Quantum field theory applications **The Final Exam**
FYSH515 **Fall 2018**

Return by 12.00, Thursday 20.12 to office FL220 (Kimmo), complete with all intermediated details of your computation.

Consider the following model for fermions with a discrete chiral symmetry in two space-time dimensions ($D = 2$):

$$\mathcal{L} = \sum_j \bar{\psi}_j i \not{\partial} \psi_j + \frac{1}{2} g^2 \Big(\sum_j \bar{\psi}_j \psi_j \Big)^2 , \tag{1}$$

where $j = 1, \ldots, N$ is the number of fermion flavours. The model resembles the familiar four dimensional $\lambda\phi^4$ theory with $O(N)$ symmetry, but for fermionic fields in two dimensions. The kinetic term of two dimensional fermions is built from matrices γ^μ that satisfy the two dimensional Dirac algebra: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ with $g^{\mu\nu} = \mathrm{diag}(+1, -1)$. These matrices can be chosen as: $\gamma^0 = \sigma^2$, $\gamma^1 = i\sigma^1$, where σ^i are Pauli matrices. We can define the two dimensional analogue of the γ_5 (which anticommutes with the γ^μ) as: $\gamma^5 = \gamma^0\gamma^1 = \sigma^3$.

Part one.

a) Show that the theory defined by Eq. (1) is invariant under the transformation

$$\psi_j \to \gamma^5 \psi_j , \tag{2}$$

and that this symmetry forbids the appearance of a fermion mass term in the theory.

b) Show that this theory is renormalizable in two dimensions (at the level of power counting) and find the divergent n-point functions.

c) Show that the functional integral for this theory can be writtn in the following form:

$$\int \mathcal{D}\psi_j \mathcal{D}\bar{\psi}_j e^{i\int \mathrm{d}^2x \mathcal{L}} = \int \mathcal{D}\psi_j \mathcal{D}\bar{\psi}_j \mathcal{D}\sigma \exp\Big(i \int \mathrm{d}^2x \{ \bar{\psi}_j i \not{\partial} \psi_j - \sigma(\bar{\psi}_j \psi_j) - \frac{1}{2g^2}\sigma^2 \} \Big) , \tag{3}$$

where $\sigma(x)$ is a new auxiliary scalar field with no kinetic terms (not to be confused with a Pauli matrices). It is evident that physically $\sigma(x)$ describes the fermion condensate.

d) Compute the leading correction to the effective potential for $\sigma(x)$ (the condensate) by integrating over the fermion fields ψ_j and $\bar{\psi}_j$. You should encounter the determinant of a Dirac operator. To evaluate it you can just take the determinant of the spinor degrees of freedom (a 2×2 matrix) and then pass to the momentum space, or you can diagonalize the matrix in the momentum space for each Fourier mode. This 1-loop contribution requires a renormalization counter term proportional to σ^2 (that is, a renormalization of the coupling constant g^2). Renormalize by modified minimal substraction or $\overline{MS}$ scheme.

e) Ignoring higher-orger contributions, minimize the effective potential you found in the previous step. Show that the $\sigma(x)$ acquires a vacuum expectation value which breaks the chiral symmetry of the model.

Part two.

Go back to the original theory as written in the Eq. (1). Compute the beta function $\beta(g)$ at the 1-loop level and show that the model is asymptotically free.

2. Decay $\mu \to e\gamma$.

The muon and electron number violating decay $\mu \to e\gamma$ (see Fig. 1) is not allowed if neutrinos are massless. However, if neutrinos do mix, this decay has a branching ratio

$$B(\mu \to e\gamma) \equiv \frac{\Gamma(\mu \to e\gamma)}{\Gamma(\mu \to e\nu\bar{\nu})} = \frac{3\alpha}{32\pi}\delta_\nu^2 \,, \tag{4}$$

where

$$\delta_\nu = \sum_{i=1}^{3} U_{ei}^* U_{\mu i}(m_i^2/M_W^2) \,. \tag{5}$$

Here M_W is the weak gauge boson mass, m_i are the neutrino mass eigenvalues and $U_{\alpha i}$ are the analogue of the CKM-mixing matrix in the neutrino sector, which defines the flavour eigenstates in terms of the mass eigenstates:

$$\nu_\alpha = \sum_{i=1}^{3} U_{\alpha i}\nu_i \,. \tag{6}$$

Your task is to compute the branching ratio Eq. (4).

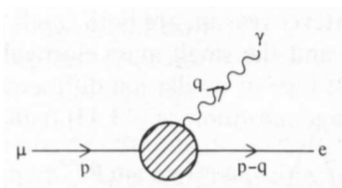

Figure 1: Generic matrix element for the decay $\mu \to e\gamma$.

Part one.

a) First argue that the most general Lorentz structure of the matrix element is

$$T_\lambda = \bar{u}_e(p')[iq^\nu \sigma_{\lambda\nu}(A + B\gamma_5) + \gamma_\lambda(C + D\gamma_5) + q_\lambda(E + F\gamma_5)]u_\mu(p)\,. \tag{7}$$

From this expression use the electromagnetic gauge invariance (why does it hold?) to show that on-shell $q^2 = 0$:

$$T(\mu \to e\gamma) = \epsilon^\lambda T_\lambda = \bar{u}_e(p')[iq^\nu \epsilon^\lambda \sigma_{\lambda\nu}(A + B\gamma_5)]u_\mu(p)\,. \tag{8}$$

We will be working in the limit $m_e = 0$. Give an argument as to why in this limit you must have $A = B$. (You will of course see this explicitly later in the calculation.) Finally, using the equations of motion for the spinors rewrite Eq. (8) in the form:

$$T(\mu \to e\gamma) = A\,\bar{u}_e(p')[(1 + \gamma_5)(2p\cdot\epsilon - m_\mu \gamma\cdot\epsilon)]u_\mu(p)\,. \tag{9}$$

b) Using Eq. (9) derive the following expression for the decay width (with $m_e = 0$):

$$\Gamma(\mu \to e\gamma) = \frac{m_\mu^3}{4\pi}|A|^2\,. \tag{10}$$

Derive (if you have time) the decay width $\Gamma(\mu \to e\nu\bar{\nu}) = G_F^2 m_\mu^5/192\pi^3$ (in the limit $m_e = 0$, $m_{\nu_i} = 0$) and use this result together with Eq. (10) to write the branching ratio as a function of $|A|^2$.

Part two. (The harder one.)

c) You now need to evaluate the amplitude A. Since our operator is known to be finite one can try to work it out in the unitary gauge[1], where there is only one relevant diagram, shown in Fig. 2. (Please use the momentum routing as defined in the left panel of the figure 2.) The relevant Lagrange function for the problem in the flavour basis is

$$\mathcal{L} = \frac{g}{2\sqrt{2}}\left(\bar{\nu}_e \gamma^\alpha(1 - \gamma_5)e\,W_\alpha^- + \bar{\nu}_\mu \gamma^\alpha(1 - \gamma_5)\mu\,W_\alpha^-\right). \tag{11}$$

To derive the relevant Feynman rules insert the expansion (6) to Eq.(11). Finally, the Feynman rule for the $\gamma_\lambda(k_1)W_\alpha^-(k_2)W_\beta^+(k_3)$-vertex, with all momenta directed *into* the vertex is:

$$-ieV_{\lambda\alpha\beta}(k_1, k_2, k_3) = -ie\left[(k_1 - k_2)_\beta g_{\lambda\alpha} + (k_2 - k_3)_\lambda g_{\alpha\beta} + (k_3 - k_1)_\alpha g_{\beta\lambda}\right]. \tag{12}$$

Now write down the matrix element for the diagram shown in figure 2 in U-gauge. Then expand the neutrino propagator in m_i^2 and show that the lowest order term vanishes. Keeping only the leading term in m_i^2 show that your matrix element is

$$T = i\frac{g^2 e}{4}\sum_i U_{ei}^* U_{\mu i} m_i^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{R}{[(k+p)^2]^2(k^2 - M_W^2)((k+q)^2 - M_W^2)}\,, \tag{13}$$

[1] This is not gurananteed to work, because U-gauge can in principle spoil even a finite operator.

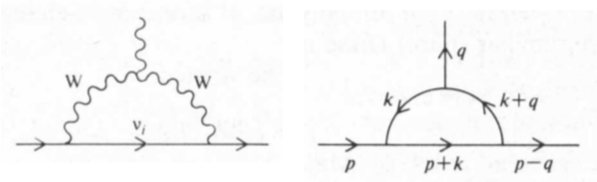

Figure 2: The 1-loop contribution to the matrix element for the decay $\mu \to e\gamma$ at 1-loop level in the unitary gauge (left). The desired momentum routing (right).

where

$$R \equiv \Delta_W^{\nu\beta}(k)\Delta_W^{\mu\alpha}(k+q)\Gamma_{\alpha\beta}\,N_{\mu\nu} \tag{14}$$

with

$$\Gamma_{\alpha\beta} \equiv \epsilon^\lambda V_{\lambda\alpha\beta}(-q, k+q, -k)\,, \tag{15}$$

$$N_{\mu\nu} \equiv \bar{u}_e(p-q)\,[\gamma_\mu(\not{p}+\not{k})\gamma_\nu(1-\gamma_5)]\,u_\mu(p) \tag{16}$$

and

$$\Delta_W^{\rho\sigma}(p) \equiv -g_{\rho\sigma} + p_\rho p_\sigma/M_W^2. \tag{17}$$

Show that out of the four terms coming from contracting Δ-tensors, only the one with $\sim pp/M_W^2$-terms from both propagators is dangerous for convergence. Show that this term vanishes because

$$(k+q)^\alpha k^\beta \epsilon^\lambda V_{\lambda\alpha\beta}(-q, k+q, -k) = 0\,. \tag{18}$$

Thus, the integral is finite and we can indeed perform the computation in U-gauge. The remaining contractions making up $R \equiv S_1 + S_2 + S_3$ then are

$$\begin{aligned} S_1 &= \Gamma^{\mu\nu}N_{\mu\nu} \\ S_2 &= -(k^\lambda\Gamma^\mu_\lambda)(k^\nu N_{\mu\nu})/M_W^2 \\ S_3 &= -((k+q)^\lambda\Gamma^\mu_\lambda)((k+q)^\nu N_{\mu\nu})/M_W^2\,. \end{aligned} \tag{19}$$

Now, introduce the Feynman parametrization using Peskin & Schröder equation (6.42) and rewrite the integral as

$$T = i3!\frac{g^2 e}{4}\sum_i U_{ei}^* U_{\mu i} m_i^2 \int z_1 \mathrm{d}z_1 \mathrm{d}z_2 \int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\frac{\tilde{S}_1+\tilde{S}_2+\tilde{S}_3}{(\ell^2-a^2)^4}\,, \tag{20}$$

where ℓ is the shifted momentum and $\tilde{S}_i$ are S_i written in terms of ℓ. Compute the effective mass a only to the leading order in M_W^2:

$$a^2 = (1-z_1)M_W^2 + ... \tag{21}$$

Obviously the next, and most tedious task is to compute the quantities $\tilde{S}_i$. However, *do not to attempt to compute them completely.* Instead, pick only the terms that eventually are proportional to $p \cdot \epsilon$. This reduces the amount of work enormously. When you are done, you should get

$$T = A \times 2p\cdot\epsilon\,\bar{u}_e(p')(1+\gamma_5)u_\mu(p) + ...\,, \tag{22}$$

where A is just the quantity you are after:

$$A = \frac{1}{64\pi^2}\frac{m_\mu}{M_W^4}\frac{g^2 e}{4}\sum_i U_{ei}^* U_{\mu i} m_i^2\,. \tag{23}$$

Rewriting this using G_F etc, and inserting back to (10) should lead to the desired result.