CONFORMAL UNIFORMIZATION OF DOMAINS BOUNDED BY QUASITRIPODS

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Abstract. We extend *Schramm's cofat uniformization theorem* to domains satisfying conditions involving *quasitripods*, i.e., quasisymmetric images of the standard tripod. If the non-point complementary components of domain $\Omega \subset \hat{\mathbb{C}}$ contain uniform quasitripods with large diameters and satisfy a *packing condition*, then there exists a conformal map $f\colon \Omega \to D$ onto a circle domain D. Moreover, f preserves the classes of point-components and non-point components. The packing condition is satisfied if Ω is *cospread*, i.e., if the complementary components contain uniform quasitripods in all scales.

1. Introduction

Koebe's conjecture ([Koe08], [HS93]) asserts that every domain in the Riemann sphere $\hat{\mathbb{C}}$ is conformally equivalent to a circle domain. In this paper we extend Schramm's *cofat uniformization theorem* [Sch95], which solves Koebe's conjecture for domains whose complementary components are *fat*, to domains whose complementary components are *spread* and satisfy a *packing condition*. We give the precise statement of Schramm's theorem after fixing some notation.

Given domain $G \subset \hat{\mathbb{C}}$, we call a connected component p of $\hat{\mathbb{C}} \setminus G$ *non-trivial* and denote $p \in \mathcal{C}_N(G)$ if $\operatorname{diam}(p \cap \mathbb{C}) > 0$[1]. Otherwise we call p a *point-component* and denote $p \in \mathcal{C}_P(G)$. Let $\hat{G} = \hat{\mathbb{C}}/\sim$, where

$$z \sim w \text{ if either } z = w \in G \text{ or } z, w \in p \text{ for some } p \in \mathcal{C}(G) := \mathcal{C}_N(G) \cup \mathcal{C}_P(G).$$

The corresponding quotient map is denoted by π_G.

Every homeomorphism $f\colon G \to G'$ has a unique homeomorphic extension $\hat{f} : \hat{G} \to \hat{G}'$. To simplify notation, we do not make a distinction between $p \in \mathcal{C}(G)$ and $\pi_G(p) \in \hat{G}$.

Recall that $A \subset \hat{\mathbb{C}}$ is τ-*fat* if for every $z_0 \in A \cap \mathbb{C}$ and every disk $\mathbb{D}(z_0, r)$ that does not contain A we have $\operatorname{Area}(A \cap \mathbb{D}(z_0, r)) \geq \tau r^2$. Domain $\Omega \subset \hat{\mathbb{C}}$ is *cofat* if there is $\tau > 0$ so that every $p \in \mathcal{C}_N(\Omega)$ is τ-fat, and a *circle domain* if every $p \in \mathcal{C}_N(\Omega)$ is a disk.

Theorem 1.1 ([Sch95]). *Let $\Omega \subset \hat{\mathbb{C}}$ be a cofat domain. Then there is a conformal map $f\colon \Omega \to D$ onto a circle domain D. Moreover, $\hat{f}(\mathcal{C}_N(\Omega)) = \mathcal{C}_N(D)$ and $\hat{f}(\mathcal{C}_P(\Omega)) = \mathcal{C}_P(D)$.*

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[1] We denote by $\operatorname{diam}(A)$ and $\operatorname{Area}(A)$ the Euclidean diameter and Lebesgue measure of $A \subset \mathbb{C}$, resp.

Theorem 1.1 and its proof involving Schramm's *transboundary modulus* have been applied to solve a variety of uniformization problems in Euclidean and metric spaces (cf. [Bon11], [Mer12], [BM13], [NY20], [Nta23a], [Nta23b]). Towards further applications, it is desirable to find minimal assumptions under which the conclusions of Theorem 1.1 hold. In this paper we consider conditions involving tripods and quasisymmetries. Recall that a homeomorphism $\phi\colon E \to F$ between subsets of $\mathbb{C}$ is *weakly H-quasisymmetric*, where H is a constant, if

$$|\phi(z_2) - \phi(z_1)| \le H|\phi(z_3) - \phi(z_1)| \quad \text{for all } z_1, z_2, z_3 \in E \text{ satisfying } |z_2 - z_1| \le |z_3 - z_1|.$$

The *standard tripod* $T_0 \subset \mathbb{C}$ is the union of segments $[0, e^{i\cdot 2j\pi/3}]$, $j = 0, 1, 2$.

Definition 1.2. We call $T \subset \mathbb{C}$ an *H-quasitripod* if there is a weakly H-quasisymmetric homeomorphism $\phi\colon T_0 \to T$.

Our main result reads as follows.

Theorem 1.3. *Let $\Omega \subset \hat{\mathbb{C}}$ be a domain containing ∞. Assume there are $H, N \ge 1$ so that*

(i) every $p \in \mathcal{C}_N(\Omega)$ contains an H-quasitripod T with $\operatorname{diam}(T) \ge \operatorname{diam}(p)/H$, and
(ii) $\operatorname{card}\{p \in \mathcal{C}_N(\Omega) : \operatorname{diam}(p) \ge r,\, p \cap \mathbb{D}(z_0, r) \ne \emptyset\} \le N$ for every $z_0 \in \mathbb{C}$ and $r > 0$.

Then there is a conformal homeomorphism $f : \Omega \to D$ onto a circle domain D. Moreover,

$$\hat{f}(\mathcal{C}_N(\Omega)) = \mathcal{C}_N(D) \quad \text{and} \quad \hat{f}(\mathcal{C}_P(\Omega)) = \mathcal{C}_P(D). \tag{1}$$

The proof of Theorem 1.3 is based on transboundary modulus estimates which are much more involved than the corresponding estimates on cofat domains. The main difficulty is that unlike cofatness, Conditions (i) and (ii) do not imply ℓ^2-summability bounds for the diameters of elements in $\mathcal{C}_N(\Omega)$. Condition (i) alone does not guarantee (1), see Section 6. We next introduce a local version of Condition (i) which leads to a Möbius invariant class of domains that satisfy the conclusions of Theorem 1.3.

Definition 1.4. We call $A \subset \hat{\mathbb{C}}$ *H-spread* if for every $z_0 \in A \cap \mathbb{C}$ and $r < \operatorname{diam}(A \cap \mathbb{C})$ there is an H-quasitripod $T \subset A \cap \mathbb{D}(z_0, r)$ with $\operatorname{diam}(T) \ge r/H$. Domain $\Omega \subset \hat{\mathbb{C}}$ is *H-cospread* if every $p \in \mathcal{C}_N(\Omega)$ is H-spread, and *cospread* if Ω is H-cospread for some H.

The class of cospread domains includes the continuum self-similar trees and uniformly branching trees considered by Bonk-Tran [BT21] and Bonk-Meyer [BM22], respectively.

Proposition 1.5. *Let $\Omega \subset \hat{\mathbb{C}}$ be an H-cospread domain. Then Conditions (i) and (ii) in Theorem 1.3 hold with H and $N = N(H)$. Moreover, if $\phi\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is α-quasi-Möbius then $\phi(\Omega)$ is H'-cospread, where H' depends only on H and α.*

The class of quasi-Möbius maps, which is defined in Section 7, contains all Möbius transformations. By Theorem 1.3 and Proposition 1.5, cospread domains admit conformal maps onto circle domains.

Corollary 1.6. *If $\Omega \subset \hat{\mathbb{C}}$ is a cospread domain, then there is a conformal homeomorphism $f : \Omega \to D$ onto a circle domain D. Moreover, $\hat{f}(\mathcal{C}_N(\Omega)) = \mathcal{C}_N(D)$ and $\hat{f}(\mathcal{C}_P(\Omega)) = \mathcal{C}_P(D)$.*

We finish the introduction by discussing possible extensions. First, our methods can be adapted to show that if every $p \in \mathcal{C}_N(\Omega)$ in Theorem 1.3 or Corollary 1.6 is the closure of a Jordan domain, then f admits a homeomorphic extension $\bar{f}\colon \overline{\Omega} \to \overline{D}$.

Another extension concerns versions of the *Brandt-Harrington theorem* for infinitely connected domains, see [Bra80], [Har82], [Sch95], [Sch96]. Although our results only concern circle domain targets, the estimates below and in the proof of [Sch95, Theorem 4.2] suggest that they can be replaced in Theorem 1.3 and Corollary 1.6 with targets D so that if $p \in \mathcal{C}_N(\Omega)$ then $\hat{f}(p) \in \mathcal{C}(D)$ is homothetic to a predetermined fat or spread set q_p.

There are fat sets that are not spread and do not even satisfy Quasitripod Condition (i) above. The proof below can be modified to show that Condition (i) can be replaced with "every $p \in \mathcal{C}_N(\Omega)$ is uniformly fat or satisfies Condition (i)" in Theorem 1.3. It would be interesting to find natural geometric conditions definining a class of domains which includes both cofat domains and the domains in Theorem 1.3. The proof below shows that it would suffice for such domains to satisfy Packing Condition (ii) and a condition on the "cost of a detour" which is strong enough to imply a version of Proposition 4.1.

This paper is organized as follows. In Section 2 we recall the definition of Schramm's transboundary modulus. In Section 3 we state our main modulus estimate, Theorem 3.1, for finitely connected domains satisfying the conditions of Theorem 1.3. We proceed to give the proof of Theorem 1.3, assuming Theorem 3.1 as well as the necessary modulus estimates on circle domains (Proposition 3.2).

We prove Theorem 1.3 by approximating Ω with a decreasing sequence of finitely connected domains $\Omega_j \supset \Omega$ satisfying $\mathcal{C}(\Omega_j) \subset \mathcal{C}_N(\Omega)$. Such an approach is standard and was also used by Schramm [Sch95]. Our new innovation and the main difficulty in the proof of Theorem 1.3 is establishing Theorem 3.1. The proof is given in Section 4.

Section 5 contains the proof of Proposition 3.2, modulus estimates on circle domains. See e.g. [Sch95], [Bon11], [Raj] for similar estimates. In Section 6 we construct an example showing that Packing Condition (ii) cannot be removed in Theorem 1.3. Finally, we prove Proposition 1.5 in Section 7.

2. Transboundary modulus

We denote the open Euclidean disk with center $a \in \mathbb{C}$ and radius $r > 0$ by $\mathbb{D}(a, r)$, and its boundary circle by $\mathbb{S}(a, r)$.

We apply the following definition due to Schramm [Sch95]. Fix a domain $G \subset \hat{\mathbb{C}}$. The *transboundary modulus* $\operatorname{mod}(\Gamma)$ of a family Γ of paths in $\hat{G}$ is

$$\operatorname{mod}(\Gamma) = \inf_{\rho \in X(\Gamma)} \int_{G \cap \mathbb{C}} \rho^2 \, dA + \sum_{p \in \mathcal{C}(G)} \rho(p)^2,$$

where $X(\Gamma)$ is the collection of *admissible functions for* Γ, i.e., Borel functions $\rho\colon \hat{G} \to [0,\infty]$ for which

$$1 \leq \int_\gamma \rho\, ds + \sum_{p \in \mathcal{C}(G) \cap |\gamma|} \rho(p) \quad \text{for all } \gamma \in \Gamma.$$

Here $|\gamma|$ denotes the image of the path γ and $\int_\gamma \rho\, ds$ is the path integral of the restriction of γ to G. More precisely, the restriction is a countable union of disjoint paths γ_j, each of which maps onto a component of $|\gamma| \setminus \mathcal{C}(G)$, and we define

$$\int_\gamma \rho\, ds = \sum_j \int_{\gamma_j} \rho\, ds.$$

Schramm worked with *transboundary extremal length* of Γ, which equals $\frac{1}{\operatorname{mod}(\Gamma)}$, and noticed that the proof of conformal invariance of classical conformal modulus can be generalized to transboundary modulus in a straightforward manner.

Lemma 2.1 ([Sch95], Lemma 1.1)**.** *Suppose $f\colon G \to G'$ is conformal. Then for every path family Γ, we have* $\operatorname{mod}(\Gamma) = \operatorname{mod}(\hat{f}(\Gamma))$*, where* $\hat{f}(\Gamma) := \{\hat{f} \circ \gamma : \gamma \in \Gamma\}$.

We will apply the following characterization of path families of non-zero modulus in Section 6. The proof follows directly from definitions and appropriate scalar multiplications of the admissible functions.

Lemma 2.2. *A family Γ of paths satisfies* $\operatorname{mod}(\Gamma) > 0$ *if and only if there exists an $M > 0$ such that for every admissible function ρ for Γ that satisfies*

$$\int_{G \cap \mathbb{C}} \rho^2\, dA + \sum_{p \in \mathcal{C}(G)} \rho(p)^2 = 1,$$

we have

$$\int_\gamma \rho\, ds + \sum_{p \in \mathcal{C}(G) \cap |\gamma|} \rho(p) \leq M \quad \textit{for some } \gamma \in \Gamma.$$

3. Proof of the main result, Theorem 1.3

The proof of our main result, Theorem 1.3, is based on the following estimate. Here we denote by π_Ω the quotient map $\hat{\mathbb{C}} \to \hat{\Omega}$, and by $\mathbb{A}(a,R)$ the annulus $\mathbb{D}(a,4R) \setminus \overline{\mathbb{D}}(a,R/2)$.

Theorem 3.1. *Let $\Omega \subset \hat{\mathbb{C}}$ be a finitely connected domain that satisfies Conditions (i) and (ii) in Theorem 1.3 with some constants H and N. Then, there is $M > 0$, depending only on H and N, so that if $a \in \bar{p} \cap \mathbb{C}$ for some $\bar{p} \in \hat{\Omega}$ and $R > 0$, then* $\operatorname{mod}\Gamma \leq M$*, where*

$$\Gamma = \{\textit{paths in } \pi_\Omega(\overline{\mathbb{A}}(a,R)) \setminus \{\pi_\Omega(\bar{p})\} \textit{ joining } \pi_\Omega(\mathbb{S}(a,4R)) \textit{ and } \pi_\Omega(\mathbb{S}(a,R/2))\}.$$

We postpone the proof of Theorem 3.1 until Section 4, and first show how it can be applied to prove Theorem 1.3. We may assume that $\operatorname{card}\mathcal{C}_N(\Omega) = \infty$, since otherwise Theorem 1.3 follows from Koebe's theorem, see e.g. [Bon11, Theorem 9.5]. We enumerate

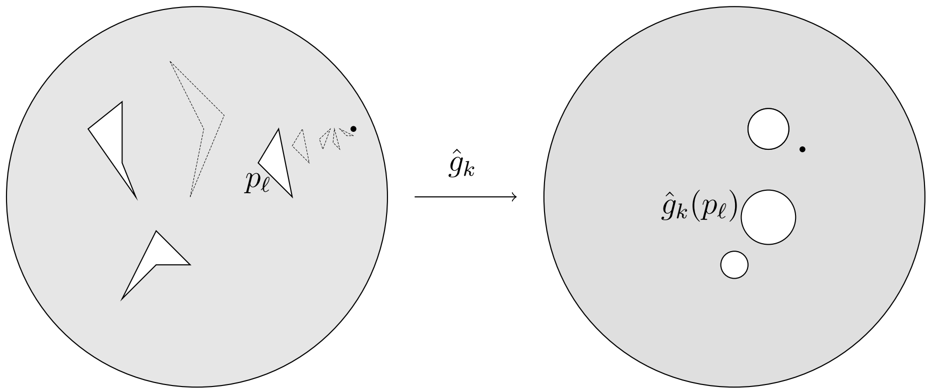

FIGURE 1. The domain Ω_k has finitely many of $p_\ell \in \mathcal{C}_N(\Omega)$ as its complement. After passing to subsequences, we can assume $\{\hat{g}_k(p_\ell)\}_k$ converge for each p_ℓ.

the elements and denote $\mathcal{C}_N(\Omega) = \{p_0, p_1, \ldots\}$. It follows directly from the definitions that if Theorem 1.3 holds for

$$\Omega' = \hat{\mathbb{C}} \setminus \overline{\bigcup_{p \in \mathcal{C}_N(\Omega)} p} \supset \Omega,$$

then the theorem also holds for Ω. In particular, we may assume that $\Omega' = \Omega$.

Recall that if $G \subset \hat{\mathbb{C}}$ is a domain and $p \in \mathcal{C}(G)$, we do not make a distinction between p and $\pi_G(p)$. In particular, if $p \subset \mathbb{C}$ then $\operatorname{diam}(\pi_G(p))$ is the Euclidean diameter of p.

Given $k \in \mathbb{N}$, let $\tilde{\Omega}_k = \hat{\mathbb{C}} \setminus (p_0 \cup p_1 \cup \cdots \cup p_k)$. By Koebe's theorem there is a conformal homeomorphism $g_k \colon \tilde{\Omega}_k \to \tilde{D}_k$ so that $q_{k,\ell} := \hat{g}_k(p_\ell)$ is a disk (with positive radius) for all $\ell = 0, 1, \ldots, k$. By postcomposing with a Möbius transformation, we may assume that

$$q_{k,0} = \hat{\mathbb{C}} \setminus \mathbb{D}(0,1) \quad \text{for all } k = 1, 2, \ldots. \tag{2}$$

For every $\ell \in \mathbb{N}$, any subsequence of $(q_{k,\ell})_k$ has a further subsequence Hausdorff converging to a limit disk or a point. Therefore we can choose a diagonal subsequence $(g_{k_j})_j$, converging locally uniformly in Ω, so that $q_{k_j,\ell} \to q_\ell$ for each ℓ. By normalization (2), the limit map f is non-constant and therefore a conformal homeomorphism from Ω onto a domain D. Each q_ℓ, $\ell \in \mathbb{N}$, is a disk or a point, and $q_0 = \hat{\mathbb{C}} \setminus \mathbb{D}(0,1)$.

Theorem 1.3 follows once we have established the following properties:

$$\operatorname{diam}(\hat{f}(p)) = 0 \quad \text{for all } p \in \mathcal{C}_P(\Omega), \tag{3}$$

$$q_\ell = \hat{f}(p_\ell) \quad \text{and} \quad \operatorname{diam}(q_\ell) > 0 \quad \text{for all } \ell = 0, 1, 2, \ldots. \tag{4}$$

We denote g_{k_j} by f_j, $\tilde{\Omega}_{k_j}$ by Ω_j, and $\tilde{D}_{k_j}$ by D_j. Moreover,

we fix $\bar{p} \in \mathcal{C}(\Omega)$ and any Jordan curve $J \subset \Omega$.

Next let $b \in \Omega \cap N_R(\bar{p})$, where $R = \operatorname{dist}(\bar{p}, J)$ and $N_\delta(A)$ is the δ-neighborhood of A in $\mathbb{C}$. Here and in what follows, all distances are Euclidean unless stated otherwise. We choose a point $a \in \partial \bar{p}$ closest to b and denote by I the segment in $\mathbb{C}$ with endpoints a and b. Given $j \geq 1$, let

$$\begin{aligned} \Gamma_j &= \{\text{paths in } \hat{\Omega}_j \setminus \{\pi_{\Omega_j}(\bar{p})\} \text{ that join } \pi_{\Omega_j}(J) \text{ and } \pi_{\Omega_j}(I)\}, \\ \Lambda_j &= \{\text{paths in } \hat{\Omega}_j \setminus \{\pi_{\Omega_j}(\bar{p})\} \text{ that separate } \pi_{\Omega_j}(J) \text{ and } \pi_{\Omega_j}(\bar{p})\}. \end{aligned}$$

In summary, here is how the proofs of (3) and (4) proceed. We use Theorem 3.1 to prove upper bounds on $\operatorname{mod} \Gamma_j$ and $\operatorname{mod} \Lambda_j$. On the other hand, estimates on circle domains D_j provide lower bounds on $\operatorname{mod} \hat{f}_j(\Gamma_j)$ and $\operatorname{mod} \hat{f}_j(\Lambda_j)$. Combined with the conformal invariance of modulus, these yield (3) and (4).

We now state the circle domain estimates; we will prove them later in Section 5.

Proposition 3.2. *The following estimates hold:*

(1) There is a homeomorphism $\varphi_a : [0,\infty) \to [0,\infty)$ so that
$$\limsup_{j\to\infty} \operatorname{mod} \hat{f}_j(\Gamma_j) \geq \limsup_{j\to\infty} \varphi_a(\operatorname{dist}(f_j(b), \hat{f}_j(\bar{p}))).$$
(2) If $\operatorname{diam}(\hat{f}(\overline{p})) = 0$ then $\lim_{j\to\infty} \operatorname{mod} \hat{f}_j(\Lambda_j) = \infty$.

We now apply Theorem 3.1 to establish modulus estimates on Γ_j, Λ_j. We first show that
$$\operatorname{mod} \Gamma_j \leq \theta_a(|b-a|), \tag{5}$$
where θ_a does not depend on j and $\theta_a(\epsilon) \to 0$ as $\epsilon \to 0$.

To prove (5), we notice that every $\gamma \in \Gamma_j$ intersects $\pi_{\Omega_j}(\mathbb{S}(a,R))$ and $\pi_{\Omega_j}(\mathbb{S}(a,|b-a|))$ but avoids $\pi_{\Omega_j}(\bar{p})$. Therefore, it suffices to show that
$$\operatorname{mod} \Gamma_j(r,R) \leq \theta(r), \quad \theta(r) \to 0 \text{ as } r \to 0, \quad \theta \text{ does not depend on } j,$$
where
$$\Gamma_j(r,R) = \{\text{paths in } \hat{\Omega}_j \setminus \{\pi_{\Omega_j}(\bar{p})\} \text{ that join } \pi_{\Omega_j}(\mathbb{S}(a,R)) \text{ and } \pi_{\Omega_j}(\mathbb{S}(a,r))\}.$$

We choose a sequence of radii R_n decreasing to zero as follows: Let $R_1 := R/10$. Then, assuming $R_1, \ldots, R_{n-1}$ are defined let
$$R_n = \frac{R_n'}{10},$$
where $R_n' \leq R_{n-1}/2$ is the smallest radius for which some $p \in \mathcal{C}_N(\Omega) \setminus \{\bar{p}\}$ intersects both $\mathbb{S}(a, R_{n-1}/2)$ and $\mathbb{S}(a, R_n')$. If no $p \in \mathcal{C}_N(\Omega) \setminus \{\bar{p}\}$ intersects $\mathbb{S}(a, R_{n-1}/2)$, we set $R_n' = R_{n-1}/2$. Then R_n does not depend on j, $R_n \to 0$ as $n \to \infty$, and both annuli
$$\mathbb{A}_n = \mathbb{D}(a, 4R_n) \setminus \overline{\mathbb{D}}(a, R_n/2), \quad n = 1, 2, \ldots,$$
and their projections $\pi_{\Omega_j}(\mathbb{A}_n)$ are pairwise disjoint (for a fixed j). Let
$$\Gamma_j(n) = \{\text{paths in } \pi_{\Omega_j}(\mathbb{A}_n) \setminus \{\pi_{\Omega_j}(\bar{p})\} \text{ joining } \pi_{\Omega_j}(\mathbb{S}(a, 4R_n)) \text{ and } \pi_{\Omega_j}(\mathbb{S}(a, R_n/2))\}.$$
Notice that if Ω satisfies Conditions (i) and (ii) in Theorem 1.3 with some H and N, then every Ω_j satisfies the same conditions. Therefore, by Theorem 3.1 we have $\operatorname{mod} \Gamma_j(n) \leq M$, where M does not depend on j or n.

We fix $N \in \mathbb{N}$ and choose for every $1 \leq n \leq N$ an admissible function ρ_n for $\Gamma_j(n)$ with

$$\int_{\Omega_j \cap \mathbb{A}_n} \rho_n^2 \, dA + \sum_{p \in \mathcal{C}(\Omega_j) \cap \pi_{\Omega_j}(\mathbb{A}_n)} \rho_n(p)^2 \leq 2M.$$

Now $\rho := \frac{1}{N} \sum_{n=1}^{N} \rho_n$ is admissible for $\Gamma_j(R_{N+1}, R)$. Moreover, since sets $\pi_{\Omega_j}(\mathbb{A}_n)$ are pairwise disjoint we have

$$\int_{\Omega_j} \rho^2 \, dA + \sum_{p \in \mathcal{C}(\Omega_j)} \rho(p)^2 \leq \frac{2MN}{N^2} = \frac{2M}{N} \to 0 \quad \text{as } N \to \infty.$$

Estimate (5) follows.

We can now prove (3): assume $\bar{p} = \{a\} \in \mathcal{C}_P(\Omega)$ and suppose towards contradiction that $\hat{f}(\bar{p}) \in \mathcal{C}_N(D)$.[2] Then there are $c > 0$ and a sequence (b_m) of points in Ω converging to a so that for every $m \in \mathbb{N}$ we have

$$\limsup_{j \to \infty} \operatorname{dist}(f_j(b_m), \hat{f}_j(\bar{p})) \geq c > 0. \tag{6}$$

Combining (5) and the first part of Proposition 3.2 with Lemma 2.1 (conformal invariance of modulus) gives a contradiction, proving (3).

Towards (4), let $\bar{p} = p_\ell$ for some $\ell \in \mathbb{N} \cup \{0\}$, and let j_ℓ be the smallest index for which $p_\ell \in \mathcal{C}_N(\Omega_{j_\ell})$. We claim that

$$\operatorname{mod} \Lambda_j \leq M_\ell < \infty \quad \text{for all } j \geq j_\ell, \tag{7}$$

where M_ℓ does not depend on j. To start the proof of (7), we fix $c \in \partial p_\ell$ and $d \in J \cap \mathbb{C}$ so that $|c - d| = \operatorname{dist}(p_\ell, J)$, and let ξ be the segment with endpoints c and d. We cover ξ with $N_1 < \infty$ disks $\mathbb{D}(z_n, r)$, where $r = \operatorname{diam}(p_\ell)/20$.

Since every $\lambda \in \Lambda_j$ separates $\pi_{\Omega_j}(\bar{p})$ and $\pi_{\Omega_j}(J)$, λ has to pass through $\pi_{\Omega_j}(\xi)$ and, consequently, through at least one $\pi_{\Omega_j}(\mathbb{D}(z_n, r))$. Furthermore, we have

$$\operatorname{diam}(\pi_{\Omega_j}^{-1}(|\lambda|)) \geq \operatorname{diam}(p_\ell),$$

which implies that if λ passes through $\pi_{\Omega_j}(\mathbb{D}(z_n, r))$ then it also passes through $\pi_{\Omega_j}(\mathbb{S}(z_n, 8r))$. Therefore,

$$\Lambda_j \subset \bigcup_{n=1}^{N_1} \Gamma_j(n), \tag{8}$$

where

$$\Gamma_j(n) = \{\text{paths in } \hat{\Omega}_j \text{ joining } \pi_{\Omega_j}(\mathbb{S}(z_n, r)) \text{ and } \pi_{\Omega_j}(\mathbb{S}(z_n, 8r))\}.$$

By Theorem 3.1, for each n, there is an admissible ρ_n for $\Gamma_j(n)$ so that

$$\int_{\Omega_j} \rho_n^2 \, dA + \sum_{p \in \mathcal{C}(\Omega_j)} \rho_n(p)^2 \leq 2M.$$

[2]Because $a \in \Omega_j$ for all j and $f_j(a)$ are singletons, one may wonder if $\hat{f}(a)$ can ever be non-trivial. However, in Section 6 we give one such example. Other examples can be found in [Nta23b] and [Raj].

By (8), the function $\rho_1 + \cdots + \rho_{N_1}$ is admissible for Λ_j. We conclude that

$$\operatorname{mod} \Lambda_j \le \int_{\Omega_j} (\rho_1 + \cdots + \rho_{N_1})^2 \, dA + \sum_{p \in \mathcal{C}(\Omega_j)} (\rho_1(p) + \cdots + \rho_{N_1}(p))^2 \le 2MN_1^2,$$

which proves (7).

We can now prove (4). The proof of the first part is similar to the proof of (3). We have $q_\ell \subset \hat{f}(p_\ell)$ by Carathéodory's kernel convergence theorem; see [Nta23b, Lemma 2.14]. Suppose towards contradiction that $q_\ell \subsetneq \hat{f}(p_\ell)$. Then there are $c > 0$ and a sequence (b_m) in Ω so that $\operatorname{dist}(b_m, p_\ell) \to 0$ as $m \to \infty$ and (6) holds with $\bar{p} = p_\ell$. Combining (5) and the first part of Proposition 3.2 with Lemma 2.1 (conformal invariance of modulus) gives a contradiction. For the second part of (4) it suffices to combine (7) and the second part of Proposition 3.2 with Lemma 2.1.

We have proved that Theorem 1.3 follows from Theorem 3.1 and Proposition 3.2.

4. Proof of Theorem 3.1

In this section we assume that $\Omega \subset \hat{\mathbb{C}}$ is as in Theorem 3.1: a *finitely connected* domain satisfying Conditions (i) and (ii) in Theorem 1.3 with some H and N.

4.1. **Costs of detours around quasitripods.**

We may assume without loss of generality that $\mathcal{C}(\Omega) = \mathcal{C}_N(\Omega)$; since Ω is finitely connected, point-components $p \in \mathcal{C}_P(\Omega)$ are isolated. Therefore, the modulus of the family of paths passing through some $p \in \mathcal{C}_P(\Omega)$ is zero.

The proof of Theorem 3.1 is based on the following technical result. Given $p \in \mathcal{C}(\Omega)$, $0 < \tau < 1/4$ and $a_p, b_p \in \mathbb{C}$, we denote

$$r_p = r_p(\tau) = \tau \operatorname{diam}(p) > 0,$$

assume that $\overline{\mathbb{D}}(a_p, 4\tau r_p) \subset \mathbb{D}(b_p, r_p)$, and let $\Gamma(a_p, b_p, \tau)$ be the family of paths $\alpha : [s_1, t_1] \to \hat{\Omega}$ for which there are $s_1 < s_2 \le t_2 < t_1$ with the following properties:

(i) $\alpha(s_2) \cup \alpha(t_2) \subset \mathbb{D}(a_p, 4\tau r_p)$,
(ii) $\alpha(t) \cap \mathbb{S}(b_p, r_p) \neq \emptyset$ for $t = s_1$ and $t = t_1$, and
(iii) $\alpha(t) \subset \mathbb{D}(b_p, r_p)$ for all $s_1 < t < s_2$ and $t_2 < t < t_1$.

Recall that we assume that every $p \in \mathcal{C}(\Omega)$ contains an H-quasitripod T with

$$\operatorname{diam}(T) \ge \operatorname{diam}(p)/H.$$

Proposition 4.1. *There are $0 < \tau < \frac{1}{1000}$, depending only on H, and map $p \mapsto (a_p, b_p) \in \mathbb{C}^2$ so that for every $p \in \mathcal{C}(\Omega)$ we have the following properties: $b_p \in p$,*

$$\mathbb{D}(a_p, 4\tau r_p) \subset \mathbb{D}(b_p, \tau^{1/2} r_p), \tag{9}$$

and if $\alpha \in \Gamma(a_p, b_p, \tau)$ then

$$\operatorname{dist}(\alpha(s_1), \alpha(t_1)) \le \operatorname{dist}(\alpha(s_1), \alpha(s_2)) + \operatorname{dist}(\alpha(t_1), \alpha(t_2)) - 200\tau r_p. \tag{10}$$

Proposition 4.1 implies that if $\alpha \in \Gamma(a_p, b_p, \tau)$ does not pass through p then it has to "take a detour" whose length is estimated from below in (10), see Figure 2. Notice that the right side of (10) does not include term $\operatorname{dist}(\alpha(s_2), \alpha(t_2))$.

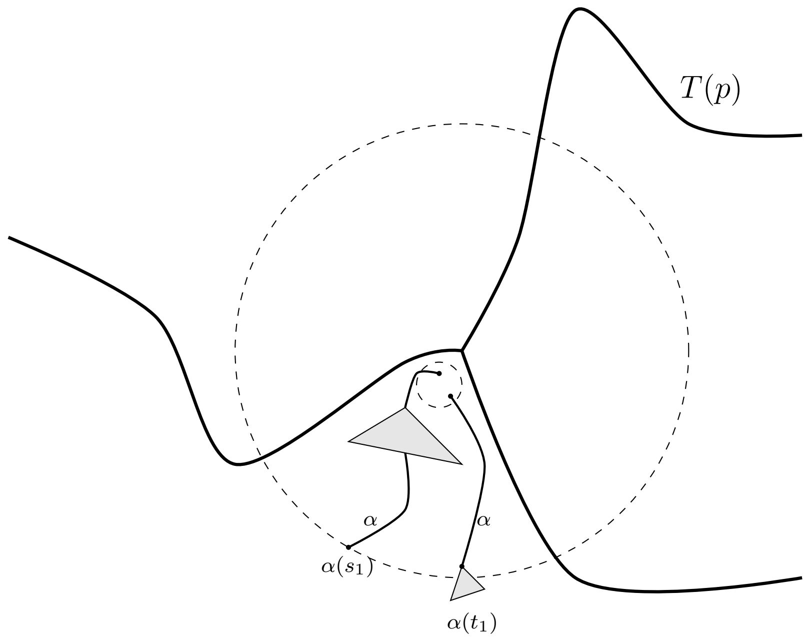

FIGURE 2. Part of a sample curve $\alpha \in \Gamma(a_p, b_p, \tau)$ depicted.

Proof. We denote the vertices of the standard tripod T_0 by z_0, z_1, z_2. By assumption there is a weakly H-quasisymmetric homeomorphism $\phi\colon T_0 \to T \subset p$ with $\operatorname{diam}(T) \geq H^{-1} \operatorname{diam}(p)$. By Väisälä's theorem [Hei01, Corollary 10.22], ϕ is in fact (strongly) quasisymmetric: there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ depending only on H so that

$$|w_1 - w_0| \leq t|w_2 - w_0| \quad \text{implies} \quad |\phi(w_1) - \phi(w_0)| \leq \eta(t)|\phi(w_2) - \phi(w_0)|.$$

Let $b_p = \phi(0) \in p$ and $R_0 = (100H)^{-1} \operatorname{diam}(p)$. Moreover, given $0 < R < R_0$ we denote

$$k_n(R) = \min\{0 < s < 1 : \phi(sz_n) \in \mathbb{S}(b_p, R)\}, \quad n \in \{0, 1, 2\}.$$

Notice that by quasisymmetry and our choice of R_0 there is $0 < s_n(R) < 1$ for every $n \in \{0, 1, 2\}$ so that $\phi(s_n(R)z_n) \in \mathbb{S}(b_p, R)$. Therefore, numbers $k_n(R)$ are well-defined. Let

$$J_n(R) = \phi([0, k_n(R)z_n]), \quad n \in \{0, 1, 2\}.$$

Then $\mathbb{D}(b_p, R)\backslash\bigcup_{n=0}^{2} J_n(R)$ is the union of pairwise disjoint connected sets $V_0(R), V_1(R), V_2(R)$ which are labelled so that $\overline{V_n(R)} \cap J_n(R) = \{b_p\}$.

We fix $0 < \delta < (100H)^{-2}$, to be determined later, and point

$$c_p = \phi(k_0(\delta R_0)z_0) \subset \mathbb{S}(b_p, \delta R_0) \cap J_0(R_0). \tag{11}$$

By standard *porosity* (see e.g. [Vä81, Proof of Theorem 4.1]) and distortion estimates on quasisymmetric maps there are $0 < C < 1$, depending only on H, and points $a_p^1, a_p^2 \in \mathbb{C}$ so that

$$\mathbb{D}(a_p^1, C\delta R_0) \subset V_1(R_0) \cap \mathbb{D}(c_p, \delta R_0) \quad \text{and} \quad \mathbb{D}(a_p^2, C\delta R_0) \subset V_2(R_0) \cap \mathbb{D}(c_p, \delta R_0). \tag{12}$$

Recall that $r_p = \tau \operatorname{diam}(p)$ by definition. Let τ be the number satisfying $C\delta R_0 = 4\tau r_p$;

$$\tau = \left(\frac{C\delta}{400H}\right)^{1/2}.$$

We can then choose δ to be small enough so that

$$2\delta R_0 < \tau^{1/2} r_p < r_p < R_0. \tag{13}$$

Standard quasisymmetric distortion estimates show that

$$\min\{\ell_1, \ell_2\} \leq (1 - C_1)\pi r_p, \tag{14}$$

where ℓ_1 and ℓ_2 are the lengths of the circular arcs bounding $V_1(r_p)$ and $V_2(r_p)$, respectively, and $0 < C_1 < 1$ depends only on H. If ℓ_1 has this property we choose a_p to be a_p^1 and otherwise we choose a_p to be a_p^2.

We have found the desired a_p, b_p and τ. By our choice of τ, (11), (12) and triangle inequality, we have

$$\mathbb{D}(a_p, 4\tau r_p) = \mathbb{D}(a_p, C\delta R_0) \subset \mathbb{D}(c_p, \delta R_0) \subset \mathbb{D}(b_p, 2\delta R_0).$$

Combining with (13) shows that (9) holds.

It remains to prove (10). We may assume without loss of generality that $\ell_1 \leq \ell_2$ in (14). We fix $\alpha \in \Gamma(a_p, b_p, \tau)$. By the definition of $\Gamma(a_p, b_p, \tau)$ (Conditions (i)-(iii) above) and since

$$\mathbb{D}(a_p, 4\tau r_p) \subset V_1(r_p) \subset V_1(R_0)$$

by (9) and (12), both $\alpha(s_1)$ and $\alpha(t_1)$ intersect the circular arc bounding $V_1(r_p)$. Therefore, by (14) we have

$$\operatorname{dist}(\alpha(s_1), \alpha(t_1)) \leq 2(1 - C_2) r_p, \tag{15}$$

where C_2 depends only on H. On the other hand, combining the definition of $\Gamma(a_p, b_p, \tau)$ with (9) and the choice of τ also shows that

$$\min\{\operatorname{dist}(\alpha(s_1), \alpha(s_2)), \operatorname{dist}(\alpha(t_1), \alpha(t_2))\} \geq r_p - \tau r_p - \tau^{1/2} r_p \geq r_p - 2\tau^{1/2} r_p,$$

and therefore

$$\operatorname{dist}(\alpha(s_1), \alpha(s_2)) + \operatorname{dist}(\alpha(t_1), \alpha(t_2)) \geq 2r_p(1 - 2\tau^{1/2}). \tag{16}$$

Combining (15) and (16) shows that (10) holds when δ and hence τ is small enough. The proof is complete. □

4.2. **Good, bad, and large components.** We continue the proof of Theorem 3.1. We fix $R > 0$ and $a \in \bar{p} \cap \mathbb{C}$ for some $\bar{p} \in \hat{\Omega}$. Our goal is to find an upper bound for the transboundary modulus of

$$\Gamma = \left\{\text{paths in } \pi_\Omega(\overline{\mathbb{A}}(a,R)) \setminus \{\pi_\Omega(\bar{p})\} \text{ joining } \pi_\Omega(\mathbb{S}(a,4R)) \text{ and } \pi_\Omega(\mathbb{S}(a,R/2))\right\}.$$

Scaling and translating Ω does not affect transboundary modulus, so we may assume that $a = 0$ and $R = 1$.

Let $P \subset \mathcal{C}(\Omega)$ be the collection of complementary components $p \neq \bar{p}$ intersecting $\mathbb{A} = \mathbb{D}(0,4) \setminus \overline{\mathbb{D}}(0,1/2)$, and let $0 < \tau < \frac{1}{1000}$ be the constant in Proposition 4.1. We denote

$$P_L = \{p \in P : \operatorname{diam}(p) \geq \tau\} \quad \text{and} \quad P_V = P \setminus P_L.$$

Then $\rho_0 : \hat{\Omega} \to [0,\infty]$, $\rho_0 = \chi_{P_L}$, is admissible for the family of paths in Γ passing through some $p \in P_L$. Notice that Γ may include constant paths which happens if p intersects both $\mathbb{S}(0,4)$ and $\mathbb{S}(0,1/2)$. We cover $\overline{\mathbb{D}}(0,4)$ with $10\tau^{-2}$ disks of radius τ and apply Packing Condition (ii) in Theorem 1.3 (with constant N) to see that the cardinality of P_L is bounded from above by $10N\tau^{-2}$. Therefore,

$$\sum_{p \in \mathcal{C}(\Omega)} \rho_0(p)^2 \leq 10N\tau^{-2}.$$

We conclude that in order to prove Theorem 3.1 it suffices to consider

$$\Gamma_2 = \{\gamma \in \Gamma : \gamma \text{ does not pass through any } p \in P_L\}. \tag{17}$$

We apply Proposition 4.1 to find a suitable partition of P_V into "good" and "bad" components. Given $p \in P_V$, let $r_p = \tau \operatorname{diam}(p)$ and $a_p \in \mathbb{C}$, $b_p \in p$, be as in Proposition 4.1. We start by choosing $p_1 \in P_V$ so that

$$\operatorname{diam}(p_1) = \max_{p \in P_V} \operatorname{diam}(p).$$

Denote $r_1 := r_{p_1}$, $a_1 := a_{p_1}$ and $b_1 := b_{p_1}$, and let

$$\begin{aligned} G_1 &= \left\{p \in P_V : \operatorname{diam}(p) \geq \tau r_1,\ \operatorname{dist}(p,p_1) \leq \tau^{-2} r_1 = \tau^{-1}\operatorname{diam}(p_1)\right\}, \text{ and} \\ B_1 &= \left\{p \in P_V : \operatorname{diam}(p) < \tau r_1,\ a_p \in \overline{\mathbb{D}}(a_1, 2\tau r_1)\right\}. \end{aligned}$$

Suppose then that $p_\ell \in P_V$ and $G_\ell, B_\ell \subset P_V$ are chosen for $1 \leq \ell \leq k$. We stop the process if $P_V \setminus \bigcup_{\ell=1}^k (G_\ell \cup B_\ell) = \emptyset$. Otherwise, we choose $p_{k+1} \in P_V \setminus \bigcup_{\ell=1}^k (G_\ell \cup B_\ell)$ so that

$$\operatorname{diam}(p_{k+1}) = \max_{p \in P_V \setminus \bigcup_{\ell=1}^k (G_\ell \cup B_\ell)} \operatorname{diam}(p).$$

We denote $r_{k+1} := r_{p_{k+1}}$, $a_{k+1} := a_{p_{k+1}}$ and $b_{k+1} := b_{p_{k+1}}$, and let

$$\begin{aligned} G_{k+1} &= \left\{p \in P_V \setminus \bigcup_{\ell=1}^k (G_\ell \cup B_\ell) : \operatorname{diam}(p) \geq \tau r_{k+1},\ \operatorname{dist}(p,p_{k+1}) \leq \tau^{-2} r_{k+1}\right\}, \text{ and} \\ B_{k+1} &= \left\{p \in P_V \setminus \bigcup_{\ell=1}^k (G_\ell \cup B_\ell) : \operatorname{diam}(p) < \tau r_{k+1},\ a_p \in \overline{\mathbb{D}}(a_{k+1}, 2\tau r_{k+1})\right\}. \end{aligned}$$

Notice that $p_{k+1} \in G_{k+1}$. Also, if $p \in B_{k+1}$ then

$$a_p \in \overline{\mathbb{D}}(a_{k+1}, 2\tau r_{k+1}) \cap \mathbb{D}(b_p, \tau^{1/2} r_p) \subset \overline{\mathbb{D}}(a_{k+1}, 2\tau r_{k+1}) \cap \mathbb{D}(b_p, \tau^{5/2} r_{k+1})$$

by (9). Since $b_p \in p$, it follows that

$$\mathrm{dist}(a_{k+1}, p) \leq \mathrm{dist}(a_{k+1}, a_p) + \mathrm{dist}(a_p, p) \leq 2\tau r_{k+1} + \tau^{5/2} r_{k+1} < 3\tau r_{k+1}.$$

Since $\mathrm{diam}(p) < \tau r_{k+1}$, we conclude that

$$p \subset \mathbb{D}(a_{k+1}, 4\tau r_{k+1}) \quad \text{for every } p \in B_{k+1}. \tag{18}$$

Since Ω is finitely connected, the process stops after $L < \infty$ steps and we have a partition of P_V into disjoint sets G_k, B_k, $k = 1, \ldots, L$, so that

$$P_V = G \cup B := \Big(\bigcup_{k=1}^{L} G_k \Big) \cup \Big(\bigcup_{k=1}^{L} B_k \Big).$$

We will construct suitable admissible functions ρ for Γ_2 which equal zero in B. The following simple estimate will be useful later on.

Lemma 4.2. *Suppose* $1 \leq m < k \leq L$ *and*

$$\mathbb{D}(a_m, 2r_m) \cap \mathbb{D}(a_k, 2r_k) \neq \emptyset. \tag{19}$$

Then $r_k < \tau r_m$.

Proof. By triangle inequality

$$\mathrm{dist}(p_m, p_k) \leq \mathrm{dist}(a_k, p_k) + \mathrm{dist}(p_m, a_m) + |a_m - a_k|. \tag{20}$$

By (9) and (19), we have

$$\mathrm{dist}(a_k, p_k) \leq r_k, \quad \mathrm{dist}(a_m, p_m) \leq r_m \quad \text{and} \quad |a_m - a_k| \leq 2(r_k + r_m). \tag{21}$$

From $m < k$ it follows that $r_k \leq r_m$. Therefore, combining (20) and (21) we have

$$\mathrm{dist}(p_m, p_k) \leq 3(r_k + r_m) \leq 6r_m. \tag{22}$$

Since $p_k \notin \bigcup_{\ell=1}^{m} (G_\ell \cup B_\ell)$, the definition of G_m and (22) show that $r_k < \mathrm{diam}(p_k) < \tau r_m$. □

4.3. **Modulus bound.** Our goal is to give an upper bound for $\mathrm{mod}\, \Gamma_2$, where Γ_2 is defined in (17). Note that if a non-negative Borel function ρ is admissible for the family of *injective* paths in Γ_2, then ρ is admissible for Γ_2. Indeed, for every rectifiable $\gamma_2 \in \Gamma_2$ there is an injective $\gamma_1 \in \Gamma_2$ so that $|\gamma_1| \subset |\gamma_2|$, see e.g. [Sem96, Proposition 15.1]. Then, if ρ is admissible for injective paths we have

$$\int_{\gamma_2} \rho \, ds \geq \int_{\gamma_1} \rho \, ds \geq 1,$$

so ρ is admissible for Γ_2.

We fix an injective $\gamma_2 \in \Gamma_2$. After reparametrization and recalling that γ_2 does not pass through any $p \in P$ with diameter greater than $\tau < \frac{1}{1000}$, we may assume that the domain of γ_2 contains $[0,1]$, $\gamma_2([0,1]) \subset \pi_\Omega(\mathbb{D}(0,3))$, and

$$\gamma_2(0) \in \Omega \cap \mathbb{D}(0,3) \setminus \mathbb{D}(0,5/2), \qquad \gamma_2(1) \in \Omega \cap \mathbb{D}(0,3/4).$$

We consider $\gamma := \gamma_2|[0,1]$ for the rest of this section. Given path $\alpha : I \to \hat{\Omega}$, we denote

$$G(\alpha) = \{t \in I :\, \alpha(t) \in G\}.$$

Proposition 4.3. *There are intervals $[c_\nu, d_\nu] \subset [0,1]$, $\nu = 1, 2, \ldots, \mu$, with non-empty and pairwise disjoint interiors so that $\gamma(t) \notin B$ for all $t \in \bigcup_{\nu=1}^{\mu}(c_\nu, d_\nu)$ and*

$$1 \leq \sum_{\nu=1}^{\mu} \operatorname{dist}(\gamma(c_\nu), \gamma(d_\nu)) + 7\tau^{-1} \sum_{t\in G(\gamma)} \operatorname{diam}(\gamma(t)). \tag{23}$$

We postpone the proof of Proposition 4.3 and first show how it yields the desired upper bound for $\operatorname{mod}(\Gamma_2)$. Let $\rho : \hat{\Omega}_j \to [0,\infty]$,

$$\rho(p) = \begin{cases} 1, & p \in \Omega \cap \mathbb{D}(0,3), \\ 8\tau^{-1}\operatorname{diam}(p), & p \in G, \\ 0, & \text{otherwise}. \end{cases}$$

We claim that ρ is admissible for Γ_2. Let γ and intervals $[c_\nu, d_\nu]$ be as in Proposition 4.3. Since $\gamma(t) \notin B$ for all $c_\nu < t < d_\nu$ and $|\gamma| \subset \mathbb{D}(0,3)$, triangle inequality gives

$$\operatorname{dist}(\gamma(c_\nu), \gamma(d_\nu)) \leq \int_{\gamma|[c_\nu,d_\nu]} \rho\, ds + \sum_{t\in G(\gamma|(c_\nu,d_\nu))} \operatorname{diam}(\gamma(t)) \tag{24}$$

for all $1 \leq \nu \leq \mu$. Recall that the integral in (24) is over the subpaths of $\gamma|[c_\nu, d_\nu]$ whose images are in Ω. Since γ is injective and intervals $[c_\nu, d_\nu]$ have disjoint interiors, summing (24) over ν gives

$$\sum_{\nu=1}^{\mu} \operatorname{dist}(\gamma(c_\nu), \gamma(d_\nu)) \leq \int_\gamma \rho\, ds + \sum_{t\in G(\gamma)} \operatorname{diam}(\gamma(t)) \leq \int_\gamma \rho\, ds + \tau^{-1} \sum_{t\in G(\gamma)} \operatorname{diam}(\gamma(t)). \tag{25}$$

Combining (25) and Proposition 4.3 shows that ρ is admissible for Γ_2.

We now estimate $\int_\Omega \rho^2\, dA + \sum_{p\in\mathcal{C}(\Omega)} \rho(p)^2$ in order to give an upper bound for $\operatorname{mod}(\Gamma_2)$. We first recall that every $p \in G_k$ satisfies

$$\tau r_k \leq \operatorname{diam}(p) \leq \operatorname{diam}(p_k) = \tau^{-1} r_k. \tag{26}$$

We pack $\mathbb{D}(a_k, \tau^{-3} r_k)$ with $10\tau^{-8}$ disks of radius τr_k. We have $p \subset \mathbb{D}(a_k, \tau^{-3} r_k)$ for all $p \in G_k$ by (26) and (9). Thus, applying the first inequality in (26) with Packing Condition (ii) in Theorem 1.3 (with constant N) shows that

$$\operatorname{card} G_k \leq 10N\tau^{-8} \quad \text{for every } 1 \leq k \leq L. \tag{27}$$

Finally,

$$\text{disks } \mathbb{D}(a_k, \tau r_k),\ 1 \leq k \leq L,\ \text{are pairwise disjoint.} \tag{28}$$

Indeed, suppose towards contradiction that there are $1 \leq m < k \leq L$ so that

$$\mathbb{D}(a_m, \tau r_m) \cap \mathbb{D}(a_k, \tau r_k) \neq \emptyset.$$

By Lemma 4.2 we have $r_k < \tau r_m$, but by the definition of B_m and since $p_k \notin B_m$ we have $a_k \notin \overline{\mathbb{D}}(a_m, 2\tau r_m)$. Combining with triangle inequality gives a contradiction, proving (28).

We are ready to estimate the energy of ρ. Clearly $\int_\Omega \rho^2\,dA \le |\mathbb{D}(0,3)| = 9\pi$, so it suffices to estimate the sum of ρ^2 over $\mathcal{C}(\Omega)$. By (26) and (27) we have

$$\sum_{p\in G_k} \rho(p)^2 = 64\tau^{-2} \sum_{p\in G_k} \operatorname{diam}(p)^2 \le \frac{640 N r_k^2}{\tau^{12}} \tag{29}$$

for every $1 \le k \le L$. On the other hand, since the pairwise disjoint disks in (28) are subsets of $\mathbb{D}(0,5)$, we have

$$\pi\tau^2 \sum_{k=1}^{L} r_k^2 = \sum_{k=1}^{L} |\mathbb{D}(a_k, \tau r_k)| \le |\mathbb{D}(0,5)| = 25\pi. \tag{30}$$

Combining (29) and (30) yields $\sum_{p\in G} \rho(p)^2 \le \frac{16000N}{\tau^{14}}$.

We have proved that Theorem 3.1 follows from Proposition 4.3.

4.4. **Proof of Proposition 4.3: Finding good subpaths.** Let $\gamma : [0,1] \to \hat{\Omega}$ be as in the proposition. We may assume that $\gamma(t) \in B$ for some $0 < t < 1$, since otherwise Proposition 4.3 follows directly from the choices of the endpoints of γ. We construct families

$$\mathcal{I}_k = \{I_1^k, I_2^k, \dots, I_{n(k)}^k\}, \quad 0 \le k \le L,$$

of subsegments of $[0,1]$ with pairwise disjoint interiors, using the following algorithm: First let $\mathcal{I}_0 = \{[0,1]\}$, then assume that $\mathcal{I}_\ell$ is defined for all $0 \le \ell \le k-1$. We define $\mathcal{I}_k$ by choosing suitable subintervals of the intervals I in $\mathcal{I}_{k-1}$.

Fix $I = [s_0, t_0] \in \mathcal{I}_{k-1}$ and denote $\alpha = \gamma|[s_0, t_0]$. We consider the following cases:

(1) If $\alpha(t) \notin B_k$ for all $s_0 < t < t_0$, then we include $[s_0, t_0]$ in $\mathcal{I}_k$.
(2) Otherwise, let (recall that B_k is a finite set and so $s_0 < s_2 \le t_2 < t_0$ below)

$$\begin{aligned} A &= \{s_0 < t < t_0 : \alpha(t) \in B_k\}, \quad s_2 = \min A \quad \text{and} \quad t_2 = \max A, \\ A_2 &= \{s_0 < t < s_2 : \alpha(t) \cap \mathbb{S}(b_k, r_k) \ne \emptyset\}, \quad \text{and} \\ A_3 &= \{t_2 < t < t_0 : \alpha(t) \cap \mathbb{S}(b_k, r_k) \ne \emptyset\}. \end{aligned}$$

(a1) If $A_2 \cup A_3 = \emptyset$, we do not include any subinterval of $[s_0, t_0]$ in $\mathcal{I}_k$.
(a2) If $A_2 \ne \emptyset$ and $A_3 = \emptyset$, we include $[s_0, s_2]$ in $\mathcal{I}_k$.
(a3) If $A_2 = \emptyset$ and $A_3 \ne \emptyset$, we include $[t_2, t_0]$ in $\mathcal{I}_k$.
(b) If $A_2 \ne \emptyset$ and $A_3 \ne \emptyset$, let $s_1 = \max A_2$ and $t_1 = \min A_3$. Notice that $s_0 < s_1 < s_2 \le t_2 < t_1 < t_0$.
(b1) if $\operatorname{diam}(\alpha(c)) \ge \tau r_k$ for $c = s_1$ or t_1, we include $[s_0, s_1]$ and $[t_1, t_0]$ in $\mathcal{I}_k$.
(b2) Otherwise we include $[s_0, s_1]$, $[s_1, s_2]$, $[t_2, t_1]$ and $[t_1, t_0]$ in $\mathcal{I}_k$.

Let $\mathcal{I}_k([s_0, t_0])$ be the family of subsegments of $[s_0, t_0] \in \mathcal{I}_{k-1}$ included in $\mathcal{I}_k$ using the above algorithm, and

$$\mathcal{I}_k = \bigcup_{[s_0,t_0]\in\mathcal{I}_{k-1}} \mathcal{I}_k([s_0, t_0]), \quad 1 \le k \le L.$$

We will show that the segments in $\mathcal{I}_L$ satisfy the requirements of Proposition 4.3. Notice that the above construction combined with a simple induction argument shows that if

$1 \le k \le L$ and $a < t < b$ for some $[a,b] \in \mathcal{I}_k$ then $\gamma(t) \notin \bigcup_{\ell=1}^{k} B_\ell$. In particular, $\gamma(t) \notin B$ for all $t \in \bigcup_{[a,b]\in\mathcal{I}_L}(a,b)$. Clearly the interiors of distinct segments in $\mathcal{I}_L$ are non-empty and pairwise disjoint. Thus, in order to prove Proposition 4.3 it suffices to show that segments in $\mathcal{I}_L$ satisfy estimate (23).

Given $1 \le k \le L$, let $\mathcal{J}_{k-1}(c) \subset \mathcal{I}_{k-1}$ be the family for which case

$$c \in \{(1), (a1), (a2), (a3), (b1), (b2)\}$$

applies, $\mathcal{J}_{k-1}(a) = \mathcal{J}_{k-1}(a1) \cup \mathcal{J}_{k-1}(a2) \cup \mathcal{J}_{k-1}(a3)$, $\mathcal{J}_{k-1}(b) = \mathcal{J}_{k-1}(b1) \cup \mathcal{J}_{k-1}(b2)$, and $\mathcal{J}(c) = \bigcup_{k=1}^{L} \mathcal{J}_{k-1}(c)$. We use notation $T(I) = \operatorname{dist}(\gamma(a),\gamma(b))$ for $I = [a,b]$. We next claim that

$$\begin{aligned} \frac{11}{10} &\le \sum_{I\in\mathcal{I}_L} T(I) + \sum_{k=1}^{L} \big(2(\operatorname{card}\mathcal{J}_{k-1}(a)) - 100\tau(\operatorname{card}\mathcal{J}_{k-1}(b2))\big)\cdot r_k \\ &+ 3\tau^{-1} \sum_{t\in G(\gamma)} \operatorname{diam}(\gamma(t)). \end{aligned} \tag{31}$$

4.5. **Proof of Proposition 4.3: Preliminary estimates.** The goal of this subsection is to establish (31).

Lemma 4.4. *Let $1 \le k \le L$ and $[s_0,t_0] \in \mathcal{J}_{k-1}(a)$. Then*

$$\operatorname{dist}(\gamma(s_0),\gamma(t_0)) \le Q([s_0,t_0]) + 2r_k,$$

where

$$Q([s_0,t_0]) = \begin{cases} 0 & \text{in Case (a1)}, \\ \operatorname{dist}(\gamma(s_0),\gamma(s_2)) & \text{in Case (a2)}, \\ \operatorname{dist}(\gamma(t_2),\gamma(t_0)) & \text{in Case (a3)}. \end{cases}$$

Proof. In Case (a1) the definitions of A_2 and A_3 show that

$$\gamma(s_0) \cap \overline{\mathbb{D}}(b_k, r_k) \ne \emptyset \quad \text{and} \quad \gamma(t_0) \cap \overline{\mathbb{D}}(b_k, r_k) \ne \emptyset. \tag{32}$$

The claim then follows by triangle inequality.

Case (a3) is similar to Case (a2). In Case (a2) the second part of (32) holds. Since $\gamma(s_2) \in B_k$, we have

$$\gamma(s_2) \subset \mathbb{D}(a_k, 4\tau r_k) \subset \mathbb{D}(b_k, \tau^{1/2} r_k)$$

by (9) and (18), and $\operatorname{diam}(\gamma(s_2)) < \tau r_k$. Therefore,

$$\begin{aligned} \operatorname{dist}(\gamma(s_0),\gamma(t_0)) &\le \operatorname{dist}(\gamma(s_0),\gamma(s_2)) + \operatorname{dist}(\gamma(s_2),\gamma(t_0)) + \operatorname{diam}(\gamma(s_2)) \\ &\le \operatorname{dist}(\gamma(s_0),\gamma(s_2)) + (1+\tau^{1/2})r_k + \tau r_k \le \operatorname{dist}(\gamma(s_0),\gamma(s_2)) + 2r_k \end{aligned}$$

by triangle inequality and since $\tau^{1/2} + \tau \le 1$. □

Lemma 4.5. *Let $1 \le k \le L$ and $[s_0,t_0] \in \mathcal{J}_{k-1}(b1)$. Then*

$$\operatorname{diam}(\gamma(c)) \ge \tau r_k \quad \textit{and} \quad \gamma(c) \in \bigcup_{\ell=1}^{k} G_\ell \tag{33}$$

for $c = s_1$ *or* $c = t_1$*. Moreover,*

$$\operatorname{dist}(\gamma(s_0), \gamma(t_0)) \leq \operatorname{dist}(\gamma(s_0), \gamma(s_1)) + \operatorname{dist}(\gamma(t_1), \gamma(t_0)) + 3\tau^{-1} D([s_0, t_0]). \tag{34}$$

Here $D([s_0, t_0]) = \sum \operatorname{diam}(p)$ *and the sum is over the* $p \in \{\gamma(s_1), \gamma(t_1)\}$ *which satisfy* (33).

Proof. Recall that both $\gamma(s_1), \gamma(t_1)$ intersect $\mathbb{S}(b_k, r_k)$ and

$$\operatorname{diam}(\gamma(c)) \geq \tau r_k \quad \text{for } c = s_1 \text{ or } t_1. \tag{35}$$

Also, recall that $s_0 < s_1 < t_1 < t_0$ and

$$\gamma(t) \notin \bigcup_{\ell=1}^{k-1} B_\ell \quad \text{for all } s_0 < t < t_0.$$

Therefore, the definition of G_k shows that if c satisfies (35) then $\gamma(c) \in \bigcup_{\ell=1}^{k} G_\ell$. By triangle inequality we have

$$\begin{aligned} \operatorname{dist}(\gamma(s_0), \gamma(t_0)) &\leq \operatorname{dist}(\gamma(s_0), \gamma(s_1)) + \operatorname{dist}(\gamma(t_1), \gamma(t_0)) + \operatorname{dist}(\gamma(s_1), \gamma(t_1)) \\ &+ \operatorname{diam}(\gamma(s_1)) + \operatorname{diam}(\gamma(t_1)). \end{aligned}$$

The last distance is bounded from above by $2r_k \leq 2\tau^{-1} D([s_0, t_0])$, and the sum of the diameters is bounded from above by $\tau r_k + D([s_0, t_0]) \leq 2D([s_0, t_0])$. Inequality (34) follows. □

Lemma 4.6. *Let* $1 \leq k \leq L$ *and* $[s_0, t_0] \in \mathcal{J}_{k-1}(b2)$*. Then*

$$\operatorname{dist}(\gamma(s_0), \gamma(t_0)) \leq \sum_{m=0}^{1} \Big[\operatorname{dist}(\gamma(s_m), \gamma(s_{m+1})) + \operatorname{dist}(\gamma(t_m), \gamma(t_{m+1})) \Big] - 100\tau r_k.$$

Proof. Recall that $p \subset \mathbb{D}(a_k, 4\tau r_k)$ for every $p \in B_k$ by (18). Therefore, Proposition 4.1 can be applied to $\alpha = \gamma|[s_1, t_1]$ and we have

$$\operatorname{dist}(\gamma(s_1), \gamma(t_1)) \leq \operatorname{dist}(\gamma(s_1), \gamma(s_2)) + \operatorname{dist}(\gamma(t_1), \gamma(t_2)) - 200\tau r_k.$$

On the other hand, $\operatorname{diam}(\gamma(s_1)) + \operatorname{diam}(\gamma(t_1)) \leq 2\tau r_k$ by assumption. The claim follows by combining the estimates with triangle inequality. □

We are ready to prove (31). We apply Lemmas 4.4, 4.5 and 4.6 to see that if $1 \leq k \leq L$ then (recall notation $T(I) = \operatorname{dist}(\gamma(a), \gamma(b))$ for $I = [a, b]$)

$$\begin{aligned} \sum_{I' \in \mathcal{I}_{k-1}} T(I') &\leq \sum_{I \in \mathcal{I}_k} T(I) + \big(2(\operatorname{card} \mathcal{J}_{k-1}(a)) - 100\tau(\operatorname{card} \mathcal{J}_{k-1}(b2))\big) \cdot r_k \\ &+ 3\tau^{-1} \sum_{I \in \mathcal{J}_{k-1}(b1)} D(I). \end{aligned} \tag{36}$$

Recalling that $T([0,1]) \geq \frac{11}{10}$ and applying induction together with (36) yields

$$\begin{aligned} \frac{11}{10} &\leq \sum_{I \in \mathcal{I}_L} T(I) + \sum_{k=1}^{L} \big(2(\operatorname{card} \mathcal{J}_{k-1}(a)) - 100\tau(\operatorname{card} \mathcal{J}_{k-1}(b2))\big) \cdot r_k \\ &+ 3\tau^{-1} \sum_{k=1}^{L} \sum_{I \in \mathcal{J}_{k-1}(b1)} D(I). \end{aligned} \tag{37}$$

Finally, it follows from the construction that each $p \in G$ satisfies (33) in Lemma 4.5 for at most one interval $[s_0, t_0] \in \mathcal{J}(b1)$. Therefore

$$\sum_{k=1}^{L} \sum_{I \in \mathcal{J}_{k-1}(b1)} D(I) = \sum_{I \in \mathcal{J}(b1)} D(I) \leq \sum_{t \in G(\gamma)} \operatorname{diam}(\gamma(t)), \tag{38}$$

recall notation $G(\gamma) = \{0 < t < 1 : \gamma(t) \in G\}$. Combining (37) and (38) proves (31).

4.6. **Proof of Proposition 4.3: Completion of the proof.** Estimate (23), which is the remaining claim in Proposition 4.3, follows by combining (31) with

$$\sum_{k=1}^{L} (\operatorname{card} \mathcal{J}_{k-1}(a)) \cdot r_k \leq \frac{1}{20} + 4\tau^{-1} \sum_{t \in G(\gamma)} \operatorname{diam}(\gamma(t)) + 12\tau \sum_{k=1}^{L} (\operatorname{card} \mathcal{J}_{k-1}(b2)) \cdot r_k. \tag{39}$$

The rest of this section is devoted to the proof of (39). The strategy is to associate to each $I \in \mathcal{J}(a1) \cup \mathcal{J}(a3)$ (resp., $\mathcal{J}(a1) \cup \mathcal{J}(a2)$) the left (resp., right) endpoint of a suitably chosen "grandparent" I' of I. We first consider the left endpoints c. We now give precise definitions.

We say that $J \in \mathcal{I}_k$ is a *child* of $I \in \mathcal{I}_{k-1}$, and I the *parent* of J, if $J \subset I$. The consequent definitions of grandchildren and grandparents are obvious. Recall that segments in $\mathcal{J}(a1)$ do not have children and every other segment in $\bigcup_{k=1}^{L} \mathcal{I}_{k-1}$ has at least one child. More precisely:

(1) If $I \in \mathcal{J}_{k-1}(1) \cup \mathcal{J}_{k-1}(a2) \cup \mathcal{J}_{k-1}(a3)$, then I has one child.
(2) If $I \in \mathcal{J}_{k-1}(b1)$, then I has two children.
(3) If $I \in \mathcal{J}_{k-1}(b2)$, then I has four children.

It follows by our choice of τ that $[0, 1] \notin \mathcal{J}(a1)$. Moreover, if $L = 1$ then (39) follows from the choice of r_1. We assume from now on that $L \geq 2$.

We next define finite sequences $S = S(c_\ell)$ of segments $I_{m-1} = [c_{m-1}, d_{m-1}]$,

$$I_{m-1} \in \mathcal{I}_{m-1}, \quad 1 \leq \ell \leq m \leq n \leq L, \quad I_{\ell-1} \supset I_\ell \supset \cdots \supset I_{n-1},$$

as follows. We fix $1 \leq \ell \leq L - 1$ so that $\ell = 1$ or $\mathcal{J}_{\ell-1}(b) \neq \emptyset$. Moreover, we fix $I_{\ell-1}$ so that

$$I_{\ell-1} = [0, 1] \quad \text{if } \ell = 1 \quad \text{and} \quad I_{\ell-1} \in \mathcal{J}_{\ell-1}(b) \quad \text{if } \ell \geq 2.$$

(1) By the above discussion $I_{\ell-1}$ has at least one child.
  (i) If $\ell = 1$, then we choose $I_1 = [c_1, d_1]$ to be any one of the children of $[0, 1]$.
  (ii) If $\ell \geq 2$, then we choose $I_\ell = [c_\ell, d_\ell]$ to be any one of the (several) children of $I_{\ell-1}$ except the one for which $c_\ell = c_{\ell-1}$.

  Our sequence will be uniquely determined by the choice of I_ℓ. Notice that if $c_\ell \neq 0$ then c_ℓ lies in the interior of $I_{\ell-1}$.
(2) Suppose that $m \geq \ell + 1$ and I_{m-1} has been defined.
  (ii) If $m = L$ or $I_{m-1} \in \mathcal{J}(a1)$, then we set $n = m$ and stop the process.
  (iii) Otherwise $\ell < m < L$ and I_{m-1} has at least one child.

(a) If $I_{m-1} \in \mathcal{J}(1) \cup \mathcal{J}(a2) \cup \mathcal{J}(a3)$, then I_{m-1} has exactly one child I. We choose $I_m := I$. Segments I_{m-1} and I_m have different left endpoints if and only if $I_{m-1} \in \mathcal{J}(a3)$.
(b) If $I_{m-1} \in \mathcal{J}(b)$, then I_{m-1} has a child I with the same left endpoint as I_{m-1}. We choose $I_m := I$.

We let $S(c_\ell)$ be the collection of all the segments I_{m-1} chosen above; this notation is valid since c_ℓ determines $S(c_\ell)$ uniquely.

Lemma 4.7. *Every $I \in \mathcal{J}(a1) \cup \mathcal{J}(a3)$ belongs to exactly one $S(c_\ell)$.*

Proof. We can identify c_ℓ as the left endpoint of the smallest grandparent I' of I (ordered by inclusion) with the property that the parent I'' of I' belongs to $\mathcal{J}(b)$ and has left endpoint different from the left endpoint of I'. If no such I' exists, then $\ell = 1$ and $c_\ell = 0$. □

We fix $S(c_\ell)$ and denote by $\ell \le m_1 < m_2 < \cdots < m_\nu \le n$ the indices for which

$$I_{m_\mu - 1} \in \mathcal{J}_{m_\mu - 1}(a1) \cup \mathcal{J}_{m_\mu - 1}(a3). \tag{40}$$

We may assume without loss of generality that $\nu \ge 1$, i.e., that there is at least one such index. Recall notation $I_{m-1} = [c_{m-1}, d_{m-1}]$.

Lemma 4.8. *Suppose that $2 \le \mu \le \nu$. Then $c_{m_\mu - 1} = c_{m_{\mu-1}}$ and*

$$\gamma(c_{m_\mu - 1}) \subset \mathbb{D}(a_{m_{\mu-1}}, 4\tau r_{m_{\mu-1}}) \cap \mathbb{D}(b_{m_\mu}, r_{m_\mu}) \subset \mathbb{D}(b_{m_{\mu-1}}, r_{m_{\mu-1}}) \cap \mathbb{D}(b_{m_\mu}, r_{m_\mu}). \tag{41}$$

Proof. The second inclusion in (41) follows from (9). Claims $\gamma(c_{m_\mu - 1}) \subset \mathbb{D}(b_{m_\mu}, r_{m_\mu})$ and $c_{m_\mu - 1} = c_{m_{\mu-1}}$ follow from the construction; notice that if $m \le n-1$ then $c_m \ne c_{m-1}$ only if $[c_{m-1}, d_{m-1}] \in \mathcal{J}_{m-1}(a3)$. To see why

$$\gamma(c_{m_\mu - 1}) = \gamma(c_{m_{\mu-1}}) \subset \mathbb{D}(a_{m_{\mu-1}}, 4\tau r_{m_{\mu-1}}) \tag{42}$$

holds, observe that we have $I_{m_{\mu-1}-1} \in \mathcal{J}_{m_{\mu-1}-1}(a3)$ and so $\gamma(c_{m_{\mu-1}}) \in B_{m_{\mu-1}}$. Now (42) follows from (18). We conclude that also the first inclusion in (41) holds. The proof is complete. □

Lemma 4.9. *Suppose that $\ell \ge 2$. Then $m_1 \ge \ell + 1$ and $c_{m_1 - 1} = c_\ell$. Moreover, if* $\operatorname{diam}(\gamma(c_{m_1-1})) < \tau r_\ell$*, then*

$$\gamma(c_{m_1 - 1}) \subset \mathbb{D}(b_\ell, (1+\tau) r_\ell) \cap \mathbb{D}(b_{m_1}, r_{m_1}).$$

Proof. We assume $\ell \ge 2$ and so $[c_{\ell-1}, d_{\ell-1}] = I_{\ell-1} \in \mathcal{J}_{\ell-1}(b)$ and $m_1 \ge \ell + 1$. Since

$$I_{m_1 - 1} \in \mathcal{J}_{m_1 - 1}(a1) \cup \mathcal{J}_{m_1 - 1}(a3), \tag{43}$$

we have $\gamma(c_{m_1-1}) \subset \mathbb{D}(b_{m_1}, r_{m_1})$ by construction. Also, since m_1 is the smallest index for which (43) holds, we have $c_{m_1-1} = c_\ell$.

Recall that $c_\ell \ne c_{\ell-1}$ by construction. More precisely, $c_\ell \in \{s_1, t_2, t_1\}$ when Case (b1) or Case (b2) is applied to $[s_0, t_0] = [c_{\ell-1}, d_{\ell-1}]$. In particular, we have

$$\gamma(c_{m_1-1}) \cap \overline{\mathbb{D}}(b_\ell, r_\ell) = \gamma(c_\ell) \cap \overline{\mathbb{D}}(b_\ell, r_\ell) \ne \emptyset. \tag{44}$$

If $\operatorname{diam}(\gamma(c_{m_1-1})) < \tau r_\ell$ then by (44) we have $\gamma(c_{m_1-1}) \subset \mathbb{D}(b_\ell, (1+\tau) r_\ell)$. □

Lemma 4.10. *Let $S(c_\ell) = \{I_{\ell-1}, \dots, I_{n-1}\}$ as in (40). Then*

$$\sum_{\mu=1}^{\nu} r_{m_\mu} \le \begin{cases} \frac{1}{160} & \text{if } \ell = 1, \\ 2\tau r_\ell & \text{if } \ell \ge 2 \text{ and } \operatorname{diam}(\gamma(c_\ell)) < \tau r_\ell, \\ 2\tau^{-1} \operatorname{diam}(\gamma(c_\ell)) & \text{if } \ell \ge 2 \text{ and } \operatorname{diam}(\gamma(c_\ell)) \ge \tau r_\ell. \end{cases} \tag{45}$$

Proof. Suppose first that $2 \le \mu \le \nu$. Combining Lemma 4.8 with (9), we see that

$$\mathbb{D}(a_{m_{\mu-1}}, 2r_{m_{\mu-1}}) \cap \mathbb{D}(a_{m_\mu}, 2r_{m_\mu}) \neq \emptyset.$$

Thus, by Lemma 4.2 we have $r_{m_\mu} \le \tau r_{m_{\mu-1}}$. Iterating the estimate yields

$$r_{m_\nu} \le \tau r_{m_{\nu-1}} \le \cdots \le \tau^{\nu-1} r_{m_1}. \tag{46}$$

If $\ell = 1$, then the upper bound in (45) follows from (46) by recalling that $r_{m_1} < \tau < \frac{1}{1000}$.

If $\ell \ge 2$ and $\operatorname{diam}(\gamma(c_\ell)) < \tau r_\ell$ then combining Lemma 4.9 with (9) and Lemma 4.2 as in the above paragraph shows that

$$r_{m_1} \le \tau r_\ell. \tag{47}$$

The upper bound in (45) follows from (46) and (47) by recalling again that $\tau < \frac{1}{1000}$. If $\ell \ge 2$ and $\operatorname{diam}(\gamma(c_\ell)) \ge \tau r_\ell \ge \tau r_{m_1}$, then the upper bound in (45) follows from (46). □

Suppose $S(c_\ell) = \{I_{\ell-1}, \dots, I_{n-1}\}$ is as in Lemma 4.10. We apply (45) to estimate sum

$$\sum_{k=1}^{L} (\operatorname{card}(\mathcal{J}_{k-1}(a1) \cup \mathcal{J}_{k-1}(a3))) \cdot r_k$$

from above. First, since $[0, 1]$ has at most four children there are at most four distinct sequences $S(c_\ell)$ for which $\ell = 1$. We denote by $\mathcal{S}^1$ the set of all intervals in $\mathcal{J}(a1) \cup \mathcal{J}(a3)$ which belong to such a sequence. Moreover, given $1 \le k \le L$, we denote

$$\mathcal{S}^1_{k-1} = \mathcal{S}^1 \cap (\mathcal{J}_{k-1}(a1) \cup \mathcal{J}_{k-1}(a3)). \tag{48}$$

By (45) we have

$$\sum_{k=1}^{L} (\operatorname{card}(\mathcal{S}^1_{k-1})) \cdot r_k \le \frac{1}{40}. \tag{49}$$

Next assume that $\ell \ge 2$. Then $I_{\ell-1} \in \mathcal{J}_{\ell-1}(b)$. We first consider the sequences $S(c_\ell)$ for which $I_{\ell-1} \in \mathcal{J}_{\ell-1}(b1)$. We denote by $\mathcal{S}^2$ the set of all intervals in $\mathcal{J}(a1) \cup \mathcal{J}(a3)$ which belong to such a sequence, and define $\mathcal{S}^2_{k-1}$ as in (48).

Each $I_{\ell-1}$ has two children, $[s_0, s_1]$ and $[t_1, t_0] = [c_\ell, d_\ell] = I_\ell$. By construction and Lemma 4.5 we have

$$\operatorname{diam}(\gamma(c)) \ge \tau r_\ell \quad \text{and} \quad \gamma(c) \in \bigcup_{k=1}^{\ell} G_k$$

for $c = s_1$ or $c = t_1$ (or both). By (45) such a c moreover satisfies

$$\sum_{\mu=1}^{\nu} r_{m_\mu} \le 2\tau^{-1} \operatorname{diam}(\gamma(c)). \tag{50}$$

Notice that since t_1, s_1 are interior points of $I_{\ell-1}$, every $0<t<1$ can have the role of c in (50) for at most one sequence $S(c_\ell)$ for which $I_{\ell-1} \in \mathcal{J}_{\ell-1}(b1)$. Therefore, summing (50) over such sequences we have

$$\sum_{k=1}^{L} (\mathrm{card}(\mathcal{S}^2_{k-1})) \cdot r_k \leq 2\tau^{-1} \sum_{t \in G(\gamma)} \mathrm{diam}(\gamma(t)). \tag{51}$$

Finally, assume that $\ell \geq 2$ and $I_{\ell-1} \in \mathcal{J}_{\ell-1}(b2)$. We denote by $\mathcal{S}^3$ the set of all intervals in $\mathcal{J}(a1) \cup \mathcal{J}(a3)$ which belong to such a sequence, and define $\mathcal{S}^3_{k-1}$ as in (48). By construction we have $\mathrm{diam}(\gamma(c_\ell)) < \tau r_\ell$. Therefore, in this case (45) gives

$$\sum_{\mu=1}^{\nu} r_{m_\mu} \leq 2\tau r_\ell. \tag{52}$$

For each $I \in \mathcal{J}(b2)$ there are at most three sequences $S(c_\ell)$ for which $I = I_{\ell-1}$. Therefore, summing (52) over all such sequences we have

$$\sum_{k=1}^{L} (\mathrm{card}(\mathcal{S}^3_{k-1})) \cdot r_k \leq 6\tau \sum_{k=1}^{L} (\mathrm{card}\, \mathcal{J}_{k-1}(b2)) \cdot r_k \tag{53}$$

By Lemma 4.7 every $I \in \mathcal{J}(a1) \cup \mathcal{J}(a3)$ belongs to some $S(c_\ell)$. Therefore, combining (49), (51) and (53) gives

$$\begin{aligned} \sum_{k=1}^{L} (\mathrm{card}(\mathcal{J}_{k-1}(a1) \cup \mathcal{J}_{k-1}(a3))) \cdot r_k &\leq \frac{1}{40} + 2\tau^{-1} \sum_{t \in G(\gamma)} \mathrm{diam}(\gamma(t)) \\ + \quad 6\tau \sum_{k=1}^{L} (\mathrm{card}\, \mathcal{J}_{k-1}(b2)) \cdot r_k. \end{aligned} \tag{54}$$

By applying an identical argument involving Case (a2) and the right endpoints d_ℓ instead of Case (a3) and the left endpoints, we have

$$\begin{aligned} \sum_{k=1}^{L} (\mathrm{card}(\mathcal{J}_{k-1}(a1) \cup \mathcal{J}_{k-1}(a2))) \cdot r_k &\leq \frac{1}{40} + 2\tau^{-1} \sum_{t \in G(\gamma)} \mathrm{diam}(\gamma(t)) \\ + \quad 6\tau \sum_{k=1}^{L} (\mathrm{card}\, \mathcal{J}_{k-1}(b2)) \cdot r_k. \end{aligned} \tag{55}$$

Combining (54) and (55) gives (39). The proof of Proposition 4.3 is complete.

5. Proofs of modulus estimates on circle domains, Proposition 3.2

We fix $\bar{p} \in \mathcal{C}(\Omega)$, Jordan curve $J \subset \Omega$, and points b, a as in the proposition. Let $j \geq 1$ if $\bar{p} \in \mathcal{C}_P(\Omega)$ and $j \geq \ell$ if $\bar{p} = p_\ell \in \mathcal{C}_N(\Omega)$. Then $\hat{f}_j(\bar{p})$ is a generalized disk or a point in $\hat{\mathbb{C}}$. In the following proof it is convenient to replace normalization (2), which was applied to guarantee the injectivity of limit map f, with a new normalization.

Namely, since transboundary modulus and generalized disks are invariant under Möbius transformations, we lose no generality by replacing sequence (f_j) with $(h \circ f_j)_j$, where h is any Möbius transformation. Therefore, by choosing h suitably we may assume that

$$\hat{f}_j(\bar{p}) \cup f_j(J) \subset \mathbb{D}(0,1), \quad \infty \in D_j, \quad \text{and} \quad f_j(J) \text{ separates } \hat{f}_j(\bar{p}) \text{ and } \infty. \tag{56}$$

We start with the first estimate in Proposition 3.2, i.e.,

$$\limsup_{j\to\infty} \operatorname{mod} \hat{f}_j(\Gamma_j) \geq \limsup_{j\to\infty} \varphi_a(\operatorname{dist}(f_j(b), \hat{f}_j(\bar{p}))). \tag{57}$$

We denote $\operatorname{dist}(f_j(b), \hat{f}_j(\bar{p}))$ by δ. Let w_0 be the point in $\hat{f}_j(\bar{p})$ closest to $f_j(b)$. After a rotation about the origin, $f_j(b) = \delta i + w_0$. Since $f_j(J)$ separates $\hat{f}_j(\bar{p})$ and ∞, it follows that every line $L_s = \{t + si + w_0 : t \in \mathbb{R}\}$, $0 < s < \delta$, has a subsegment $I_s \subset U \subset \mathbb{D}(0,1)$ so that $\pi_{D_j}(I_s) \in \hat{f}_j(\Gamma_j)$. Here U is the bounded component of $\hat{\mathbb{C}} \setminus f_j(J)$.

Recall that $\mathcal{C}(D_j)$ consists of disks. Let ρ be admissible for $\hat{f}_j(\Gamma_j)$. Then

$$1 \leq \int_{I_s \cap D_j} \rho \, ds + \sum_{q \in \mathcal{C}^s(D_j)} \rho(q) \quad \text{for all } 0 < s < \delta, \tag{58}$$

where $\mathcal{C}^s(D_j) = \{q \in \mathcal{C}(D_j) : \, I_s \cap q \neq \emptyset\}$. Combining (58) with Fubini's theorem yields

$$\delta \leq \int_{D_j \cap U} \rho \, dA + \sum_{q \in \mathcal{C}_U} \operatorname{diam}(q)\rho(q), \tag{59}$$

where $\mathcal{C}_U = \{q \in \mathcal{C}(D_j) : q \subset U\}$. By Hölder's inequality (since $U \subset \mathbb{D}(0,1)$) we have

$$\int_{D_j \cap U} \rho \, dA \leq \operatorname{Area}(U)^{1/2} \Big(\int_{D_j} \rho^2 \, dA \Big)^{1/2} \leq \pi^{1/2} \Big(\int_{D_j} \rho^2 \, dA \Big)^{1/2} \quad \text{and}$$

$$\sum_{q \in \mathcal{C}_U} \operatorname{diam}(q)\rho(q) \leq \Big(\sum_{q \in \mathcal{C}_U} \operatorname{diam}(q)^2 \Big)^{1/2} \Big(\sum_{q \in \mathcal{C}(D_j)} \rho(q)^2 \Big)^{1/2} \leq 2 \Big(\sum_{q \in \mathcal{C}(D_j)} \rho(q)^2 \Big)^{1/2}.$$

Combining with (59) and taking infimum with respect to admissible functions shows that

$$\operatorname{mod} \hat{f}_j(\Gamma_j) \geq \left(\frac{\delta}{\pi^{1/2} + 2} \right)^2.$$

In particular, (57) holds.

We now consider the second estimate in Proposition 3.2, i.e.,

$$\text{If } \operatorname{diam}(\hat{f}(\bar{p})) = 0 \text{ then } \lim_{j\to\infty} \operatorname{mod} \hat{f}_j(\Lambda_j) \to \infty. \tag{60}$$

Notice that the first claim in (4) does not depend on (60), so by (57) and the proof given in Section 3 we already know that $\hat{f}(p_\ell) = q_\ell$ for every $\ell = 1, 2, \ldots$. In particular, generalized disks q_ℓ are pairwise disjoint.

We construct a sequence of annuli as follows (compare to the proof of (5)): By our assumption and Normalization (56) we have $\hat{f}(\bar{p}) = \{w_0\}$, where $w_0 \in \mathbb{C}$. Let r_1 be the number satisfying $\operatorname{dist}(f(J),w_0)=10r_1$. Since $f_j\to f$ locally uniformly in Ω, we may assume that $\operatorname{dist}(f_j(J),w_0)\geq 5r_1$ for all j. Assuming $r_1,\dots,r_{n-1}$ are defined, let

$$r_n=\frac{r_n'}{10},$$

where $r_n'\leq r_{n-1}/2$ is the smallest radius for which some q_ℓ intersects both $\mathbb{S}(w_0,r_{n-1}/2)$ and $\mathbb{S}(w_0,r_n')$. If no q_ℓ intersects $\mathbb{S}(w_0,r_{n-1}/2)$ then we set $r_n'=r_{n-1}/2$. We let

$$\mathbb{A}_n=\mathbb{D}(w_0,4r_n)\setminus\overline{\mathbb{D}}(w_0,r_n/2),\quad n=1,2,\dots.$$

Now fix $M\geq 1$, Jordan curve $J'\subset\Omega$ surrounding $\bar{p}$, and j_M so that

$$f_j(J')\subset\mathbb{D}(w_0,r_M/10)\quad\text{for all } j\geq j_M;$$

such choices are possible since $\hat{f}(\bar{p})=\{w_0\}$. By uniform convergence and our choices of radii r_n we may also assume that

$$\pi_{D_j}(\mathbb{A}_n)\cap\pi_{D_j}(\mathbb{A}_m)=\emptyset\quad\text{for all } 1\leq n,m\leq M \text{ and } j\geq j_M. \tag{61}$$

Let $1\leq n\leq M$. Given $r_n/2<t<4r_n$, we denote by $\tilde{\gamma}_t$ the circle $\mathbb{S}(w_0,t)$ parametrized by arclength, $\gamma_t=\pi_{D_j}\circ\tilde{\gamma}_t$, and

$$\Phi_j(n)=\{\gamma_t:\ r_n/2<t<4r_n\}.$$

Then $\Phi_j(n)\subset\hat{f}_j(\Lambda_j)$. We next prove a lower bound for $\operatorname{mod}(\Phi_j(n))$. Let ρ be admissible for $\Phi_j(n)$ and $r_n/2<t<4r_n$. Then

$$1\leq\int_{\mathbb{S}(w_0,t)\cap D_j}\rho\,ds+\sum_{q\in\mathcal{C}^t(D_j)}\rho(q), \tag{62}$$

where $\mathcal{C}^t(D_j)=\{q\in\mathcal{C}(D_j):\ q\cap\mathbb{S}(w_0,t)\neq\emptyset\}$. We divide both sides of (62) by t and integrate from $r_n/2$ to $4r_n$ to conclude

$$\log 8\leq\int_{\mathbb{A}_n\cap D_j}\frac{\rho(z)}{|z|}\,dA(z)+\frac{2}{r_n}\sum_{q\cap\mathbb{A}_n\neq\emptyset}\min\{\operatorname{diam}(q),4r_n\}\rho(q). \tag{63}$$

We apply Hölder's inequality to estimate the integral on the right:

$$\begin{aligned}\int_{\mathbb{A}_n\cap D_j}\frac{\rho(z)}{|z|}\,dA(z) &\leq \Big(\int_{\mathbb{A}_n\cap D_j}\frac{dA(z)}{|z|^2}\Big)^{1/2}\Big(\int_{\mathbb{A}_n\cap D_j}\rho(z)^2\,dA(z)\Big)^{1/2}\\ &\leq (2\pi\log 8)^{1/2}\Big(\int_{\mathbb{A}_n\cap D_j}\rho(z)^2\,dA(z)\Big)^{1/2}.\end{aligned}$$

To estimate the sum in (63), we denote

$$\begin{aligned}\mathcal{Q}_L &= \{q\in\mathcal{C}(D_j):\ q\cap\mathbb{A}_n\neq\emptyset,\ \operatorname{diam}(q)\geq r_n\},\\ \mathcal{Q}_S &= \{q\in\mathcal{C}(D_j):\ q\cap\mathbb{A}_n\neq\emptyset,\ \operatorname{diam}(q)<r_n\}.\end{aligned}$$

Then

$$\operatorname{card}\mathcal{Q}_L\leq 100\quad\text{and}\quad q\subset\mathbb{D}(w_0,5r_n)\text{ for all } q\in\mathcal{Q}_S, \tag{64}$$

and

$$\frac{2}{r_n}\sum_{q\cap\mathbb{A}_n\neq\emptyset}\min\{\operatorname{diam}(q),4r_n\}\rho(q)\le 8\sum_{q\in\mathcal{Q}_L}\rho(q)+\frac{2}{r_n}\sum_{q\in\mathcal{Q}_S}\operatorname{diam}(q)\rho(q).$$

By Cauchy-Schwarz and (64) we have

$$\sum_{q\in\mathcal{Q}_L}\rho(q)\le 10\Big(\sum_{q\in\mathcal{Q}_L}\rho(q)^2\Big)^{1/2}.$$

Since disks q are pairwise disjoint, Cauchy-Schwarz and (64) also yield

$$\begin{aligned}\sum_{q\in\mathcal{Q}_S}\rho(q) &\le \Big(\sum_{q\in\mathcal{Q}_S}\operatorname{diam}(q)^2\Big)^{1/2}\Big(\sum_{q\in\mathcal{Q}_S}\rho(q)^2\Big)^{1/2}\\ &\le \Big(\frac{4\operatorname{Area}(\mathbb{D}(w_0,5r_n))}{\pi}\Big)^{1/2}\Big(\sum_{q\in\mathcal{Q}_S}\rho(q)^2\Big)^{1/2}\le 10r_n\Big(\sum_{q\in\mathcal{Q}_S}\rho(q)^2\Big)^{1/2}.\end{aligned}$$

Combining the estimates with (63) and taking infimum over all ρ shows that

$$\operatorname{mod}(\Phi_j(n))\ge 10^{-4}\quad\text{for all } j\ge j_M \text{ and } 1\le n\le M. \tag{65}$$

Since $\Phi_j(n)\subset\hat{f}_j(\Lambda_j)$, combining (61) and (65) shows that

$$\operatorname{mod}(\hat{f}_j(\Lambda_j))\ge 10^{-4}M$$

for all $j\ge j_M$. Letting $M\to\infty$ proves (60). The proof of Proposition 3.2 is complete.

6. Necessity of the packing condition in Theorem 1.3

We construct a countably connected domain $\Omega\subset\hat{\mathbb{C}}$ containing ∞ and satisfying Quasitripod Condition (i) (but not Packing Condition (ii)) in Theorem 1.3 so that $\{0\}\in\mathcal{C}(\Omega)$ and $\operatorname{diam}(\hat{f}(\{0\}))>0$ for every conformal $f\colon\Omega\to D$ onto a circle domain D.

We describe the elements of $\mathcal{C}(\Omega)$. First, $\{0\}$ is the only element of $\mathcal{C}_P(\Omega)$. Collection $\mathcal{C}_N(\Omega)$ is parametrized as follows: Given $k\in\mathbb{N}$, we denote by W_k the collection of finite words $w=w_1\cdots w_k$, where $w_j\in\{0,1\}$ for every $1\le j\le k$. Moreover, let $W_0=\{\emptyset\}$ and $W=\bigcup_{k=0}^{\infty}W_k$. We then have

$$\mathcal{C}_N(\Omega)=\{p_w:\, w\in W\}.$$

Words $w\in W$ are ordered so that $0<1<00<01<10<11<000\ldots$. We denote the order of w by $\ell(w)$.

Each p_w is the union of radial segments I_w, J_w and subarcs S_w, T_w of circles centered at the origin. If $w=\bar{w}w_k$, $w_k\in\{0,1\}$, then I_w is a segment of length $2^{-\ell-2}-\epsilon_\ell$, $\ell=\ell(\bar{w})$, in annulus

$$\mathbb{A}_\ell=\overline{\mathbb{D}}(0,2^{-\ell})\setminus\mathbb{D}(0,2^{-\ell-1}),$$

where $\epsilon_\ell>0$ is a small number. Segments $I_{\bar{w}0}$ and $I_{\bar{w}1}$ are subsets of the same half-line starting at the origin.

Arc S_w is attached to the middle of I_w and has length $\frac{1}{24}$ times the length of the full circle. Arc T_w is roughly a half-circle, attached to an end of I_w, and lies in $\mathbb{S}(0,3\cdot 2^{-\ell-2})$

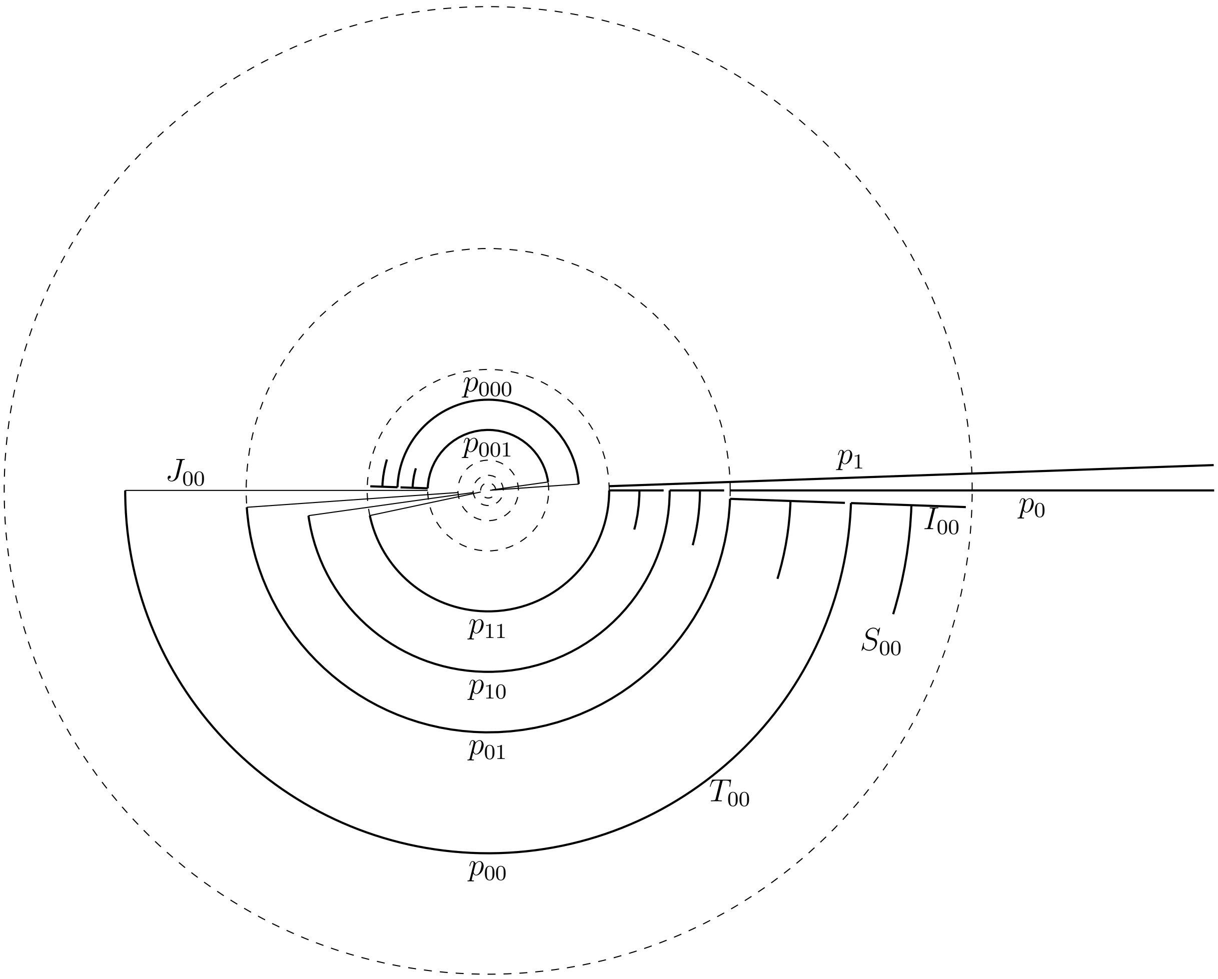

FIGURE 3. Some complementary components of Ω

if $w_k = 0$ and in $\mathbb{S}(0, 2^{-\ell-1})$ if $w_k = 1$. Segment J_w is attached to an end of T_w. The other end of J_w lies at circle $\mathbb{S}(0, 2^{-\ell(w)-1})$. The distance between I_w and $J_{\bar{w}}$ is less than ϵ_ℓ.

Figure 3 shows segments I_{00}, J_{00}, arcs S_{00}, T_{00}, components $p_{00}, p_{01}, p_{10}, p_{11}, p_{000}, p_{001}$, and parts of components p_0, p_1. Sequence $(\epsilon_\ell)_\ell$ can be chosen so that elements p_w have the following properties.

(1) For every $w \in W$ there is $c_w > 0$ so that p_w is the image of $c_w T_0 = \{c_w z : z \in T_0\}$ under a 10^6-biLipschitz map. In particular, each p_w an 10^{12}-quasitripod.
(2) For every $\epsilon > 0$ there is $k_\epsilon \geq 1$ so that if word length $|w| = k \geq k_\epsilon$ then $p_w \subset \mathbb{D}(0, \epsilon)$.
(3) For every $w = \bar{w} w_k$, $w_k \in \{0, 1\}$, there is a family Γ_w of paths connecting $p_{\bar{w}}$ and p_w in Ω so that $\operatorname{mod}(\Gamma_w) \geq 4^k$. More precisely, Γ_w consists of short subarcs of circles in $\mathbb{A}_\ell$ centered at the origin.

Since Ω is countably connected, the He-Schramm theorem [HS93] guarantees the existence of a conformal $f\colon \Omega \to D$ onto a circle domain D. Moreover, f is unique up to postcomposition by a Möbius transformation. To show that $\hat{f}(\{0\}) \in \mathcal{C}_N(D)$, we denote by Γ the family of paths in $\hat{\Omega}$ joining $p_\emptyset$ and $\{0\}$.

Towards contradiction, assume $\hat{f}(\{0\})$ is a point-component. Then $\mathrm{mod}(\hat{f}(\Gamma)) = 0$, which can be proved by applying [Sch95, Theorem 6.1(2)] to a sequence of annuli (or by modifying the proof of (5) in the special case of circle domains). Since transboundary modulus is conformally invariant (Lemma 2.1), then also $\mathrm{mod}(\Gamma) = 0$. The desired contradiction thus follows if we can prove that

$$\mathrm{mod}(\Gamma) > 0. \tag{66}$$

We denote by W_∞ the collection of infinite words $w_1 w_2 \cdots$, where $w_j \in \{0,1\}$. We equip W_∞ with the unique probability measure μ satisfying $\mu(A_w) = 2^{-k}$ for all $k \geq 1$ and $w \in W_k$. Here

$$A_w = \{w_\infty \in W_\infty : \, w_\infty = w w_{k+1} w_{k+2} \cdots\}.$$

Let $\rho : \hat{\Omega} \to [0,\infty]$ be an arbitrary Borel function satisfying

$$\int_\Omega \rho^2 \, dA + \sum_{w \in W} \rho(w)^2 = 1. \tag{67}$$

We will find a $v_\infty = v_1 v_2 \cdots \in W_\infty$ so that

$$\sum_{k=1}^{\infty} \rho(p_{\bar{v}_k}) \leq 1. \tag{68}$$

Here $\bar{v}_k = v_1 v_2 \cdots v_k$. We first notice that

$$\int_{W_\infty} \sum_{k=1}^{\infty} \rho(p_{\bar{w}_k}) \, d\mu(w_\infty) = \sum_{k=1}^{\infty} \sum_{w \in W_k} \mu(A_w) \rho(p_w) = \sum_{k=1}^{\infty} 2^{-k} \sum_{w \in W_k} \rho(p_w) =: S.$$

Cauchy-Schwarz yields (notice that $\mathrm{card}\, W_k = 2^k$)

$$S \leq \sum_{k=1}^{\infty} 2^{-k/2} \Big(\sum_{w \in W_k} \rho(p_w)^2 \Big)^{1/2} \leq \Big(\sum_{k=1}^{\infty} 2^{-k} \Big)^{1/2} \Big(\sum_{w \in W} \rho(p_w)^2 \Big)^{1/2} \leq 1,$$

where the last inequality follows from (67). Combining the estimates shows that there indeed exists $v_\infty = v_1 v_2 \cdots \in W_\infty$ satisfying (68).

Recall that for each $\bar{v}_k = v_1 v_2 \cdots v_k$, $k = 1, 2, \ldots$, there is a family $\Gamma_{\bar{v}_k}$ of paths connecting $p_{\bar{v}_{k-1}}$ and $p_{\bar{v}_k}$ in Ω so that $\mathrm{mod}(\Gamma_{\bar{v}_k}) \geq 4^k$. Now (67) implies that for every k there is $\gamma_k \in \Gamma_{\bar{v}_k}$ so that

$$\int_{\gamma_k} \rho \, ds \leq 2^{-k}. \tag{69}$$

Concatenating paths $\pi_\Omega \circ \gamma_k$, $k = 1, 2, \ldots$, yields a path $\gamma \in \Gamma$ so that $|\gamma| \cap \mathcal{C}(\Omega)$ only contains $\{0\}$, $p_\emptyset$, and elements $p_{\bar{v}_k}$, $k = 1, 2, \ldots$. Combining (68) and (69) gives

$$\int_{\gamma \cap \Omega} \rho \, ds + \sum_{k=1}^{\infty} \rho(p_{\bar{v}_k}) \leq 2. \tag{70}$$

We have proved that for every ρ satisfying (67) there is $\gamma \in \Gamma$ satisfying (70). Lemma 2.2 now shows that (66) holds. We conclude that Ω has all the desired properties.

Remark 6.1. It is also possible to construct a countably connected domain $\Omega \subset \hat{\mathbb{C}}$ which satisfies Packing Condition (ii) (but not Quasitripod Condition (i)) in Theorem 1.3, so that $\{0\} \in \mathcal{C}(\Omega)$ and $\operatorname{diam}(\hat{f}(\{0\})) > 0$ for every conformal homeomorphism $f\colon \Omega \to D$ onto a circle domain D. The details will appear elsewhere.

7. Cospread domains, Proof of Proposition 1.5

To start the proof of Proposition 1.5, we notice that the definition of cospread domains already contains Quasitripod Condition (i) in Theorem 1.3. We state the remaining claims of Proposition 1.5 as the following two propositions.

Proposition 7.1. *Let $\Omega \subset \hat{\mathbb{C}}$ be an H-cospread domain. There is N depending only on H so that*

$$\operatorname{card}\{p \in \mathcal{C}_N(\Omega) : \operatorname{diam}(p) \geq r,\ p \cap \mathbb{D}(z_0, r) \neq \emptyset\} \leq N \quad \textit{for every } z_0 \in \mathbb{C} \textit{ and } r > 0. \tag{71}$$

Proposition 7.2. *Let $\Omega \subset \hat{\mathbb{C}}$ be an H-cospread domain and $\phi\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ an α-quasi-Möbius map. Then $\phi(\Omega)$ is H'-cospread, where H' depends only on H and α.*

We recall the definitions of quasi-Möbius and quasisymmetric maps. The *cross-ratio* of distinct points $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ is $[z_1, z_2, z_3, z_4] := \dfrac{q(z_1, z_2)q(z_3, z_4)}{q(z_1, z_3)q(z_2, z_4)}$, where q is the chordal distance defined by

$$q(z, w) = \frac{|z - w|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}}, \quad z, w \in \mathbb{C}, \qquad q(z, \infty) = \frac{1}{\sqrt{1 + |z|^2}}.$$

Homeomorphism $\phi\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is *quasi-Möbius* if there is a homeomorphism $\alpha\colon [0, \infty) \to [0, \infty)$ so that

$$[\phi(z_1), \phi(z_2), \phi(z_3), \phi(z_4)] \leq \alpha([z_1, z_2, z_3, z_4]) \quad \text{for all distinct } z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}. \tag{72}$$

To emphasize the role of α, we use the term *α-quasi-Möbius*. Notice that Möbius transformations are quasi-Möbius maps with $\alpha(t) = t$.

Homeomorphism $\phi\colon E \to F$ between subsets of $\mathbb{C}$ is (strongly) *η-quasisymmetric* if there is a homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$ so that

$$|\phi(z_2) - \phi(z_1)| \leq \eta(t)|\phi(z_3) - \phi(z_1)| \quad \text{for all } z_1, z_2, z_3 \in E \text{ satisfying } |z_2 - z_1| \leq t|z_3 - z_1|.$$

It follows from the definitions that compositions and inverses of quasi-Möbius (resp., quasisymmetric) maps are quasi-Möbius (resp., quasisymmetric), quantitatively. If $E \subset \mathbb{C}$ is connected, then by Väisälä's theorem [Hei01, Corollary 10.22] weakly H-quasisymmetric maps $\phi\colon E \to F$ are η-quasisymmetric with η depending only on H.

7.1. Proof of the packing condition, Proposition 7.1.

We fix $z_0 \in \mathbb{C}$ and $r > 0$, and denote

$$\mathcal{P} := \{p \in \mathcal{C}_N(\Omega) : \operatorname{diam}(p) \geq r,\ p \cap \mathbb{D}(z_0, r) \neq \emptyset\}.$$

Given $p \in \mathcal{P}$, we choose $z_p \in p \cap \mathbb{D}(z_0, r)$. Since $r \leq \operatorname{diam}(p)$ and p is H-spread, there is an H-quasitripod $T_p \subset p \cap \mathbb{D}(z_p, r)$ with $\operatorname{diam}(T_p) \geq r/H$. Clearly $T_p \subset \mathbb{D}(z_0, 2r)$. Since quasitripods T_p are pairwise disjoint, the sought (71) is an immediate consequence of the next lemma.

Lemma 7.3. *Let $M, H \geq 1$ and suppose that $\mathcal{T}$ is a collection of pairwise disjoint H-quasitripods $T \subset \mathbb{D}(z_0, Mr)$ satisfying* $\operatorname{diam}(T) \geq r$*. Then* $\operatorname{card}\mathcal{T} \leq N$*, where N depends only on M and H.*

Proof. Given $T \in \mathcal{T}$, recall that there is an η-quasisymmetric homeomorphism $\phi_T\colon T_0 \to T$. We call $\phi_T(0)$ the *center* 0_T of T and the components of $T \setminus 0_T$ *the branches* of T.

We fix $0 < \delta < 1$ to be chosen later and cover $\mathbb{D}(z_0, Mr)$ with disks $D_1, \ldots, D_n$ of radius δr so that $n \leq 100(M\delta^{-1})^2$. Given $1 \leq k \leq n$, we denote by $\mathcal{T}_k$ the collection of elements $T \in \mathcal{T}$ for which $0_T \in D_k$. Since $\mathcal{T} = \bigcup_k \mathcal{T}_k$, the lemma follows if we can choose δ depending only on H so that for some $N = N(H)$ that depends on H,

$$\operatorname{card}\mathcal{T}_k \leq N \quad \text{for all } 1 \leq k \leq n. \tag{73}$$

Towards (73), a straightforward application of quasisymmetry shows that if $T \in \mathcal{T}_k$ and if δ is small enough depending on H, each of the branches $J_1(T), J_2(T), J_3(T)$ of T must leave $B_k = 2D_k$. Here $2D_k$ the disk with the center of D_k and twice the radius. Let $\alpha_s^T(t), 0 \leq t \leq 1$, be a homeomorphic parametrization of $J_s(T)$ with $\alpha_s^T(0) = 0_T$. We denote $a_s^T = \alpha_s^T(t_s)$, where

$$t_s := \inf\{t : \ \alpha_s^T(t) \in \partial B_k\}.$$

Points a_1^T, a_2^T, a_3^T partition ∂B_k into subarcs $S_1(T), S_2(T), S_3(T)$. Another straightforward application of quasisymmetry shows that their lengths satisfy

$$\ell(S_s(T)) \geq \theta r, \quad \text{for all } s \in \{1,2,3\}, \tag{74}$$

where $\theta > 0$ depends only on H.

We fix $S_T \in \{S_1(T), S_2(T), S_3(T)\}$ so that $\ell(S_T) \leq \ell(S_s(T))$ for $s \in \{1,2,3\}$. We replace $\mathcal{T}_k$ with a finite subcollection if needed, and enumerate the elements $T_1, T_2, \ldots, T_L$ so that $\ell(S_{T_1}) \leq \ell(S_{T_2}) \leq \cdots \leq \ell(S_{T_L})$. We denote S_{T_m} and $\ell(S_{T_m})$ by S_m and ℓ_m, respectively.

Next, notice that there is $s \in \{1,2,3\}$ so that $a_{s'}^{T_1} \in S_s(T_2)$ for every $s' = 1,2,3$. In particular, by our choice of subarcs S_T and the enumeration of quasitripods T_j, either

(1) $S_1 \cap S_2 = \emptyset$, or
(2) S_2 contains S_1 and another subarc $S_{s'}(T_1)$.

Using (74) we see that in both cases $\ell(S_1 \cup S_2) \geq \theta r + \ell(S_1)$. Similar reasoning shows that if $2 \leq m \leq L$ then there are $1 \leq m' \leq m$ and $s \in \{1,2,3\}$ so that

$$S_s(T_{m'}) \subset S_m \setminus \Big(\bigcup_{l=1}^{m-1} S_l\Big) \quad \text{and so} \quad \ell\Big(\bigcup_{l=1}^{m} S_l\Big) \geq \theta r + \ell\Big(\bigcup_{l=1}^{m-1} S_l\Big). \tag{75}$$

Applying (75) and induction yields

$$L\theta r \leq \ell\Big(\bigcup_{l=1}^{L} S_l\Big) \leq \ell(\partial B_k) = 4\delta\pi r. \tag{76}$$

Since (76) holds for all finite subcollections of $\mathcal{T}_k$ and θ depends only on H, the desired bound (73) holds. The proof is complete. $\square$

7.2. **Proof of quasi-Möbius invariance, Proposition 7.2.** We will apply the following estimate. The proof is a straightforward application of quasisymmetry.

Lemma 7.4. *Let $\nu\colon \overline{\mathbb{D}}(z_0,r) \to \nu(\overline{\mathbb{D}}(z_0,r))$ be η-quasisymmetric and $A \subset \mathbb{D}(z_0,r)$ a set satisfying*

$$\operatorname{diam}(\nu(A)) \geq \delta \min_{z\in\mathbb{S}(z_0,r)} |\nu(z)-\nu(z_0)|.$$

Then $\operatorname{diam}(A) \geq \delta' r$, where δ' depends only on δ and η.

Let $\Omega \subset \hat{\mathbb{C}}$ be H-cospread and $\phi\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ an α-quasi-Möbius map. Let $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation so that $g = \varphi \circ \phi$ fixes infinity. Testing quasi-Möbius condition (72) with quadruple z_1, z_2, z_3, ∞ then shows that $g|_{\mathbb{C}}$ is α-quasisymmetric. Therefore, since $\phi = \varphi^{-1} \circ g$ it suffices to show the claim for quasisymmetric maps and Möbius transformations.

We fix $p \in \mathcal{C}_N(\phi(\Omega))$, $z_0 \in p \cap \mathbb{C}$ and $r \leq \operatorname{diam}(p)$. Our goal is to show that $p \cap \mathbb{D}(z_0,r)$ contains a quasitripod with diameter comparable to r, under the assumption that ϕ is a quasisymmetric map or a Möbius transformation.

First, let ϕ be η-quasisymmetric and denote $\nu = \phi^{-1}$ and $\ell = \min_{z\in\mathbb{S}(z_0,r)} |\nu(z)-\nu(z_0)|$. Since $\nu(p)$ is H-spread by assumption, there is an H-quasitripod $T \subset \mathbb{D}(\nu(z_0),\ell) \cap \nu(p)$ with $\operatorname{diam}(T) \geq \ell/H$. Then, since compositions of quasisymmetric maps are quasisymmetric, $\phi(T) \subset p \cap \mathbb{D}(z_0,r)$ is an H_1-quasitripod, where H_1 depends only on H and η. Moreover, since inverses of quasisymmetric maps are quasisymmetric, Lemma 7.4 shows that $\operatorname{diam}(\phi(T)) \geq r/H_2$, where H_2 depends only on H and η. We conclude that $\phi(\Omega)$ is $\max\{H_1,H_2\}$-cospread.

We now show that $\phi(\Omega)$ is cospread when ϕ is a Möbius transformation. If ϕ fixes infinity then the claim is obvious. It therefore suffices to prove the claim for inversion $\phi(z) = z^{-1}$. The following lemma follows directly from the definition of quasisymmetry.

Lemma 7.5. *Let $\phi(z) = z^{-1}$ and suppose that $s > 0$ and $w_0 \in \mathbb{C}$ satisfy $|w_0| \geq 2s$. Then $\phi|_{\overline{\mathbb{D}}(w_0,s)}$ is η-quasisymmetric with $\eta(t) = 3t$.*

Now, if point $z_0 \in p \cap \mathbb{C}$ above satisfies $|z_0| \geq r/10$ then $\phi^{-1} = \phi$ is quasisymmetric on $\mathbb{D}(z_0, r/20)$ by Lemma 7.5. On the other hand, if $|z_0| \leq r/10$ then we choose any $w_0 \in p \cap \mathbb{S}(z_0, r/2)$ (such a w_0 exists since $\operatorname{diam}(p) \geq r$) and notice that $|w_0| \geq r/10$. Lemma 7.5 then shows that h is quasisymmetric on $\mathbb{D}(w_0, r/20) \subset \mathbb{D}(z_0, r)$.

Since $\phi^{-1}(p)$ is spread by assumption, applying quasisymmetry and Lemma 7.4 as above shows that

$$p \cap \mathbb{D}(k_0, r/20) \subset p \cap \mathbb{D}(z_0, r)$$

contains an H'-quasitripod with diameter bounded from below by r/H'. Here $k_0 = z_0$ if $|z_0| \geq r/10$ and $k_0 = w_0$ otherwise, and H' depends only on H. It follows that p is H'-spread. The proofs of Propositions 7.2 and 1.5 are complete.

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