Quantum field theory applications **The Final Exam**
FYSH515 **Fall 2024**

Return to Kimmo by 10.30, Thursday 19.12.2024.

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1 SM vacuum stability and triviality

Perform the BPHZ-decomposition for the quartic Higgs vertes in the Minimal Standard Model (SM), and show that the renormalized coupling can be written as

$$\lambda = \tilde{\lambda} Z_\lambda^{-1}(\tilde{\lambda}, \tilde{g}, \tilde{g}', \tilde{y}_t) Z_\phi^2(\tilde{\lambda}, \tilde{g}, \tilde{g}', \tilde{y}_t)\,, \tag{1}$$

where $\tilde{\lambda} \equiv \lambda_0 \mu^{-\epsilon}$ and $\tilde{f} = f_0 \mu^{-\epsilon/2}$ for $f = \tilde{g}, \tilde{g}', \tilde{y}_t$, and $Z_\lambda = 1 + \delta_\lambda$ and $Z_\phi = 1 + \delta_\phi$ are the Higgs vertex and wave function renormalization constants. All other fermion Yukawas are so small that they can be neglected. Draw the relevant 1-loop diagrams for the Higgs boson propagator and for the quartic scalar vertex function involving these couplings and from these extract the counter terms δ_λ and δ_ϕ that are necessary to compute the β-function to order λ^2, λf^2 and f^4. Use the $\overline{\text{MS}}$-subtraction scheme and only find the appropriate poles. (You might want to use the Landau gauge as ghosts do not couple to scalars in L-gauge[1]. Note that you take $p = 0$ in the external legs in the quartic operator.) That is, show that at 1-loop level the dominant RGE for λ is:

$$\mu \frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = \beta(\lambda, ...)\,, \tag{2}$$

where

$$16\pi^2 \beta(\lambda, ...) = 24\lambda^2 + 12 y_t^2 \lambda - 6 y_t^4 - 3\lambda(3g^2 + g'^2) + \frac{3}{8}(2g^4 + (g^2 + g'^2)^2)\,. \tag{3}$$

Convert the μ-dependence in the above equation to physical scale-dependence $\lambda(Q^2)$ as we did in lectures with other models. You should find all necessary Feynman rules in the appendix.

New physics/triviality

First note that for large enough Higgs mass (how large?) $\beta > 0$, so that λ grows with the increasing scale. Show that in this region, you can eventually neglect all couplings in the β-function except λ, and derive an analytic solution for $\lambda(Q^2)$ in this approximation. Show that in this case the coupling becomes infinite at a finite energy scale and compute the position of this Landau pole $Q^2 \equiv \Lambda^2$ assuming that you can neglect all other couplings all the way to the weak scale. Parametrize the pole with $\lambda(Q_{\text{weak}}) \equiv m_H^2/2v^2$ and plot its position (along the x-axis) as a function of the Higgs mass (in the y-axis). Estimate, or compute exactly (it is possible!) how the position of the Landau pole is changed when you include other couplings in the game. (For top quark in particular use values around 175 GeV.)

[1] In L-gauge the gauge-loop contribution to Higgs self-energy is cumbersome on the other hand, but you only need to extract the p^2-dependent part of it to compute Z_ϕ. Alternatively, you can use the Feynman gauge throughout with the ghost contributions included.

Vacuum stability

For very light Higgs bosons (small $\lambda(Q_{\rm weak})$) the top-quark coupling may dominate and make beta function negative and effectively λ-independent. If this is the case, then the 1-loop quartic coupling may be driven negative at large energies. Compute at what scales this happens (if it does) for different values of m_H and plot these limiting scales to your earlier plot with Landau poles.

If we require that SM remains valid as such for arbitrary large scales and that its vacuum is absolutely stable, we must exclude the region above the Landau pole and in the region where quartic coupling becomes negative. Show that this argument excludes all Higgs masses except a narrow window around $m_H \approx 130$ GeV. Interesting, right? (To play this game properly one should obviously perform a simultaneous RGE-analysis of the top-quark Yukawa and gauge couplings.)

2 Higgs decay to two photons

Precise knowledge of the decay patterns of the Higgs boson is instrumental in testing the SM. In this problem you will calculate the higgs decay to two real photons $H \to \gamma\gamma$ in the SM and beyond.

Generic reduction using Ward identity

The higgs decay to two photons is generically proportional to some rank-two tensor $M_{\mu\nu}$:

$$\mathcal{M}_{H\to\gamma\gamma} = M_{\mu\nu}\epsilon_1^{*\nu}\epsilon_2^{*\mu}\,, \tag{4}$$

where $\epsilon_i^{*\alpha} \equiv \epsilon^{*\alpha}(k_i)$ are the outgoing photon polarization vectors. Instead of directrly attacking the full one-loop $M_{\mu\nu}$ in the SM, let us first use the gauge invariance to save us some work.

Show, by use of Ward-identitites, that on-shell $M_{\mu\nu}$ contributing to the decay can be written as:

$$M_{\mu\nu} = \frac{g_{H\gamma\gamma}}{v}\Big(k_1\cdot k_2 g_{\mu\nu} - k_{1\nu}k_{2\mu}\Big) \equiv g_{H\gamma\gamma}\frac{k_1\cdot k_2}{v}P_{\mu\nu}\,. \tag{5}$$

Given this result show that

$$\Gamma_{H\to\gamma\gamma} = \frac{m_H^3}{64\pi v}(g_{H\gamma\gamma})^2. \tag{6}$$

Show also that the on-shell photon amplitude gives rise to an effective interaction Lagrangian:

$$\mathcal{L}_{H\gamma\gamma} = -\frac{g_{H\gamma\gamma}}{4v}hF_{\mu\nu}F^{\mu\nu}\,. \tag{7}$$

That is, show that $\int {\rm d}^4x\langle\gamma\gamma|\hat{\mathcal{L}}_{H\gamma\gamma}(x)|h\rangle = (2\pi)^4\delta^4(p_H - k_1 - k_2)\mathcal{M}_{H\to\gamma\gamma}$, where you understand $\hat{\mathcal{L}}_{H\gamma\gamma}$ as composed of field operators with appropriate creation and annihilation operators. (Remember the method I used to get matrix elements in connection with anomalies. See also the note at the end of the exam.).

SM-one loop result and new physics limits

In the SM, a one loop calculation gives the rather simple looking result:

$$g^{\rm SM}_{H\gamma\gamma} = \frac{\alpha}{2\pi}\left|F_1(\tau_W) + \sum_f N_c^f Q_f^2 F_{1/2}(\tau_f)\right|, \tag{8}$$

where

$$\begin{aligned} F_{1/2}(\tau_f) &\equiv -2\tau_f\left[1+(1-\tau_f)f(\tau_f)\right], && (9)\\ F_1(\tau_W) &\equiv 2+3\tau_W+3\tau_W(2-\tau_W)f(\tau_W), && (10)\end{aligned}$$

with

$$f(\tau_j) \equiv \begin{cases} \arcsin^2\frac{1}{\sqrt{\tau_j}} & \text{if } \tau_j \geq 1 \\ -\frac{1}{4}\left[\log\frac{1+\sqrt{1-\tau_j}}{1-\sqrt{1-\tau_j}} - i\pi\right]^2 & \text{if } \tau_j < 1 \end{cases},$$

and $\tau_j \equiv 4m_j^2/m_H^2$. Your task is to prove these results as far as you can. However, before you embark on this, let us work out some of the consequences.

Work out the limit $\tau \to \infty$ of the functions $f(\tau)$ and $F_{1/2}(\tau_f)$. Check numerically the accuracy of this limiting formula for top quark with $m_t = 175$ GeV when $m_H = 125$ GeV.

The latest observational result to the Higgs total width and the branching ratio to two photons are:

$$\Gamma_{\rm TOT} = 3.7^{+1.9}_{-1.7}\ {\rm MeV} \qquad \text{and} \qquad Br_{H\to\gamma\gamma} = (2.50\pm 0.2)\times 10^{-3} \tag{11}$$

Now assume that you have N_F new heavy fermionic degrees of freedom, denoted by F, which couple to Higgs field with strength $c_F m_F/v$ and to photons with a charge eQ_F. Compute the correction to $g^{SM}_{H\gamma\gamma}$ from these fields as a function of the parameters N_F, c_F and Q_F. Assume that all these states are heavy and find the asymptotic expression for the new $g_{H\gamma\gamma}$. Estimate the limit on the new physics mass and couplings you get this using (11).

Fermion loops, computation

There are two triangle-diagrams contributing to $M_{\mu\nu}$ from each fermion, similar to the ones contributing to axial anomaly. Choose the momentum routings such that the fermion propagator between the two photon vertices has the loop momentum p, as this will lead to least cumbersome expressions. Perform traces keeping track only of the $g_{\mu\nu}$- and $p_\mu p_\nu$-terms, so that you can write (you need to keep track of $p_\mu p_\nu$ term because it eventually gives an extra $g_{\mu\nu}$-term in the symmetric integration):

$$M^f_{\mu\nu} = \int\frac{{\rm d}^d p}{(2\pi)^d}\frac{a_f g_{\mu\nu} + b_f p_\mu p_\nu + \cdots}{D_{f0}D_{f1}D_{f2}}, \tag{12}$$

where

$$D_{f0} \equiv p^2 - m_f^2 + i\epsilon; \quad D_{f1} \equiv (p+k_1)^2 - m_f^2 + i\epsilon \quad \text{and} \quad D_{f2} \equiv (p-k_2)^2 - m_f^2 + i\epsilon, \tag{13}$$

with $m_f = y_f v/\sqrt{2}$. Even though the final result will be finite, integrals need regularization. Perform the Feynman parametrization and evaluate the symmetric momentum integrals, keeping track of only the $g_{\mu\nu}$ terms. After integrating out also one of the Feynman parameters you will encounter an integral of the form:

$$S(\beta) = \int_0^1 \frac{\mathrm{d}x}{x} \log\left[1 - x(1-x)\beta - i\epsilon\right] , \tag{14}$$

You work this integral out by use of the the Spence, or dilogartihm functions:

$$\mathbf{Li}_2(x) = -\int_0^x \mathrm{d}t \frac{\log(1-t)}{t} . \tag{15}$$

Prove that these functions obey the identity:

$$\mathbf{Li}_2(z) + \mathbf{Li}_2(\frac{z}{z-1}) = -\frac{1}{2}\log^2(1-z) . \tag{16}$$

Using this identity you can find an analytic expression for $S(\beta)$, which in fact is just $S(4/\tau) = -2f(\tau)$. After this it is just a matter of writing down the full solution for $M^f_{\mu\nu}$. Summing over all fermions you should find, consistently with (8-9):

$$M^f_{\mu\nu} = \frac{m_H^2 \alpha}{4\pi v}\Big(\sum_f N_c^f Q_f^2 F_{1/2}(\tau_f)\Big) \times P_{\mu\nu} . \tag{17}$$

This is more or less where I expect you to get if you do very well. Think of the next part as more like a bonus question, and do it if you still have time and steam left.

Gauge loops, computation

Because the result is finite, the calculation could be done in Unitary gauge with only two different charged gauge boson loops. However, there are some issues with keeping the gauge invariance in U-gauge, so it is easier to do the calculation in the Feynman gauge despite the large number of diagrams involved. You find the The Feynman rules for the necessary vertices (involving photons A_μ, Higgs h, gauge bosons $W^\pm$, charged Goldstone modes $\phi^\pm$ and charged ghosts $c^\pm$) in the appendix. Note that the direction of momentum and the charge assignments in vertices are crucial, and that I wrote the vertices without any symmetry factors (unlike is the case for example in Cheng and Li), so you have to work out the symmetry factors for the graphs.

You should find 20 triangle diagrams, ten of which are related to ten others by crossing of external legs. Furthermore, of the remaining ten diagrams three are related by three others by charge conjugation. So there are only seven independent triangles with a multiplicity factor 4 in three and 2 in four of them.

Draw the *seven* independent triangle diagrams and indicate their multiplicity factors.

You should also find six diagrams with a bubble in one of the external legs. However, four of these are again related either by crossing or by charge conjugation.

Draw the *three* different bubble diagrams with the appropriate multiplicity factors.

Consider first the triangle graphs. Note that in Feynman gauge all propagators have the same denominator. Use the same momentum routing as in the fermionic case and compute the coefficients of the tensor structures $g_{\mu\nu}$ and $p_\mu p_\nu$ for each diagram by use of the above Feynman rules. In this way you obtain:

$$M_{\mu\nu}^{W,\text{triangles}} = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\sum_{i=1}^{7}(a_i g_{\mu\nu} + b_i p_\mu p_\nu)}{D_{W0}D_{W1}D_{W2}}\,, \tag{18}$$

where D_{Wi} are given by (13) with $m_f \to m_W$. Do not attempt to compute these integrals yet! Instead, consider now the bubble diagrams. In each diagram, choose the loop momenta so that you can easily expand the denominator to the form appearing in the triangle case. In this way you obtain effective coefficients a_i and b_i also for the bubble diagrams. Now form the grand total nominator:

$$P_{\mu\nu}^W \equiv \sum_{i=1}^{10} \Big(a_i g_{\mu\nu} + b_i p_\mu p_\nu \Big) = a_{\text{TOT}}^W g_{\mu\nu} + b_{\text{TOT}}^W p_\mu p_\nu \,. \tag{19}$$

Only at this point perform the Feynman parametrizations, shift p and do the symmetric momentum and FP-integrations as you did in the fermionic case. No new tricks that will be needed and the outcome should be just:

$$M_{\mu\nu}^W = \frac{m_H^2 \alpha}{4\pi v} F_1(\tau_W) \times P_{\mu\nu} \,. \tag{20}$$

This is not an exam question, but a note on how to simply derive F-rules needed in this problem

Apart from the standard Higgs and gauge self-coupling vertices and the ghost-vertices, the necessary interactions arise from the covariant derivative term. For the decay problem the relevant part of the covariant term is just:

$$|D_\mu \Phi|^2 = \left| \begin{pmatrix} \partial_\mu - ieA_\mu & -\frac{i}{\sqrt{2}} g W_\mu^+ \\ -\frac{i}{\sqrt{2}} g W_\mu^- & \partial_\mu \end{pmatrix} \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}(\eta + v) \end{pmatrix} + \ldots \right|^2 \tag{21}$$

Consider for example term $-ieA_\mu(\phi^+\partial^\mu\phi^- - \phi^-\partial^\mu\phi^+)$ included in (21). To find the F-rule, replace each field with the appropriate field operator and compute the matrix element of this operator between vacuum and the *incoming* one-particle states

$$\langle \Omega | i(-ie)\hat{A}_\alpha(\hat{\phi}^+\partial^\alpha\hat{\phi}^- - \hat{\phi}^-\partial^\alpha\hat{\phi}^+)|k,\mu;p_+;p_-\rangle_{\text{amp}} = -ie(p_-^\mu - p_+^\mu)\,. \tag{22}$$

This is in fact a shorthand for computing the tree level S-matrix element:

$$\begin{aligned} &\int \mathrm{d}^4x \;{}_{\text{out}}\langle \Omega | e\hat{A}_\alpha(x)(\hat{\phi}^+(x)\partial^\alpha\hat{\phi}^-(x) - \cdots)|k,\mu;p_+;p_-\rangle_{\text{in}} \\ &\qquad = (2\pi)^4\delta^4(k - p_+ - p_-)\Big[-ie(p_-^\mu - p_+^\mu)\,\epsilon_\mu(k)\Big]\,. \end{aligned} \tag{23}$$

The quantity in brackets is the lowest order $-iT$-matrix element, from which you amputate the photon wave function to get the vertex rule (for scalars the wave functions are just ones). To remind, eg:

$$\hat{A}_\alpha(x) = \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3 2\omega_\mathbf{q}} (\epsilon_{\alpha\mathbf{q}} \hat{a}_{\alpha\mathbf{q}} e^{-iq\cdot x} + \epsilon^*_{\alpha\mathbf{q}} \hat{a}^\dagger_{\alpha\mathbf{q}} e^{iq\cdot x})\,.$$

Note that in this prescription all momenta in graphs are flowing *into* the vertex. Note also that by nature this derivation accounts for all possible ways of contracting the fields, so the rule you get contains the symmetry factor. To get rules without symmetry factors, you have to divide it out explicitly. This is what I have done in the graphs with identical particles in the appendix.

The standard gauge vertices shown in figure 2 can be obtained by a similar technique. You just have to note that a contraction of $\hat{A}_\alpha$ and $\hat{a}_\beta$ gives a $g_{\alpha\beta}$ in addition to the usual momentum delta function. Given the SM-ghost lagrangian in Feynman gauge, the same techniques can be used to derive the associated Feynman rules.

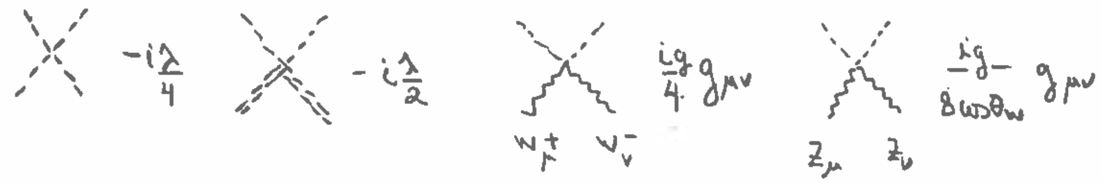
Feynman rules

For problem 1:

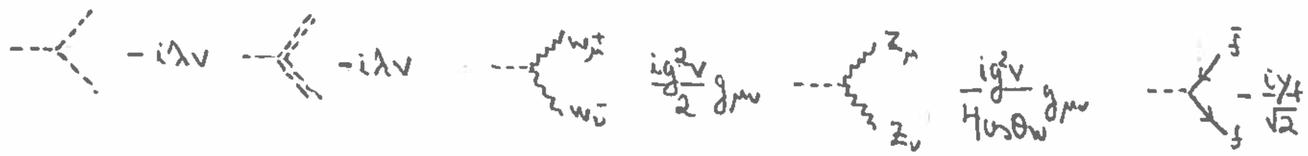
(Landau gauge)

h propagator: $\dfrac{i}{p^2-m_h^2+i\epsilon}$; goldstones: $\dfrac{i}{p^2+i\epsilon}$

$V = W, Z$: $\dfrac{-i\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)}{k^2-M_V^2+i\epsilon}$; f: $\dfrac{i}{\not p-m_f+i\epsilon}$



$-i\dfrac{\lambda}{4}$; $-i\dfrac{\lambda}{2}$; $\dfrac{ig}{4}g_{\mu\nu}$; $\dfrac{ig}{8\cos\theta_W}g_{\mu\nu}$



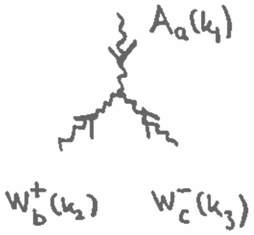
$-i\lambda v$; $-i\lambda v$; $\dfrac{ig^2 v}{2}g_{\mu\nu}$; $\dfrac{ig^2 v}{4\cos\theta_W}g_{\mu\nu}$; $-\dfrac{iy_f}{\sqrt 2}$

Note: none of these rules contain any symmetry factors! (Also for problem 2) So they differ from what you find e.g. in Cheng & Li.

For problem 2:

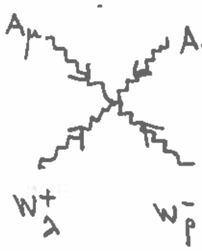
(Feynman gauge)

h: $\dfrac{i}{p^2-m_h^2+i\epsilon}$; $\phi^\pm$: $\dfrac{i}{p^2-M_W^2+i\epsilon}$; $c_\pm$: $\dfrac{i}{p^2-M_W^2+i\epsilon}$

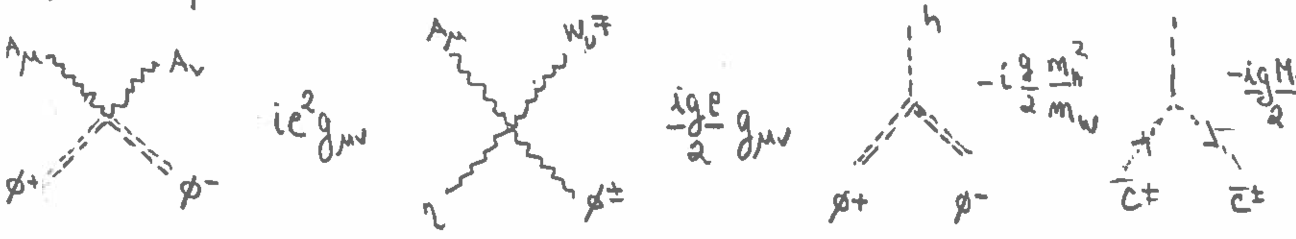


$$-ie\left((k_1-k_2)_c g_{ab}+(k_2-k_3)_a g_{bc}+(k_3-k_1)_b g_{ca}\right)$$

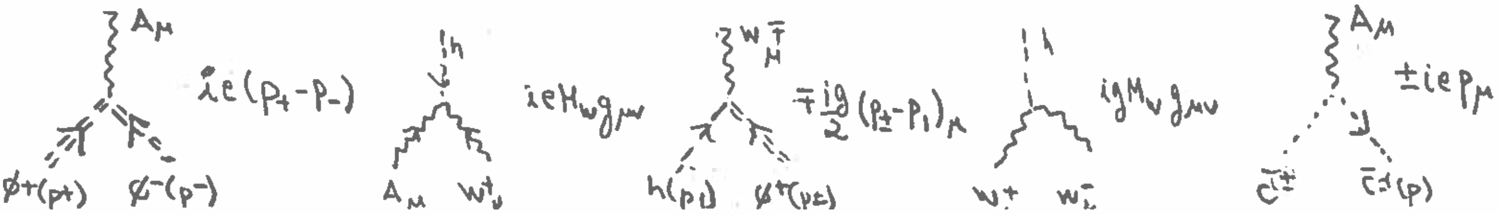
Note: All momenta directed into the vertex.



$$-\frac{ie^2}{2}\left(2g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}\right) \equiv -\frac{ie^2}{2}S_{\mu\nu,\lambda\rho}$$



$ie^2 g_{\mu\nu}$; $\dfrac{ige}{2}g_{\mu\nu}$; $-i\dfrac{g}{2}\dfrac{m_h^2}{m_W}$; $-\dfrac{igM_W}{2}$



$ie(p_+-p_-)$; $ieM_W g_{\mu\nu}$; $\mp\dfrac{ig}{2}(p_\pm-p_1)_\mu$; $igM_W g_{\mu\nu}$; $\pm iep_\mu$