Quasiconformal mappings in the plane, Exercise set 4
Due 14.10. 2016

1. Show that the inverse of a quasisymmetric map is quasisymmetric.

2. Show that the composition of two quasisymmetric maps is quasisymmetric.

3. Let $\mathcal{C} : (0,1) \to (0,1)$ be the standard Cantor function, and

$$f : (0,1)^2 \to (0,1) \times (0,2),\ f(x+iy) = x + i(y + \mathcal{C}(x)).$$

Show that f is a homeomorphism satisfying $H(z,f) = 1$ at almost every $z \in (0,1)^2$ but NOT quasiconformal.

A Borel measure μ on $\mathbb{R}^n$ is *doubling*, if there exists a constant $C > 0$ such that $\mu(B(x,2r)) \leq C\mu(B(x,r))$ for every $x \in \mathbb{R}^n$ and $r > 0$.

4. Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 1$, be quasisymmetric, and define μ by $\mu(E) = |f(E)|$. Show that μ is doubling (notice that the definition of quasisymmetry extends to maps between $\mathbb{R}^n$).

5. Let μ be a doubling measure on $\mathbb{R}$. Show that f is quasisymmetric, where

$$f(t) = \begin{cases} \mu([0,t]) & t \geq 0, \\ -\mu([t,0]) & t < 0 \end{cases}$$

(Theorem 4.6 extends to $n = 1$, so it suffices to show weak quasisymmetry).

6. Let $F \subset \mathbb{R}$ be compact, and $\epsilon > 0$. Show that there exists $\delta > 0$ such that for every $0 < r < \delta$ there are intervals $I_j = (s_j - r, s_j + r)$, $j = 1, \ldots, p$, for which

1. $F \subset \cup_j I_j$,
2. $\sum_j \chi_{I_j}(y) \leq 2$ for every $y \in \mathbb{R}$,
3. $p \cdot r < |F| + \epsilon$.