Quasiconformal mappings in the plane, Exercise set 9
Due 18.11. 2016

Recall: the winding number of a smooth $\gamma : T(0,1) \to \mathbb{C}$ around $z_0 \notin |\gamma|$ is

$$W_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0} \in \mathbb{Z}.$$

1. Extend the definition to continuous paths as follows: Show that there is $n \in \mathbb{Z}$ such that if (γ_j) is a sequence of smooth paths converging uniformly to γ, then $W_{\gamma_j}(z_0) = n$ for all $z_0 \notin |\gamma|$ and j large enough (recall the complex logarithm).

Let $f : \Omega \to \mathbb{C}$ be continuous, $G \subset\subset \Omega$ a positively oriented (see Section III.2) domain bounded by finitely many disjoint Jordan curves γ_k, and $w_0 \notin f(\partial G)$. The *degree* of f at w_0 with respect to G is $\deg(w_0, f, G) = \sum_k W_{f \circ \gamma_k}(w_0)$.

A continuous $f : \Omega \to \mathbb{C}$ is *sense-preserving*, if $\deg(w_0, f, G) > 0$ for all G and all $w_0 \in f(G) \setminus f(\partial G)$. The degree has the following properties:

(i) Let $H_t : \Omega \times [0,1] \to \mathbb{C}$ be continuous, and $w_0 \notin H(\partial G \times [0,1])$. Then $\deg(w_0, H_0, G) = \deg(w_0, H_1, G)$.

(ii) If $L : \mathbb{C} \to \mathbb{C}$ is linear and $\det(L) \neq 0$, then for all G and all $w_0 \in L(G)$, $\deg(w_0, L, G) = \det(L)/|\det(L)|$.

(iii) If $w_0 \notin f(\overline{G})$, then $\deg(w_0, f, G) = 0$.

(iv) If w_0 and w_1 belong to the same component of $\mathbb{C} \setminus f(\partial G)$, then $\deg(w_0, f, G) = \deg(w_1, f, G)$

2. Let $f : \Omega \to \mathbb{C}$ be continuous. Moreover, assume that f is differentiable at z_0 with $\det(Df(z_0)) > 0$. Show that $\deg(f(z_0), f, \mathbb{D}(z_0, \epsilon)) = 1$ when ϵ is small (hint: Apply (i) and (ii) to $H_t(z) = (1-t)f(z) + t(Df(z_0)(z - z_0) + f(z_0)))$.

3. Give a continuous $f : \mathbb{C} \to \mathbb{C}$ such that neither f nor $\bar{f}$ is sense-preserving.

4. Let $f : \Omega \to \Omega'$ be quasiconformal. Show that f is sense-preserving (hint: Recall that f is differentiable and $J(z, f) > 0$ a.e.. Apply Problem 2, the definition of degree, and properties (iii) and (iv)).

5. Let $f : \mathbb{C} \to \mathbb{C}$ be a sense-preserving local homeomorphism. Assume that there exists a constant $M > 0$ such that $|f(z) - z| \leq M$ for all $z \in \mathbb{C}$. Show that f is a homeomorphism.