**Quasiconformal mappings in the plane, Exercise 1**
**Due 23.9. 2016**

The $z$- and $\bar{z}$-derivatives of $f = (u, v)$ can be expressed in polar coordinates $(r, \theta)$:

$$f_z = \frac{1}{2}e^{-i\theta}\Big(f_r - \frac{i}{r}f_\theta\Big), \quad f_{\bar{z}} = \frac{1}{2}e^{i\theta}\Big(f_r + \frac{i}{r}f_\theta\Big). \quad \text{Here } f_r = u_r + iv_r,\ f_\theta = u_\theta + iv_\theta.$$

**1.** Suppose $f$ is of the form $f(r, \theta) = r^t e^{is\theta}$, where $t, s > 0$. Evaluate $\mu_f$.

**2.** Let $0 < \varphi < 2\pi$. Construct a conformal map from the upper half plane onto

$$D_\varphi = \{(r, \theta) : r > 0, 0 < \theta < \varphi\} \quad \text{(hint: Exercise 1).}$$

A homeomorphism $g : \Omega \to \Omega'$ is $L$-*biLipschitz*, if

$$\frac{|z - z'|}{L} \le |g(z) - g(z')| \le L|z - z'| \quad \text{for every } z, z' \in \Omega.$$

**3.**

(i) Let $f : \Omega \to \Omega'$ be conformal, and $z_0 \in \Omega$. Show that there exist $L \ge 1$ and $r > 0$ depending on $z_0$ such that the restriction of $f$ to the disc $D(z_0, r)$ is $L$-biLipschitz (recall: conformal maps are diffeomorphisms).

(ii) Construct a bi-Lipschitz map $f : \mathbb{C} \to \mathbb{C}$ that is not conformal.

**4.**

(i) Suppose $f$ is a (smooth) bi-Lipschitz map. Show that $f$ is quasiconformal.

(ii) Construct a quasiconformal map that is not bi-Lipschitz near the origin.

For $0 < \alpha < \beta$, $A(\alpha, \beta) = \{(r, \theta) : \alpha < r < \beta\}$ is an annulus.

**5.** Suppose $f : A(s, S) \to A(t, T)$ is a (smooth) $K$-quasiconformal homeomorphism, and $T/t \ge S/s$. Show that

$$K_f \ge \frac{\log \frac{T}{t}}{\log \frac{S}{s}}$$

(hint: Argue as in the Grötzsch problem, replacing $u$ by $\log|f|$ and applying polar coordinates).

**6.** Suppose $f : A(s, S) \to A(t, T)$ is a (smooth) $K$-quasiconformal homeomorphism, and $T/t \le S/s$. Show that

$$K_f \ge \frac{\log \frac{S}{s}}{\log \frac{T}{t}}.$$