Quasiconformal mappings in the plane, Exercise set 2
Due 30.9. 2016

Let $\varphi : \mathbb{C} \to [0, \infty)$ be the smoothing kernel in Theorem 7.1 of the lectures, $u \in L^1_{\text{loc}}(\Omega)$, and $u^\epsilon = u \star \varphi_\epsilon$, $\epsilon > 0$.

1. Apply Dominated Convergence to show that u^ϵ has partial derivatives that satisfy $\partial_j u^\epsilon = u \star \partial_j \varphi_\epsilon$. Similarly, show that $u^\epsilon \in C^\infty$.

2. Let $V \subset\subset \Omega$.

(i) Change variables to show that $u^\epsilon(z) = \int_{\mathbb{D}} \varphi(w) u(z - \epsilon w)\, dA(w)$ for all $z \in V$ and $\epsilon > 0$ small enough.

(ii) Let u be continuous. Apply (i) and continuity to show that $u^\epsilon \to u$ uniformly in V.

3. Let $u \in L^p(\Omega)$ and $V \subset\subset \Omega$.

(i) Apply Exercise 2(i) and Hölder's inequality to show

$$|u^\epsilon(z)| \le \Big(\int_{\mathbb{D}} \varphi(w) |u(z - \epsilon w)|^p \, dA(w) \Big)^{1/p} \tag{1}$$

for all $z \in V$ and $\epsilon > 0$ small enough.

(ii) Apply (1) to conclude that $||u^\epsilon||_{p,V} \le ||u||_{p,\Omega}$.

(iii) Real analysis fact: For every $\delta > 0$ there exists a continuous $v : \Omega \to \mathbb{R}$ such that

$$\int_{\Omega} |u - v|^p \, dA < \delta.$$

Combine with (ii) and Exercise 2(i) to show that $u^\epsilon \to u$ in $L^p(V)$.

4. Let $u \in W^{1,p}_{\text{loc}}(\Omega)$.

(i) Let $z \in \Omega$, $\mathrm{d}(z, \partial\Omega) \ge \epsilon$. Apply Exercise 1 to show that the weak derivatives of u satisfy $\partial_j u^\epsilon = \varphi_\epsilon \star D_j u$.

(ii) Apply (i) and Exercise 3 to show that $u^\epsilon \to u$ in $W^{1,p}_{\text{loc}}(\Omega)$.

5. Another real analysis fact (Lebesgue differentiation theorem): Let $u \in L^1_{\text{loc}}(\Omega)$. Then

$$\lim_{r \to 0} r^{-2} \int_{\mathbb{D}(z,r)} u(z) - u(w) \, dA(w) = 0$$

for almost every $z \in \Omega$. Show that $u^\epsilon(z) \to u(z)$ for almost every $z \in \Omega$.