**Quasiconformal mappings in the plane, Exercise set 8**
**Due 11.11. 2016**

**1.** Let $f$ be quasiconformal. Show that

$$(f^{-1})_{\bar{w}}(w) = \frac{-f_{\bar{z}}(z)}{J(z,f)} \quad \text{and} \quad \overline{(f^{-1})_w(w)} = \frac{f_z(z)}{J(z,f)},$$

where $z = f^{-1}(w)$ (hint: apply Exercise 7.2).

**2.** Let $f : \Omega \to \Omega'$ and $g : \Omega \to \Omega''$ be quasiconformal with Beltrami coefficients $\mu_f$ and $\mu_g$, respectively. Show that

$$\mu_{g\circ f^{-1}}(w) = \frac{\mu_g(z) - \mu_f(z)}{1 - \mu_g(z)\overline{\mu_f(z)}} \Big(\frac{f_z(z)}{|f_z(z)|}\Big)^2$$

where $z = f^{-1}(w)$. Conclude that if $f$ and $g$ solve the same Beltrami equation, then $g \circ f^{-1}$ is conformal.

**3.** Evaluate $\mathcal{C}\chi_{\mathbb{D}(0,r)}$ and $S\chi_{\mathbb{D}(0,r)}$ (hint: Apply $(\mathcal{C}\phi)_{\bar{z}} = \phi$, $(\mathcal{C}\phi)_z = S\phi$).

**4.** *The logarithmic potential* $\mathcal{L}$ of $\phi \in C_0^\infty(\mathbb{C},\mathbb{C})$ is

$$(\mathcal{L}\phi)(z) = \frac{2}{\pi}\int_{\mathbb{C}} \phi(w)\log|z-w|\, dA(w).$$

Show that

$$(\mathcal{L}\phi)_z(z) - (\mathcal{L}\phi)_z(0) = (\mathcal{L}\phi_z)(z) - (\mathcal{L}\phi_z)(0) = (\mathcal{C}\phi)(z).$$

**5.** Let $p, q \in C^\infty(\mathbb{C},\mathbb{C})$ satisfy $p_{\bar{z}} = q_z$. Show that there exists $f \in C^\infty(\mathbb{C},\mathbb{C})$ such that $f_z = p$ and $f_{\bar{z}} = q$ (notice that the converse is also true, since $p_{\bar{z}} = f_{z\bar{z}} = f_{\bar{z}z} = q_z$).