Compute the classic result for QCD corrected jet production in $e^- e^+$-annihilation

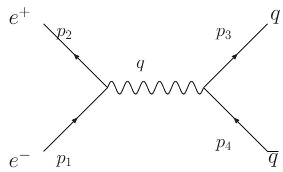
$$R(\sqrt{s}) \equiv \frac{\sigma_{\rm tot}\,(e^+ e^- \to \text{hadrons})}{\sigma_{\rm tot}\,(e^+ e^- \to \mu^+ \mu^-)} = 3 \sum_{\text{flavors}} Q_f^2 \left(1 + \frac{\alpha_s}{\pi}\right)$$

where Q_f is the electric charge of the quark of flavor f, and $\alpha_s \equiv g_s^2/4\pi$ is the strong coupling constant. All quark masses are conventionally taken to be zero above their production treshold $\sqrt{s} \geq 2m_f$. The QCD-corrected total cross-section can be divided in three parts (bare expansion):

$$\sigma_{tot} = \sigma_0 Z_2^2 + \sigma_V + \sigma_R,$$

where σ_0 is the tree-level cross-section, Z_2 the quark wave function renormalization factor, σ_V comes from the vertex correction and σ_R from the gluon radiation. The wave function normalization correction to the tree level result comes from the LSZ-normalization. [An aside note: (only) in the zero mass limit this can also be seen as arising from adding gluon loop corrections to the external quark propagators.] There is no *need* for new renormalization counter terms as QCD-corrections cannot affect the running of the electric charge. (In BPHZ-scheme you *would* add counter terms and hence no Z_2^2-correction to σ_0, but same term would be recreated by then necessary vertex counterterm.) Here are some instructions and intermediate results for the computation in $D = 4 - \epsilon$ dimensions.

- Consider first the tree-level diagram for $e^- e^+ \to q\bar{q}$ process. The squared and spin



summed matrix element can easily be arranged in the form

$$\overline{|\mathcal{M}|^2} \equiv \frac{1}{4} \sum_{\text{spins}} \sum_{\text{colors}} |\mathcal{M}|^2 = \frac{1}{4} \frac{e^4}{q^4} L_{\mu\nu} H^{\mu\nu},$$

where $L_{\mu\nu}$ is the electron tensor and $H^{\mu\nu}$ is the quark tensor. At the tree level these are given by

$$L^{\mu\nu} = 4\,[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - g^{\mu\nu}(p_1 \cdot p_2)]\ , \quad H^{\mu\nu} = 12\,Q_f^2\,[p_3^\mu p_4^\nu + p_3^\nu p_4^\mu - g^{\mu\nu}(p_3 \cdot p_4)]\,.$$

The fact that the squared matrix element can be written as above remains true also for many more complicated diagrams, especially for those where there is gluon radiation from the quarks. The form of $L_{\mu\nu}$ and $H_{\mu\nu}$ is tightly restricted by the Ward identity $q^\mu L_{\mu\nu} = q^\mu H_{\mu\nu} = 0$. To obtain the total cross-section we always integrate $H^{\mu\nu}$ over the phase-space $d\Pi$, and the the only 4-vector that the result can depend on is q. Therefore, one must have the relation

$$\int d\Pi \, H^{\mu\nu} = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) H(q^2),$$

where $H(q^2)$ is a scalar, in order to satisfy the Ward identity. In D dimensions, this implies

$$L_{\mu\nu} \int d\Pi \, H^{\mu\nu} = (g^{\mu\nu} L_{\mu\nu}) \frac{1}{D-1} \int d\Pi \, (g_{\eta\sigma} H^{\eta\sigma}).$$

This shows that the quantity that is really interesting is $\int d\Pi \, (g_{\eta\sigma} H^{\eta\sigma})$.

- In D dimensions the two-body phase-space element reads

$$d\Pi_2^{(D)} = \frac{d^{D-1} p_3}{(2\pi)^{D-1} 2E_3} \frac{d^{D-1} p_4}{(2\pi)^{D-1} 2E_4} (2\pi)^D \delta^{(D)} \left(p_1 + p_2 - p_3 - p_4 \right).$$

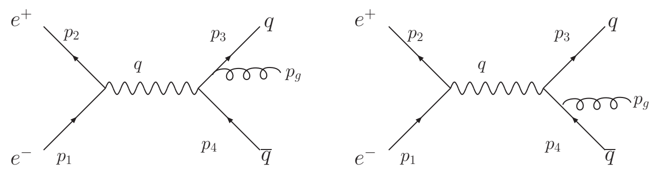
At tree-level the $(g_{\eta\sigma} H^{\eta\sigma})$ depends only on q^2 and one can do the phase-space integral above without any reference to the $(g_{\eta\sigma} H^{\eta\sigma})$:

$$\int d\Pi_2^{(D)} = \frac{1}{8\pi} \left(\frac{16\pi}{q^2} \right)^{\epsilon/2} \frac{1}{1-\epsilon} \frac{\sqrt{\pi}}{\Gamma(\frac{1-\epsilon}{2})}.$$

Using this, one ends up, at tree-level, with

$$\int d\Pi \, (g_{\eta\sigma} H^{\eta\sigma})_{\text{tree}} = \frac{-3q^2 Q_f^2}{4\pi} \frac{2-\epsilon}{1-\epsilon} \left(\frac{16\pi}{q^2} \right)^{\epsilon/2} \frac{\sqrt{\pi}}{\Gamma(\frac{1-\epsilon}{2})}.$$

- Gluon radiation $e^+ e^- \to q\bar{q}g$. The quark tensor convoluted with the metric tensor



takes now the form

$$g^{\mu\nu} \Gamma^{\text{R}}_{\mu\nu} = -g_D^2 Q_f^2 C_F \, 4(2-\epsilon) \left[\frac{x_3^2 + x_4^2 - \frac{\epsilon}{2} x_g^2}{(1-x_3)(1-x_4)} \right],$$

where $C_F = 4$, g_D is the strong coupling constant in D dimensions, and x_3, x_4, x_g are dimensionless parameters

$$x_3 = \frac{2p_3 \cdot q}{q^2} \quad x_4 = \frac{2p_4 \cdot q}{q^2} \quad x_g = \frac{2p_g \cdot q}{q^2}$$

with p_3, p_4, p_g being the momenta of quark, antiquark, and gluon. The three-body phase-space can be computed, up to x_3 and x_4 integrations to be

$$d\Pi_3^{(D)} = \frac{1}{16\pi^2}\frac{q^2}{8\sqrt{\pi}}\frac{1}{1-\epsilon}\left(\frac{16\pi}{q^2}\right)^{\epsilon}\frac{dx_3dx_4}{\Gamma(\frac{1-\epsilon}{2})\Gamma(\frac{2-\epsilon}{2})}(x_3x_4)^{-\epsilon}(1-z^2)^{-\epsilon/2},$$

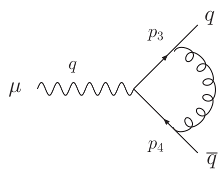
where $z \equiv 1 + 2\frac{1-x_3-x_4}{x_3x_4}$. Doing the remaining x_3 and x_4 integrals gives finally

$$\int d\Pi_3^{(D)} g^{\mu\nu}\Gamma^{\rm R}_{\mu\nu} = \left(\int d\Pi_2^{(D)} g^{\mu\nu} H_{\mu\nu}\right)\cdot\frac{2\alpha_s}{3\pi}\left(\frac{4\pi\mu^2}{q^2}\right)^{\epsilon/2}\frac{\Gamma^2(\frac{2-\epsilon}{2})}{\Gamma^2(\frac{2-3\epsilon}{2})}\left[\frac{8}{\epsilon_{\rm IR}^2}+\frac{6}{\epsilon_{\rm IR}}+\frac{19}{2}\right],$$

where $\int d\Pi_2^{(D)} g^{\mu\nu} H_{\mu\nu}$ is the tree-level quantity above. Therefore

$$\sigma_R = \sigma_0\cdot\frac{2\alpha_s}{3\pi}\left(\frac{4\pi\mu^2}{q^2}\right)^{\epsilon/2}\frac{\Gamma^2(\frac{2-\epsilon}{2})}{\Gamma(\frac{2-3\epsilon}{2})}\left[\frac{8}{\epsilon_{\rm IR}^2}+\frac{6}{\epsilon_{\rm IR}}+\frac{19}{2}\right].$$

- To compute the vertex correction, one can consider the graph



where external quarks are on-shell. The contribution from this diagram turns out to reduce simply to the modification of the tree-level vertex factor

$$-ieQ_f\gamma_\mu \to -ieQ_f\gamma_\mu\left[1+\Gamma\right],$$

and therefore the vertex correction contributes to the total cross-section simply as

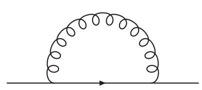
$$\sigma_V = \sigma_0|1+\Gamma|^2.$$

where

$$\Gamma(q^2<0) = \frac{g_s^2}{16\pi^2}C_2(N)\left(\frac{4\pi\mu^2}{-q^2}\right)^{\epsilon/2}\frac{\Gamma(\frac{2+\epsilon}{2})\Gamma^2(\frac{2-\epsilon}{2})}{\Gamma(2-\epsilon)}\left[\frac{2}{\epsilon_{\rm UV}}-\frac{8}{\epsilon_{\rm IR}^2}-2\right],$$

with $C_2(N) = 4/3$. The restriction $q^2 < 0$ above is crucial. During the computation of this diagram one is forced to take $q^2 < 0$ in order to avoid singularitites in the momentum integration. In the end one should then analytically continue back to the physical, time-like region $q^2 > 0$.

- The renormalization factor Z_2 coming from the quark self-energy diagram



turns out to be

$$Z_2 = 1 + \frac{g_s^2}{16\pi^2}C_2(N)\left(\frac{2}{\epsilon_{IR}}-\frac{2}{\epsilon_{UV}}\right).$$