UNIFORMIZATION OF PLANAR DOMAINS BY EXHAUSTION

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Abstract. We study the method of finding conformal maps onto circle domains by approximating with finitely connected subdomains. Every domain $D \subset \hat{\mathbb{C}}$ admits *exhaustions*, i.e., increasing sequences of finitely connected subdomains D_j whose union is D. By Koebe's theorem, each D_j admits a conformal map f_{D_j} from D_j onto a circle domain $f_{D_j}(D_j)$. Assuming $f_{D_j} \to f$, our goal is to find out if $f(D)$ is also a circle domain.

We present a countably connected D with an exhaustion (D_j) so that (f_{D_j}) has a limit whose image is *not a circle domain*, and a domain Ω with an exhaustion (Ω_j) so that (f_{Ω_j}) has a limit whose image has *uncountably many non-point complementary components*.

On the other hand, we prove that every exhaustion (D_j) of a countably connected D admits a *refinement* so that the image of the corresponding limit map is a circle domain. Our result extends the He-Schramm theorem on the uniformization of countably connected domains and provides a new proof.

1. Introduction

1.1. **Background.** The long-standing *Koebe conjecture* [15] predicts that every domain $D \subset \hat{\mathbb{C}}$ admits a conformal map onto a *circle domain*, i.e., a domain whose set of complementary components consists of closed disks and points. See [10] for an overview. Koebe himself proved this to be the case for finitely connected domains, cf. [7, Theorem 5.1]. Koebe's theorem has been extended to cover finitely connected targets with varying boundary shapes, the most general results being those by Brandt [5] and Harrington [9]. See [20] for further information.

A major breakthrough was made by He and Schramm [10], who showed that the Koebe conjecture holds for countably connected domains. Soon after Schramm [19] introduced the *transboundary extremal length* (or *transboundary modulus*), and applied it to give a simplified proof to the He-Schramm theorem as well as a generalization to uncountably connected "co-fat" domains. See also [11], [12], [13]. Recently, results related to the Koebe conjecture have been established in [2], [14], [16], [18], [21], and [22].

The proofs by He-Schramm and Schramm apply approximation of a given domain from *outside* by a decreasing sequence of finitely connected domains together with Koebe's theorem to construct a sequence of conformal maps whose limit has circle domain image. In this paper, we study a modification of this method where a given domain is approximated from *inside* by *exhaustions*, i.e., increasing sequences of finitely connected subdomains.

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Our approach is motivated by the fact that exhaustions offer more flexibility than approximations from outside. They can potentially be applied to gain a better understanding of the Koebe conjecture and related problems. The challenge is in finding exhaustions with the desired properties among all the exhaustions of a given domain.

Theorems 1.1 and 1.2 below show that an arbitrary exhaustion does not work in general; the image of the limit map is not always a circle domain. However, our main result, Theorem 1.3, shows that any exhaustion of a countably connected domain admits a *refinement* so that the image of the corresponding limit map is a circle domain. We now describe our results in detail.

1.2. **Main results.** An *exhaustion* Φ of a domain $D \subset \hat{\mathbb{C}}$ is a sequence of domains $D_j \subset D$, each bounded by finitely many disjoint Jordan curves in D, such that

$$D_j \subset D_{j+1} \text{ for all } j = 1, 2, \dots \text{ and } D = \cup_j D_j.$$

We fix disjoint points $a_0, a_1, a_2 \in D_1$. Then by Koebe's theorem there are unique conformal maps $f_j : D_j \to \tilde{D}_j$ onto circle domains $\tilde{D}_j \subset \hat{\mathbb{C}}$ so that $f_j(a_k) = a_k$ for $k = 0, 1, 2$. The sequence (f_j) has a subsequence converging locally uniformly to a conformal $f : D \to f(D)$. We denote

$$\mathcal{F}_\Phi = \{f : D \to f(D) : f \text{ is the limit of a subsequence of } (f_j)\}.$$

If $\mathcal{F}_\Phi$ contains only one map f, i.e., if (f_j) converges, we denote $f = f_\Phi$. The use of this notation always contains the implicit assumption that $f_j \to f_\Phi$.

THEOREM 1.1. *There is a countably connected domain $D \subset \hat{\mathbb{C}}$ with exhaustion Φ such that $f_\Phi(D)$ is not a circle domain.*

We denote the set of complementary components of domain G by $\mathcal{C}(G)$. We say that $p \in \mathcal{C}(G)$ is *non-trivial* if $\operatorname{diam}(p) > 0$.

THEOREM 1.2. *There is a domain $D \subset \hat{\mathbb{C}}$ with exhaustion Φ such that $\mathcal{C}(f_\Phi(D))$ contains uncountably many non-trivial elements.*

Theorems 1.1 and 1.2 are in sharp contrast to [7, Theorem 2.1] on *slit domains*, i.e., domains whose sets of complementary components consist of vertical segments and points; if Φ is an exhaustion of D and if the targets $\tilde{D}_j$ above are slit domains so that $f_j \to f$, then $f(D)$ is always a slit domain.

In view of Theorems 1.1 and 1.2, in order to produce a limit map onto a circle domain it is necessary to modify, or refine, a given exhaustion. Let $\Phi = (D_j)$ and $\Phi' = (D'_j)$ be exhaustions of D. We say that Φ is a *refinement* of Φ', if every $p \in \mathcal{C}(D_j)$ is an element of $\mathcal{C}(D'_{j(p)})$ for some $j(p) \geqslant j$. Our main result reads as follows.

THEOREM 1.3. *Every exhaustion of a countably connected domain $D \subset \hat{\mathbb{C}}$ has a refinement Φ such that $f_\Phi(D)$ is a circle domain.*

Since every domain admits an exhaustion, Theorem 1.3 gives a new proof to the He-Schramm theorem. Our main tools are transfinite induction, which was also used by He-Schramm and Schramm, and Schramm's transboundary modulus.

2. Proof of Theorem 1.3

Let $G \subset \hat{\mathbb{C}}$ be a domain and $\hat{G} = \hat{\mathbb{C}}/\sim$, where

$$x \sim y \text{ if either } x = y \in G \text{ or } x, y \in p \text{ for some } p \in \mathcal{C}(G).$$

The corresponding quotient map is $\pi_G : \hat{\mathbb{C}} \to \hat{G}$. Identifying each $x \in G$ and $p \in \mathcal{C}(G)$ with $\pi_G(x)$ and $\pi_G(p)$, respectively, we have

$$\hat{G} = G \cup \mathcal{C}(G).$$

A homeomorphism $f : G \to G'$ has a homeomorphic extension $\hat{f} : \hat{G} \to \hat{G}'$.

Let $\Phi' = (D'_j)$ be an exhaustion of a countably connected domain D. We consider the following property: If $q_1 \in \mathcal{C}(D'_{j_1})$ and $q_2 \in \mathcal{C}(D'_{j_2})$, $j_1 \geqslant j_2$, and if $q_1 \cap q_2 \neq \emptyset$, then

$$\text{either } q_1 = q_2 \text{ or } q_1 \text{ lies in the interior of } q_2. \tag{1}$$

It is not difficult to see that any exhaustion Φ'' of D has a refinement Φ' satisfying (1). Since any refinement of Φ' is also a refinement of Φ'', we conclude that it suffices to prove Theorem 1.3 for exhaustions satisfying (1).

We prove Theorem 1.3 using transfinite induction (cf. [6]) and the following result. In this paper, *we allow closed disks to have zero diameter.* For instance, in the following proposition a disk $q \in \mathcal{C}(D)$ may be a point component.

Proposition 2.1. *Let $D \subset \hat{\mathbb{C}}$ be a countably connected domain. Fix an exhaustion $\Phi' = (D'_j)$ of D satisfying* (1)*, $p \in \mathcal{C}(D)$, and an open neighborhood U of p in $\hat{\mathbb{C}}$ such that $\overline{U} \in \mathcal{C}(D'_n)$ for some index n. Moreover, suppose every $f \in \mathcal{F}_{\Phi'}$ satisfies*

$$\hat{f}(q) \textit{ is a disk for all } q \in \mathcal{C}(D) \setminus \{p\},\ q \subset U. \tag{2}$$

Then Φ' has a refinement $\Phi_p = (D_j(p))$ such that

$$D_j(p) \setminus U = D'_j \setminus U \quad \textit{for all } j \in \mathbb{N} \quad \textit{and} \tag{3}$$

(4) *if Φ is any refinement of Φ_p, then $\hat{g}(p)$ is a disk for all $g \in \mathcal{F}_{\Phi}$.*

2.1. Transfinite induction.

Transfinite induction. Suppose $D \subsetneq \hat{\mathbb{C}}$ is a countably connected domain. We lose no generality by assuming that the number of complementary components of D if infinite. We denote $E_0 = \hat{D} \setminus D$. For any compact non-empty $E \subset E_0$, let

$$E^* = \{p \in E : p \text{ is not isolated in } E\}.$$

By the Baire category theorem, $E^* \subsetneq E$. We can now use transfinite induction to define a well ordered set of subsets E_α of E_0 as follows: Given an ordinal $\alpha > 0$, we define

$$E_\alpha = \begin{cases} (E_\beta)^*, & \text{if } \alpha = \beta + 1 \text{ is a successor ordinal,} \\ \cap_{\beta<\alpha} E_\beta, & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

It follows that each E_α is compact and $E_\alpha \subsetneq E_\beta$ if $\alpha > \beta$ and $E_\beta \neq \emptyset$. There is an α_L so that E_{α_L} is finite and non-empty, thus $E_{\alpha_L+1} = \emptyset$.

We now show how Theorem 1.3 follows from Proposition 2.1.

Proposition 2.2. *Let $\Phi(0) = (D_j(0))$ be an exhaustion of D satisfying* (1). *For every ordinal $0 \leqslant \alpha \leqslant \alpha_L + 1$ there is an exhaustion $\Phi(\alpha) = (D_j(\alpha))$ of D so that*

(i) *if $0 \leqslant \beta \leqslant \alpha$, then $\Phi(\alpha)$ is a refinement of $\Phi(\beta)$, and*
(ii) *if Φ is any refinement of $\Phi(\alpha)$, then*

$$\hat{f}(q) \text{ is a disk for all } f \in \mathcal{F}_\Phi \text{ and } q \in E_0 \setminus E_\alpha. \tag{5}$$

Suppose $\Phi(\alpha_L + 1) = (D_j)$ satisfies (5), and let (f_{j_k}) be a subsequence of the corresponding (f_j) converging to some f. Choosing $\Phi = (D_{j_k})$ and $f = f_\Phi$ shows that Theorem 1.3 follows from Proposition 2.2.

Proof of Proposition 2.2 assuming Proposition 2.1. First, we enumerate the elements $p = p(k) \in \mathcal{C}(D)$, and denote $p(k) \prec p(\ell)$ if $k < \ell$. This should not be confused with the ordering of the sets E_α. Each p belongs to $E_\alpha \setminus E_{\alpha+1}$ for exactly one $0 \leqslant \alpha \leqslant \alpha_L$. Fix such an α. Then each $p \in E_\alpha \setminus E_{\alpha+1}$ admits an open neighborhood $U_p \subset \hat{\mathbb{C}}$ so that $\overline{U}_p \in \mathcal{C}(D_j(0))$ for some j,

(6) $\pi_D(\overline{U}_p) \cap E_{\alpha+1} = \emptyset$, and
(7) $\overline{U}_p \cap \overline{U}_q = \emptyset$ if $q \in E_\alpha \setminus (E_{\alpha+1} \cup \{p\})$ or if $q \in E_0 \setminus E_\alpha$ satisfies $q \prec p$.

We apply transfinite induction. The claims of the proposition clearly hold for $\alpha = 0$ with the given exhaustion $\Phi(0)$. We assume that the claims hold for all $\beta < \alpha$ and verify them for α.

Let $\alpha = \beta + 1$ be a successor ordinal. By the induction assumption, (6) and (7), Condition (2) in Proposition 2.1 is satisfied with $\Phi' = \Phi(\beta)$, $p \in E_0 \setminus E_\beta$, and $U = U_p$. The proposition combined with our choice of U_p then gives a refinement $\Phi(\alpha) = (D_j(\alpha))$ of $\Phi(\beta) = (D_j(\beta))$ so that

$$D_j(\alpha) \setminus \bigcup_{p \in E_\beta \setminus E_\alpha} U_p = D_j(\beta) \setminus \bigcup_{p \in E_\beta \setminus E_\alpha} U_p \tag{8}$$

and so that (5) holds for all $p \in E_\beta \setminus E_\alpha$. Notice again that if Φ' is a refinement of Φ and if Φ'' is a refinement of Φ', then Φ'' is a refinement of Φ. The claims follow.

Now let $\alpha = \cap_{\beta < \alpha} \beta$ be a limit ordinal. We define $\Phi(\alpha) = (D_j(\alpha))$ as follows: first, let

$$D_j(\alpha) \setminus \Big(\bigcup_{p \in E_0 \setminus E_\alpha} U_p \Big) = D_j(0) \setminus \Big(\bigcup_{p \in E_0 \setminus E_\alpha} U_p \Big). \tag{9}$$

Fix $p \in E_0 \setminus E_\alpha$. Each $q \in E_0$ belongs to some $E_{\beta(q)} \setminus E_{\beta(q)+1}$. With this notation, we have $\beta(p) < \alpha$.

By (7) there are only finitely many $q \in E_{\beta(p)} \setminus E_\alpha$ such that

$$\overline{U}_p \cap \overline{U}_q \neq \emptyset. \tag{10}$$

Moreover, since each such $\overline{U}_q$ belongs to $\mathcal{C}(D_j(0))$, (6) and (7) show that

$$U_p \subset U_q \subset U_{q'} \tag{11}$$

if both $\overline{U}_q$ and $\overline{U}_{q'}$ satisfy (10) and $\beta(q) \leqslant \beta(q')$.

Among the elements q for which (10) holds, let $q(p)$ be the one with the maximal $\beta(q)$. Then $\beta(p) \leqslant \beta(q(p)) < \alpha$. We set

$$D_j(\alpha) \cap U_p = D_j(\beta(q(p))) \cap U_p, \quad p \in E_0 \setminus E_\alpha. \tag{12}$$

Then (9) and (12) define $\Phi(\alpha) = (D_j(\alpha))$. Furthermore, (8), (11), and the induction assumption show that $\Phi(\alpha)$ is a refinement of every $\Phi(\beta)$, $\beta \leqslant \alpha$, and that (5) holds. The proof is complete, modulo Proposition 2.1. □

2.2. **Transboundary modulus.** We will apply the following generalization of conformal modulus, first introduced by Schramm [19]. In addition to its importance in classical uniformization problems, this method has played a central role in recent developments on the uniformization of fractal metric spaces, cf. [1], [3], [4], [8], [17].

Let $G \subset \hat{\mathbb{C}}$ be a domain. The *transboundary modulus* $\mathrm{mod}(\Gamma)$ of a family Γ of paths in $\hat{G}$ is

$$\mathrm{mod}(\Gamma) = \inf_{\rho \in X(\Gamma)} \int_G \rho^2 \, dA + \sum_{p \in \mathcal{C}(G)} \rho(p)^2,$$

where $X(\Gamma)$ consists of all Borel functions $\rho : \hat{G} \to [0, \infty]$ for which

$$1 \leqslant \int_\gamma \rho \, ds + \sum_{p \in \mathcal{C}(G) \cap |\gamma|} \rho(p) \quad \text{for all } \gamma \in \Gamma.$$

Here $\int_\gamma \rho \, ds$ is the path integral of the restriction of γ to G. More precisely, this restriction is a countable union of disjoint paths γ_j, each of which maps onto a component of $|\gamma| \setminus \mathcal{C}(G)$, and we define

$$\int_\gamma \rho \, ds = \sum_j \int_{\gamma_j} \rho \, ds.$$

As noticed in [19], the transboundary modulus is a conformal invariant.

Lemma 2.3. *Suppose $f : G \to G'$ is conformal. Then for every path family Γ and $\hat{f}(\Gamma) = \{\hat{f} \circ \gamma : \gamma \in \Gamma\}$ we have*

$$\mathrm{mod}(\hat{f}(\Gamma)) = \mathrm{mod}(\Gamma).$$

The proof is a straightforward modification of the proof of the corresponding result for conformal modulus.

We will prove Proposition 2.1 by applying the following estimate. Given a domain $G \subset \hat{\mathbb{C}}$ and disjoint sets $A, B \subset \hat{\mathbb{C}}$, we denote

$$\begin{aligned} \Gamma(A, B; G) &= \{\text{paths in } \hat{G} \text{ joining } \pi_G(A) \text{ and } \pi_G(B)\}, \\ \mathrm{mod}(A, B; G) &= \mathrm{mod}(\Gamma(A, B; G)). \end{aligned}$$

Proposition 2.4. *Let $D \subset \hat{\mathbb{C}}$ be a countably connected domain. Fix an exhaustion $\Phi' = (D'_j)$ of D, $p \in \mathcal{C}(D)$, and an open neighborhood U of p in $\hat{\mathbb{C}}$ such that $\overline{U} \in \mathcal{C}(D'_n)$ for some index n. Moreover, suppose every $q \in \mathcal{C}(D) \setminus \{p\}$, $q \subset U$, is a disk. Then Φ' has a refinement $\Phi_p = (D_j(p))$ satisfying*

$$D_j(p) \setminus U = D'_j \setminus U \quad \textit{for all } j \in \mathbb{N}$$

so that if $\Phi = (D_j)$ is any refinement of Φ_p, then

$$\lim_{r \to 0} \limsup_{j \to \infty} \operatorname{mod}(S(z,r) \setminus p_j, \partial U; D_j) = 0 \tag{13}$$

for every $z \in p$, where p_j is the element of $\mathcal{C}(D_j)$ containing p.

We postpone the proof of Proposition 2.4 until Section 2.4. We next show that Proposition 2.1 follows from Proposition 2.4.

2.3. **From Proposition 2.4 to Proposition 2.1.** Fix $\Phi' = (D'_j)$, p, and U as in Proposition 2.1. Replacing D with $f(D)$ and Φ' with $(f(D'_j))$ for some $f \in \mathcal{F}_{\Phi'}$ if necessary, we may assume that the assumptions of Proposition 2.4 are valid. It then suffices to show that (13) implies (4) in Proposition 2.1: if Φ is any refinement of Φ_p, then $\hat{g}(p)$ is a disk for all $g \in \mathcal{F}_\Phi$.

Fix a refinement $\Phi = (D_j)$ of Φ_p. As before, let $f_j : D_j \to \tilde{D}_j$ be the associated conformal maps onto circle domains $\tilde{D}_j$. Fix $g \in \mathcal{F}_\Phi$. By passing to a subsequence if necessary, we may assume that $f_j \to g$.

Taking another subsequence if necessary, we may assume that $(\hat{f}_j(p_j))$ Hausdorff converges to a closed disk B, where p_j is the element of $\mathcal{C}(D_j)$ containing p (recall that B may have zero radius).

Since $f_j \to g$, we have $B \subset \hat{g}(p)$. We will prove that in fact $B = \hat{g}(p)$. This implies (4).

Applying suitable Möbius transformations if necessary, we may assume that $U, f_j(\partial U), g(\partial U)$ are all subsets of the unit disk in the complex plane. Towards contradiction, suppose that $B \subsetneq \hat{g}(p)$. Then

$$\operatorname{dist}(w_0, B) \geqslant 2\delta \quad \text{for some } w_0 \in \partial \hat{g}(p) \text{ and } \delta > 0,$$

where dist is the euclidean distance. It follows that there are sequences (j_k) and (z_k) such that $j_k > k$ and $z_k \in \partial p_k \subset D_{j_k}$ for all $k \in \mathbb{N}$, and

$$\operatorname{dist}(f_j(z_k), \hat{f}_j(p_j)) \geqslant \delta \quad \text{for all } j \geqslant j_k.$$

By passing to another subsequence if necessary, we may assume that

$$z_k \to z \in p.$$

Fix $k \in \mathbb{N}$ and $j \geqslant j_k$. We construct a suitable path family $\Gamma(j,k)$ and estimate its modulus to arrive at a contradiction. Let $w \in \mathbb{C}$ be the point in $\hat{f}_j(p_j)$ closest to $f_j(z_k)$, and denote

$$I = (w, f_j(z_k)), \quad \ell = \text{ the line containing segment } I.$$

Let V_j be the bounded component of $\mathbb{C} \setminus f_j(\partial U)$, and denote the $f_j(z_k)$- and w-components of $\overline{V}_j \cap (\ell \setminus I)$ by P' and Q', respectively. Moreover, let

$$P = \hat{f}_j^{-1}(\pi_{f_j(D_j)}(P')),\ Q = \hat{f}_j^{-1}(\pi_{f_j(D_j)}(Q')) \subset \hat{D}_j.$$

There are unique points $a, b \in \partial U$ so that $\pi_{D_j}(a) \in P$ and $\pi_{D_j}(b) \in Q$. Let J_1, J_2 be the connected components of $\partial U \setminus \{a, b\}$, and let

$$\Gamma(j,k) = \{\text{paths joining } \pi_{D_j}(J_1) \text{ and } \pi_{D_j}(J_2) \text{ in } \pi_{D_j}(U) \setminus (P \cup Q)\}.$$

Then every $\gamma \in \Gamma(j,k)$ passes through $\pi_j(B(z, |z - z_k|))$, so if we denote $r_k = |z - z_k|$ and choose k large enough so that $S(z, r_k) \subset U$, we have

$$\Gamma(j,k) \subset \Gamma(S(z, r_k) \setminus p_j, \partial U; D_j)$$

(observe that $\pi_{D_j}(p_j) \in Q$). Thus,

$$\operatorname{mod}(\Gamma(j,k)) \leqslant \operatorname{mod}(S(z, r_k) \setminus p_j, \partial U; D_j)$$

so by (13),

$$\lim_{k \to \infty} \limsup_{j \to \infty} \operatorname{mod}(\Gamma(j,k)) = 0. \tag{14}$$

Lemma 2.5. *We have*

$$\operatorname{mod}(\hat{f}_j \Gamma(j,k)) \geqslant M > 0 \quad \textit{for all } k \in \mathbb{N} \textit{ and } j \geqslant j_k, \tag{15}$$

where $\hat{f}_j \Gamma(j,k) = \{\hat{f}_j \circ \gamma : \gamma \in \Gamma(j,k)\}$ *and* M *does not depend on* j *or* k.

Combining (14) with Lemmas 2.3 and 2.5 leads to a contradiction, so once Lemma 2.5 has been proved we know that Proposition 2.1 follows from Proposition 2.4.

Proof of Lemma 2.5. We consider the subfamily Γ of $\hat{f}_j \Gamma(j,k)$ consisting of projections of segments orthogonal to ℓ. More precisely, denote by T the length of I, $T = |w - f_j(z_k)|$, and let $\eta(s) = (1 - \frac{s}{T})w + \frac{s}{T} f_j(z_k)$, $0 < s < T$, be an arc-length parametrization of I. Notice that $T \geqslant \delta$.

Fix $0 < s < T$, and denote by ℓ_s the line orthogonal to ℓ passing through $\eta(s)$. Then there is a component I_s of $\ell_s \cap \overline{V}_j$ with endpoints $m_1 \in f_j(J_1)$ and $m_2 \in f_j(J_2)$ (recall that V_j is the bounded component of $\mathbb{C} \setminus f_j(\partial U)$). Choose a parametrization γ_s of $\pi_{D_j}(I_s)$, and let

$$\Gamma = \{\gamma_s : 0 < s < T\}.$$

Then $\Gamma \subset \hat{f}_j \Gamma(j,k)$, so it suffices to prove (15) with $\hat{f}_j \Gamma(j,k)$ replaced by Γ.

Fix $\rho \in X(\Gamma)$, and denote by $\mathcal{D}_j$ the family of disks $\tau \in \hat{f}_j(\mathcal{C}(D_j))$ satisfying $\tau \subset \pi_{D_j}(V_j)$. Then

$$1 \leqslant \int_{I_s} \rho \, ds + \sum_{q \in \mathcal{D}_j \cap |\gamma_s|} \rho(q) \quad \text{for all } 0 < s < T.$$

Integrating from 0 to T and applying Fubini's theorem and Hölder's inequality yields

$$\begin{aligned}\delta &\leqslant T \leqslant \int_{f_j(U\cap D_j)} \rho\, dA + \sum_{\tau\in\mathcal{D}_j} \operatorname{diam}(\tau)\rho(\tau) \\ &\leqslant |V_j|^{1/2}\Big(\int_{f_j(U\cap D_j)} \rho^2\, dA\Big)^{1/2} + \Big(\sum_{\tau\in\mathcal{D}_j} \operatorname{diam}(\tau)^2\Big)^{1/2}\Big(\sum_{\tau\in\mathcal{D}_j} \rho(\tau)^2\Big)^{1/2} \\ &\leqslant \Big(\frac{4}{\pi}|V_j|\Big)^{1/2}\Big(\Big(\int_{f_j(U\cap D_j)} \rho^2\, dA\Big)^{1/2} + \Big(\sum_{\tau\in\mathcal{D}_j} \rho(\tau)^2\Big)^{1/2}\Big),\end{aligned}$$

where the last inequality follows since the disks τ are disjoint subsets of V_j.

The uniform convergence of (f_j) guarantees that there is $N>0$ independent of j such that $\operatorname{diam}(V_j) \leqslant N$ for all j. Combining with the estimate above leads to

$$\frac{\pi\delta^2}{16N^2} \leqslant \int_{f_j(U\cap D_j)} \rho^2\, dA + \sum_{\tau\in\mathcal{D}_j} \rho(\tau)^2.$$

Since this holds for all $\rho \in X(\Gamma)$, we have $\operatorname{mod}(\Gamma) \geqslant \pi\delta^2/(16N^2)$. □

2.4. **Proof of Proposition 2.4.** We use the following notation: if $G, V \subset \hat{\mathbb{C}}$ are domains, then

$$\mathcal{C}(G,V) = \{q \in \mathcal{C}(G) : q \subset V\}.$$

Lemma 2.6. *Suppose D, Φ', p and U are as in Proposition 2.4. Then Φ' has a refinement $\Phi_p = (D_j(p))$ so that $D_j(p)\setminus U = D'_j \setminus U$ and*

$$\mathcal{C}(D_j(p),U) = \hat{\mathcal{C}}_{e,j} \cup \hat{\mathcal{C}}_{d,j} \cup \{\hat{p}_j\}$$

for all $j\in\mathbb{N}$, where $\hat{p}_j \supset p$ and $\hat{p}_j \notin \hat{\mathcal{C}}_{e,j}\cup\hat{\mathcal{C}}_{d,j}$,

$$\sum_{\hat{q}(j)\in\hat{\mathcal{C}}_{d,j}} \operatorname{diam}(\hat{q}(j)) \leqslant 2^{-j-1}, \tag{16}$$

and for every $\hat{q}(j) \in \hat{\mathcal{C}}_{e,j}$ there is $q = \overline{B}(x,t) \in \mathcal{C}(D,U)$, $t>0$, such that

$$\overline{B}(x,t) \subset \hat{q}(j) \subset B(x,t+s), \quad s = \min\left\{\frac{t}{100}, \frac{\operatorname{dist}(\hat{q}(j),p)}{100}\right\}. \tag{17}$$

Proof. We have $\mathcal{C}(D,U) = \mathcal{C}_e \cup \mathcal{C}_d \cup \{p\}$, $p\notin \mathcal{C}_e\cup\mathcal{C}_d$, where $\mathcal{C}_e$ is a family of disks with positive radius and $\mathcal{C}_d$ a family of point components. We enumerate the elements of $\mathcal{C}_d$:

$$\mathcal{C}_d = \{q_1, q_2, \ldots\}.$$

We define $\Phi_p = (D_j(p))$ as follows: First, let $D_j(p)\setminus U = D'_j\setminus U$, $j\in\mathbb{N}$. To describe the sets $D_j(p)\cap U$, assume that $j=1$ or $j\geqslant 2$, and $D_k(p)$ has been defined for all $k\leqslant j-1$.

We denote by $\hat{p}_j$ the element of $\mathcal{C}(D'_j,U)$ containing p. We lose no generality by assuming that $\hat{p}_1 \subset U$. Then $\mathcal{C}(D_j, U\setminus \hat{p}_j)$ is non-empty for all $j\geqslant 1$.

Each $q \in \mathcal{C}(D,U)$ is contained in some $\hat{q}(j)$ such that

(i) $\hat{p}(j) = \hat{p}_j$,
(ii) $\hat{q}(j) \in \mathcal{C}(D'_{j'}, U \setminus \hat{p}_j)$ for some $j' \geqslant j$,
(iii) if $j \geqslant 2$ then $\hat{q}(j) \subset \hat{q}(j-1)$ for some $\hat{q}(j-1) \in \mathcal{C}(D_{j-1}(p), U)$,
(iv) if $q = q_m \in \mathcal{C}_d$, then

$$\operatorname{diam}(\hat{q}_m(j)) \leqslant 2^{-j-m-1},$$

(v) if $q = \overline{B}(x,t) \in \mathcal{C}_e$, then $\hat{q}(j)$ satisfies (17).

Denote $\mathcal{Q}_j = \{\hat{q}(j) : q \in \mathcal{C}(D,U)\}$. If $\hat{q}(j), \hat{q}'(j) \in \mathcal{Q}_j$, then either $\hat{q}(j) \cap \hat{q}'(j) = \emptyset$ or one is contained in the other. Thus we can define $D_j(p) \cap U$ as the domain for which $\mathcal{C}(D_j(p), U)$ is the set of maximal elements in $\mathcal{Q}_j$.

Properties (i)–(iii) guarantee that $\{D_j(p)\}$ is a refinement of $\{D'_j\}$. Moreover, every $\hat{q}(j)$ satisfies (iv) or (v). We define

$$\begin{aligned}
\hat{\mathcal{C}}_{d,j} &= \{\hat{q}(j) \in \mathcal{C}(D_j(p), U) \setminus \{\hat{p}_j\} : \hat{q}(j) \text{ satisfies (iv)}\},\\
\hat{\mathcal{C}}_{e,j} &= \{\hat{q}(j) \in \mathcal{C}(D_j(p), U) \setminus \{\hat{p}_j\} : \hat{q}(j) \text{ satisfies (v)}\}.
\end{aligned}$$

□

We complete the proof of Proposition 2.4 by showing that any refinement $\Phi = (D_j)$ of the Φ_p in Lemma 2.6 satisfies the remaining estimate (13), i.e.,

$$\lim_{r \to 0} \limsup_{j \to \infty} \operatorname{mod}(S(z,r) \setminus p_j, \partial U; D_j) = 0 \quad \text{for every } z \in p.$$

Lemma 2.7. *Every refinement $\Phi = (D_j)$ of Φ_p satisfies* (13).

Proof. Fix $z \in p$ and let v be the largest integer such that

$$B(z, e^v) = B(z, R) \subset U.$$

It suffices to show that if j is large enough, then

$$\operatorname{mod}(S(z,r) \setminus p_j, S(z,R); D_j) \leqslant \epsilon(r) \to 0 \quad \text{as } r \to 0, \tag{18}$$

where $\epsilon(r)$ does not depend on j. We will do this by first constructing a suitable sequence of disjoint annuli, and then applying them to find admissible functions.

First, let $v_1 = v$. Then, fix $n \geqslant 1$ and assume that $v_n < v_{n-1} < \cdots < v_1$ have been defined. Denote $R_k = e^{v_k}$ and $A_k = B(z, R_k) \setminus \overline{B}(z, R_k/e)$, and let $v_{n+1} < v_n$ be the largest integer such that

$$\overline{B}(z, R_{n+1}) \cap \overline{B}(x, t+s) = \emptyset \quad \text{for all } q = \overline{B}(x,t) \in \mathcal{C}(D,U),\ q \cap A_n \neq \emptyset,$$

where s is as in (17).

Recall from Lemma 2.6 that

$$\mathcal{C}(D_j(p), U) = \hat{\mathcal{C}}_{e,j} \cup \hat{\mathcal{C}}_{d,j} \cup \{\hat{p}_j\}.$$

We denote $\hat{\mathcal{C}}_{e,j} = \hat{\mathcal{C}}_{b,j} \cup \hat{\mathcal{C}}_{s,j}$, where

$$\hat{\mathcal{C}}_{b,j} = \{\hat{q}(j) \in \hat{\mathcal{C}}_{e,j} : \operatorname{diam}(\hat{q}(j)) \geqslant \operatorname{dist}(\hat{q}(j), z)\},$$
$$\hat{\mathcal{C}}_{s,j} = \{\hat{q}(j) \in \hat{\mathcal{C}}_{e,j} : \operatorname{diam}(\hat{q}(j)) < \operatorname{dist}(\hat{q}(j), z)\}.$$

Moreover, let $\mathcal{C}_j = \mathcal{C}_{d,j} \cup \mathcal{C}_{b,j} \cup \mathcal{C}_{s,j}$, where

$$\mathcal{C}_{d,j} = \cup_{m \geqslant j} \hat{\mathcal{C}}_{d,m}, \quad \mathcal{C}_{b,j} = \cup_{m \geqslant j} \hat{\mathcal{C}}_{b,m}, \quad \mathcal{C}_{s,j} = \cup_{m \geqslant j} \hat{\mathcal{C}}_{s,m}.$$

Fix a refinement $\Phi = (D_j)$ of Φ_p, and $u < v - 100$. We denote $r = e^u$. Let j be large enough so that $2^{-j+1} < r/e$, and p_j the element of $\mathcal{C}(D_j, U)$ containing p. Since Φ is a refinement of Φ_p, we have $\mathcal{C}(D_j, U \setminus p_j) \subset \mathcal{C}_j$. In particular,

$$\mathcal{C}(D_j, U \setminus p_j) = \mathcal{D}_j \cup \mathcal{B}_j \cup \mathcal{S}_j, \quad \text{where } \mathcal{D}_j \subset \mathcal{C}_{d,j},\ \mathcal{B}_j \subset \mathcal{C}_{b,j},\ \mathcal{S}_j \subset \mathcal{C}_{s,j}.$$

By Lemma 2.6 and the definition of the above sets, the following hold: First,

$$\sum_{q(j) \in \mathcal{D}_j} \operatorname{diam}(q(j)) \leqslant 2^{-j} < \frac{r}{2e}. \tag{19}$$

Secondly, denoting $\mathcal{B}_j(n) = \{q(j) \in \mathcal{B}_j : q(j) \cap A_n \neq \emptyset\}$, we have $\mathcal{B}_j(n) \cap \mathcal{B}_j(n') = \emptyset$ if $n \neq n'$. Moreover, since every $q(j) \in \mathcal{B}_j(n)$ contains a disk whose area is comparable to the area of A_n, the cardinality of $\mathcal{B}_j(n)$ has an absolute bound;

$$|\mathcal{B}_j(n)| \leqslant 30 \quad \text{for all } n \in \mathbb{N}. \tag{20}$$

Finally, every $q(j) \in \mathcal{S}_j$ satisfies

$$\operatorname{diam}(q(j))^2 \leqslant 2 \operatorname{Area}(q(j)). \tag{21}$$

Moreover, denoting $\mathcal{S}_j(n) = \{q(j) \in \mathcal{S}_j : q(j) \cap A_n \neq \emptyset\}$, we have $\mathcal{S}_j(n) \cap \mathcal{S}_j(n') = \emptyset$ if $n \neq n'$.

We construct an admissible function

$$\rho \in X(\Gamma(S(z,r) \setminus p_j, S(z,R); D_j)) \tag{22}$$

as follows: let m be the largest integer such that $v_{m+1} \geqslant u$, and $1 \leqslant n \leqslant m$. Define $\rho_n : \hat{D}_j \to [0, \infty]$,

$$\rho_n(w) = \begin{cases} \frac{1}{m}, & w \in \mathcal{B}_j(n), \\ \frac{2e \operatorname{diam}(w)}{m R_n} & w \in \mathcal{S}_j(n), \\ \frac{2}{m|w-z|}, & w \in A_n \cap D_j, \end{cases}$$

and $\rho_n(w) = 0$ otherwise. We claim that

$$\frac{1}{m} \leqslant \int_\gamma \rho_n \, ds + \sum_{q \in \mathcal{C}_j \cap |\gamma|} \rho_n(q) \tag{23}$$

for all $\gamma \in \Gamma(S(z,r) \setminus p_j, S(z,R); D_j)$. Fix such a γ, and denote

$$\begin{aligned}\Omega_1 &= \{R_n/e < T < R_n : T = |y-z| \text{ for some } y \in |\gamma| \cap D_j\},\\ \Omega_2 &= \{R_n/e < T < R_n : T = |y-z| \text{ for some } y \in w,\ w \in |\gamma| \cap \mathcal{S}_j(n)\},\\ \Omega_3 &= \{R_n/e < T < R_n : T = |y-z| \text{ for some } y \in w,\ w \in |\gamma| \cap \mathcal{D}_j(n)\}.\end{aligned}$$

We may assume that γ does not intersect any $w \in \mathcal{B}_j(n)$, otherwise (23) follows directly from the definition of ρ_n. We then have

$$\int_{\Omega_1} \frac{dT}{T} + \int_{\Omega_2} \frac{dT}{T} + \int_{\Omega_3} \frac{dT}{T} \geqslant 1,$$

which combined with (19) yields

$$\int_{\Omega_1} \frac{dT}{T} + \int_{\Omega_2} \frac{dT}{T} \geqslant \frac{1}{2}.$$

The definition of ρ_n in $A_n \cap D_j$ yields

$$\int_\gamma \rho_n \, ds \geqslant \frac{2}{m} \int_{\Omega_1} \frac{dT}{T}.$$

On the other hand, combining the definition of ρ_n in $\mathcal{S}_j(n)$ with inequality

$$\frac{e(\beta-\alpha)}{R_n} \geqslant \log\beta - \log\alpha, \qquad \frac{e}{R_n} \leqslant \alpha \leqslant \beta,$$

yields

$$\sum_{q \in \mathcal{S}_j(n) \cap |\gamma|} \rho_n(q) \geqslant \frac{2}{m} \int_{\Omega_2} \frac{dT}{T}.$$

Combining the estimates yields (23). In particular, $\rho = \sum_{n=1}^m \rho_n$ satisfies (22), i.e., ρ is admissible for $\Gamma(S(z,r) \setminus p_j, S(z,R); D_j)$.

We prove (18) by estimating the energy

$$\int_{D_j \cap U} \rho^2 \, dA + \sum_{w \in \mathcal{C}_j} \rho(w)^2 \tag{24}$$

from above. First, we have

$$\int_{D_j \cap U} \rho^2 \, dA \leqslant \frac{4}{m^2} \sum_{n=1}^m \int_{D_j \cap A_n} \frac{dA(w)}{|w-z|^2} \leqslant \frac{8\pi}{m}. \tag{25}$$

In order to estimate the sum in (24), we recall that each $w \in \mathcal{B}_j \cup \mathcal{S}_j$ intersects at most one A_n. By (20),

$$\sum_{w \in \mathcal{B}_j} \rho(w)^2 \leqslant \sum_{n=1}^m \frac{|\mathcal{B}_j(n)|}{m^2} \leqslant \frac{30}{m}. \tag{26}$$

Finally, since every $w \in \mathcal{S}_j(n)$ is a subset of $B(z, 2R_n)$, (21) yields

$$\begin{aligned}\sum_{w \in \mathcal{S}_j} \rho(w)^2 &\leqslant \frac{4e^2}{m^2} \sum_{n=1}^m \sum_{w \in \mathcal{S}_j(n)} \frac{\operatorname{diam}(w)^2}{R_n^2}\\ &\leqslant \frac{8e^2}{m^2} \sum_{n=1}^m \frac{\operatorname{Area}(B(z,2R_n))}{R_n^2} = \frac{32\pi e^2}{m}.\end{aligned} \tag{27}$$

Combining (25), (26) and (27), we conclude

$$\int_{D_j \cap U} \rho^2 \, dA + \sum_{w \in \mathcal{C}_j} \rho(w)^2 \leqslant \frac{1000}{m} \to 0 \quad \text{as } r \to 0,$$

and (18) follows. The proof is complete. □

3. Proof of Theorem 1.1

3.1. **Construction of the domain.** We will construct a countably connected *square domain*[1] $D \subset \hat{\mathbb{C}}$ so that $\{0\} \in \mathcal{C}(D)$, and an exhaustion Φ of D so that $\hat{f}_\Phi(\{0\})$ is non-trivial. The following result, which follows from the modulus estimate in [19, Theorem 6.2], then shows that $f_\Phi(D)$ cannot be a circle domain.

Proposition 3.1. *If f is a conformal map from domain $D \subset \hat{\mathbb{C}}$ with the above properties onto a circle domain, then $\hat{f}(\{0\})$ is a point-component.*

We start the construction of D with a sequence of disjoint squares

$$Q_k = [a_k - R_k, a_k + R_k] \times [-R_k, R_k], \quad R_1 = 1, \, R_k < a_k,$$

where $(a_k)_{k=1}^\infty$, $(R_k)_{k=1}^\infty$ are decreasing sequences converging to zero, so that

$$D_k := \operatorname{dist}(Q_k, Q_{k+1}) = a_k - (a_{k+1} + R_k + R_{k+1}) = 2^{-k} R_{k+1}. \tag{28}$$

Each Q_k, $1 \leqslant k \leqslant j$, is surrounded by a sequence $(Q_{k,j})$ of inflated squares

$$Q_{k,j} = [a_k - T_{k,j}, a_k + T_{k,j}] \times [-T_{k,j}, T_{k,j}], \quad T_{k,j} = R_k + 2^{-j-1} D_k.$$

We also denote

$$Q_{0,j} = [-T_j, T_j] \times [-T_j, T_j], \quad T_j = a_{j+1} + R_{j+1} + D_j/2.$$

Then

$$\cup_{k=j+1}^\infty Q_j \subset \operatorname{int}(Q_{0,j}), \quad \cup_{k=1}^j Q_{k,j} \cap Q_{0,j} = \emptyset \quad \text{for every } j \in \mathbb{N}.$$

Next, for $m \in \mathbb{N}$ and $1 \leqslant \ell \leqslant M_m$ (M_m will be chosen later), let

$$q_{m,\ell} = [(\ell-1)(s_m + d_m), (\ell-1)s_m + \ell d_m] \times [0, d_m], \tag{29}$$

where d_m, s_m are positive numbers so that

$$(M_m - 1)s_m + M_m d_m = 1 \quad \text{and} \quad d_m \geqslant s_m.$$

In particular, $d_m \leqslant M_m^{-1}$. For a fixed $m \in \mathbb{N}$, the sets $q_{m,\ell}$ are evenly spaced squares of sidelength d_m inside the rectangle $[0,1] \times [0, d_m]$.

For each $m \in \mathbb{N}$ and $1 \leqslant k \leqslant m$, let $\phi_{k+1,m}$ be the Möbius transformation so that $\phi_{k+1,m}(\infty) = \infty$,

$$\begin{aligned} \phi_{k+1,m}(0,0) &= (a_{k+1} + (1 - s_m)T_{k+1,m-1}, -R_{k+1}) \quad \text{and} \\ \phi_{k+1,m}(1,0) &= (a_{k+1} + (1 - s_m)T_{k+1,m-1}, R_{k+1}). \end{aligned}$$

We denote

$$t_{k,m} = (a_k - T_{k,m-1}) - (a_{k+1} + (1 - 2s_m)T_{k+1,m-1}) + d_m, \tag{30}$$

[1]The construction of D is flexible in terms of the shapes of the complementary components. In particular, there are circle domains D satisfying the requirements of Theorem 1.1. We use squares in our construction for convenience of presentation.

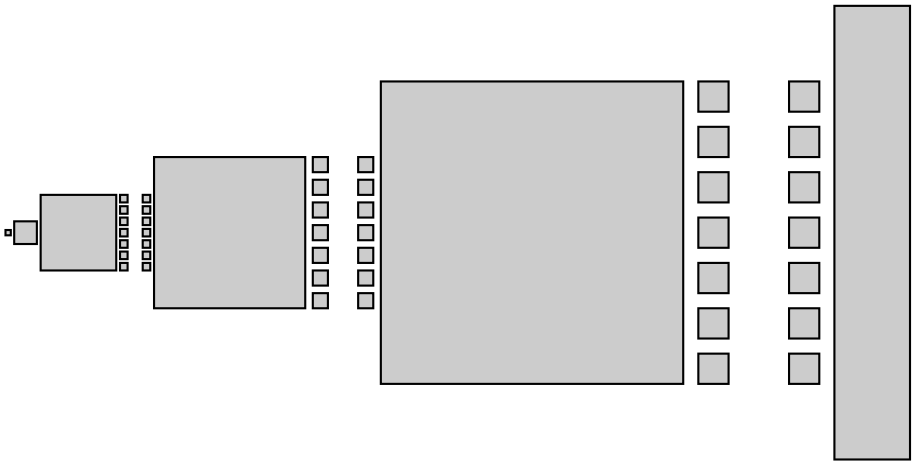

FIGURE 1. Part of the complement of D

and define

$$\begin{aligned} q^e_{k+1,m,\ell} &= \phi_{k+1,m}(q_{m,\ell}), \\ q^w_{k,m,\ell} &= \phi_{k+1,m}(q_{m,\ell}) + (t_{k,m}, 0), \quad 1 \leqslant \ell \leqslant M_m. \end{aligned}$$

The squares $q^e_{k+1,m,\ell}, q^w_{k,m,\ell}$ lie "between" Q_{k+1} and Q_k, and we can choose M_m large enough so that

$$q^e_{m+1,m,\ell} \subset \operatorname{int}(Q_{0,m}),$$

$$q^e_{k+1,m,\ell} \subset \operatorname{int}(Q_{k+1,m-1}) \setminus Q_{k+1,m} \quad \text{for all } 1 \leqslant k \leqslant m-1, \tag{31}$$

$$q^w_{k,m,\ell} \subset \operatorname{int}(Q_{k,m-1}) \setminus Q_{k,m} \quad \text{for all } 1 \leqslant k \leqslant m. \tag{32}$$

We define D by

$$\hat{\mathbb{C}} \setminus D = \{0\} \cup \Big(\bigcup_{k=1}^{\infty} Q_k \cup \Big(\bigcup_{m=1}^{\infty} \bigcup_{\ell=1}^{M_m} \bigcup_{k=1}^{m} (q^e_{k+1,m,\ell} \cup q^w_{k,m,\ell}) \Big) \Big).$$

3.2. **Construction of the exhaustion.** We define exhaustion $\Phi_0 = (D_j)$ of D. First, every $\mathcal{C}(D_j)$ includes

$$Q_{0,j} \quad \text{and} \quad Q_{k,j}, \quad 1 \leqslant k \leqslant j.$$

To describe the rest of the elements of $\mathcal{C}(D_j)$, we first define

$$q^e_{k+1,m,\ell,j} = (1+\epsilon(j)) q^e_{k+1,m,\ell} \quad 1 \leqslant k \leqslant j-1, \tag{33}$$

$$q^w_{k,m,\ell,j} = (1+\epsilon(j)) q^w_{k,m,\ell} \quad 1 \leqslant k \leqslant j \tag{34}$$

for all $k \leqslant m \leqslant j$ and $1 \leqslant \ell \leqslant M_m$, i.e., squares with same center and $(1+\epsilon(j))$ times the sidelength of $q^e_{k+1,m,\ell}$ and $q^w_{k,m,\ell}$, respectively. Here $(\epsilon(j))$ is a strictly decreasing sequence converging to zero, and $\epsilon(1)$ is small enough such that for any fixed $j \in \mathbb{N}$ we have

(i) none of the squares intersect each other, and
(ii) (31) holds for $q^e_{k+1,m,\ell,j}$ and (32) holds for $q^w_{k,m,\ell,j}$.

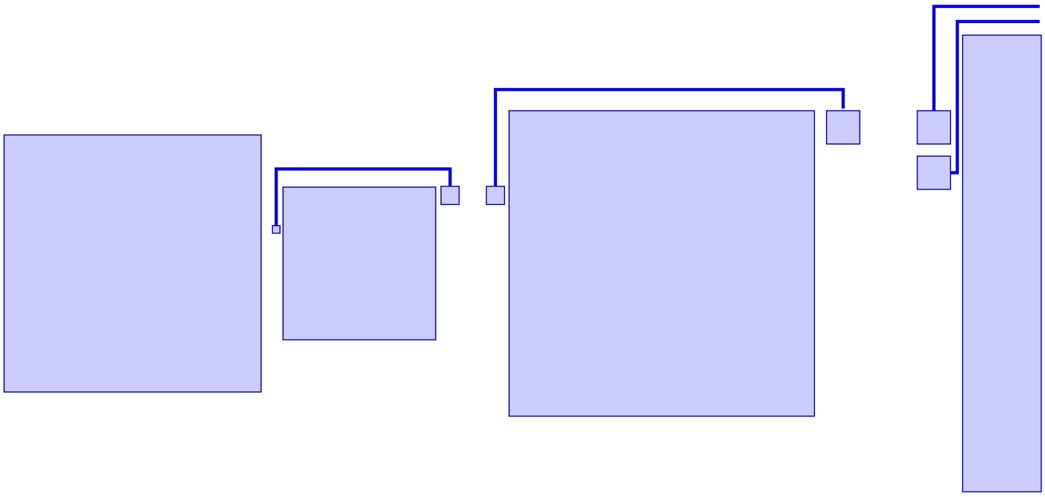

FIGURE 2. Some of the sets $\overline{U}_{k,j,\ell}$

We let $\mathcal{C}(D_j)$ include all the squares in (33) and (34) for $k \leqslant m \leqslant j-1$ and $1 \leqslant \ell \leqslant M_m$. Notice that the squares for which $m = j$ are not included.

The remaining elements of $\mathcal{C}(D_j)$ will be components $\overline{U}_{k,j,\ell}$ which "surround" $Q_{k,j}$ and contain both $q^e_{k,j,\ell,j}$ and $q^w_{k,j,\ell,j}$. More precisely, fix $2 \leqslant k \leqslant j$, and let $U_{k,j,\ell}$, $1 \leqslant \ell \leqslant M_j$, be Jordan domains so that

(i) the sets $\overline{U}_{k,j,\ell}$ are pairwise disjoint,
(ii) $\overline{U}_{k,j,\ell} \subset \operatorname{int}(Q_{k,j-1}) \setminus Q_{k,j}$,
(iii) $\overline{U}_{k,j,\ell}$ contains both $q^e_{k,j,\ell,j}$ and $q^w_{k,j,\ell,j}$,
(iv) if $(x,y) \in \partial U_{k,j,\ell}$ has the largest x-coordinate among all points of $\partial U_{k,j,\ell}$, then $(x,y) \in q^e_{k,j,\ell,j}$,
(v) if $(x,y) \in \partial U_{k,j,\ell}$ has the smallest x-coordinate among all points of $\partial U_{k,j,\ell}$, then $(x,y) \in q^w_{k,j,\ell,j}$.

We conclude the definition of $\mathcal{C}(D_j)$ by including $\overline{U}_{1,j,\ell} := q^w_{1,j,\ell,j}$ and

$$\overline{U}_{k,j,\ell} \quad 2 \leqslant k \leqslant j,\ 1 \leqslant \ell \leqslant M_j.$$

Then D_j is the set for which $\hat{\mathbb{C}} \setminus D_j = \cup\{p \in \mathcal{C}(D_j)\}$, and $\Phi_0 = (D_j)$.

Proposition 3.2. *There is $\delta > 0$ such that*

$$\operatorname{mod}(Q_{0,j_0}, Q_{1,j_0}; D_j) \geqslant \delta \quad \textit{for all } j_0 \in \mathbb{N} \textit{ and } j \geqslant j_0.$$

We postpone the proof of Proposition 3.2 and first show how it implies Theorem 1.1. Choose any subsequence $\Phi = (D_{j_n})$ of (D_j) so that (f_{j_n}) converges to f_Φ. By Proposition 3.1 it suffices to show that $\hat{f}_\Phi(\{0\})$ is non-trivial. But this follows directly by combining Proposition 3.2 with Proposition 4.2 below. Here the latter proposition can be applied with $E = Q_{1,1}$ since every $j_0 \geqslant 1$ satisfies

$$\operatorname{mod}(Q_{0,j_0}, Q_{1,j_0}; D_j) \leqslant \operatorname{mod}(Q_{0,j_0}, Q_{1,1}; D_j).$$

Thus, Theorem 1.1 follows once we have proved these propositions.

3.3. **Proof of Proposition 3.2.** Fix j_0 and $j \geqslant j_0$, and let F_j be the projection of $\cup_{\ell=1}^{M_j} q_{j,\ell}$ to the real axis, recall the definition in (29). We construct a family of paths Γ parametrized by F_j, so that each $\gamma \in \Gamma$ connects $\pi_{D_j}(Q_{1,j_0})$ and $\pi_{D_j}(Q_{0,j_0})$ in $\hat{D}_j$. We then give a lower bound for $\mathrm{mod}(\Gamma)$.

Fix $\tau \in F_j$, and denote

$$\begin{aligned} z^e_{k+1}(\tau) &= \phi_{k+1,j}((\tau, d_j/2)), \quad 1 \leqslant k \leqslant j-1, \\ z^w_k(\tau) &= \phi_{k+1,j}((\tau, d_j/2)) + (t_{k,j}, 0), \quad 1 \leqslant k \leqslant j, \end{aligned}$$

where $t_{k,j}$ is the number in (30) and $\phi_{k+1,j}$ the Möbius transformation defined before (30). Then

$$z^e_{k+1}(\tau) \in q^e_{k+1,j,\ell,j} \subset \overline{U}_{k+1,j,\ell} \quad \text{and} \quad z^w_k(\tau) \in q^w_{k,j,\ell,j} \subset \overline{U}_{k,j,\ell},$$

where $\ell = \ell(j,\tau)$ is the index for which $(\tau, 0) \in q_{j,\ell}$.

We denote

$$\overline{U}_{k,j,\ell} =: \overline{U}_k(\tau),$$

and let $I_k(\tau)$ be the horizontal line segment in $\mathbb{C}$ which connects Q_{1,j_0} to $z^e_2(\tau)$, $z^w_k(\tau)$ to $z^e_{k+1}(\tau)$ if $2 \leqslant k \leqslant j-1$, and $z^w_j(\tau)$ to $Q_{0,j}$. Then

$$J(\tau) = (\cup_{k=1}^{j} I_k(\tau)) \cup (\cup_{k=2}^{j} \overline{U}_k(\tau))$$

is a continuum connecting Q_{1,j_0} and Q_{0,j_0} in $\mathbb{C}$. Moreover, $\pi_{D_j}(J(\tau))$ is a rectifiable curve in $\hat{D}_j$, with arc-length parametrization γ_τ. We define

$$\Gamma = \{\gamma_\tau : \tau \in F_j\}.$$

We now estimate the modulus of Γ. Let $\rho \in X(\Gamma)$ be an admissible function and $\tau \in F_j$. We denote by $\mathcal{A}_j$ all the sets in $\mathcal{C}(D_j)$ of the form $\overline{U}_{k,j,\ell}$, and by $\mathcal{B}_j$ all the other squares in $\mathcal{C}(D_j)$ of the form (33) or (34). Then

$$\begin{aligned} (35) \qquad 1 &\leqslant \int_{\gamma_\tau} \rho\, ds + \sum_{q \in \mathcal{C}(D_j) \cap |\gamma_\tau|} \rho(q) \\ &= \sum_{k=1}^{j} \int_{I_k(\tau) \cap D_j} \rho\, ds + \sum_{k=1}^{j} \rho(\overline{U}_k(\tau)) + \sum_{q \in \mathcal{B}_j \cap |\gamma_\tau|} \rho(q). \end{aligned}$$

Given $1 \leqslant k \leqslant j$, let A_k be the smallest rectangle containing all the segments $I_k(\tau) \cap D_j$, $\tau \in F_j$. Then by (28),

$$(36) \qquad \mathrm{Area}(A_k) \leqslant 2 D_k R_{k+1} \leqslant 2^{1-k} R_{k+1}^2.$$

To estimate the modulus, we integrate both sides of (35) over τ and apply change of variables and Fubini's theorem to get

$$\begin{aligned} (37) \qquad \ell(F_j) &\leqslant \sum_{k=1}^{j} (2R_{k+1})^{-1} \int_{A_k \cap D_j} \rho\, dA + \int_{F_j} \sum_{k=1}^{j} \rho(\overline{U}_k(\tau))\, d\tau \\ &+ \int_{F_j} \sum_{q \in \mathcal{B}_j \cap |\gamma_\tau|} \rho(q)\, d\tau = S_1 + S_2 + S_3. \end{aligned}$$

We apply Hölder's inequality and (36) to estimate S_1 as follows:

$$\begin{aligned} S_1 &\leqslant \sum_{k=1}^{j}(2R_{k+1})^{-1}\operatorname{Area}(A_k)^{1/2}\Big(\int_{A_k\cap D_j}\rho^2\,dA\Big)^{1/2} \\ &\leqslant \Big(\sum_{k=1}^{j}2^{-1-k}\Big)^{1/2}\Big(\int_{D_j}\rho^2\,dA\Big)^{1/2} \leqslant \Big(\int_{D_j}\rho^2\,dA\Big)^{1/2}. \end{aligned} \tag{38}$$

To estimate S_2 and S_3, we choose M_m so that

$$M_m \geqslant m2^{m+1} \quad \text{for all } m\in\mathbb{N}. \tag{39}$$

We notice that the length of the set of parameters τ for which a given $\overline{U}_{k,j,\ell}\in\mathcal{A}_j$ is $\overline{U}_k(\tau)$ equals d_j. We have $d_jM_j\leqslant 1$ by construction. Thus, Hölder's inequality and (39) yield

$$\begin{aligned} S_2 &= d_j\sum_{k=1}^{j}\sum_{\ell=1}^{M_j}\rho(\overline{U}_{k,j,\ell}) \leqslant d_j(jM_j)^{1/2}\Big(\sum_{k=1}^{j}\sum_{\ell=1}^{M_j}\rho(\overline{U}_{k,j,\ell,j})^2\Big)^{1/2} \\ &\leqslant \Big(\sum_{\overline{U}\in\mathcal{A}_j}\rho(\overline{U})^2\Big)^{1/2}. \end{aligned} \tag{40}$$

Next, we notice that the length of the set of parameters τ for which a given $q=q^{y}_{k,m,\ell,j}\in\mathcal{B}_j\cap|\gamma_\tau|$ is at most M_m^{-1}. Here $y=e$ or w. As before, Hölder's inequality yields

$$S_3\leqslant\sum_{q\in\mathcal{B}_j}M_m^{-1}\rho(q)\leqslant\Big(\sum_{q\in\mathcal{B}_j}M_m^{-2}\Big)^{1/2}\Big(\sum_{q\in\mathcal{B}_j}\rho(q)^2\Big)^{1/2}. \tag{41}$$

We estimate the first sum from above by summing over all $q\in\mathcal{B}_j$ and applying (39) to have

$$\sum_{q\in\mathcal{B}_j}M_m^{-2}\leqslant 2\sum_{m=1}^{j}\sum_{k=1}^{m}\sum_{\ell=1}^{M_m}M_m^{-2}=2\sum_{m=1}^{j}mM_m^{-1}\leqslant\sum_{m=1}^{\infty}2^{-m}=1. \tag{42}$$

We have $\ell(F_j)\geqslant\frac{1}{2}$ by construction. Combining with (37), (38), (40), (41), and (42), yields

$$\begin{aligned} \frac{1}{2} &\leqslant \Big(\int_{D_j}\rho^2\,dA\Big)^{1/2}+\Big(\sum_{\overline{U}\in\mathcal{A}_j}\rho(\overline{U})^2\Big)^{1/2}+\Big(\sum_{q\in\mathcal{B}_j}\rho(q)^2\Big)^{1/2} \\ &\leqslant 3\Big(\int_{D_j}\rho^2\,dA+\sum_{q\in\mathcal{C}(D_j)}\rho(q)^2\Big)^{1/2}. \end{aligned} \tag{43}$$

Since (43) holds for all $\rho\in X(\Gamma)$, we conclude that

$$\operatorname{mod}(Q_{0,j_0},Q_{1,j_0};D_j)\geqslant\operatorname{mod}(\Gamma)\geqslant\frac{1}{36}.$$

The proof is complete.

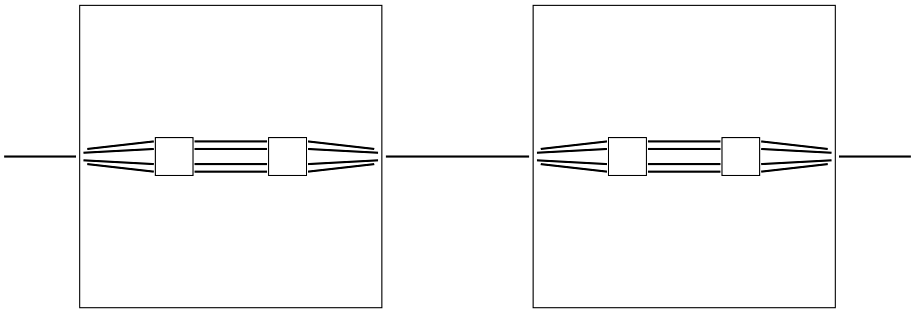

FIGURE 3. First steps in the construction of D

4. Proof of Theorem 1.2

4.1. **Construction of the domain.** The set $\mathcal{C}(D)$ of complementary components of D, which we now describe, consists of countably many segments and a Cantor set.[2] Let $\mathcal{W}_0 = \{e\}$, $\mathcal{Y}_0 = \{(e,e)\}$, and for $k = 1, 2, \ldots$, let

$$\begin{aligned} \mathcal{W}_k &= \{w = w_1 w_2 w_3 \cdots w_k : w_\ell \in \{0,1\} \text{ for } 1 \leqslant \ell \leqslant k\}, \\ \mathcal{W}_\infty &= \{\bar{w} = w_1 w_2 w_3 \cdots : w_\ell \in \{0,1\} \text{ for } \ell = 1, 2, \ldots\}, \quad \text{and} \\ \mathcal{Y}_k &= \{(w,v) : w \in \mathcal{W}_k,\ v = v_1 v_2 v_3 \cdots v_k,\ v_\ell \in \{0,1,2,3\} \text{ for } 1 \leqslant \ell \leqslant k\}. \end{aligned}$$

If $\bar{w} = w_1 w_2 \cdots \in \mathcal{W}_\infty$ and $k \in \mathbb{N}$, we denote $\bar{w}(k) = w_1 \cdots w_k$.

Next, let (R_k) be a sequence of positive real numbers so that $R_{k+1} < R_k/2$ for all $k = 0, 1, 2, \ldots$. Moreover, given such a k let

$$\mathcal{Q}_k = \{Q_w = [x_w - R_k, x_w + R_k] \times [-R_k, R_k] : w \in \mathcal{W}_k\} \tag{44}$$

be a family of disjoint, congruent closed squares in $\mathbb{C}$ with centers on the real axis so that

$$\text{if } w \in \mathcal{W}_k \text{ and } a \in \{0,1\}, \text{ then } Q_{wa} \subset \operatorname{int}(Q_w).$$

The intersection

$$K = \bigcap_{k=0}^{\infty} \Big(\bigcup_{w \in \mathcal{W}_k} Q_w \Big) \tag{45}$$

is a Cantor set on the real axis. It is the Cantor set part of $\mathcal{C}(D)$. Each $p = p_{\bar{w}} \in K$ is uniquely determined by the $\bar{w} \in \mathcal{W}_\infty$ that satisfies

$$\{p_{\bar{w}}\} = \cap_{k=1}^{\infty} Q_{\bar{w}(k)}.$$

We now inductively define the segments in $\mathcal{C}(D)$. The definition involves a sequence (ϵ_k) of positive real numbers converging rapidly to zero. We initially require that $\epsilon_k < R_{k-2}/10$. The segments are of the form

$$I_m(w,v) = [a_m(w,v), b_m(w,v)], \quad m = 1, 2, 3,$$

where $a_m(w,v), b_m(w,v) \in \mathbb{C}$ and $(w,v) \in \mathcal{Y}_k$ for some $k = 0, 1, 2, \ldots$. We denote by $\pi_1 : \mathbb{C} \to \mathbb{R}$ the projection to the real axis.

We first choose

$$[a_1, b_1] = I_1 = I_1(e,e),\ [a_2, b_2] = I_2 = I_2(e,e),\ [a_3, b_3] = I_3 = I_3(e,e) \tag{46}$$

[2]The size of the Cantor set is not relevant for our construction. For instance, the construction can be carried out so that $\hat{\mathbb{C}} \setminus D$ has σ-finite length.

of length larger than ϵ_1 in $Q_e \setminus (Q_0 \cup Q_1)$, so that

$$\begin{aligned} \pi_1(a_1) &< \pi_1(b_1) < x_0 - R_1 < \pi_1(b_1) + \epsilon_2/10, \\ \pi_1(a_2) - \epsilon_2/10 &< x_0 + R_1 < \pi_1(a_2), \\ \pi_1(a_2) &< \pi_1(b_2) < x_1 - R_1 < \pi_1(b_2) + \epsilon_2/10, \\ \pi_1(a_3) - \epsilon_2/10 &< x_1 + R_1 < \pi_1(a_3) < \pi_1(b_3). \end{aligned}$$

We can also require the segments to be horizontal, but this is not necessary and such a requirement cannot be made below when $k \geqslant 1$.

Next fix $k \geqslant 1$ and assume that $I_m(w', v')$ and ϵ_ℓ are defined for $(w', v') \in \mathcal{Y}_\ell$, $0 \leqslant \ell \leqslant k-1$, so that

$$\text{if } B_1 \in \mathcal{B}_{\ell_1} \text{ and } B_2 \in \mathcal{B}_{\ell_2},\ B_1 \neq B_2,\ \text{then } \overline{B}_1 \cap \overline{B}_2 = \emptyset. \tag{47}$$

Here

$$\mathcal{B}_\ell = \{B(z, \epsilon_\ell) :\ z \text{ endpoint of } I_m(w,v),\ (w,v) \in \mathcal{Y}_\ell,\ m = 1,2,3\}. \tag{48}$$

Let

$$I_1(w,v),\ I_2(w,v),\ I_3(w,v) \subset \operatorname{int}(Q_w) \setminus (Q_{w0} \cup Q_{w1}), \quad (w,v) = (w'\alpha, v'\beta) \in \mathcal{Y}_k,$$

be disjoint segments with the following properties: if we denote $a_m(w,v) = a_m$ and $b_m(w,v) = b_m$, then

$$\begin{aligned} \pi_1(a_1) &< \pi_1(b_1) < x_{w0} - R_{k+1} < \pi_1(b_1) + \epsilon_{k+1}/10, \\ \pi_1(a_2) - \epsilon_{k+1}/10 &< x_{w0} + R_{k+1} < \pi_1(a_2), \\ \pi_1(a_2) &< \pi_1(b_2) < x_{w1} - R_{k+1} < \pi_1(b_2) + \epsilon_{k+1}/10, \\ \pi_1(a_3) - \epsilon_{k+1}/10 &< x_{w1} + R_{k+1} < \pi_1(a_3) < \pi_1(b_3). \end{aligned}$$

We also require that (See Figure 4) if we denote

$$\begin{aligned} r_v = \epsilon_k,\ R_v = (\epsilon_{k-1}\epsilon_k)^{1/2} \quad &\text{if } \beta = 0 \text{ or } 1, \\ r_v = (\epsilon_{k-1}\epsilon_k)^{1/2},\ R_v = \epsilon_{k-1} \quad &\text{if } \beta = 2 \text{ or } 3, \end{aligned} \tag{49}$$

then for each $r_v < r < R_v$ there are arcs

$$\begin{aligned} J_1(r,w,v) \subset S(b_m(w',v'), r), \quad m = \alpha + 1, \\ J_3(r,w,v) \subset S(a_m(w',v'), r), \quad m = \alpha + 2, \end{aligned}$$

whose relative interiors are disjoint and do not intersect any segment $I_m(\tilde{w}, \tilde{v})$, $(\tilde{w}, \tilde{v}) \in \mathcal{Y}_\ell$, $0 \leqslant \ell \leqslant k$, so that

(i) the endpoints of $J_1(r,w,v)$ lie in $I_{\alpha+1}(w',v')$ and $I_1(w,v)$.
(ii) the endpoints of $J_3(r,w,v)$ lie in $I_{\alpha+2}(w',v')$ and $I_3(w,v)$.

We are now ready to define D; it is the domain for which

$$\mathcal{C}(D) = K \cup \{I_m(w,v) :\ m = 1,2,3,\ (w,v) \in \mathcal{Y}_k,\ k = 0,1,2,\ldots\},$$

where K is the Cantor set in (45).

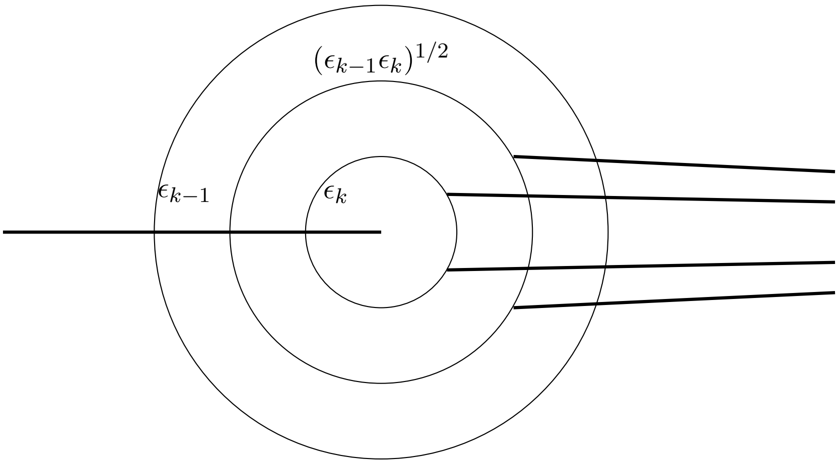

FIGURE 4. Positioning of the segments $I_m(w,v)$

4.2. **Construction of the exhaustion.** We now construct an exhaustion $\Phi_0 = (D_j)$ of D. We fix $k \in \mathbb{N}$ and $(w,v) \in \mathcal{Y}_k$. First, let $U(w,v)$ be a Jordan domain so that if we denote $I(w,v) = I_1(w,v) \cup I_2(w,v) \cup I_3(w,v)$ then

$$I(w,v) \subset U(w,v) \subset \overline{U}(w,v) \subset \operatorname{int}(Q_w) \setminus (Q_{w0} \cup Q_{w1}).$$

We also require that if $B \in \cup_\ell \mathcal{B}_\ell$ where $\mathcal{B}_\ell$ is the family of balls in (48), then either $U(w,v) \cap B = \emptyset$ or $I(w,v) \cap B \neq \emptyset$ and

$$U(w,v) \cap B \subset N_k(I(w,v)) \cap B. \tag{50}$$

Here $N_k(I(w,v))$ is the set of those $x \in \mathbb{C}$ for which

$$\operatorname{dist}(x, I(w,v)) < \frac{\min\{\epsilon_{k+1}, \operatorname{dist}(I(w,v), \mathbb{C} \setminus (D \cup I(w,v))\}}{100}. \tag{51}$$

Next, for $m = 1,2,3$ denote

$$U_m(k+1,w,v) = U(w,v)$$

and let

$$U_m(j,w,v), \quad j = k+2, k+3, \ldots$$

be Jordan domains so that

$$\overline{U}_m(k+2,w,v) \cap \overline{U}_{m'}(k+2,w,v) = \emptyset \quad \text{if } m \neq m',$$

and, with $N_j(I_m(w,v))$ defined as in (51),

$$I_m(w,v) \subset U_m(j,w,v) \subset \overline{U}_m(j,w,v) \subset U_m(j-1,w,v) \cap N_j(I_m(w,v)).$$

We denote

$$\mathcal{A}_j = \{\overline{U}_m(j,w,v) : m = 1,2,3, (w,v) \in \mathcal{Y}_k, 0 \leqslant k \leqslant j-1\},$$

and define D_j by

$$\mathcal{C}(D_j) = \mathcal{Q}_j \cup \mathcal{A}_j,$$

where $\mathcal{Q}_j$ is the family of squares in (44). Then $\Phi_0 = (D_j)$ is an exhaustion of D. Theorem 1.2 follows by combining the two propositions below and choosing any $\Phi = (D_{j_n})$ so that (f_{j_n}) converges.

Proposition 4.1. *There is $\delta > 0$ such that if $p = p_{\overline{w}} \in K$, then*

$$\operatorname{mod}(I_1, Q_{\overline{w}(j_0)}; D_j) \geqslant \delta \quad \textit{for all } j_0 \in \mathbb{N} \textit{ and } j > j_0. \tag{52}$$

Here I_1 is the segment in (46).

Recall that, given a domain $D \subset \hat{\mathbb{C}}$, $p \in \mathcal{C}(D)$, and an exhaustion $\Phi = (D_j)$ of D, we denote by p_ℓ the component in $\mathcal{C}(D_\ell)$ containing p. With this notation, $Q_{\overline{w}(j_0)} = p_{j_0}$ in (52).

Proposition 4.2. *Suppose $D \subset \hat{\mathbb{C}}$ is a domain with exhaustion $\Phi = (D_j)$. Fix $p \in \mathcal{C}(D)$ and a compact set $E \subset \hat{\mathbb{C}}$ such that $E \cap p = \emptyset$. If*

$$\lim_{\ell\to\infty} \liminf_{j\to\infty} \operatorname{mod}(E, p_\ell; D_j) > 0, \tag{53}$$

then $\hat{f}(p)$ is non-trivial for all $f \in \mathcal{F}_\Phi$.

4.3. **Proof of Proposition 4.1.** Fix $p = p_{\overline{w}} \in K$, $j_0 \in \mathbb{N}$, and $j > j_0$. Let $\mathcal{V}_0 = \{e\}$, and for $k = 1, 2, \ldots$, let

$$\mathcal{V}_k = \{v = v_1 v_2 \cdots v_k : v_\ell = \{0, 1, 2, 3\} \text{ for all } 1 \leqslant \ell \leqslant k\},$$

so that $\mathcal{Y}_k = \mathcal{W}_k \times \mathcal{V}_k$. We consider the family of continua

$$\eta(v,t) \subset \hat{D}_j,\ v \in \mathcal{V}_{j-1},\ 1/4 < t < 3/4,$$

defined as follows: if $v = v_1 v_2 \ldots v_{j-1}$, let $\eta(v,t) = A_j(v) \cup B_j(v,t)$, where

$$\begin{aligned} A_j(v) &= \cup\{\overline{U}_m(j, w, v_k) \in \mathcal{A}_j : m = 1,2,3,\ w \in \mathcal{W}_k,\ 0 \leqslant k \leqslant j-1\}, \text{ and} \\ B_j(v,t) &= \cup\{J_m(r[t,k], w, v_k) : m = 1,3,\ w \in \mathcal{W}_k,\ 1 \leqslant k \leqslant j-1\}. \end{aligned}$$

Here $r[t,k] = R_{v_k}^t r_{v_k}^{1-t}$ and R_{v_k}, r_{v_k} are the radii in (49).

Each $\eta(v,t)$ is a continuum joining $\overline{U}_1(j,e,e)$ and $\overline{U}_3(j,e,e)$ in $\hat{D}_j$. Moreover, each $\eta(v,t)$ intersects $Q_{\overline{w}(j_0)}$. By (50), we have $\eta(v,t) \setminus A_j(v) \subset D_j$. It is important to notice that the continua $\eta(v,t)$ do not intersect any of the squares in $\mathcal{Q}_j \subset \mathcal{C}(D_j)$.

Let $\gamma_{v,t}$ be an arc-length parametrization of $\eta(v,t)$, and

$$\Gamma_j = \{\gamma_{v,t} : v \in \mathcal{V}_{j-1},\ 1/4 < t < 3/4\}.$$

In view of the comments above, (52) follows if we can prove a lower bound for $\operatorname{mod}\Gamma_j$ independent of j. Fix $\rho \in X(\Gamma_j)$;

$$1 \leqslant \sum_{q \in A_j(v)} \rho(q) + \sum_{J \in B_j(v,t)} \int_J \rho\, ds.$$

Integrating both sides over $1/4 < t < 3/4$ and summing over $v \in \mathcal{V}_{j-1}$ yields

$$\frac{4^{j-1}}{2} \leqslant \frac{1}{2} \sum_{v \in \mathcal{V}_{j-1}} \sum_{q \in A_j(v)} \rho(q) + \sum_{v \in \mathcal{V}_{j-1}} \int_{1/4}^{3/4} \sum_{J \in B_j(v,t)} \int_J \rho\, ds\, dt = S_1 + S_2.$$

We estimate the sums S_1, S_2 from above. First, changing the order of summation yields

$$2S_1 = \sum_{k=0}^{j-2} 4^{j-1-k} \sum_{\substack{(w,v') \in \mathcal{Y}_k \\ m=1,2,3}} \rho(\overline{U}_m(j,w,v')) + \sum_{(w,v) \in \mathcal{Y}_{j-1}} \rho(\overline{U}(j,w,v)) = \sum_{k=0}^{j-1} S'_k.$$

Hölder's inequality yields

$$S_k' \leqslant 4^{j-k-1}(3\cdot 2^k\cdot 4^k)^{1/2}\Big(\sum_{q\in\mathcal{C}(D_j)}\rho(q)^2\Big)^{1/2} \leqslant 2^{2j-k/2-1}\Big(\sum_{q\in\mathcal{C}(D_j)}\rho(q)^2\Big)^{1/2}$$

for all $0\leqslant k\leqslant j-1$. Thus, summing over k we have

$$S_1 \leqslant 4^j\sum_{k=0}^{j-1}2^{-k/2}\Big(\sum_{q\in\mathcal{C}(D_j)}\rho(q)^2\Big)^{1/2} \leqslant 4^{j+1}\Big(\sum_{q\in\mathcal{C}(D_j)}\rho(q)^2\Big)^{1/2}.$$

We now estimate S_2. First, we denote by $\mathcal{Z}_\ell$ the set of centers z in the definition of $\mathcal{B}_\ell$ in (48). Fubini's theorem and (47) yield

$$S_2 \leqslant \sum_{k=1}^{j-1}4^{j-k-1}\sum_{z\in\mathcal{Z}_k}\int_{1/4}^{3/4}\int_{S(z,r[t,k])}\rho\,ds\,dt = \sum_{k=1}^{j-1}T_k. \tag{54}$$

We apply change of variables to the integral in (54) to conclude that

$$T_k \leqslant 4^{j-k}\Big(\log\frac{\epsilon_{k-1}}{\epsilon_k}\Big)^{-1}\sum_{z\in\mathcal{Z}_k}\int_{B(z,\epsilon_{k-1})\setminus\overline{B}(z,\epsilon_k)}\frac{\rho(x)}{|x|}\,dA(x). \tag{55}$$

Applying Hölder's inequality to the integral in (55) yields

$$T_k \leqslant (2\pi)^{1/2}4^{j-k}\Big(\log\frac{\epsilon_{k-1}}{\epsilon_k}\Big)^{-1/2}\sum_{z\in\mathcal{Z}_k}\Big(\int_{B(z,\epsilon_{k-1})}\rho(x)^2\,dA(x)\Big)^{1/2}.$$

Since $\operatorname{card}(Z_k)\leqslant 6\cdot 8^{k-1}\leqslant 8^k$ for all $0\leqslant k\leqslant j-1$, we moreover have

$$T_k \leqslant (2\pi)^{1/2}4^j\cdot 2^{-k/2}\Big(\log\frac{\epsilon_{k-1}}{\epsilon_k}\Big)^{-1/2}\Big(\int_{D_j}\rho(x)^2\,dA(x)\Big)^{1/2}.$$

Thus, if we require that $\epsilon_k\leqslant\epsilon_{k-1}/e$ for all k, we have

$$S_2 \leqslant (2\pi)^{1/2}4^j\sum_{k=1}^{j-1}2^{-k/2}\Big(\int_{D_j}\rho(x)^2\,dA(x)\Big)^{1/2} \leqslant 4^{j+2}\Big(\int_{D_j}\rho(x)^2\,dA(x)\Big)^{1/2}.$$

Combining the estimates yields

$$\begin{aligned}4^{j-2} &\leqslant 4^{j+1}\Big(\sum_{q\in\mathcal{C}(D_j)}\rho(q)^2\Big)^{1/2} + 4^{j+2}\Big(\int_{D_j}\rho(x)^2\,dA(x)\Big)^{1/2}\\ &\leqslant 4^{j+3}\Big(\int_{D_j}\rho(x)^2\,dA(x) + \sum_{q\in\mathcal{C}(D_j)}\rho(q)^2\Big)^{1/2}.\end{aligned}$$

We conclude that $\operatorname{mod}(\Gamma_j)\geqslant 4^{-10}$. The proof is complete.

4.4. **Proof of Proposition 4.2.** By taking a subsequence of (D_j), we may assume $f_j\to f$. Suppose towards contradiction that $\hat{f}(p)$ is a point component. We lose no generality by assuming $\hat{f}(p)=\{0\}$.

Lemma 4.3. *Suppose $\hat{f}(p)=\{0\}$. For every $R>0$ there are $r>0$ and $m\in\mathbb{N}$ so that if $j\geqslant m$ and if $q\in\mathcal{C}(D_j)$ satisfies $\hat{f}_j(q)\cap S(0,R)\neq\emptyset$, then $\hat{f}_j(q)\cap S(0,r)=\emptyset$.*

Proof. Suppose towards contradiction that there is $R > 0$ and a sequence (q_{n_j}), $q_{n_j} \in \mathcal{C}(D_{n_j})$, so that each $\hat{f}_{n_j}(q_{n_j})$ intersects both $S(0,R)$ and $S(0,2^{-j})$. By passing to a subsequence if necessary, we may assume $n_j = j$.

For each $j \in \mathbb{N}$, fix a point $x_j \in q_j$. Since $\hat{\mathbb{C}} \setminus D$ is compact, (x_j) has a subsequence converging to $x_0 \in q_0$ for some $q_0 \in \mathcal{C}(D)$. We may assume that $x_j \to x_0$. It follows that if $k \in \mathbb{N}$ and if $q_0(k)$ is the element of $\mathcal{C}(D_k)$ containing q_0, then

$$q_j \subset q_0(k) \quad \text{for all } j \geqslant j_k.$$

In particular, since $\hat{f}_j(q_j)$ intersects both $S(0,R)$ and $S(0,2^{-j})$, so does $\hat{f}_j(q_0(k))$. We conclude that $\hat{f}(q_0(k))$ contains both the origin and a point in $S(0,R)$. But this holds for all k, so also $\hat{f}(q_0)$ contains both the origin and a point in $S(0,R)$. This contradicts our assumption, that $\hat{f}(p) = \{0\}$. The proof is complete. □

We use Lemma 4.3 to construct a decreasing sequence (R_n) of positive real numbers and an increasing sequence j_n of indices as follows (compare to the proof of Lemma 2.7): First, choose R_1, j_1 so that $\hat{f}_j(E) \cap B(0,2R_1) = \emptyset$ for all $j \geqslant j_1$. Here E is the compact set in the statement of the proposition.

Then, assuming that R_n, j_n have been constructed, choose $R_{n+1} < R_n/2$ and $j_{n+1} \geqslant j_n$ such that if $q \in \mathcal{C}(D_j)$, $j \geqslant j_{n+1}$, and $\hat{f}_j(q) \cap S(0,R_n) \neq \emptyset$, then $\hat{f}_j(q) \cap S(0,2R_{n+1}) = \emptyset$.

Given $k \in \mathbb{N}$, let N be the largest number for which there is $j'_N \geqslant k$ so that $\hat{f}_j(p_k) \subset B(0,R_N)$ for all $j \geqslant j'_N$. We may assume that $j'_N = j_N$. Then

$$\operatorname{mod}(\hat{f}_j(E), \hat{f}_j(p_k); f_j(D_j)) \leqslant \operatorname{mod}(S(0,2R_1), S(0,R_N); f_j(D_j)) \tag{56}$$

for all $j \geqslant j_N$ (here the modulus on the left is over all paths connecting $\hat{f}_j(E)$ and $\hat{f}_j(p_k)$ in $\widehat{f_j(D_j)}$, a slight abuse of earlier terminology). Fix such a j. We construct a test function ρ as follows: First, let $1 \leqslant n \leqslant N$. We denote $A(R,r) = B(0,R) \setminus \overline{B}(0,r)$ and define

$$\rho_n(x) = \begin{cases} \frac{1}{|x|\log 2}, & x \in D_j \cap A(2R_n, R_n) \\ \frac{\operatorname{diam}(x)}{R_n \log 2}, & x \in \mathcal{C}(f_j(D_j)),\ x \cap A(2R_n,R_n) \neq \emptyset,\ \operatorname{diam}(x) \leqslant \operatorname{dist}(x,0), \\ 1, & x \in \mathcal{C}(f_j(D_j)),\ x \cap A(2R_n,R_n) \neq \emptyset,\ \operatorname{diam}(x) > \operatorname{dist}(x,0), \end{cases}$$

and $\rho_n(x) = 0$ otherwise. As in the proof of Lemma 2.7, we have

$$\rho = \frac{1}{N}\sum_{n=1}^{N} \rho_n \in X(S(0,2R_1), S(0,R_N); f_j(D_j)) \quad \text{for all } j \geqslant j_N.$$

For each $q \in \mathcal{C}(D_j)$ there is at most one n such that $\rho_n(q) \neq 0$. Moreover, for every n there are at most 30 elements (disks) $q \in \mathcal{C}(f_j(D_j))$ such that $q \cap A(2R_n,R_n) \neq \emptyset$ and $\operatorname{diam}(q) > \operatorname{dist}(q,0)$. Thus we can estimate

$$\int_{f_j(D_j)} \rho_n^2 \, dA + \sum_{q \in \mathcal{C}(f_j(D_j))} \rho_n(q)^2 \leqslant \frac{1}{(\log 2)^2} \int_{A(2R_n,R_n)} \frac{dA}{|x|^2}$$

$$+ \quad \frac{\operatorname{Area}(B(0,4R_n))}{R_n^2 (\log 2)^2} + 30 \leqslant \frac{2\pi}{\log 2} + \frac{16\pi}{(\log 2)^2} + 30 \leqslant 1000,$$

and, since we have chosen j_k so that every $q \in \mathcal{C}(f_j(D_j))$ satisfies $\rho_n(q) \neq 0$ for at most one n,

$$\int_{f_j(D_j)} \rho^2 \, dA + \sum_{q \in \mathcal{C}(f_j(D_j))} \rho(q)^2 \leqslant \frac{1000N}{N^2} = \frac{1000}{N}. \tag{57}$$

Since $N \to \infty$ as $k \to \infty$, combining (57) with (56) yields

$$\lim_{k\to\infty} \liminf_{j\to\infty} \operatorname{mod}(\hat{f}_j(E), \hat{f}_j(p_k); f_j(D_j)) = 0. \tag{58}$$

But (58) and the conformal invariance of the modulus contradict our assumption (53). The proof is complete.

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