Quasiconformal mappings in the plane, Exercise set 5
Due 21.10. 2016

1. Let $f_j : \Omega \to \Omega_j$ be η-quasisymmetric homeomorphisms converging locally uniformly to a non-constant $f : \Omega \to f(\Omega)$. Show that f is η-quasisymmetric.

2. Let f_j, f be as in Problem 1, and $D \subset \Omega$ a closed disc. Show that

$$\lim_{j\to\infty} |f_j(D)| = |f(D)|$$

(hint: Apply measure theory. Notice that
$(f(D) \cap f_j(D) \cap f_{j+1}(D)...) \subset f_j(D) \subset (f(D) \cup f_j(D) \cup f_{j+1}(D)...))$

3. A continuous function $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a *quasiminimizer* (of 2-energy), if there exists $M \geq 1$ such that, for all domains $G \subset\subset \Omega$ and all continuous $v \in W^{1,2}_{\mathrm{loc}}(\Omega)$ with $v|_{\partial G} = u|_{\partial G}$,

$$\int_G |\nabla u|^2 \leq M \int_G |\nabla v|^2.$$

Show that if $f : \Omega \to \Omega'$ is quasiconformal and u a quasiminimizer on Ω', then $u \circ f$ is a quasiminimizer on Ω.

We denote

$$\mathcal{F}_K = \{f : \mathbb{C} \to \mathbb{C} |\, f \text{ is } K-\text{quasiconformal},\ f(0) = 0,\ f(1) = 1\}.$$

4. Show that $\mathcal{F}_K$ is a normal family (hint: apply quasisymmetry).

5. Let (f_j) be a sequence in $\mathcal{F}_K$. Suppose that each f_j is K_j-quasiconformal and $K_j \to 1$ as $j \to \infty$. Show that $f_j \to Id$ locally uniformly (hint: recall that Id is the only 1-quasiconformal (ie, conformal) map in $\mathcal{F}_K$).

6. Let $f \in \mathcal{F}_K$. Show that there exists a constant $C(K)$ such that $|f(z)| \geq C(K)|z|^{1/K}$ for every z with $|z| \geq 2$ (hint: apply the proof of Theorem 7.1.)