Quasiconformal mappings in the plane, Exercise set 10
Due 25.11. 2016

Let Γ be a family of rectifiable paths in $\mathbb{C}$. The *conformal modulus* $\mathrm{mod}(\Gamma)$ of Γ is $\mathrm{mod}(\Gamma) = \inf_{\rho \in X(\Gamma)} \int_{\mathbb{C}} \rho^2 \, dA$, where

$$X(\Gamma) = \Big\{ \rho : \mathbb{C} \to [0, \infty] \text{ Borel measurable} : \int_\gamma \rho \, ds \geq 1 \quad \forall \gamma \in \Gamma \Big\}.$$

1. Let $f : \Omega \to \Omega'$ be conformal. Show that $\mathrm{mod}(\Gamma) = \mathrm{mod}(f\Gamma)$ for all families Γ of paths in Ω, where $f\Gamma = \{f \circ \gamma : \gamma \in \Gamma\}$ (hint: since the inverse of f is also conformal, it suffices to show "$\leq$". Given $\rho \in X(f\Gamma))$, show that $\rho' = ||Df||(\rho \circ f) \in X(\Gamma))$.

Let $A \subset \hat{\mathbb{C}}$ be a domain homeomorphic to $A(r, R) = \mathbb{D}(0, R) \setminus \overline{\mathbb{D}}(0, r)$ and bounded by Jordan curves η_1 and η_2. We denote $\mathrm{mod}(A) := \mathrm{mod}(\Gamma_A)$, where

$$\Gamma_A = \{\gamma : [a, b] \to A \text{ rectifiable} : \gamma(a) \in \eta_1, \, \gamma(b) \in \eta_2\}.$$

2. Show that $\rho(z) = |z|^{-1} \log^{-1}(R/r) \chi_{A(r,R)} \in X(\Gamma_{A(r,R)})$, and

$$\mathrm{mod}(A(r, R)) \leq \int_{\mathbb{C}} \rho^2 \, dA = 2\pi \Big(\log \frac{R}{r} \Big)^{-1} \tag{1}$$

(hint: For the first claim, reparametrize $\gamma \in \Gamma_{A(r,R)}$ by arc length to get $\tilde{\gamma} : [r, r + \ell(\gamma)] \to A(r, R)$. Notice that $\ell(\gamma) \geq R - r$ and $|\tilde{\gamma}(t)| \leq t$ for every t).

3. Show that (1) is an equality (hint: Fix $\rho \in X(A(r, R))$. Then $1 \leq \int_{I(\theta)} \rho \, ds$ for all radial segments $I(\theta)$. Apply polar coordinates and Fubini's theorem).

4. Let $0 < a < 1$ and $\mathbb{H}_+$ the upper half plane. Show that

$$\mathrm{mod}(\mathbb{H}_+ \setminus \overline{\mathbb{D}}(i, a)) = 2\pi \Big(\log \frac{1 + a + \sqrt{1 - a^2}}{1 + a - \sqrt{1 - a^2}} \Big)^{-1}.$$

(hint: Find a Möbius transformation of form $T(z) = (z - bi)/(z + bi)$ mapping $\mathbb{H}_+ \setminus \overline{\mathbb{D}}(i, a)$ onto a concentric annulus. Then apply Problems 1 and 3).

Fact: A homeomorphism f is K-quasiconformal if and only if $K^{-1} \mathrm{mod}(\Gamma) \leq \mathrm{mod}(f\Gamma) \leq K \, \mathrm{mod}(\Gamma)$ for all Γ (one direction is basically Problem 1, the other direction is proved using the methods in Exercise Set 3 and II.5 of Lectures).

5. Does there exist a 2-quasiconformal $f : A(1, 2) \to \mathbb{H}_+ \setminus \overline{\mathbb{D}}(i, 1/2)$?

6. Show that there exists a conformal map $f : A \to A(1, R)$ if (and only if) $\mathrm{mod}(A) = 2\pi \log^{-1}(R)$ (hint: after translating the origin, you can map A onto a “vertical strip"V with the *multivalued* complex logarithm. Apply the Riemann mapping theorem to map V onto $(a, b) \times \mathbb{R}$, then show that composing with the exponential function gives a conformal map).