Quasiconformal mappings in the plane, Exercise set 3
Due 7.10. 2016

You will prove the following Theorem: Let $f : \mathbb{C} \to \Omega$ be K-quasiconformal. Then $\Omega = \mathbb{C}$, and f is weakly H-quasisymmetric with H depending only on K.

Recall: $T(z_0, t)$ is the circle with center z_0 and radius t.

1. Fix $\mathbb{D}(z_0, r)$ and $a > 1$. Denote $E = \{z \in \mathbb{D}(z_0, ar) : m < |f(z)| < M\}$, and assume $[m, M] \subset |f|(T(z_0, t))$ for every $r < t < ar$. Show that

$$(a-1)r \log \frac{M}{m} \leq \int_E \frac{||Df||}{|f|}\, dA \tag{1}$$

(hint: apply the fundamental theorem of calculus to $\log |f|$).

2. Apply (1), Hölder's inequality, and change of variables to show

$$\log \frac{M}{m} \leq \frac{2\pi^2 K a^2}{(a-1)^2}. \tag{2}$$

3. Fix $s > 0$. Let a, b be distinct points in $T(0, s)$. Show that there exist $z_0 \in \mathbb{D}(0, s)$ and $0 < r \leq s/2$ such that, for every $r < t < \sqrt{3}r$, either

(i) $0, b \in \mathbb{D}(z_0, t)$ and $a \notin \mathbb{D}(z_0, t)$, or

(ii) $a, b \in \mathbb{D}(z_0, t)$ and $0 \notin \mathbb{D}(z_0, t)$

(hint: Draw a triangle with vertices $0, a, b$. Divide to two cases depending on whether or not $|a - b| \leq s = |b|$).

4. Let s, a, b, z_0 and r be as above. Let E be connected set joining 0 to a, and F a connected set joining b to $T(0, 2s)$. Show that, for every $r < t < \sqrt{3}r$, $T(z_0, t)$ intersects both E and F (it suffices to give an intuitive argument).

5. Show that there exists $H = H(K)$ such that $H_f(z, s) \leq H$ for every $z \in \mathbb{C}$ and $s > 0$ (hint: You may assume $z = 0 = f(z)$. Draw the smallest disc D_M centered at 0 containing $f(\mathbb{D}(0, s))$, and similarly the largest disc D_m contained in it. Choose $E = f^{-1}\overline{D}_m$, $F = \mathbb{C} \setminus f^{-1}(D_M)$, and apply Problems 2 and 4).

6. Show that $\Omega = \mathbb{C}$ (hint: If not, there exists a $w \in \partial f(\mathbb{C})$. Apply weak quasisymmetry of f to a suitable disc to get a contradiction).