B}$ and A_i.

Let $\varepsilon = \delta/2$, $E = \bigcup_{B\in\mathcal{U}} \partial V_B$, and $F = \bigcup_{B\in\mathcal{C}_L(\Omega)} \widetilde{B}$. By the choice of $\widetilde{B}$, we have $E \cap F = \emptyset$. Moreover, $F \subset \mathbb{C} \setminus \bigcup_{i=1}^m A_i$. By Lemma 2.4, we obtain a homeomorphism $\phi\colon \mathbb{C} \to \mathbb{C}$ such that ϕ is $(\delta/2)$-close to the identity map, and $\phi(E)$ is disjoint from $\bigcup_{i=1}^m A_i$ and $\bigcup_{B\in\mathcal{C}_L(\Omega)} \widetilde{B}$. For $B \in \mathcal{U}$ define $q_B = \overline{\phi(V_B)}$; see Figure 2. We will show that the collection q_B, $B \in \mathcal{B}$, has the desired properties.

Note that (1) and (3) are immediate by the properties of ϕ and the fact that the regions V_B, $B \in \mathcal{U}$, are pairwise disjoint. For $B \in \mathcal{U}$, by the definition of $\mathcal{U}$, we have $\overline{V_B} \cap N_{10\delta}(\Omega_{j-1}) = \emptyset$. Since $q_B \subset N_{\delta/2}(\overline{V_B})$, we conclude that $q_B \cap \overline{\Omega_{j-1}} = \emptyset$, as required in (2). For (4), let p be a component of $\hat{\mathbb{C}} \setminus \Omega$. By Lemma 2.3, $N_{10\delta}(p) \subset \bigcup_{B\in\mathcal{U}} \overline{V_B}$. Using a homotopy argument, based on the fact that ϕ is $(\delta/2)$-close to the identity map, one can show that each point of p is surrounded by $\phi(\partial N_{10\delta}(p))$, so $p \subset \bigcup_{B\in\mathcal{U}} q_B$.

Let $B = \overline{\mathbb{D}}(z_B, r_B) \in \mathcal{U}$. Suppose first that $B \in \mathcal{B}'$. Since ϕ is $(\delta/2)$-close to the idenity map and $r_B = \delta$, we conclude (by considering a linear homotopy from ϕ to the identity) that

$$\frac{1}{2}B \subset \phi(B).$$

Combining this with Lemma 2.2 (7) we obtain

$$\overline{\mathbb{D}}(z_B, \frac{1}{2}r_B) \subset \phi(B) \subset \phi(V_B) \subset q_B \subset N_{\delta/2}(\overline{V_B}) \subset \overline{\mathbb{D}}(z_B, r_B + 5\delta).$$

If $B = \overline{\mathbb{D}}(z_B, r_B) \in \mathcal{C}_L(\Omega)$, then $r_B \geqslant \delta/4$ and $B \subset V_B$. As we proved in part (1), ∂q_B is disjoint from B, so we either have $B \subset \operatorname{int} q_B$ or B is not surrounded by ∂q_B. In the latter case, let $\gamma(t,z) = t(\phi(z) - z) + z$, $t \in [0,1]$, and note that a point $z \in \partial V_B$ must satisfy $\gamma(t,z) = z_B$ for some $t \in (0,1)$. Then, the quantity $|\gamma(1,z) - \gamma(0,z)| = |\phi(z) - z|$ is the length of a segment passing through z_B with endpoints outside B. Hence, $|\phi(z) - z| \geqslant \delta/2$, a contradiction. Therefore $B \subset \operatorname{int} q_B$. Combining this with Lemma 2.2 (7), we obtain

$$\overline{\mathbb{D}}(z_B, r_B) \subset q_B \subset N_{\delta/2}(\overline{V_B}) \subset \overline{\mathbb{D}}(z_B, r_B + 5\delta) \subset \overline{\mathbb{D}}(z_B, 21 r_B),$$

given that $\delta \leqslant 4r_B$. This completes the proof of part (5).

Finally, we show part (6). Suppose $B \in \mathcal{U}$ and $q_B \setminus \Omega \neq \emptyset$. By (5), we have $q_B \subset N_{6\delta}(\hat{\mathbb{C}} \setminus \Omega)$ when $B \in \mathcal{C}_L(\Omega)$ and $q_B \subset N_{13\delta}(\hat{\mathbb{C}} \setminus \Omega)$ when $B \in \mathcal{B}$. The choice of δ in (2.2) implies that $q_B \subset N_{1/j}(\hat{\mathbb{C}} \setminus \Omega)$. $\square$

We are ready to construct the domain Ω_j. Let $\mathcal{V} = \{B \in \mathcal{U} : q_B \setminus \Omega \neq \emptyset\}$. By Lemma 2.5 (4), the set $\hat{\mathbb{C}} \setminus \Omega$ is covered by the collection $\{q_B : B \in \mathcal{V}\}$. Also by (2) and (6), each q_B is contained in $\hat{\mathbb{C}} \setminus \overline{\Omega_{j-1}}$ and in $N_{1/j}(\hat{\mathbb{C}} \setminus \Omega)$. We conclude that

$$\hat{\mathbb{C}} \setminus \Omega \subset \bigcup_{B \in \mathcal{V}} q_B \subset \hat{\mathbb{C}} \setminus \overline{\Omega_{j-1}} \quad \text{and} \quad \bigcup_{B \in \mathcal{V}} q_B \subset N_{1/j}(\hat{\mathbb{C}} \setminus \Omega). \tag{2.5}$$

Note also that each q_B is 42-quasiround by (5) and $\partial q_B \subset \Omega$ by (1). Thus, if q_B, $B \in \mathcal{V}$, were the complementary components of a domain Ω_j, then (2.1) would be satisfied and the proof would be completed. Although the sets q_B have disjoint interiors by (3), their boundaries might intersect, so they are not necessarily the complementary components of a domain.

We hence modify each q_B slightly to amend this. Namely, we consider a closed Jordan region $p_B \subset \operatorname{int} q_B$ such that p_B is 43-quasiround, $\partial p_B \subset \Omega$, and (2.5) is true with p_B in place of q_B. Then Ω_j is the domain for which

$$\mathcal{C}(\Omega_j) = \{p_B : B \in \mathcal{V}\}.$$

The proof of Theorem 2.1 is completed.

3. Quasiround exhaustions and limit maps

Let $\Omega \subset \hat{\mathbb{C}}$ be a circle domain with $\infty \in \Omega$, and $(\Omega_j)_{j\in\mathbb{N}}$ an exhaustion of Ω. Moreover, let $f_j : \Omega_j \to D_j$, $j \in \mathbb{N}$, be the normalized conformal maps in Theorem 1.2; that is, each f_j fixes three prescribed points $a_1, a_2, a_3 \in \Omega_1$ and each D_j is a finitely connected circle domain. Recalling that $(f_j)_{j\in\mathbb{N}}$ has a converging subsequence and that a subsequence of an exhaustion of Ω is also an exhaustion of Ω, the first claim in Theorem 1.2 follows from Theorem 2.1 and the following result.

THEOREM 3.1. *Suppose that* $(\Omega_j)_{j\in\mathbb{N}}$ *is quasiround and that* $(f_j)_{j\in\mathbb{N}}$ *converges locally uniformly in* Ω *to a conformal homeomorphism* $f:\Omega\to D$ *for some domain* $D\subset\hat{\mathbb{C}}$. *Then* $(\hat{f}(p_j(\bar{p})))_{j\in\mathbb{N}}$ *converges to* $\hat{f}(\bar{p})$ *in the Hausdorff sense for every* $\bar{p}\in\mathcal{C}(\Omega)$. *In particular, D is a circle domain.*

Recall that $p_j(\bar{p})$ denotes the unique element of $\mathcal{C}(\Omega_j)$ that contains $\bar{p}$. The proof is based on transboundary modulus estimates on quasiround domains; see e.g. [Sch95, Bon11, Raj] for similar estimates.

We recall the definition of transboundary modulus, as introduced by Schramm [Sch95]. Let $G\subset\hat{\mathbb{C}}$ be a domain. Let $\rho\colon\hat{G}\to[0,\infty]$ be a Borel function and $\gamma\colon[a,b]\to\hat{G}$ be a curve. Then $\gamma^{-1}(\pi_G(G))$ has countably many components $O_j\subset[a,b]$, $j\in J$. For $j\in J$ define $\gamma_j=\gamma|_{O_j}$ and $\alpha_j=\pi_G^{-1}\circ\gamma_j$. We define

$$\int_\gamma \rho\,ds=\sum_{j\in J}\int_{\alpha_j}\rho\circ\pi_G\,ds,$$

where the integral is understood to be infinite if one of the curves α_j is not locally rectifiable. Let Γ be a family of curves in $\hat{\Omega}$. We say that a Borel function $\rho\colon\hat{G}\to[0,\infty]$ is *admissible* for Γ if

$$\int_\gamma \rho\,ds+\sum_{\substack{p\in\mathcal{C}(G)\\ |\gamma|\cap p\neq\emptyset}}\rho(p)\geqslant 1$$

for each $\gamma\in\Gamma$. Here $|\gamma|$ denotes the image of γ. The *transboundary modulus* of Γ with respect to the domain G is defined to be

$$\operatorname{mod}_G\Gamma=\inf_\rho\left\{\int_G(\rho\circ\pi_G)^2\,dA+\sum_{p\in\mathcal{C}(G)}\rho(p)^2\right\},$$

where the infimum is taken over all admissible functions ρ. It was observed by Schramm that transboundary modulus is invariant under conformal maps. Specifically, if $f\colon G\to G'$ is a conformal map between domains $G,G'\subset\hat{\mathbb{C}}$, then for every curve family Γ in $\hat{G}$ we have $\operatorname{mod}_G\Gamma=\operatorname{mod}_{G'}\hat{f}(\Gamma)$.

The rest of this section is devoted to the proof of Theorem 3.1. Fix $\bar{p}\in\mathcal{C}(\Omega)$ and let $J\subset\Omega_1$ be a Jordan curve that separates $p_1(\bar{p})$ and ∞. Denote the bounded component of $\hat{\mathbb{C}}\setminus J$ by U. Recall that f_j extends to a homeomorphism $\hat{f}_j:\hat{\Omega}_j\to\hat{D}_j$. Consider a compact set $q(\bar{p})\subset\mathbb{C}$ that is the Hausdorff limit of a subsequence of $(\hat{f}_j(p_j(\bar{p})))_{j\in\mathbb{N}}$. Then $q(\bar{p})$ is a disk or a point, and $q(\bar{p})\subset\hat{f}(\bar{p})$, as a consequence of Carathéodory's kernel convergence theorem; see [Nta23, Lemma 2.14]. Theorem 1.2 follows if we can show that $q(\bar{p})=\hat{f}(\bar{p})$.

Towards contradiction, suppose that $f(\bar{p})\setminus q(\bar{p})\neq\emptyset$. Then there are $\delta>0$ and a sequence of points $(z_m)_{m\in\mathbb{N}}$ so that $z_m\in\partial p_m(\bar{p})$ and

$$\liminf_{j\to\infty}\operatorname{dist}(f_j(z_m),\hat{f}_j(p_j(\bar{p})))\geqslant 2\delta\quad\text{for every } m=1,2,\dots. \tag{3.1}$$

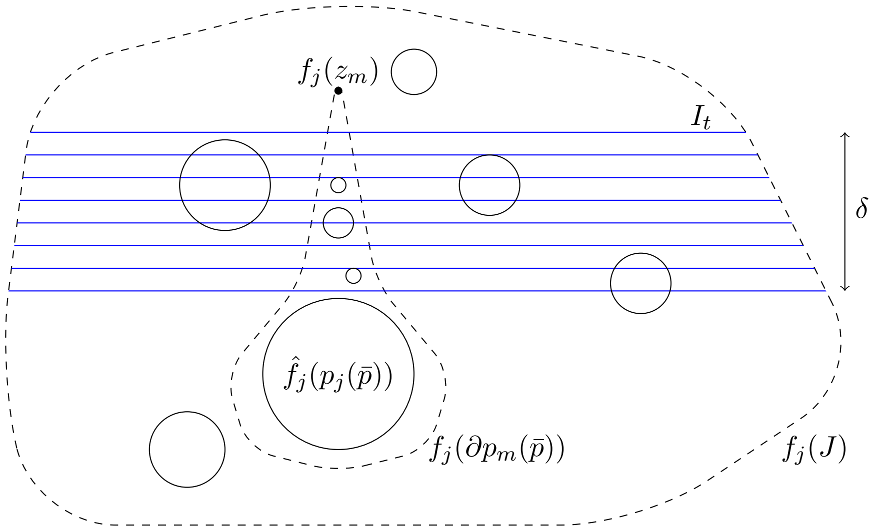

FIGURE 3. Illustration of (3.2) and of the proof of Lemma 3.2.

See Figure 3 for an illustration. Passing to a subsequence if necessary, we may assume that $z_m \to z_0 \in \partial\bar{p}$ and that $W_m \subset U$ for every $m \geqslant 1$, where

$$W_m = \overline{\mathbb{D}}(z_0, |z_m - z_0|).$$

Fix $j(m) > m$ so that $z_m \in \Omega_j$ and

$$\operatorname{dist}(f_j(z_m), \hat{f}_j(p_j(\bar{p}))) > \delta \quad \text{for every } j \geqslant j(m), \tag{3.2}$$

and let $\Gamma(j,m)$ be the family of curves in $\hat{\Omega}_j$ joining J and $\pi_{\Omega_j}(W_m)$ without intersecting $p_j(\bar{p})$.

Lemma 3.2. *There exists $N > 0$ such that*

$$\operatorname{mod}_{D_j} \hat{f}_j(\Gamma(j,m)) \geqslant N > 0 \quad \textit{for every } j \geqslant j(m) \textit{ and } m \geqslant 1. \tag{3.3}$$

Proof. The claim is almost the same as [Raj, Lemma 2.5]. We therefore only give a rough outline here. By (3.2) there is a line ℓ in $\mathbb{C}$, orthogonal to $w \in \mathbb{S}(0,1)$, so that for every $0 < t < \delta$ there is a parametrized line segment I_t with image in $\ell + tw$ so that $\gamma_t = \pi_{D_j} \circ I_t$ connects $f_j(J)$ and $\hat{f}_j(\pi_{\Omega_j}(W_m))$ and belongs to $\hat{f}_j(\Gamma(j,m))$; see Figure 3. Here $\ell + tw = \{z + tw : z \in \ell\}$.

Then, integrating an arbitrary admissible function ρ of $\hat{f}_j(\Gamma(j,m))$ over each γ_t and using a variation of the standard length-area method together with the assumption that D_j is a circle domain gives (3.3). Here the lower bound N depends on δ and the choice of J, but not on j or m. □

In view of Lemma 3.2 and the conformal invariance of transboundary modulus, a contradiction to (3.1) follows if we can show that

$$\lim_{m\to\infty} \lim_{j\to\infty} \operatorname{mod}_{\Omega_j} \Gamma(j,m) = 0. \tag{3.4}$$

To prove (3.4), we may assume that $z_0 = 0$ and $\mathrm{dist}(0, J) = 1$. We fix a sequence

$$\mathbb{A}_k = \mathbb{D}(0, R_k) \setminus \overline{\mathbb{D}}(0, R_k/2), \quad k = 1, 2, \ldots,$$

of annuli, where radii R_k are chosen as follows: $R_1 = 1$, and if R_{k-1} is defined then R_k is the largest number so that $R_k \leqslant R_{k-1}/4$ and so that no $p \in \mathcal{C}(\Omega)$ other than $\bar{p}$ intersects both $\mathbb{S}(0, R_{k-1}/2)$ and $\mathbb{D}(0, 2R_k)$.

Lemma 3.3. *There exists $M > 0$ such that*

$$\limsup_{j\to\infty} \mathrm{mod}_{\Omega_j} \Lambda(j,k) \leqslant M \quad \textit{for every } k = 1, 2, \ldots, \tag{3.5}$$

where $\Lambda(j,k)$ is the family of curves joining

$$\pi_{\Omega_j}(\mathbb{S}(0, R_k)) \quad \textit{and} \quad \pi_{\Omega_j}(\mathbb{S}(0, R_k/2)) \quad \textit{in} \quad \hat{\Omega}_j \setminus \{\pi_{\Omega_j}(p_j(\bar{p}))\}.$$

Proof. Fix $k \geqslant 1$ and notice that if $\mathcal{B}$ is a family of pairwise disjoint disks of radius larger than $R_k/10$ that intersect $\overline{\mathbb{D}}(0, R_k)$, then $\#\mathcal{B} \leqslant 400$. Thus, at most 400 elements $p \in \mathcal{C}(\Omega \cup \bar{p})$ can have radius larger than $R_k/10$ and intersect $\overline{\mathbb{A}_k}$. We conclude that there exists $j_0 = j_0(k)$ so that if $j \geqslant j_0$ then there are at most 400 elements $p \in \mathcal{C}(\Omega_j \cup p_j(\bar{p}))$ with diameter greater than $R_k/4$ intersecting $\mathbb{A}_k$. We denote the collection of such elements by $P_L = P_L(j)$. Also, let $P_S = P_S(j)$ be the elements of $\mathcal{C}(\Omega_j \cup p_j(\bar{p})) \setminus P_L$ that intersect $\mathbb{A}_k$. We define

$$\rho(p) = \begin{cases} 1, & p \in P_L, \\ 2R_k^{-1} \,\mathrm{diam}\, p, & p \in P_S, \\ 2R_k^{-1}, & p \in \Omega_j \cap \mathbb{A}_k. \end{cases}$$

Then ρ is admissible for $\Lambda(j,k)$, and

$$\int_{\Omega_j} \rho^2 \, dA + \sum_{p \in P_L} \rho(p)^2 \leqslant 4\pi + 400.$$

To estimate the sum of values $\rho(p)^2$ over P_S, we recall that Ω_j is K-quasiround for some $K \geqslant 1$ by assumption. In particular,

$$(\mathrm{diam}\, p)^2 \leqslant \frac{4K^2 \,\mathrm{Area}(p)}{\pi} \quad \text{for every } p \in P_S,$$

so

$$\sum_{p \in P_S} \rho(p)^2 \leqslant \sum_{p \in P_S} \frac{16K^2 \,\mathrm{Area}(p)}{\pi R_k^2} \leqslant \frac{16K^2 \,\mathrm{Area}(\mathbb{D}(0, 2R_k))}{\pi R_k^2} \leqslant 64K^2. \tag{3.6}$$

Combining the estimates and letting $j \to \infty$ gives (3.5). □

We remark that (3.6) is the only estimate in the proof of Theorem 3.1 which depends on the quasiroundness assumption.

We are now ready to prove (3.4). By Lemma 3.3, given $\ell \in \mathbb{N}$ there is $j'(\ell)$ so that if $1 \leqslant k \leqslant \ell$ and $j \geqslant j'(\ell)$, then $\mathrm{mod}_{\Omega_j} \Lambda(j,k) \leqslant 2M$ and no $p \in \mathcal{C}(\Omega_j)$ other than $p_j(\bar{p})$ intersects both $\mathbb{A}_k$ and $\mathbb{A}_{k+1}$. Let $\rho_{j,k}$ be admissible for $\Lambda(j,k)$ and

$$\int_{\Omega_j} \rho_{j,k}^2 \, dA + \sum_{p \in \mathcal{C}(\Omega_j)} \rho_{j,k}(p)^2 \leqslant 3M.$$

Now, since $\operatorname{diam} W_m \to 0$ as $m \to \infty$, there is $m(\ell)$ so that if $m \geqslant m(\ell)$ then $\rho = \ell^{-1} \sum_{k=1}^{\ell} \rho_{j,k}$ is admissible for $\Gamma(j,m)$ for all $j \geqslant \max\{j'(\ell), j(m)\}$. Therefore,

$$\operatorname{mod}_{\Omega_j} \Gamma(j,m) \leqslant \int_{\Omega_j} \rho^2 \, dA + \sum_{p \in \mathcal{C}(\Omega_j)} \rho(p)^2 \leqslant 3M\ell^{-1}$$

for all $j \geqslant \max\{j'(\ell), j(m)\}$. Now, (3.4) follows by letting $j \to \infty$, then $m \to \infty$, and then $\ell \to \infty$. The proof of Theorem 3.1 is complete.

4. Definition of quasiconformality

We will show that the limiting map f of Theorem 1.2 is conformal using a variant of a recent characterization of quasiconformality due to the first-named author [Nta].

Let $A \subset \mathbb{C}$ be a bounded open set. The *eccentricity* $E(A)$ of A is the infimum of all numbers $H \geqslant 1$ for which there exists an open ball B such that $B \subset A \subset HB$. Let $g \colon U \to V$ be a homeomorphism between open sets $U, V \subset \mathbb{C}$. The *eccentric distortion of* g at a point $x \in U$, denoted by $E_g(x)$, is the infimum of all values $H \geqslant 1$ such that there exists a sequence of open sets $A_n \subset U$, $n \in \mathbb{N}$, containing x with $\operatorname{diam} A_n \to 0$ as $n \to \infty$ and with the property that $E(A_n) \leqslant H$ and $E(g(A_n)) \leqslant H$ for each $n \in \mathbb{N}$.

THEOREM 4.1. *Let $g \colon U \to V$ be an orientation-preserving homeomorphism between open sets $U, V \subset \mathbb{C}$. Let $G \subset U$ be a set with the property that*

$$\mathcal{H}^1(g(|\gamma| \cap G)) = 0$$

for a.e. horizontal and a.e. vertical line segment γ in U. Suppose that there exists $H \geqslant 1$ such that $E_g(x) \leqslant H$ for each point $x \in U \setminus G$. Then g is quasiconformal in U, quantitatively. Moreover, if G is measurable and $H = 1$, then g is conformal in U.

Recall that $|\gamma|$ denotes the image of the path γ. Also, see [Hei01, §8.3] for the definition of Hausdorff 1-measure $\mathcal{H}^1$ and Hausdorff 1-content $\mathcal{H}^1_\infty$. We recall the definition of 2-modulus. Let Γ be a family of curves in $\mathbb{C}$. A Borel function $\rho \colon \mathbb{C} \to [0,\infty]$ is *admissible* for Γ if

$$\int_\gamma \rho \, ds \geqslant 1$$

for all rectifiable curves $\gamma \in \Gamma$. The 2-modulus of Γ is defined to be

$$\operatorname{mod} \Gamma = \inf_\rho \int \rho^2 \, dA,$$

where the infimum is taken over all admissible functions. The proof of Theorem 4.1 is a slight modification of the proof of [Nta, Theorem 3.3].

Proof. By the assumption regarding E_g and [Nta, Theorem 3.1], there exists a curve family Γ_0 with $\operatorname{mod}\Gamma_0 = 0$ and a Borel function $\rho_g\colon U \to [0,\infty]$ with $\rho_g \in L^2_{\mathrm{loc}}(U)$ such that for all curves $\gamma \notin \Gamma_0$ with trace in U we have

$$\mathcal{H}^1_\infty(g(|\gamma| \setminus G)) \leqslant \int_\gamma \rho_g\, ds \tag{4.1}$$

and for each Borel function $\rho\colon V \to [0,\infty]$ we have

$$\int_U (\rho \circ g) \cdot \rho_g^2\, dA \leqslant c(H) \int_V \rho\, dA. \tag{4.2}$$

We will show that for each open rectangle Q with $Q \subset \overline{Q} \subset U$ with sides parallel to the coordinate axes we have

$$\operatorname{mod}\Gamma(Q) \leqslant c(H) \operatorname{mod} g(\Gamma(Q)), \tag{4.3}$$

where $\Gamma(Q)$ denotes the family of curves joining the horizontal (resp. vertical) sides of Q. By a result of Gehring–Väisälä [GV61, Theorem 2], this implies that g is quasiconformal, quantitatively. Let $Q \subset \overline{Q} \subset U$ be a rectangle with sides parallel to the coordinate axes and $\rho\colon V \to [0,\infty]$ be a Borel function that is admissible for $g(\Gamma(Q))$.

Without loss of generality, suppose that $\Gamma(Q)$ is the family of curves joining the left to the right side of Q. For a.e. horizontal line segment $\gamma \in \Gamma(Q)$, $\gamma\colon [a,b] \to Q$, and for $[s,t] \subset [a,b]$ we have

$$\begin{aligned}|g(\gamma(t)) - g(\gamma(s))| &\leqslant \mathcal{H}^1_\infty(g(\gamma([s,t]))) = \mathcal{H}^1_\infty(g(\gamma([s,t]) \setminus G))\\ &\leqslant \int_{\gamma|_{[s,t]}} \rho_g\, ds < \infty,\end{aligned}$$

where use used the assumption on the set G, (4.1), and Fubini's theorem. This implies that (e.g., see [Väi71, Theorem 5.3])

$$\int_\gamma (\rho \circ g) \cdot \rho_g\, ds \geqslant \int_\gamma \rho\, ds \geqslant 1.$$

Therefore, $(\rho \circ g) \cdot \rho_g$ is admissible for a family containing a.e. horizontal line segment of $\Gamma(Q)$, which can be seen to have the same modulus as $\Gamma(Q)$ with a straightforward argument. Hence by (4.2),

$$\operatorname{mod}\Gamma(Q) \leqslant \int_U (\rho \circ g)^2 \cdot \rho_g^2\, dA \leqslant c(H) \int_V \rho^2\, dA.$$

This implies the desired (4.3) and proves the first part of the theorem.

If the set G is measurable then $g(G)$ is also measurable by quasiconformality. We set $f = g^{-1} = u + iv$, which is quasiconformal, and we have

$$\int_{g(G)} |\nabla v|\, dA = \int \mathcal{H}^1(v^{-1}(t) \cap g(G))\, dt$$

by the coarea formula for Sobolev functions [MSZ03]. By assumption, for a.e. $t \in \mathbb{R}$ we have $\mathcal{H}^1(v^{-1}(t) \cap g(G)) = 0$. Hence $|\nabla v|\chi_{g(G)} = 0$ a.e. By quasiconformality, we cannot have $|\nabla v| = 0$ on a set of positive measure, since this would imply that $J_f = 0$ on a set of positive measure. Hence, $\mathrm{Area}(g(G)) = 0$. Again, by quasiconformality, $\mathrm{Area}(G) = 0$. Therefore, if $H = 1$, then $E_g(x) = 1$ for a.e. $x \in U$. This now implies that f is conformal, as shown in [Nta23b, Lemma 2.5]. □

5. Regularity of limiting map

As in the first conclusion of Theorem 1.2, let $\Omega \subset \hat{\mathbb{C}}$ be a circle domain and let $(\Omega_j)_{j\in\mathbb{N}}$ be a quasiround exhaustion of Ω such that $(f_j)_{j\in\mathbb{N}}$ converges locally uniformly to a conformal homeomorphism f from Ω onto a circle domain $D \subset \hat{\mathbb{C}}$. In addition suppose that $\partial\Omega$ has 2-measure zero. Let $g = f^{-1}$ and $g_j = f_j^{-1}$, $j \in \mathbb{N}$. We consider the derivative $|Dg|\colon D \to (0,\infty)$ in the Riemannian metric of $\hat{\mathbb{C}}$. If $z, g(z) \in \mathbb{C}$, then

$$|Dg|(z) = \frac{1+|z|^2}{1+|g(z)|^2}|g'(z)|.$$

Throughout the section, we use the spherical metric σ and measure Σ on $\hat{\mathbb{C}}$, even if this is not explicitly stated. In particular, line integrals and 2-modulus are computed with respect to the spherical metric. Our main goal in this section is to show the next statement, under the above assumptions.

Proposition 5.1. *There exists a family of curves Γ_0 in $\hat{\mathbb{C}}$ with $\operatorname{mod}\Gamma_0 = 0$ such that for all curves $\gamma\colon [a,b] \to \hat{\mathbb{C}}$ outside Γ_0 with $\gamma(a), \gamma(b) \in D$ we have*

$$\sigma(g(\gamma(a)), g(\gamma(b))) \leqslant \int_\gamma |Dg|\chi_D\, ds + \sum_{\substack{q\in\mathcal{C}(D)\\ q\cap|\gamma|\neq\emptyset}} \operatorname{diam}\hat{g}(q).$$

We will need several preparatory statements. In the next statements, closed disks can be degenerate, i.e., they can have radius equal to zero.

Lemma 5.2 ([Nta23, Lemma 4.14])**.** *For each $n \in \mathbb{N}$, let $q_{i,n}$, $i \in I \cap \{1,\dots,n\}$, where $I \subset \mathbb{N}$, be a collection of pairwise disjoint closed disks in $\hat{\mathbb{C}}$. Suppose that there exists a collection of pairwise disjoint closed disks q_i, $i \in \mathbb{N}$, with the property that*

$$\lim_{n\to\infty} q_{i,n} = q_i$$

for each $i \in I$, in the Hausdorff sense. Then for each non-negative sequence $(\lambda_i)_{i\in I} \in \ell^2(\mathbb{N})$ there exists a family of curves Γ_0 in $\hat{\mathbb{C}}$ with $\operatorname{mod}\Gamma_0 = 0$ such that for all curves $\gamma \notin \Gamma_0$ we have

$$\limsup_{n\to\infty} \sum_{i:q_{i,n}\cap|\gamma|\neq\emptyset} \lambda_i \leqslant \sum_{i:q_i\cap|\gamma|\neq\emptyset} \lambda_i.$$

Lemma 5.3. *For each $n \in \mathbb{N}$, let $q_{i,n}$, $i \in I_n$, be a collection of pairwise disjoint closed disks on $\hat{\mathbb{C}}$ and $(\lambda_{i,n})_{i\in I_n}$ be a non-negative sequence with*

$$\lim_{n\to\infty} \sum_{i\in I_n} \lambda_{i,n}^2 = 0 \quad \Big(\textit{resp.} \sum_{i\in I_n} \lambda_{i,n}^2 < \infty \textit{ for each } n \in \mathbb{N}\Big).$$

Then there exists a family of curves Γ_0 *in* $\hat{\mathbb{C}}$ *with* $\operatorname{mod}\Gamma_0 = 0$ *such that for all curves* $\gamma \notin \Gamma_0$ *we have*

$$\lim_{n\to\infty} \sum_{i:q_{i,n}\cap|\gamma|\neq\emptyset} \lambda_{i,n} = 0 \quad \Big(\textit{resp.} \sum_{i:q_{i,n}\cap|\gamma|\neq\emptyset} \lambda_{i,n} < \infty \textit{ for each } n \in \mathbb{N}\Big).$$

Proof. Note that the conclusion is true for constant curves, as

$$\lim_{n\to\infty} \sup_{i\in I_n} \lambda_{i,n} = 0 \quad \big(\text{resp. } \sup_{i\in I_n} \lambda_{i,n} < \infty.\big) \tag{5.1}$$

Let $J_n \subset I_n$ be the set of indices $i \in I_n$ such that $\operatorname{diam} q_{i,n} > 0$. For $n \in \mathbb{N}$ define

$$\phi_n = \sum_{i\in J_n} \frac{\lambda_{i,n}}{\operatorname{diam} q_{i,n}} \chi_{2q_{i,n}},$$

where $2q_{i,n}$ denotes the disk with the same center as $q_{i,n}$ and twice the radius (in the spherical metric). By a variation of Bojarski's lemma ([Boj88], [Nta23, Lemma 2.7]), we have

$$\|\phi_n\|^2_{L^2(\hat{\mathbb{C}})} \leqslant C \sum_{i\in J_n} \lambda_{i,n}^2.$$

By assumption, we have $\phi_n \to 0$ in $L^2(\hat{\mathbb{C}})$ (resp. $\phi_n \in L^2(\hat{\mathbb{C}})$). Hence, by Fuglede's lemma [HKST15, p. 131], there exists a curve family Γ_1 with $\operatorname{mod}\Gamma_1 = 0$ such that

$$\lim_{n\to\infty} \int_\gamma \phi_n \, ds = 0 \quad \Big(\text{resp. } \int_\gamma \phi_n \, ds < \infty\Big) \tag{5.2}$$

for $\gamma \notin \Gamma_1$. Let $\gamma \notin \Gamma_1$ be a non-constant curve. Observe that there exists $N \in \mathbb{N}$, depending on γ, such that the set

$$K_n = \{i \in J_n : \operatorname{diam}(2q_{i,n}) \geqslant \operatorname{diam}(|\gamma|) > 0\}$$

has at most N elements; to see this, compare the area of $\bigcup_{i\in K_n} q_{i,n}$ with the area of $\hat{\mathbb{C}}$. Thus, for $i \in J_n \setminus K_n$ with $q_{i,n} \cap |\gamma| \neq \emptyset$ we have

$$\lambda_{i,n} \leqslant \int_\gamma \frac{\lambda_{i,n}}{\operatorname{diam} q_{i,n}} \chi_{2q_{i,n}} \, ds.$$

It follows that

$$\sum_{\substack{i\in J_n\setminus K_n \\ q_{i,n}\cap|\gamma|\neq\emptyset}} \lambda_{i,n} \leqslant \int_\gamma \phi_n \, ds.$$

Also,

$$\sum_{\substack{i\in K_n \\ q_{i,n}\cap|\gamma|\neq\emptyset}} \lambda_{i,n} \leqslant N \sup_{i\in I_n} \lambda_{i,n}.$$

By (5.1) and (5.2), we conclude that

$$\lim_{n\to\infty} \sum_{\substack{i\in J_n \\ q_{i,n}\cap|\gamma|\neq\emptyset}} \lambda_{i,n} = 0 \quad \Big(\text{resp. } \sum_{i:q_{i,n}\cap|\gamma|\neq\emptyset} \lambda_{i,n} < \infty\Big)$$

whenever $\gamma \notin \Gamma_1$ and γ is non-constant.

Finally, note that the family Γ_2 of non-constant curves passing through the countably many points $q_{i,n}$, $i \in I_n \setminus J_n$, has 2-modulus zero [Väi71, §7.9]. This implies the statement for $\Gamma_0 = \Gamma_1 \cup \Gamma_2$. $\square$

In the next lemma, $g_j = f_j^{-1} : D_j \to \Omega_j$, $j = 1, 2, \dots$, are the maps defined in the beginning of this section.

Lemma 5.4. *If we pass to a subsequence of $(g_j)_{j\in\mathbb{N}}$, there exists a family of curves Γ_0 in $\hat{\mathbb{C}}$ with $\operatorname{mod}\Gamma_0 = 0$ such that for all curves $\gamma \notin \Gamma_0$ we have*

$$\limsup_{j\to\infty} \sum_{\substack{q\in\mathcal{C}(D_j)\\ q\cap|\gamma|\neq\emptyset}} \operatorname{diam} \hat{g}_j(q) \leqslant \sum_{\substack{q\in\mathcal{C}(D)\\ q\cap|\gamma|\neq\emptyset}} \operatorname{diam} \hat{g}(q).$$

Proof. We enumerate the components of $\mathcal{C}_N(\Omega)$ as p_i, $i \in I$, where $I \subset \mathbb{N}$, and let $q_i \in \mathcal{C}(D)$ be such that $\hat{g}(q_i) = p_i$. Let $n \in \mathbb{N}$ and consider a large enough $j(n) \geqslant n$ so that for $i \in I \cap \{1, \dots, n\}$ there are components $p_{i,j(n)} \in \mathcal{C}(\Omega_{j(n)})$ that are pairwise disjoint and

$$p_i \subset p_{i,j(n)} \subset (1 + 1/n)p_i.$$

Consider $q_{i,n} \in \mathcal{C}(D_{j(n)})$ such that $\hat{g}_{j(n)}(q_{i,n}) = p_{i,j(n)}$.

For each $i \in I$, by Theorem 3.1, $q_{i,n}$ converges to q_i as $n \to \infty$. By Lemma 5.2, we have

$$\limsup_{n\to\infty} \sum_{i:q_{i,n}\cap|\gamma|\neq\emptyset} \operatorname{diam} \hat{g}(q_i) \leqslant \sum_{i:q_i\cap|\gamma|\neq\emptyset} \operatorname{diam} \hat{g}(q_i) \leqslant \sum_{\substack{q\in\mathcal{C}(D)\\ q\cap|\gamma|\neq\emptyset}} \operatorname{diam} \hat{g}(q)$$

for all curves γ outside a family Γ_0 of 2-modulus zero. Also,

$$\operatorname{diam} \hat{g}_{j(n)}(q_{i,n}) = \operatorname{diam} p_{i,j(n)} \leqslant (1 + 1/n) \operatorname{diam} \hat{g}(q_i)$$

so we obtain

$$\limsup_{n\to\infty} \sum_{i:q_{i,n}\cap|\gamma|\neq\emptyset} \operatorname{diam} \hat{g}_{j(n)}(q_{i,n}) \leqslant \sum_{\substack{q\in\mathcal{C}(D)\\ q\cap|\gamma|\neq\emptyset}} \operatorname{diam} \hat{g}(q).$$

It remains to treat the sum of $\lambda_n(q) = \operatorname{diam} \hat{g}_{j(n)}(q)$ over $q \in I_n = \mathcal{C}(D_{j(n)}) \setminus \{q_{i,n} : i \in I \cap \{1, \dots, n\}\}$ with $q \cap |\gamma| \neq \emptyset$, and show that the limit is zero for all curves γ outside another exceptional family of 2-modulus zero. This is an immediate consequence of Lemma 5.3, upon verifying that

$$\lim_{n\to\infty} \sum_{q\in I_n} \lambda_n(q)^2 = 0.$$

Since $\partial\Omega$ has area zero, for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\Sigma(N_\delta(\partial\Omega)) < \varepsilon.$$

We claim that there exists $N \in \mathbb{N}$ such that for $n > N$ and $q \in I_n$ we have $\hat{g}_{j(n)}(q) \subset N_\delta(\partial\Omega)$. Assuming this, since $(\Omega_j)_{j\in\mathbb{N}}$ is a quasiround exhaustion of Ω, we have

$$\sum_{q\in I_n} \lambda_n(q)^2 \leqslant C \sum_{q\in I_n} \Sigma(\hat{g}_{j(n)}(q)) \leqslant C\Sigma(N_\delta(\partial\Omega)) < C\varepsilon$$

for $n > N$. This completes the proof.

Now we prove the claim. Since $(\Omega_j)_{j\in\mathbb{N}}$ is an exhaustion of Ω, there exists $N_0 \in \mathbb{N}$ such that $\hat{g}_{j(n)}(q) \subset N_\delta(\hat{\mathbb{C}} \setminus \Omega)$ for $q \in \mathcal{C}(D_{j(n)})$ and $n > N_0$. In particular,

$$\hat{g}_{j(n)}(q) \cap \overline{\Omega} \subset N_\delta(\partial\Omega) \quad \text{for } q \in \mathcal{C}(D_{j(n)}) \text{ and } n > N_0.$$

Also, since Ω is a circle domain, there exists $N > N_0$ such that if $i \in I$ and $i > N$, then $\operatorname{diam} p_i < \delta$. If $q \in I_n$, then $\hat{g}_{j(n)}(q)$ does not intersect $p_1, \dots, p_n$. Hence, if $n > N$ and $q \in I_n$, then each point $z \in \hat{g}_{j(n)}(q) \setminus \overline{\Omega}$ lies in some $p_{i(z)}$ with $i(z) > n > N$. By our choice of N we have $\operatorname{diam} p_{i(z)} < \delta$, and therefore $p_{i(z)} \subset N_\delta(\partial p_{i(z)}) \subset N_\delta(\partial\Omega)$. This completes the proof of the claim. □

Proof of Proposition 5.1. Recall that $g_j = f_j^{-1}\colon D_j \to \Omega_j$, $j \in \mathbb{N}$. Since g_j is a conformal map between finitely connected domains, we have

$$\sigma(g_j(\gamma(a)), g_j(\gamma(b))) \leqslant \int_\gamma |Dg_j|\chi_{D_j}\, ds + \sum_{\substack{q\in\mathcal{C}(D_j)\\ q\cap|\gamma|\neq\emptyset}} \operatorname{diam} \hat{g}_j(q) \tag{5.3}$$

for every rectifiable curve $\gamma\colon [a,b] \to \hat{\mathbb{C}}$ with $\gamma(a), \gamma(b) \in D_j$.

Since $g_j \to g$ locally uniformly in D, we have $|Dg_j| \to |Dg|$ locally uniformly in D. In fact, $|Dg_j|\chi_{D_j}$ also converges strongly in $L^2(\hat{\mathbb{C}})$ to $|Dg|\chi_D$. To see this, let $\varepsilon > 0$ and $K \subset \Omega$ be a compact set with $\Sigma(\Omega \setminus K) < \varepsilon$. Then by kernel convergence, $f_j(K)$ is contained in a compact set K' that is contained in D_j for all large j. We have that $|Dg_j|\chi_{K'} \to |Dg|\chi_{K'}$ strongly in $L^2(\hat{\mathbb{C}})$ and

$$\int |Dg_j|^2 \chi_{D_j\setminus K'}\, d\Sigma \leqslant \Sigma(\Omega_j \setminus K) \leqslant \Sigma(\Omega \setminus K) < \varepsilon$$

for large j. The integral of $|Dg|^2\chi_{D\setminus K'}$ is also less than ε. This implies the claim regarding strong convergence.

By Fuglede's lemma [HKST15, p. 131], there exists a curve family Γ_1 of 2-modulus zero such that for $\gamma \notin \Gamma_1$ we have

$$\int_\gamma |Dg_j|\chi_{D_j}\, ds \to \int_\gamma |Dg|\chi_D\, ds.$$

By Lemma 5.4, if we pass to a subsequence, there exists a curve family Γ_2 of 2-modulus zero such that for $\gamma \notin \Gamma_2$ we have

$$\limsup_{j\to\infty} \sum_{\substack{q\in\mathcal{C}(D_j)\\ q\cap|\gamma|\neq\emptyset}} \operatorname{diam} \hat{g}_j(q) \leqslant \sum_{\substack{q\in\mathcal{C}(D)\\ q\cap|\gamma|\neq\emptyset}} \operatorname{diam} \hat{g}(q).$$

For $\gamma \notin \Gamma_0 = \Gamma_1 \cup \Gamma_2$ we combine the above with (5.3) to obtain the desired conclusion. □

6. Conformal extension to the sphere

In this ultimate section we complete the proof of Theorem 1.2 by proving the following more general result.

THEOREM 6.1. *Let $g\colon D \to \Omega$ be a conformal homeomorphism between circle domains $D, \Omega \subset \hat{\mathbb{C}}$ that satisfies the conclusion of Proposition 5.1. Then g is the restriction of a Möbius transformation of $\hat{\mathbb{C}}$.*

The proof follows closely the proof of [Nta23b, Theorem 1.2]. Let $g\colon D \to \Omega$ be a map as in Theorem 6.1. We establish some preliminary statements.

Lemma 6.2. *The map g has an extension to a homeomorphism from $\overline{D}$ onto $\overline{\Omega}$.*

Proof. The conclusion of Proposition 5.1 is exactly the same as the conclusion of [Nta23b, Theorem 3.1]. This conclusion is the main assumption for the considerations in [Nta23b, Section 4], which imply that g is a *packing-conformal map* in the sense of [Nta23]. As shown in [Nta23b, Section 4] this implies that g has a homeomorphic extension to the closures; see also Theorems 6.1 and 7.1 in [Nta23]. □

Lemma 6.3. *For each (anti-)Möbius transformation $T\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and for a.e. horizontal and a.e. vertical line segment γ in $\mathbb{C}$ we have*

$$\mathcal{H}^1((g \circ T^{-1})(|\gamma| \cap T(\partial D))) = 0.$$

Proof. By Lemma 6.2, g has a homeomorphic extension to the closure of D. This implies that

$$\operatorname{diam} \hat{g}(q) = 0 \text{ for } q \in \mathcal{C}(D) \setminus \mathcal{C}_N(D). \tag{6.1}$$

Let Γ_0 be the curve family with $\operatorname{mod} \Gamma_0 = 0$ given by Proposition 5.1. Then $\operatorname{mod} T(\Gamma_0) = 0$. Therefore, for a.e. horizontal (resp. vertical) line segment $\gamma\colon [a,b] \to \mathbb{C}$ (with an injective parametrization), the inequality in Proposition 5.1 is true for $\widetilde{\gamma} = T^{-1} \circ \gamma$ and for all of its subcurves. Moreover, by the very last inequality of Lemma 5.3, and upon enlarging the exceptional curve family Γ_0 if necessary, we may assume that the right-hand side of the inequality of Proposition 5.1 is finite for such $\widetilde{\gamma}$. Finally, if we enlarge again Γ_0, we may have $\mathcal{H}^1(|\widetilde{\gamma}| \cap \partial q) = 0$ for each $q \in \mathcal{C}(D)$. In particular, the circular arc $|\widetilde{\gamma}|$ intersects each circle ∂q in at most two points.

Let $A \subset \mathcal{C}_N(D)$ be a finite set and $B \subset D$ be a compact set. The set $[a,b] \setminus \widetilde{\gamma}^{-1}(B \cup \bigcup\{q : q \in A\})$ is a countable union of disjoint intervals O_j,

$j \in J$. Let $\gamma_j = \widetilde{\gamma}|_{O_j}$, $j \in J$, and observe that

$$|\widetilde{\gamma}| \cap \partial D \subset \left(\bigcup_{j \in J} |\gamma_j| \cap \partial D \right) \cup \left(\bigcup_{q \in A} \partial q \cap |\widetilde{\gamma}| \right),$$

where the latter union is a finite set. Therefore,

$$\mathcal{H}^1_\infty(g(|\widetilde{\gamma}| \cap \partial D)) \leqslant \sum_{j \in J} \operatorname{diam} g(|\gamma_j| \cap \partial D).$$

Note that the curves γ_j, $j \in J$, are pairwise disjoint subarcs of a circle in $\hat{\mathbb{C}}$ and each disk $q \in \mathcal{C}_N(D)$ can intersect at most one of them. Applying the inequality of Proposition 5.1 to each γ_j, $j \in J$, and using (6.1), we obtain

$$\begin{aligned} \mathcal{H}^1_\infty(g(|\widetilde{\gamma}| \cap \partial D)) &\leqslant \sum_{j \in J} \left(\int_{\gamma_j} |Dg| \chi_D \, ds + \sum_{\substack{q \in \mathcal{C}_N(D) \\ q \cap |\gamma_j| \neq \emptyset}} \operatorname{diam} \hat{g}(q) \right) \\ &\leqslant \int_{\widetilde{\gamma}} |Dg| \chi_{D \setminus B} \, ds + \sum_{\substack{q \in \mathcal{C}_N(D) \setminus A \\ q \cap |\widetilde{\gamma}| \neq \emptyset}} \operatorname{diam} \hat{g}(q). \end{aligned}$$

As the set A increases to $\mathcal{C}_N(D)$ and the set B increases to D, the right-hand side converges to zero by dominated convergence. Hence

$$0 = \mathcal{H}^1_\infty(g(|\widetilde{\gamma}| \cap \partial D)) = \mathcal{H}^1(g(|\widetilde{\gamma}| \cap \partial D)) = \mathcal{H}^1((g \circ T^{-1})(|\gamma| \cap T(\partial D))).$$

This completes the proof. □

Proof of Theorem 6.1. Without loss of generality we assume that $\infty \in D$ and $g(\infty) = \infty$. With the aid of Lemma 6.2, we extend g to a homeomorphism of $\hat{\mathbb{C}}$ through reflections across the boundary circles of D. A detailed proof can be found in [NY20, Section 7.1]. Here we highlight the important features of the extension procedure.

We denote by S_i, $i \in I$, the collection of circles in ∂D, by $B_i \subset \hat{\mathbb{C}} \setminus \overline{D}$ the open ball bounded by S_i, and by R_i the reflection across the circle S_i, $i \in I$. Here, we regard I as a subset of $\mathbb{N}$. Consider the free discrete group generated by the family of reflections $\{R_i : i \in I\}$. This is called the *Schottky group* of D and is denoted by $\Gamma(D)$. Each $T \in \Gamma(D)$ that is not the identity can be expressed uniquely as $T = R_{i_1} \circ \cdots \circ R_{i_k}$, where $i_j \neq i_{j+1}$ for $j \in \{1, \dots, k-1\}$. We also note that $\Gamma(D)$ contains countably many elements.

By Lemma 6.2, g extends to a homeomorphism between $\overline{D}$ and $\overline{\Omega}$. Hence, there exists a natural bijection between $\Gamma(D)$ and $\Gamma(\Omega)$, induced by g. Namely, if R_i^* is the reflection across the circle $S_i^* = g(S_i)$, then for $T = R_{i_1} \circ \cdots \circ R_{i_k}$ we define $T^* = R_{i_1}^* \circ \cdots \circ R_{i_k}^*$. By [NY20, Lemma 7.5], there exists a unique extension of g to a homeomorphism $\widetilde{g}$ of $\hat{\mathbb{C}}$ with the property that $T^* = \widetilde{g} \circ T \circ \widetilde{g}^{-1}$ for each $T \in \Gamma(D)$. We will verify that $\widetilde{g}$ is conformal. For simplicity, we use the notation g instead of $\widetilde{g}$.

For each point $x \in \hat{\mathbb{C}}$ we have the following trichotomy; see Lemma 7.2 and Corollary 7.4 in [NY20].

(I) (Interior type) $x \in T(D)$ for some $T \in \Gamma(D)$.
(II) (Boundary type) $x \in T(\partial D)$ for some $T \in \Gamma(D)$.
(III) (Buried type) There exists a sequence of indices $(i_j)_{j\in\mathbb{N}}$ with $i_j \neq i_{j+1}$ and disks $D_0 = B_{i_1}$, $D_k = R_{i_1} \circ \cdots \circ R_{i_k}(B_{i_{k+1}})$ such that $D_{k+1} \subset D_k$ for each $k \geqslant 0$ and $\{x\} = \bigcap_{k=0}^{\infty} D_k$.

At each point x of interior type (I) the map g is conformal, so it maps infinitesimal balls centered at x to infinitesimal balls centered at $g(x)$. In particular, $E_g(x) = 1$; recall the definition from Section 4. If x is of buried type (III), then there exists a sequence of balls D_k, $k \in \mathbb{N}$, shrinking to x such that $g(D_k)$, $k \in \mathbb{N}$, are balls shrinking to $g(x)$. It follows that $E_g(x) = 1$.

Finally, we treat points of boundary type (II). By Lemma 6.3, for each $T \in \Gamma(D)$ and for a.e. horizontal and a.e. vertical line segment γ in $\mathbb{C}$, we have

$$\mathcal{H}^1((g \circ T^{-1})(|\gamma| \cap T(\partial D))) = 0.$$

Since $(T^{-1})^* \circ g = g \circ T^{-1}$, and $(T^{-1})^*$ is bi-Lipschitz, we obtain

$$\mathcal{H}^1(g(|\gamma| \cap T(\partial D))) = \mathcal{H}^1(((T^{-1})^* \circ g)(|\gamma| \cap T(\partial D))) = 0.$$

Therefore, for a.e. horizontal and a.e. vertical line segment γ in $\mathbb{C}$ we have

$$\mathcal{H}^1(g(|\gamma| \cap G)) = 0$$

where G is the countable union $\bigcup_{T\in\Gamma(D)} T(\partial D)$. Observe that the Euclidean metric is comparable to the spherical metric on G so the above statement is true with respect to either metric. By Theorem 4.1, since G is measurable, we conclude that g is conformal in $\hat{\mathbb{C}}$. □

7. No infinitesimally round exhaustions

It would be useful to have an improvement of Theorem 2.1 in which quasiround exhaustions of a circle domain $\Omega \subset \hat{\mathbb{C}}$ are replaced with *infinitesimally round* exhaustions $(\Omega_j)_{j\in\mathbb{N}}$ for which each Ω_j is K_j-quasiround and $K_j \to 1$ as $j \to \infty$. However, such an improvement does not hold in general. We construct circle domains which do not admit infinitesimally round exhaustions.

Consider the closed disk centered at $(0, 2)$ with radius 1 and the closed disk centered at $(0, 2/3)$ with radius $1/3$. The two disks are tangent to each other and we call them *central*. We also consider two *lateral* disks of radius $2/9$ that are tangent to both central disks as shown in Figure 4. We consider scaled copies of all those disks so that we obtain a packing as in the figure, with central disks of radii $\dots, 9, 3, 1, 1/3, 1/9, \dots$ and lateral disks of radii $\dots, 6, 2, 2/3, 2/9, 2/27, \dots$. The choice of the radii guarantees that

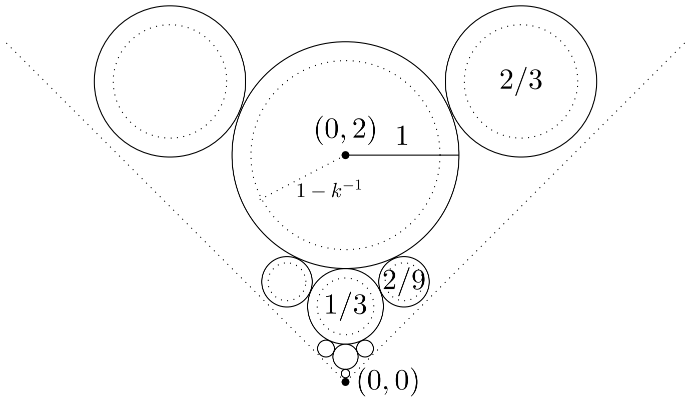

FIGURE 4. Construction of the domain G_k.

all disks are contained in the region $|x| < y$. We rotate this packing by multiples of $\pi/2$ so that we obtain four disjoint packings, each contained in a complementary region of the lines $y = |x|$.

We denote by D_m, $m \in \mathbb{Z}$, the family of all the closed disks chosen above. Moreover, let

$$\begin{aligned} G_k &= \mathbb{C} \setminus \Big(\{(0,0)\} \cup \bigcup_{m\in\mathbb{Z}} (1-k^{-1})D_m\Big), \quad k = 1, 2, \dots, \\ G_\infty &= \mathbb{C} \setminus \Big(\{(0,0)\} \cup \bigcup_{m\in\mathbb{Z}} D_m\Big). \end{aligned}$$

Recall that if $D = \overline{\mathbb{D}}(a, r)$ then $\delta D = \overline{\mathbb{D}}(a, \delta r)$. See the dotted disks in Figure 4 for the construction of the domains G_k and the larger disks for the construction of G_∞ The complementary components of G_k in $\mathbb{C}$ are by definition the point $(0, 0)$ and the disks $(1 - k^{-1})D_m$, so G_k is a circle domain.

We claim that there exists $k_0 \in \mathbb{N}$ so that if $k \geqslant k_0$ then the circle domain G_k does not have infinitesimally round exhaustions. Suppose towards contradiction that there is a subsequence $(G_{k_\ell})_{\ell\in\mathbb{N}} =: (U_\ell)_{\ell\in\mathbb{N}}$ of $(G_k)_{k\in\mathbb{N}}$ so that each U_ℓ has an infinitesimally round exhaustion. It follows that for each $\ell \in \mathbb{N}$ there are a closed disk $\overline{\mathbb{D}}(a_\ell, R_\ell)$ and a Jordan curve $\tilde{J}_\ell \subset U_\ell$ separating $(0, 0)$ from ∞ that bounds a Jordan region W_ℓ so that

$$0 < R_\ell < \ell^{-1} \quad \text{and} \quad \overline{\mathbb{D}}(a_\ell, R_\ell) \subset W_\ell \subset \overline{\mathbb{D}}(a_\ell, (1 + \ell^{-1})R_\ell). \tag{7.1}$$

We scale each $\tilde{J}_\ell$ by 3^{s_ℓ}, $s_\ell \in \mathbb{N}$, so that the diameter of the scaled curve J_ℓ satisfies $1/3 < \operatorname{diam} J_\ell \leqslant 1$ and J_ℓ separates $(0, 0)$ from ∞. Notice that $J_\ell \subset U_\ell$ by the scaling invariance of U_ℓ.

After possibly taking a subsequence, the curves J_ℓ converge in the Hausdorff sense. By (7.1), the limit is a circle that we denote by $\mathbb{S}_\infty$ and satisfies $1/3 \leqslant \operatorname{diam} \mathbb{S}_\infty \leqslant 1$. Since each J_ℓ surrounds the origin, we have $(0, 0) \in D_\infty$, where D_∞ is the closed disk bounded by $\mathbb{S}_\infty$. Moreover, since each J_ℓ is a subset of U_ℓ, we have $\mathbb{S}_\infty \subset \overline{G_\infty}$.

It follows that there exists $b_0 \in \mathbb{S}_\infty$ that lies in one of the coordinate axes but is not the origin. Since $\mathbb{S}_\infty \subset \overline{G_\infty}$, b_0 is a point where two central disks meet. We denote these disks by D_1 and D_2 and assume that D_1 is the one with larger diameter. Since D_∞ contains the origin and the distance of D_1 to the origin is equal to its diameter, we conclude that

$$\operatorname{diam} D_\infty \geqslant \operatorname{dist}((0,0), D_1) = \operatorname{diam} D_1 > \operatorname{diam} D_2.$$

Consider the lateral disks D_3, D_4 that are tangent to D_1 and D_2. Since $\mathbb{S}_\infty \subset \overline{G_\infty}$, there exist distinct points $b_1, b_2 \in \mathbb{S}_\infty$ so that b_1 lies in D_3 and in one of D_1 or D_2, and b_2 lies in D_4 and in one of D_1 or D_2.

We now arrive at a contradiction by considering all possible cases. First, if b_0, b_1, b_2 all lie on the boundary of D_1, then $\mathbb{S}_\infty = \partial D_1$. This is a contradiction since D_∞ contains the origin. Next, if b_1 (resp., b_2) lies on the boundary of D_2, then since $\operatorname{diam} \mathbb{S}_\infty > D_2$, the shorter subarc of $\mathbb{S}_\infty$ connecting b_0 an b_1 (resp., b_0 and b_2) passes through the interior of D_2. This is a contradiction because $\mathbb{S}_\infty$ is a subset of $\overline{G_\infty}$. The proof is complete.

References

[Boj88] B. Bojarski, *Remarks on Sobolev imbedding inequalities*, Complex analysis, Joensuu 1987, Lecture Notes in Math., Springer, Berlin, 1988, pp. 52–68.

[Bon11] M. Bonk, *Uniformization of Sierpiński carpets in the plane*, Invent. Math. **186** (2011), no. 3, 559–665.

[Bon16] M. Bonk, *Uniformization by square domains*, J. Anal. **24** (2016), no. 1, 103–110.

[Cou50] R. Courant, *Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces*, Interscience Publishers, Inc., New York, N.Y., 1950. Appendix by M. Schiffer.

[Dav86] R. J. Daverman, *Decompositions of manifolds*, Pure and Applied Mathematics, vol. 124, Academic Press, Inc., Orlando, FL, 1986.

[GV61] F. W. Gehring and J. Väisälä, *On the geometric defintion for quasiconformal mappings*, Comment. Math. Helv. **36** (1961), 19–32.

[Hei01] J. Heinonen, *Lectures on analysis on metric spaces*, Universitext, Springer-Verlag, New York, 2001.

[HK90] D. A. Herron and P. Koskela, *Quasiextremal distance domains and conformal mappings onto circle domains*, Complex Variables Theory Appl. **15** (1990), no. 3, 167–179.

[HKST15] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. T. Tyson, *Sobolev spaces on metric measure spaces: An approach based on upper gradients*, New Mathematical Monographs, vol. 27, Cambridge University Press, Cambridge, 2015.

[HS93] Z.-X. He and O. Schramm, *Fixed points, Koebe uniformization and circle packings*, Ann. of Math. (2) **137** (1993), no. 2, 369–406.

[HS94] Z.-X. He and O. Schramm, *Rigidity of circle domains whose boundary has σ-finite linear measure*, Invent. Math. **115** (1994), no. 2, 297–310.

[HS95] Z.-X. He and O. Schramm, *Koebe uniformization for "almost circle domains"*, Amer. J. Math. **117** (1995), no. 3, 653–667.

[HvdM09] S. Hildebrandt and H. von der Mosel, *Conformal mapping of multiply connected Riemann domains by a variational approach*, Adv. Calc. Var. **2** (2009), no. 2, 137–183.

[Koe08] P. Koebe, *Über die Uniformisierung beliebiger analytischer Kurven III*, Nachr, Ges. Wiss. Gott. (1908), 337–358.

[Kur68] K. Kuratowski, *Topology. Vol. II*, New, Academic Press, New York-London; Państwowe Wydawnictwo Naukowe [Polish Scientific Publishers], Warsaw, 1968. Translated from the French by A. Kirkor.

[Moo25] R. L. Moore, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc. **27** (1925), no. 4, 416–428.

[MSZ03] J. Malý, D. Swanson, and W. P. Ziemer, *The co-area formula for Sobolev mappings*, Trans. Amer. Math. Soc. **355** (2003), no. 2, 477–492.

[Nta23a] D. Ntalampekos, *Conformal uniformization of planar packings by disk packings*, Adv. Math. **428** (2023), Paper No. 109159, pp. 58.

[Nta23b] D. Ntalampekos, *Rigidity and continuous extension for conformal maps of circle domains*, Trans. Amer. Math. Soc. **376** (2023), no. 7, 5221–5239.

[Nta] D. Ntalampekos, *Metric definition of quasiconformality and exceptional sets*, Math. Ann., to appear.

[NY20] D. Ntalampekos and M. Younsi, *Rigidity theorems for circle domains*, Invent. Math. **220** (2020), no. 1, 129–183.

[Raj] K. Rajala, *Uniformization of planar domains by exhaustion*, Amer. J. Math., to appear.

[Sch95] O. Schramm, *Transboundary extremal length*, J. Anal. Math. **66** (1995), 307–329.

[SV20] A. Yu. Solynin and N. C. Vidanage, *Uniformization by rectangular domains: a path from slits to squares*, J. Math. Anal. Appl. **486** (2020), no. 2, 123927, 12.

[Väi71] J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Mathematics, vol. 229, Springer-Verlag, Berlin-New York, 1971.

[You16] M. Younsi, *Removability, rigidity of circle domains and Koebe's conjecture*, Adv. Math. **303** (2016), 1300–1318.

Mathematics Department, Stony Brook University, Stony Brook, NY 11794, USA.

Email address: dimitrios.ntalampekos@stonybrook.edu

Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), FI-40014, University of Jyväskylä, Finland.

Email address: kai.i.rajala@jyu.fi