Quasiconformal mappings in the plane, Exercise set 6
Due 28.10. 2016

Let $\Gamma \subset \mathbb{C}$ be a Jordan curve (ie, set homeomorphic to the unit circle $T(0,1)$). Points $a, b \in \Gamma$ divide Γ into components $\Gamma^{a,b}$ and $\Gamma_{a,b}$ such that $\operatorname{diam} \Gamma^{a,b} \geq \operatorname{diam} \Gamma_{a,b}$.

1. Suppose $h : T(0,1) \to \Gamma$ is η-quasisymmetric. Show that there exists $C > 0$ depending only on η such that $\operatorname{diam}(h^{-1}(\Gamma^{a,b})) \geq C$ for every $a, b \in \Gamma$.

2. Let Γ be a quasicircle. Show that there exists $A \geq 1$ such that for every $a, b \in \Gamma$ and $c \in \Gamma_{a,b}$,

$$|a - c| + |c - b| \leq A|a - b| \tag{1}$$

(hint: Apply quasisymmetry and Problem 1).

Fact: A Jordan curve is a quasicircle if and only if it satisfies (1) (one direction is Problem 2, the other direction is more difficult to show)

3. Draw the von Koch snowflake. Assuming the above fact, convince yourself that the von Koch snowflake is a quasicircle (you do not need to write down your arguments).

A subset E of $\mathbb{C}$ is δ-*porous*, if for every $z \in \mathbb{C}$ and $r > 0$ there is $w \in \mathbb{C}$ such that $\mathbb{D}(w, \delta r) \subset \mathbb{D}(z, r)$ and $\mathbb{D}(w, \delta r) \cap E = \emptyset$.

4. Let Γ be a K-quasicircle. Show that Γ is δ-porous, where δ depends only on K (hint: fix a K-quasiconformal map f of the plane mapping $T(0,1)$ onto Γ, then apply quasisymmetry of f to suitable discs).

5. Let $E \subset \mathbb{C}$ be δ-porous. Show that the Hausdorff dimension of E is at most $2 - \epsilon$, where $\epsilon > 0$ depends on δ. Conclude that the Hausdorff dimension of every quasicircle is strictly less than 2 (hint: you may assume $E \subset [0,1]^2$. Cover E by smaller and smaller congruent squares, and apply porosity to count the number of squares needed for covering).