Quasiconformal mappings in the plane, Exercise set 7
Due 4.11. 2016

1. Show that the complex derivatives satisfy

$$\overline{(f_z)} = (\overline{f})_{\bar{z}}, \quad \overline{(f_{\bar{z}})} = (\overline{f})_z.$$

2. Show that the complex derivatives satisfy

$$(g \circ f)_z = (g_\zeta \circ f) f_z + (g_{\bar{\zeta}} \circ f)\overline{(f_{\bar{z}})}, \quad (g \circ f)_{\bar{z}} = (g_\zeta \circ f) f_{\bar{z}} + (g_{\bar{\zeta}} \circ f)\overline{(f_z)}$$

(hint: Express $Df(z)h = f_z(z)h + f_{\bar{z}}(z)\bar{h}$, $Dg(w)k = g_w(w)k + g_{\bar{w}}(w)\bar{k}$).

3. Show that the complex integrals satisfy

$$\int_\gamma f\,dz = \int_\gamma f\,dx + i\int_\gamma f\,dy, \quad \int_\gamma f\,d\bar{z} = \int_\gamma f\,dx - i\int_\gamma f\,dy.$$

4. Let $f : \mathbb{C} \to \mathbb{C}$ be continuous. Show that for every $z \in \mathbb{C}$

$$\lim_{\epsilon \to 0} \int_{T(z,\epsilon)} \frac{f(w)\,dw}{w - z} = 2\pi i f(z), \quad \lim_{\epsilon \to 0} \int_{T(z,\epsilon)} \frac{f(w)\,d\bar{w}}{w - z} = 0$$

(the circles are positively oriented).

5. Let Ω be a bounded Jordan domain. Prove the *isoperimetric inequality*

$$|\Omega| \leq \frac{\ell(\partial\Omega)^2}{4\pi}$$

as follows: Use Green's formula to conclude

$$\int_{\partial\Omega}\int_{\partial\Omega} \frac{\bar{w} - \bar{z}}{w - z}\,dz\,dw = -2i \int_{\partial\Omega}\int_{\Omega} \frac{1}{w - z}\,dA(z)\,dw,$$

then apply winding numbers on the right.

6. Let $\Omega = (0,1)^2 \subset \mathbb{C}$, and $g \in W^{1,1}(\Omega, \mathbb{C}) \cap C(\overline{\Omega})$. Prove one part of Green's formula (see the proof of Theorem 2.1):

$$\int_\Omega g_{\bar{z}}\,dA = \frac{-i}{2}\int_{\partial\Omega} g\,dz.$$