Quantum Field Theory Aplications Fall 2024

Exercise 1 Return by 12.00 on Tuesday 10.9.

1. Let us consider the renormalizability of a scalar theory described by Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{m^2}{2}\phi^2 + \frac{\lambda_k}{k!}\phi^k$$

in d-dimensions. Prove first that if $k = 2d/(d-2)$ the theory is renormalizable, that is, only finite number of n-point functions are divergent. Furthermore, show that the renormalizability depends on the dimension of the coupling constant:

- If λ_k is dimensionless, theory is renormalizable.
- If λ_k has negative dimension of mass, $[\lambda_k] = M^\alpha$, $\alpha < 0$, theory is non-renormalizable: infinite number of n-point functions are divergent.
- If λ_k has positive dimension of mass, $[\lambda_k] = M^\alpha$, $\alpha > 0$, theory is super-renormalizable: only finite number of Feynman diagrams are divergent.

Categorize the cases $(d = 3, k = 6)$, $(d = 6, k = 3)$, $(d = 3, k = 4)$ and $(d = 4, k = 3)$ to renormalizable, non-renormalizable, super-renormalizable theories. Draw then a diagram in (d, k)-plane which indicates the regions of different degrees of renormalizability.

2. Compute very carefully the following two integrals in the dimensional regularization method:

$$A_0(m^2) \equiv \mu^{4-d} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 - m^2 + i\varepsilon}$$

$$B_0(m_1^2, m_2^2, p^2) \equiv \mu^{4-d} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m_1^2 + i\varepsilon)((k-p)^2 - m_2^2 + i\varepsilon)}$$

I want you to work out both the divergent parts of the integrals (the poles in $1/\epsilon$ where $\epsilon = 4 - d$ and the finite part. Throw out the $\mathcal{O}(\epsilon)$ corrections. Note that the latter integral needs to be handled differently in the three different domains: $p^2 < 0$, $0 < p^2 < (m_1 + m_2)^2$ and $p^2 > (m_1 + m_2)^2$. In the last domain it obtains a complex part, which is well defined because of the ε-prescription in the denominator.

(These two integrals suffice for 1-loop renormalization of $\lambda\phi^4$-theory and get us a long way into renormalizing QED as well.)