Perturbation theory anomalies

In some cases radiative corrections can destroy symmetries of classical equations of motion. Prime example involves chiral symmetry.

In free massless theory, as well as in massless QED and QCD the axial vector current is conserved

$$\partial_\mu j^\mu_5 = 0 \quad ; \quad j^\mu_5 = \bar{\Psi}(x)\gamma^\mu\gamma^5\psi(x)$$

In massive case the classical current is broken

$$\partial_\mu j^\mu_5 = 2mP(x) = 2im\bar{\Psi}(x)\gamma^5\psi(x)$$

These hold even in local theories like QED & QCD. Eg in QED, the eom. are $(D_\mu \equiv \partial_\mu + ieA_\mu)$

$$\begin{aligned} \not{\partial}\psi &= -im\psi - ie\not{A}\psi \\ \bar{\Psi}\overleftarrow{\not{\partial}} &= im\bar{\Psi} + ie\bar{\Psi}\not{A} \end{aligned} \qquad (1)$$

$$\Rightarrow \quad \partial_\mu j^\mu = \partial_\mu \bar{\Psi}\gamma^\mu\psi = (\bar{\Psi}\overleftarrow{\not{\partial}})\psi + \bar{\Psi}(\not{\partial}\psi) = 0 \quad \text{obviously}$$

$$\begin{aligned} \partial_\mu j^\mu_5 = (\bar{\Psi}\overleftarrow{\not{\partial}})\gamma^5\psi - \bar{\Psi}\gamma^5(\not{\partial}\psi) &= +im\bar{\Psi}\gamma^5\psi + ie\bar{\Psi}\cancel{\not{A}\gamma^5}\psi \\ &+ im\bar{\Psi}\gamma^5\psi + ie\bar{\Psi}\cancel{\gamma^5\not{A}}\psi = 2mi\bar{\Psi}\gamma^5\psi . \end{aligned}$$

j^μ-conservation is due to invariance under $\psi \to e^{i\alpha}\psi$. In $m=0$-limit j^μ_5-conservation is due to invariance under $\psi \to e^{i\gamma^5\theta}\psi$.

However, at quantum level the axial current conservation breaks even when $m=0$. There are many ways to see that, and axial current anomalies have several important consequences. We start by analysing the axial current more closely at q-level.

Quantum axial current

The key issue is that products of field operators are typically singular. Therefore we write the quantum counterpart of j^μ_5 as

$$\langle j^\mu_5 \rangle = \underset{\epsilon \to 0}{\text{symmlim}} \langle 0 | T\Big(\bar{\Psi}(x+\tfrac{\epsilon}{2}) \gamma^\mu \gamma^5 \underbrace{e^{-ie\int_{x-\epsilon/2}^{x+\epsilon/2} dz^\mu A_\mu(z)}}_{\text{Wilson line } U(x+\frac{\epsilon}{2}, x-\frac{\epsilon}{2})} \psi(x-\tfrac{\epsilon}{2}) \Big) | 0 \rangle \quad (2)$$

Some comments are in order here.

① Wilson line is necessary to make current gauge invariant. In general

$$\bar{\Psi}(y)\psi(x) \xrightarrow{\psi \to e^{i\alpha}\psi} e^{-i(\alpha(y)-\alpha(x))} \bar{\Psi}(y)\psi(x) \neq \bar{\Psi}(y)\psi(x)$$

So we need to generalize $\bar{\Psi}(y)\psi(x) \to \bar{\Psi}(y)\, U(y,x)\, \psi(x)$ where

$$U(y,x) \longrightarrow e^{i(\alpha(y)-\alpha(x))} U(y,x)$$

Wilson line does just that : $e^{-ie\int_x^y dz^\mu A_\mu(z)} \to e^{-ie\int_x^y dz^\mu (A_\mu(z) - \frac{1}{e}\partial_\mu \alpha)}$

$$= U(y,x) \exp\big(i(\alpha(y)-\alpha(x))\big) \quad //.$$

(Rem

$\psi \to \psi' = e^{i\alpha}\psi$; $\bar{\Psi}\not{D}\psi \to \bar{\Psi}'\not{D}'\psi' = \bar{\Psi} e^{-i\alpha}(\partial_\mu + ieA'_\mu) e^{i\alpha}\psi = \bar{\Psi}(\partial_\mu + ieA'_\mu + i\partial_\mu\alpha)\psi$

$\equiv \bar{\Psi}\not{D}\psi \Rightarrow ieA'_\mu = ieA_\mu - i\partial_\mu\alpha \Rightarrow A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\alpha$)

② Time ordering is necessary because we have time-separation between points.

(x^μ is a 4-vector : "ϵ" then has a direction)

③ Symmetric limit is necessary to ensure that resulting $\langle j^\mu_5\rangle$ transforms as a pseudo-vector under L-group. You may imagine that

$$\underset{\epsilon\to 0}{\text{symm lim}} \langle j^\mu_5\rangle_\epsilon \equiv \lim_{\epsilon\to 0} \frac{1}{2\pi^2}\int d\Omega_n \langle j^\mu_5\rangle_{\epsilon^\mu = n^\mu\epsilon} \qquad (3)$$

This then implies that (these can also be viewed as a regularization method.)

$$\underset{\epsilon\to 0}{\text{symm lim}} \frac{\epsilon^\mu}{\epsilon^2} = 0$$

$$(4)$$

$$\underset{\epsilon\to 0}{\text{symm lim}} \frac{\epsilon^\mu\epsilon^\nu}{\epsilon^2} = \frac{1}{d} g^{\mu\nu}$$

We can now compute the divergence of (2):

$$\partial_\mu\langle j^\mu_5\rangle = \underset{\epsilon\to 0}{\text{symm lim}} \langle 0|T\Big(\big(\bar\Psi(x+\tfrac{\epsilon}{2})\overleftarrow{\not{\partial}}_x\big)\ \gamma^5 U\,\psi(x-\tfrac{\epsilon}{2})$$

$$- \bar\Psi(x+\tfrac{\epsilon}{2})\gamma^5 U\big(\not{\partial}_x\psi(x-\tfrac{\epsilon}{2})\big)$$

$$+ \bar\Psi(x+\tfrac{\epsilon}{2})\gamma^\mu\gamma^5\big(-ie\,\epsilon^\nu\partial_\mu A_\nu(x)\big)U\,\psi(x-\tfrac{\epsilon}{2})\Big)|0\rangle$$

Where $U\equiv U(x+\frac{\epsilon}{2},x-\frac{\epsilon}{2})$ and we used

$$\partial_\mu U = \Big(\underbrace{-ie\,\partial^x_\mu\int_{x-\frac{\epsilon}{2}}^{x+\epsilon/2} dz^\lambda A_\lambda(z)}_{\simeq\ \partial^x_\mu\epsilon^\lambda A_\lambda(x)}\Big)U \simeq -ie\,\epsilon^\lambda\partial_\mu A_\lambda(x)\,\overbrace{U(x+\tfrac{\epsilon}{2},x-\tfrac{\epsilon}{2})}^{\to 1 \text{ to } 1^{st} \text{ order in } \epsilon}$$

(integral path is just one step to first order in ϵ.

Then, using the e.o.m's (1), we get $\left(\begin{array}{l} \not{\partial}\psi = -im\psi - ie\not{A}\psi \\ \bar{\psi}\overleftarrow{\not{\partial}} = im\bar{\psi} + ie\bar{\psi}\not{A} \end{array} \right)$

$$\partial_\mu \langle j^\mu_5 \rangle \simeq \operatorname{symm}\lim_{\epsilon \to 0} \langle 0 | T\Big(ie\bar{\psi}(x+\tfrac{\epsilon}{2})\not{A}(x+\tfrac{\epsilon}{2})\gamma^5\psi(x-\tfrac{\epsilon}{2}) + im\bar{\psi}(x+\tfrac{\epsilon}{2})\gamma^5 U\psi(x-\tfrac{\epsilon}{2})$$

$$+ ie\bar{\psi}(x+\tfrac{\epsilon}{2})\gamma^5\not{A}(x-\tfrac{\epsilon}{2})\psi(x-\tfrac{\epsilon}{2}) + im\bar{\psi}(x+\tfrac{\epsilon}{2})\gamma^5 U\psi(x-\tfrac{\epsilon}{2})$$

$$+\bar{\psi}(x+\tfrac{\epsilon}{2})\gamma^\mu\gamma^5\left(-ie\epsilon^\nu\partial_\mu A_\nu(x)\right)\psi(x-\tfrac{\epsilon}{2})\Big)|0\rangle$$

Expanding $\not{A}(x \pm \frac{\epsilon}{2}) \simeq \not{A}(x) \pm \frac{\epsilon^\lambda}{2}\partial_\lambda \not{A}(x) + \cdots$ we see that all lowest order $\not{A}$-terms cancel. Also m-terms combine to give the quantum pseudoscalar current. (This result is exact and this is why I kept U in those terms.) Thus

$$\partial_\mu \langle j^\mu_5 \rangle = 2m\langle \bar{\psi} i\gamma^5\psi \rangle$$

$$+ \operatorname{symm}\lim_{\epsilon \to 0} \langle 0 | T\Big(\bar{\psi}(x+\tfrac{\epsilon}{2})\left(-ie\gamma^\mu\epsilon^\nu\right)\overbrace{\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)}^{=F_{\mu\nu}}\gamma^5\psi(x-\tfrac{\epsilon}{2})\Big)|0\rangle$$

$$= 2m\langle\bar{\psi} i\gamma^5\psi\rangle - ieF_{\mu\nu}\operatorname{symm}\lim_{\epsilon\to 0}\epsilon^\nu\langle 0|T\Big(\bar{\psi}(x+\tfrac{\epsilon}{2})\gamma^\mu\gamma^5\psi(x-\tfrac{\epsilon}{2})\Big)|0\rangle \qquad (5)$$

Now observe that in general

$$\lim_{x\to y}\langle 0|T(\bar{\psi}(y)\gamma^\mu\gamma^5\psi(x))|0\rangle = -\lim_{x\to y}\operatorname{Tr}\Big[\gamma^\mu\gamma^5\overbrace{\langle 0|T(\psi(x)\bar{\psi}(y))|0\rangle}^{\text{propagator } iS}\Big]$$

$$= +\lim_{x\to y}\operatorname{Tr}\left[\gamma^\mu\gamma^5\frac{1}{i\not{\partial}-m}\delta^4(x-y)\right].$$

In the last step we used formally $(i\not{\partial}-m)iS_\psi(x,y) = -\delta^4(x-y)$.

Now expand the propagator $\left(i\not{D} = i(\not{\partial} + ie\not{A}) = i\not{\partial} - e\not{A} \right)$

$$\frac{1}{i\not{D}-m} = \frac{1}{i\not{\partial}-m} + \frac{1}{i\not{\partial}-m}\, e\not{A}\, \frac{1}{i\not{\partial}-m} + \ldots$$

The first term vanishes when traced over with $\gamma^\mu \gamma^5$. This would have been the most singular term. Higher-order terms are not singular and vanish at $\epsilon \to 0$ - limit (not proven here). Consider second term:

$$\lim_{x\to y} \mathrm{Tr}\left[\gamma^\mu \gamma^5 \frac{1}{i\not{\partial}_x - m} e\not{A}(x) \frac{1}{i\not{\partial}_x - m} \delta(x-y) \right] \Big|_{x-y=\epsilon}$$

$$= \lim_{x\to y} e \int d^4z\, A_\lambda(z)\, \mathrm{Tr}\left[\gamma^\mu \gamma^5 \frac{1}{i\partial_x - m} \gamma^\lambda \underbrace{\delta^4(x-z)}_{= \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-z)}} \frac{1}{i\not{\partial}_x - m} \underbrace{\delta(x-y)}_{\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}} \right] \Big|_{x-y=\epsilon}$$

$$= \lim_{x\to y} e \int \frac{d^4q}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \overbrace{\int d^4z\, A_\lambda(z)\, e^{+iq\cdot z}}^{= A_\lambda(q)}\, \underbrace{\mathrm{Tr}\left[\gamma^\mu \gamma^5 \frac{1}{\not{q}+\not{p}-m} \gamma^\lambda \frac{1}{\not{p}-m} \right]}_{= \frac{\mathrm{Tr}(\gamma^\mu\gamma^5 \not{q} \gamma^\lambda \not{p})}{[(q+p)^2-m^2](p^2-m^2)} = \frac{-4i\epsilon^{\mu\lambda\alpha\beta} q_\alpha p_\beta}{[(q+p)^2-m^2](p^2-m^2)}} e^{-ip\cdot\epsilon - iq\cdot x}$$

$$= \lim_{\epsilon\to 0} -4ie\,\epsilon^{\mu\lambda\alpha\beta} \underbrace{\int \frac{d^4q}{(2\pi)^4} q_\alpha A_\lambda e^{+iq\cdot x}}_{\to +i\partial_\alpha A_\lambda(x) \text{ if const}} \underbrace{\int \frac{d^4p}{(2\pi)^4} \frac{p_\beta}{((q+p)^2-m^2)(p^2-m^2)} e^{-ip\cdot\epsilon}}_{\checkmark}$$

Now the last term needs to be singular $\propto \frac{\epsilon_\beta}{\epsilon^2}$, for there to be any contribution to the current divergence. Certainly for any finite cutoff $p^2 < \Lambda^2$ there is no such divergence

$$\int^{\Lambda} \frac{d^4p}{(2\pi)^4} \frac{p_\beta}{(\,)(\,)} e^{-ip\cdot\epsilon} \sim \text{const}$$

A possible divergence then comes from $p \to \infty$ limit. We can thus rewrite the last integral as

$$= i\int_\Lambda \frac{d^4p_E}{(2\pi)^4}\,\frac{-p_\beta}{p^4}\, e^{+ip_E\cdot\epsilon_E} + \text{const} \quad ; \quad p_E\cdot\epsilon = p\epsilon\cos\theta$$

by suitable choice of coord. system.

$$= i\frac{\pi^2}{(2\pi)^4}(-i)\frac{\partial}{\partial\epsilon_\beta}\int_\Lambda^\infty dp\,\frac{1}{p}\int_{-1}^{1} d\cos\theta\; e^{ip_E\epsilon_E\cos\theta}$$

$$=(-i)\frac{i}{16\pi^2}\frac{\partial}{\partial\epsilon_\beta}\int_\Lambda^\infty dp\,\frac{1}{p}\,\frac{2i}{p\epsilon}\sin p\epsilon_E$$

; (From this expression we see that only modes $p\to\infty$ contribute to integral when $\epsilon \to 0$.)

$$= \frac{+i}{8\pi^2}\frac{\epsilon_\beta}{\epsilon_E}\int_\Lambda^\infty dp\left(-\frac{\sin p\epsilon_E}{(p\epsilon)^2}+\frac{\cos p\epsilon}{p\epsilon}\right)_E$$

$$= \frac{+i}{8\pi^2}\frac{\epsilon_\beta}{\underbrace{\epsilon_E^2}_{=-\epsilon_M^2}}\int_{\epsilon\Lambda}^\infty dx\,\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{-i}{8\pi^2}\frac{\epsilon_\beta^M}{\epsilon_M^2}\Big|_0^\infty \frac{\sin x}{x} = +\frac{i}{8\pi^2}\frac{\epsilon_\beta}{\epsilon^2}$$

$\hookrightarrow 0$ for any finite Λ when $\epsilon \to 0$!!

That is

$$\lim_{x\to y}\mathrm{Tr}\left[\gamma^\mu\gamma^5\frac{1}{i\not{D}-m}\delta^4(x-y)\right]_{|x-y=\epsilon} = +\frac{ie}{2\pi^2}\frac{\epsilon_\beta}{\epsilon^2}\overbrace{\epsilon^{\mu\lambda\alpha\beta}\partial_\alpha A_\lambda}^{=\frac{1}{2}\overbrace{\epsilon^{\mu\lambda\alpha\beta}}^{=-\epsilon^{\alpha\lambda\mu\beta}}F_{\alpha\lambda}}$$

./.

All together then

$$\partial_\mu j^\mu_5 = 2mP + \frac{e^2}{4\pi^2}F_{\mu\nu}\overbrace{\epsilon^{\mu\lambda\alpha\beta}F_{\alpha\lambda}}^{=-\tilde{F}^{\mu\beta}}\;\overbrace{\underset{\epsilon\to 0}{\text{symm}\lim}\frac{\epsilon_\beta\epsilon^\nu}{\epsilon^2}}^{=\frac{1}{4}\delta_\beta^{\ \nu}}$$

$$= 2mP - \frac{e^2}{16\pi^2}\tilde{F}_{\mu\nu}F^{\mu\nu}$$

↑ Anomalous term not present at tree level. This is exact result.

Ward identity for QED thru current algebra

Remember:

- $\{\psi_a(\vec{x},t), \psi_b^\dagger(\vec{y},t)\} = \delta^3(\vec{x}-\vec{y})\,\delta_{ab}$; $\{\psi_a, \bar\psi_b\} = \delta^3(\vec{x}-\vec{y})\,\gamma^0_{ba}$
- $\{\psi_a, \psi_b\} = \{\psi_b^\dagger, \psi_b^\dagger\} = 0$

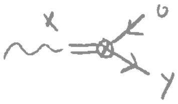
and

$$J^\mu(x) = \bar\psi(x)\gamma^\mu\psi(x) = \psi(x)^\dagger\gamma^0\gamma^\mu\psi(x) \quad ; \quad \boxed{\partial_\mu J^\mu = 0}$$

Noether.

$$\Rightarrow \quad J^0(x) = \psi^\dagger(x)\psi(x) .$$

Consider: (current-field-field)



$$G_\mu(p,q) = -\int d^4x\, d^4y\, e^{+iq\cdot x + ip\cdot y}\langle 0|T(J_\mu(x)\overbrace{\psi(y)\bar\psi(0)}^{-\bar\psi\psi})|0\rangle$$

Contract by q^μ: $(= i\partial^\mu_x$ on l.h.s$)$

← bosonic!

hits T-ordering $\theta(x_0-y_0)$ & $\theta(x_0)$ functions!

$$q^\mu G_\mu(p,q) = \int d^4x\, d^4y\, e^{+iq\cdot x + ip\cdot y}(+i)\partial^\mu_x \langle 0|T(\overbrace{J_\mu(x)}^{=0 \quad \partial_\mu J^\mu = 0}\psi_\alpha(y)\bar\psi(0))|0\rangle$$

$$= -i\int d^4x\, d^4y\, e^{+iq\cdot x + ip\cdot y}\Big(\langle 0|T(\delta(x_0-y_0)[J_0(x),\psi(y)]\bar\psi(0))|0\rangle$$

$$+ \langle 0|T(\delta(x_0)\psi(y)[J_0(x),\bar\psi(0)])|0\rangle\Big)$$

but

- $[J_0(x),\psi(y)] = [\psi^\dagger(x)\psi(x),\psi(y)] = \psi^\dagger(x)\psi(x)\psi(y) - \underbrace{\psi(y)\psi^\dagger(x)}\psi(x)$

$\psi^\dagger(x)\psi(y)\psi(x)$

$-\delta^3(\vec{x}-y)\psi(x)$

$\bar\psi_\alpha = \psi^\dagger\gamma^0$

$$= -\delta^3(\vec{x}-\vec{y})\psi(x)$$

- $[J_0(x),\bar\psi(0)] = \psi^\dagger(x)\psi(x)\bar\psi(0) - \bar\psi_a(0)\psi_b^\dagger(x)\psi_b(x)$

$$= +\psi_b^\dagger(x)\,\delta^3(\vec{x})\,\gamma^0_{ab} = +\bar\psi(0)$$

$$\Rightarrow \underline{\underline{q^\mu G_\mu(p,q)}} = -i\int d^4x\, d^4y\, e^{+iq\cdot x+ip\cdot y}\Big(-\delta^4(\vec{x}-\vec{y})\langle 0|T(\psi(x)\bar{\psi}(0))|0\rangle$$

$$+\,\delta^4(\vec{x})\,\langle 0|T(\psi(y)\bar{\psi}(0))|0\rangle\Big)$$

$$= +i\int d^4y\, e^{+i(q+p)\cdot y}\langle 0|T(\psi(y)\bar{\psi}(0))|0\rangle$$

$$-\,i\int d^4y\, e^{+ip\cdot y}\langle 0|T(\psi(y)\bar{\psi}(0))|0\rangle =$$

$$= \underline{\underline{i\big(S(q+p)-S(p)\big).}} \quad \boxtimes$$

This holds because of the gauge invariance.

Rem. all these were derived for bare fields. For renormalized fields the canonical comm. relation get Z_4 on the r.h.s. $\Rightarrow$ eventually $\underline{Z_1 = Z_2}$ as derived earlier.

How about the axial one:

$$q^\mu G^5_\mu(q,p) = -i\int d^4x\, d^4y\, e^{+iq\cdot x+ip\cdot y}\,\overbrace{\partial^\mu_x\langle 0|T(J^5_\mu(x)}^{=2imP(x)}\,\psi_\alpha(y)\,\bar{\psi}_\beta(0))|0\rangle$$

$$= 2m\,G^5(p,q) - i\int d^4x\, d^4y\, e^{+iq\cdot x+ip\cdot y}\Big(\langle 0|T(\delta(x_0-y_0)\,[\overset{\psi^+_\epsilon\gamma^5_{\epsilon\delta}\psi_\delta}{J^5_0(x)},\psi_\alpha(y)]\,\bar{\psi}_\beta(0))|0\rangle$$

$$+\,\langle 0|T(\delta(x_0)\,\psi_\alpha(y)\,[J^5_0(x),\bar{\psi}_\beta(0)])|0\rangle\Big)$$

- $[J^5_0(x),\psi_a(y)] = [\psi^+_\epsilon(x)\gamma^5_{\epsilon\delta}\psi(x)_\delta,\psi_\alpha(y)] = -\delta^3(\vec{x}-y)\,\gamma^5_{\epsilon\delta}\,\psi(\vec{x})$
- $[J^5_0(x),\bar{\psi}_\beta(0)] = [\psi^+_\epsilon(x)\gamma^5_{\epsilon\delta}\psi_\delta(x),\ \psi^+_\alpha(0)\gamma^0_{\alpha\beta}] = \delta^3(\vec{x})\,\psi^+_\epsilon\,\gamma^5_{\epsilon\alpha}\gamma^0_{\alpha\beta} = -\delta^3(x)\,\bar{\psi}\gamma^5$

$$\Rightarrow \underline{q^\mu G^5_\mu(q,p) = 2m\,G^5(p,q) + i\big(\gamma^5 S(p+q) + S(p)\gamma^5\big)}$$

Let us now consider 3-point functions

- $T_{\mu\nu\lambda}(k_1,k_2,q) \doteq i\int d^4x_1\, d^4x_2 \langle 0|T(V_\mu(x_1)\,V_\nu(x_2)\,A_\lambda(0))|0\rangle\, e^{+ik_1\cdot x_1+ik_2\cdot x_2}$

- $T_{\mu\nu}(k_1,k_2,q) \equiv i\int d^4x_1\, d^4x_2 \langle 0|T(V_\mu(x_1)\,V_\nu(x_2)\,P(0))|0\rangle\, e^{+ik_1\cdot x_1+ik_2\cdot x_2}$

↑ pseudoscalar.

where

$$V_\mu(x) \equiv \bar{\psi}(x)\gamma^\mu\psi(x)$$
$$A_\mu(x) \equiv \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)$$
$$P(x) \equiv \bar{\psi}(x)\gamma^5\psi(x)$$

Using classical relations $\partial^\mu V_\mu=0$ and $\partial^\mu A_\mu = 2imP$ we now get

$$\underline{k_1^\mu T_{\mu\nu\lambda}(k_1,k_2,q)} = \int d^4x_1\, d^4x_2 \left[\partial^\mu_{x_1} \overbrace{\langle 0|T(V_\mu}^{0}(x_1)V_\nu(x_2)\,A_\lambda(0))|0\rangle\right] e^{+ik_1\cdot x_1+ik_2\cdot x_2}$$

$$= \int d^4x_1\, d^4x_2 \Big(\delta(x_0-y_0)\langle 0|T([J_0(x_1),V_\nu(x_2)]\,A_\lambda(0))|0\rangle + \delta(x_0)\langle 0|T(V_\mu(x_1)\,[J_0(x_1),A_\lambda(0)])|0\rangle\Big)$$

$$\underline{= 0 = k_2^\nu T_{\mu\nu\lambda}(k_1,k_2,q)}$$

Because

$$[J_0(x),V_\nu(y)] = [J_0(x),\bar{\psi}(y)\gamma_\nu\psi(x)]$$
$$= [J_0(x),\bar{\psi}(y)]\gamma_\nu\psi(x) + \bar{\psi}(y)\gamma_\nu[J_0(x),\psi(y)]$$
$$= \delta^3(\vec{x}-\vec{y})(\bar{\psi}(y)\gamma_\nu\psi(x) - \bar{\psi}(y)\gamma_\nu\psi(x)) = 0$$

and similarly $[J_0(x),A_\lambda(0)] = 0$.

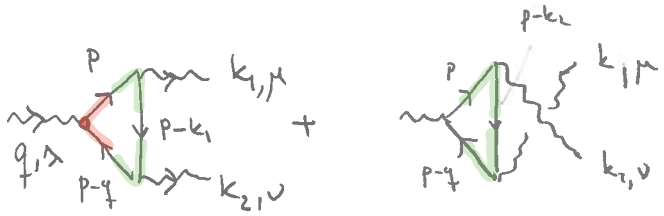
This is the vector ward identity. Now, at classical level similarly

$$q^\lambda T_{\mu\nu\lambda}(k_1,k_2,q) = \int d^4x\, d^4y \left[\overbrace{\partial_y^\lambda \langle 0 | T(V_\mu(0)\, V_\nu(x)\, A_\lambda(y)) | 0 \rangle}^{2im P(y)} \right] e^{+ik_1\cdot x + iq\cdot y}$$

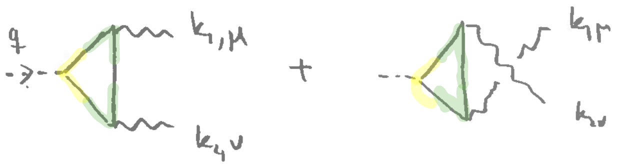
$$= 2im \int d^4x\, d^4y\, \langle 0 | T(V_\mu(0)\, V_\nu(x)\, P_\lambda(y)) | 0 \rangle\, e^{+ik_1\cdot x + iq\cdot y}$$

$$= 2im\, T_{\mu\nu}(k_1,k_2,q)$$

This, Axial Ward identity relates $T_{\mu\nu\lambda}$ and $T_{\mu\nu}$. It is broken by quantum effects however. To see this let us consider 1-loop corrections to $T_{\mu\nu\lambda}$:



$\propto T_{\mu\nu\lambda}$



$\propto T_{\mu\nu}$

We find:

$$T_{\mu\nu\lambda}(k_1,k_2,q) = -\int \frac{d^4p}{(2\pi)^4}\, \mathrm{Tr}\left[\frac{i}{\not p - m}\, \gamma_\lambda \gamma_5\, \frac{i}{\not p - \not q - m}\, \gamma_\nu\, \frac{i}{\not p - \not k_1 - m}\, \gamma_\mu \right] + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

$$T_{\mu\nu}(k_1,k_2,q) = -\int \frac{d^4p}{(2\pi)^4}\, \mathrm{Tr}\left[\frac{i}{\not p - m}\, \gamma^5\, \frac{i}{\not p - \not q - m}\, \gamma_\nu\, \frac{i}{\not p - \not k_1 - m}\, \gamma_\mu \right] + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

Do these functions satisfy the ward identities found above?

details of calculation

These integrals are divergent and the ~~answer~~ depend~~s to a degree~~ on the choice of the regulator, For example :

- $*$ cut-off -regulator ; breaks at least one identity
- $*$ D-reg ; breaks axial ward identity.

Axial ward identity in d-reg. $(q^\mu T_{\mu\nu\lambda})$

Write:

$$\not{q}\gamma^5 = \gamma^5(\not{p}-\not{q}-m) + (\not{p}-m)\gamma_5 + \underbrace{2m\gamma^5}_{\text{classical part}} \qquad (1)$$

Whence immediately:

$$q^\lambda T_{\mu\nu\lambda} = 2m\,T_{\mu\nu}$$
$$+ \int \frac{d^4p}{(2\pi)^4}\Bigg(\mathrm{Tr}\left[\frac{i}{\not{p}-m}\gamma^5\gamma_\nu\frac{i}{\not{p}-\not{k}_1-m}\gamma_\mu\right] - \mathrm{Tr}\left[\frac{i}{\not{p}-\not{k}_1-m}\gamma^5\gamma_\mu\frac{i}{\not{p}-\not{q}-m}\gamma_\nu\right]$$
$$+ \mathrm{Tr}\left[\frac{i}{\not{p}-m}\gamma^5\gamma_\nu\frac{i}{\not{p}-\not{k}_2-m}\gamma_\nu\right] - \mathrm{Tr}\left[\frac{i}{\not{p}-\not{k}_2-m}\gamma^5\gamma_\nu\frac{i}{\not{p}-\not{q}-m}\gamma_\mu\right]\Bigg) \qquad (2)$$

Now, at first sight the extra terms cancel, since setting $p \to p+k_1$ $(p \to p+k_2)$ in second (fourth) term makes that – the third (first) term. However, as these are linearly divergent integrals this operation is not legitimate.

In dim. regularization all these integrals are finite and then the cancellation should be ok !? Almost, but not quite. The problem is that γ^5 is inherently a 4-D object that cannot be continuously continued to arbitrary dimension.

To be precise we define also in d-dimension

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

It then follows that $\{\gamma^5, \gamma^\mu\} = 0$ $\mu = 0,1,2,3$ but $[\gamma^5, \gamma^\alpha] = 0$ for $\mu \notin \{0,1,2,3\}$. So we can go to d-dimensions and shift the momenta, but we must be careful with the ordering of γ^5! Thus we need to rewrite the identity (1) in d-dimensions as

$$\not{q}\gamma^5 = \gamma^5(\not{p} - \not{q} - m) + (\not{p} - m)\gamma^5 + 2m\gamma^5 + \underbrace{2\not{p}_\perp\gamma^5}_{\text{extra piece}} \qquad (3)$$

where we wrote $\not{p} = \not{p}_\parallel + \overset{\text{extra dim}}{\not{p}_\perp}$ and used $\{\not{p}_\parallel, \gamma_5\} = 0$ and $[\not{p}_\perp, \gamma^5] = 0$ (Note that q_μ and $k_{i\mu}$ are 4-d-objects, so no distinction is needed for them.)

Now the "normal pieces" written in (2) do cancel in d-reg. However, we are left with only the extra piece $\propto \not{p}_\perp$:

$$q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu} + \int \frac{d^d p}{(2\pi)^d} \mathrm{Tr}\left[\frac{i}{\not{p} - m} 2\not{p}_\perp\gamma^5 \frac{i}{\not{p} - \not{q} - m}\gamma_\nu \frac{i}{\not{p} - \not{k}_1 - m}\gamma_\mu\right] + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

To reduce the trace we observe that:

i) one of the $\not{p}$:s in propagators must be $\not{p}_\perp$ (odd. integrals)

ii) we need one term $\sim \not{q}$ and one $\sim \not{k}_1$ from propagators to be associated with γ^5 (along with γ^μ & γ^ν). [need 4 different momenta $\not{p}_i$ with γ_5]

⇒ all mass-terms vanish.

all terms with one ℓ associated with γ^5 vanish (either odd or symmetrical × anti tensor product)

Then observe that
$$\frac{1}{(p^2-m^2)((p-q)^2-m^2)((p+k_1)^2-m^2)} = \int_0^1 dx \int_0^{1-x} dy \, \frac{2!}{\left[(p-xq-yk_1)^2-\Delta^2\right]^3}$$
with
$$\Delta \equiv m^2 - x(1-x)q^2 - y(1-y)\,k_1^2$$
Shifting
$$\ell \equiv p - xq - yk_1$$
we then get
$$q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu} + 4\int_0^1 dx \int_0^{1-y} dy \int \frac{d^d\ell}{(2\pi)^d}\frac{\ell_\perp^2}{(\ell^2-\Delta)^3}\Big(\mathrm{tr}\big([\overbrace{(x-1)\not{q}+y\not{k}_1]\gamma_\nu[x\not{q}+(y-1)\not{k}_1}^{k}]\gamma_\mu\gamma_5\big)$$
$$- \mathrm{tr}\big((x\not{q}+y\not{k}_1)\gamma_\nu[x\not{q}+(y-1)\not{k}_1]\gamma_\mu\gamma_5\big)$$
$$- \mathrm{tr}\big((x\not{q}+y\not{k}_1)((x-1)\not{q}+y\not{k}_1)\gamma_\nu\gamma_\mu\gamma_5\big)\Big)$$
$$+ \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

Now
$$\int \frac{d^d\ell}{(2\pi)^d}\frac{\overset{\ell^2-\ell_\parallel^2}{\widetilde{\ell_\perp^2}}}{(\ell^2-\Delta^2)^3} = \frac{d-4}{d}\int\frac{d^d\ell}{(2\pi)^4}\frac{\ell^2}{(\ell^2-\Delta)^3}$$
$$= \frac{d-4}{2}\cdot\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(3)\,\Delta^{2-d/2}} \xrightarrow{d\to 4} \underline{\underline{-\frac{i}{32\pi^2}}} \quad \text{indep. of } m, x \text{ etc}$$

$$\Rightarrow q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu} - \frac{i}{8\pi^2}\int_0^1 dx\int_0^{1-x} dy \Big[(x-1)(y-1)\,\overset{-\mathrm{tr}(\not{k}_1\not{k}_2\gamma_\mu\gamma_\nu\gamma_5)}{\widetilde{\mathrm{tr}(\not{k}_2\gamma_\nu\not{k}_1\gamma_\mu\gamma_5)}} + xy\,\mathrm{tr}(\not{k}_1\gamma_\nu\not{k}_2\gamma_\mu\gamma_5)$$
$$-x(y-1)\,\mathrm{tr}(\not{k}_2\gamma_\nu\not{k}_1\gamma_\mu\gamma_5) - yx\,\mathrm{tr}(\not{k}_1\gamma_\nu\not{k}_2\gamma_\mu\gamma_5)$$
$$-y(x-1)\,\underset{-\mathrm{tr}(\not{k}_1\not{k}_2\gamma_\mu\gamma_\nu\gamma^5)}{\underbrace{\mathrm{tr}(\not{k}_1\not{k}_2\gamma_\nu\gamma_\mu\gamma^5)}} - xy\,\mathrm{tr}(\not{k}_2\not{k}_1\gamma_\nu\gamma_\mu\gamma_5)\Big]$$

$$= 2mT_{\mu\nu} - \frac{i}{8\pi^2} \cdot 4i\epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta \cdot \int_0^1 dx \int_0^{1-x} dy \Big((1-\overset{y}{\cancel{x}})(y-1) + \overset{0}{\cancel{xy}} + \cancel{x}(y-1) - \underset{0}{\cancel{yx}}$$

$$+ y(\underset{0}{\cancel{x}}-1) - \overset{0}{\cancel{xy}} \Big) + \left(\begin{matrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{matrix} \right)$$

$$= \int_0^1 dx \int_0^{1-x} (\cancel{y} - 1 - \cancel{y}) = -\int_0^1 dx (1-x) = -1 + \frac{1}{2} = -\frac{1}{2}$$

$$= 2mT_{\mu\nu} - \frac{1}{2\pi^2} \epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta$$

↑

anomalous part. Independent of both m and q^2.

Physical significance. Observe that to 1-loop, $T_{\mu\nu\lambda}$ gives the amplitude to create 2 photons out of the current $j^{\mu}_{5}(z)$:

$$\int d^4z\, e^{+iq\cdot z} \langle k_{1,\mu}; k_{2,\nu} | j_{5\lambda}(z) | 0\rangle$$

$$= (-ie)^2 \epsilon_1^{\mu} \epsilon_2^{\nu}\, i \int d^4z\, d^4x_1\, d^4x_2\, e^{+iq\cdot z - ik_1x_1 - ik_2\cdot x_1} \langle 0 | T(j_\mu(x_1) j_\nu(x_2) j_{5\lambda}(z)) | 0\rangle$$

$$= -e^2 (2\pi)^4 \overset{4}{\delta}(q-k_1-k_2)\, \epsilon_1^{\mu} \epsilon_2^{\nu}\, T_{\mu\nu\lambda}$$

Then, our computation shows that

$$\int d^4z\, e^{iq\cdot z} \langle k_{1,\mu}; k_{2}\nu | \partial^{\lambda} j_{5\lambda}(z) | 0\rangle = e^2 (2\pi)^4 \delta^4(q-k_1-k_2)\, \epsilon_1^{\mu} \epsilon_2^{\nu}\, q^{\lambda} T_{\mu\nu\lambda}$$

$$= -e^2 (2\pi)^4 \delta^4(q-k_1-k_2)\, \epsilon_1^{\mu} \epsilon_2^{\nu} \left(2m T_{\mu\nu} - \frac{1}{2\pi^2} \epsilon_{\mu\nu\alpha\beta} k_1^{\alpha} k_2^{\beta} \right)$$

The first term on r.h.s follows from using the classical current equation on l.h.s, but the second term is new. However, when we add the anomaly in the current, we get a new contribution on the l.h.s.:

$$\int d^4z\, e^{iq\cdot z} \langle k_{1,\mu}; k_{2,\nu} | -\frac{e^2}{16\pi^2} \underbrace{\epsilon^{\alpha\beta\rho\sigma} \hat{F}_{\alpha\beta} \hat{F}_{\rho\sigma}}_{= 4\epsilon^{\alpha\beta\rho\sigma} \partial_\alpha \hat{A}_\beta \partial_\rho \hat{A}_\sigma} | 0\rangle$$

$$\rightsquigarrow$$

$$= -\frac{e^2}{4\pi^2} \epsilon^{\alpha\beta\rho\sigma} \int d^4z\, e^{iq\cdot z} \langle 0 | a_{k_1\mu} a_{k_2\nu} \int \frac{d^3q_1}{(2\pi)^3 2\omega_{q_1}} \int \frac{d^3q_2}{(2\pi)^3 2\omega_{q_2}} (-iq_1^{\alpha} \epsilon_{\beta}^{q_1})(-iq_2^{\rho} \epsilon_{\sigma}^{q_2})\, e^{-i(q_1+q_2)\cdot z} \underset{+\text{cross}}{} \times a^{\dagger}_{q_1\alpha} a^{\dagger}_{q_2\beta} + \ldots | 0\rangle$$

↑ terms giving 0

$$= (2\pi)^4 \delta^4(q-k_1-k_2) \cdot \underbrace{\frac{e^2}{2\pi^2} \epsilon^{\alpha\beta\rho\sigma} k_1^{\alpha} k_2^{\rho} \epsilon_1^{\beta} \epsilon_2^{\sigma}}$$

$$= +\frac{e^2}{2\pi^2} \epsilon_{\alpha\beta\rho\sigma} k_1^{\alpha} k_2^{\rho} \epsilon_1^{\beta} \epsilon_2^{\sigma} \quad \%.$$

JUST THE RIGHT THING!

Chiral anomalies in practice. Pions & their decays

First look into symmetries of QCD-Lagrangian

$$\mathcal{L}_{QCD} = \bar{u}(i\not{D}-m_u)u + \bar{d}(i\not{D}-m_d)d \quad \left(-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}\right)$$

If $m_u = m_d$ $\mathcal{L}_{QCD}$ is symmetric under unitary transformations

$$Q_L = \begin{pmatrix}u\\d\end{pmatrix}_L \to U_L \begin{pmatrix}u\\d\end{pmatrix}_L \quad \& \quad Q_R = \begin{pmatrix}u\\d\end{pmatrix}_R \to U_R \begin{pmatrix}u\\d\end{pmatrix}_R$$

where (unitary) $U_{L,R}$ include separate $U(1)_{L,R}$ & $SU(2)_{L,R}$- symmetries.

(overall phase) ↳ ok even when $m_u \neq m_d$

(special unitary) only for $m_u = m_d$

The symmetry currents are

$$j^\mu_{L,R} = \bar{Q}_{L,R}\gamma^\mu Q_{L,R}$$

and

$$j^{\mu a}_{L,R} = \bar{Q}_{L,R}\gamma^\mu\tau^a Q_{L,R}$$

where $\tau^a = \sigma^a/2$. The sums of these are the conserved baryon number and isospin currents:

$$j^\mu = \bar{Q}\gamma^\mu Q \quad \text{and} \quad j^{\mu a} = \bar{Q}\gamma^\mu\tau^a Q$$

(conserved) (conserved when $m_u = m_d$)

These lead to the known baryon and isospin conservation laws of QCD. The differences on the other hand give axial vector currents

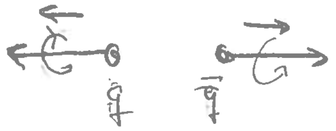
$$j^\mu_5 = \bar{Q}\gamma^\mu\gamma^5 Q \quad \text{and} \quad j^{\mu a}_5 = \bar{Q}\gamma^\mu\gamma^5\tau^a Q$$

These do not correspond to any obvious symmetry of strong interactions. Nambu and Jona-Lasinio suggested that these are (assuming the massless limit $m_u = m_d = 0$) accurate symmetries of QCD, which are spontaneously broken. If this is so there should be 4 (nearly) massless Goldstone bosons associated with these symmetries. (Nearly, because these are not really exact symmetries.)

3 such candidates exist, namely the pions. 4th does not, and we shall soon learn why.

Spontaneous breaking of chiral symmetry (qualitatively)

In QCD quarks & antiquarks have strong attractive interactions. In analogy with superconductivity this attraction can create a condensate of quark-antiquark pairs in vacuum. Such condensate should have zero momentum and zero angular momentum;



; same helicity

Since particles and antiparticles of same helicity have opposite chiralities such condensate has a nonzero chiral charge

$$\langle 0|\bar{Q}Q|0\rangle = \langle 0|\bar{Q}_L Q_R + \bar{Q}_R Q_L|0\rangle \neq 0$$

Thus this vacuum breaks the $U_L \otimes U_R$-symmetry down to $U = U_L = U_R$, just as we desired $\Rightarrow$ expect 4 goldstones.

Now $j_5^{\mu a}$ and j_5^{μ} are odd under parity. Since pions are also (known to be) odd under parity, they remain good candidates for axial isotriplet GB's. Let's suppose they are. They can then be created from $j_5^{\mu a}$: Let us parametrize

$$\langle 0 | j_5^{\mu a}(x) | \pi^b(p) \rangle = -i p^\mu \underset{\substack{\uparrow \\ \text{pion decay constant.}}}{f_\pi} \delta^{ab} e^{-ip\cdot x}$$

Determination of f_π: $\pi^\pm \to \ell^\pm \nu_\ell$-decay.

The effective Lagrangian for semileptonic weak interactions read

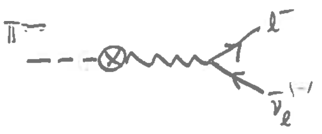
$$\mathcal{L} = \frac{G_F}{\sqrt{2}} \left(\bar{\ell}\, \gamma^\mu (1-\gamma_5) \nu_\ell\right) \left(\bar{u}\, \gamma_\mu (1-\gamma_5)\, d\right)$$

Noting that

$$\bar{u}\, \gamma_\mu (1-\gamma_5)\, d = \bar{Q}\, \gamma^\mu (1-\gamma^5) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Q = \bar{Q}\, \gamma^\mu (1-\gamma_5)(\tau_1 + i\tau_2)\, Q$$

$$= j^{\mu 1} + i j^{\mu 2} - j_5^{\mu 1} - i j_5^{\mu 2} = \sqrt{2}\left(j^{\mu -} - j_5^{\mu -}\right)$$

Thus the π^--decay



$$\sim \langle \bar{\nu}_\ell \ell \underbrace{| W^- \rangle \langle W}_{+1/M_W^2} | \pi^- \rangle \quad \leftarrow \frac{g}{2\sqrt{2}} \sqrt{2} \langle 0 | j_5^{\mu -} | \pi(q) \rangle$$

$$\sim \langle \bar{\nu}_\ell \ell | j_L^\mu | 0 \rangle \langle 0 | j_5^{\mu -} | \pi \rangle$$

$$= \frac{i g^2}{8 M_W^2} \sqrt{2} \left(-i q^\mu f_\pi\right) \bar{u}_\ell(p) \gamma^\mu (1-\gamma_5) v_\nu(k)$$

$$= \underline{G_f f_\pi \bar{u}(\not{p}) \not{q} (1-\gamma_5) v(k)}$$

Then straightforwardly

$$\Gamma_{\pi^- \to \ell \bar{\nu}} = \frac{\overline{|M|^2}}{16\pi m_\pi^3} \lambda^{1/2}(m_\pi^2, m_\nu^2, 0) \quad ; \qquad q \equiv (m_\pi, \vec{0}) \ ; \ k = (K, 0, 0 - p) \qquad p = (E_\ell, 0, 0, -P)$$

$$\overline{|M|}^2 = \frac{1}{4} G_F^2 f_\pi^2 \, \mathrm{Tr}\left(\not{p}\not{q}(1-\gamma_5)\not{k}\not{q}(1-\gamma_5)\right) = G_F^2 f_\pi^2 \left(2k\cdot q\, p\cdot q - k\cdot p\, q^2\right)$$

$$= G_F^2 f_\pi^2 m_\pi^2 \left(2 k_0 p_0 - k\cdot p\right) = G_F^2 f_\pi^2 m_\pi^2 \, p (p_0 - p) .$$

$$\text{Now : } \left.\begin{aligned} p &= \frac{\lambda^{1/2}(m_\pi^2, m_\ell^2, 0)}{2 m_\pi} = \frac{m_\pi^2 - m_\ell^2}{2 m_\pi} \\ p_0 &= \sqrt{m_\ell^2 + p^2} = \frac{m_\pi^2 + m_\ell^2}{2 m_\pi} \end{aligned}\right\} \Rightarrow p_0 - p = \frac{m_\ell^2}{m_\pi}$$

$$\Rightarrow \underline{\Gamma_{\pi^- \to \ell^- \bar{\nu}}} = \frac{1}{16\pi m_\pi^3} G_F^2 f_\pi^2 \cancel{m_\pi^2} \frac{(m_\pi^2 - m_\ell^2)^2}{2\cancel{m_\pi}} \cdot \frac{m_\ell^2}{\cancel{m_\pi}} = \underline{\frac{G_F^2 f_\pi^2 m_\pi m_\ell^2}{16\pi} \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2}$$

One observes that π^- dominantly decays to muon.

$$\frac{\Gamma_{\pi^- \to e^- \bar{\nu}_e}}{\Gamma_{\pi^- \to \mu^- \bar{\nu}_\mu}} = \frac{m_e^2}{m_\mu^2} \frac{\left(1 - \frac{m_e^2}{m_\pi^2}\right)^2}{\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2} \simeq 10^{-4}$$

Putting in the numbers: $\tau_{\pi^-} = 2.6 \cdot 10^{-8}$ sec, $m_{\pi^-} = 140$ MeV and $m_\mu = 106$ MeV, along with $G_F = 1.116 \cdot 10^{-11} \text{MeV}^{-2}$ one finds

$$\boxed{f_\pi = 93 \text{ MeV}}$$

Diggression: If we restore quark-masses in the eom, we get

$$i\not{D}Q = \bar{m}Q \qquad \& \qquad -i\bar{Q}\overleftarrow{\not{D}} = \bar{Q}\bar{m}$$

where

$$\bar{m} \equiv \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$$

The single-flavour axial current relation now simply generalizes to

$$\partial_\mu j_5^{\mu a} = i\bar{Q}\{\bar{m},\tau^a\}\gamma^5 Q$$

This then implies

$$\begin{aligned}\langle 0|\partial_\mu j_5^{\mu a}(0)|\pi^b(q)\rangle &= -q^2 f_\pi \delta_{ab} \\ &= i\underbrace{\langle 0|\bar{Q}_i \overbrace{\{\bar{m},\tau^a\}^{ij}}^{\text{matrix}} \gamma^5 Q_j|\pi^b\rangle}_{\text{has to be } \sim(\tau^b)_{ij} \times \text{invariant } M^2} \\ &\equiv \mathrm{Tr}(\{\bar{m},\tau^a\},\tau^b)\, M^2 \\ &= \tfrac{1}{2}\delta^{ab}(m_a+m_b)M^2\end{aligned}$$

(this is the condensate)

Equating q^2 with π-mass we get

$$m_\pi^2 = (m_u+m_d)\frac{M^2}{f_\pi}$$

So, if $m_{u,d}=0$, then m_π is strictly zero. However, when $m_{ud}\neq 0$ $m_\pi \neq 0$ and not necessarily related to their size. In fact it turns out $M \sim 400\,\text{MeV}$

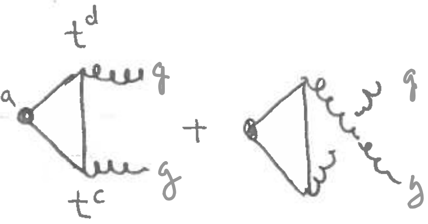
$$\Rightarrow \quad \frac{m_\pi}{100\,\text{MeV}} \sim \frac{2\sqrt{(m_u+m_d)/\text{MeV}}}{5}$$

That is a few-MeV m_u & m_d suffice to 'seed' the observed m_π. Thus most of the mass of pion comes from effective quark masses arising from the SSB. Note that we do not need $m_u = m_d$ to have the effective mass for pion. Thus the isospin symmetry of QCD does not come from the fundamental (weak) symmetry linking u and d.

- Now, is there a fourth GB associated with j_5^μ-conservation? The answer is no, because this current is anomalous.

Indeed, while

$$\partial_\mu j_5^{\mu a} = \tau^a$$



$$= -\frac{g^2}{16\pi^2}\tilde{F}^c_{\mu\nu}F^{\mu\nu,d}\cdot\frac{1}{2}\mathrm{Tr}\Big[\overbrace{\tau^a,\overbrace{\{t^c,t^d\}}^{\propto\delta^{cd}}}^{\propto \mathrm{Tr}\,\tau^a = 0}\Big]$$

SU(2)-isospin generator

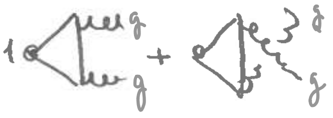
SU(3)-generators

QCD-generators

$$= 0$$

so the axial isovector current is anomaly-free. However $\left(d = \sum_f \frac{1}{2}\mathrm{Tr}(1\cdot\{t^c,t^d\}) = \frac{1}{2}n_f \right)$

$$\partial_\mu j_5^\mu = 1$$



$$= -\frac{g^2 n_f}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F^a_{\mu\nu}F^a_{\alpha\beta}$$

because $\mathrm{tr}[1\, t^c t^d] = \delta^{cd} \neq 0$, and each quark-flavour adds the same anomaly. Thus there is no isosinglet axial symmetry and hence no light isosinglet pseudoscalar. (There is a pseudoscalar, η, which is much heavier $\sim$ 400 MeV.)

Neutral pion decay

We are interested in the amplitude

$$\langle \gamma(k_1,\epsilon_1)\gamma(k_2,\epsilon_2)|\pi^0(q)\rangle \equiv (2\pi)^4 i\,\delta^4(q-k_1-k_2)\,\overbrace{\epsilon^\mu(k_1)\epsilon^\nu(k_2)\,T_{\mu\nu}(k_1,k_2,q)}^{=\mathcal{M}_{\pi\to\gamma\gamma}}$$

where

$$T_{\mu\nu} = e^2\int d^4x_1\,d^4x_2\,e^{ik_1\cdot x_1+ik_2\cdot x_2}\langle 0|T(j_\mu(x_1)\,j_\mu(x_2))|\pi^0(q)\rangle$$

This is related to amplitude we encountered earlier:

$$T^a_{\mu\nu\lambda}(k_1,k_2q) = \int d^4x\,d^4y\,e^{ik_1\cdot x+ik_2\cdot x}\langle 0|T(j^a_{5\lambda}(x)\,j_\mu(y)\,j_\nu(0))|0\rangle.$$

Indeed

isospin τ^3. (Not weak)

due to anomaly. Remains finite when $q^2\to 0$!

- $$\langle \gamma(k_1,\epsilon_1)\gamma(k_2,\epsilon_2)|\partial^\lambda j^3_{5\lambda}|0\rangle = \underbrace{e^2}_{\text{anomaly}}(2\pi)^4 i\,\delta^4(q-k_1-k_2)\,\epsilon^\mu(k_1)\epsilon^\nu(k_2)\,\underbrace{q^\lambda T^3_{\mu\nu\lambda}(k_1,k_2,q)}.$$

Let us rewrite the l.h.s as (pion resonance will dominate the amplitude)

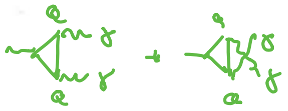
- $$\underbrace{\langle\gamma\gamma|\pi^0\rangle}_{\text{the unknown}}\underbrace{\langle\pi^0|\partial^\lambda j^3_{5\lambda}|0\rangle}_{\text{easy}} + \ldots$$

Let us for the moment assume that π is GB in this calculation and use $\langle\pi^0|j^3_{5\lambda}|0\rangle = -iq_\lambda f_\pi$

$$\Rightarrow \text{l.h.s} = \langle\gamma\gamma|\pi^0\rangle\,\frac{i}{\cancel{q^2}}(-i\cancel{q^2}f_\pi) \Rightarrow \mathcal{M}^{(q^2\to 0)}_{\pi\to\gamma\gamma} = \frac{e^2}{f_\pi}\left(q^\lambda T^3_{\mu\nu\lambda}\right)_{\text{anomaly}}\epsilon^\mu_1\epsilon^\nu_2$$

That is: if there were no anomaly, then $\mathcal{M}_{\pi\to\gamma\gamma}(0) = 0$, and π^0 could not decay to two photons.

Now, even though there was no QCD-anomaly in $j_5^{\mu 3}$, there is one in QED. Indeed, quite easily



$$\left(q^\lambda T^3_{\mu\nu\lambda}\right)_{\text{anomaly}} = -\frac{1}{2\pi^2}\epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta \, \frac{1}{2}\sum_f \operatorname{tr}\left(\tau^3\{Q,Q\}\right)$$

where $\tau^3 = \frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and $Q = \begin{pmatrix}2/3&0\\0&-1/3\end{pmatrix} \Rightarrow \operatorname{Tr}(\tau^3\overbrace{\{Q,Q\}}^{2Q^2}) = \operatorname{Tr}\begin{pmatrix}4/9&0\\0&-1/9\end{pmatrix} = 1/3.$

We then get

$$\mathcal{M}^{(q^2\to 0)}_{\pi\to\gamma\gamma} = -\frac{e^2 n_{fl}}{12\pi^2 f_\pi}\epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta \epsilon_1^\mu \epsilon_2^\nu$$

It is then simple to compute

$$\Gamma_{\pi^0\to\gamma\gamma} = \left(\frac{\alpha n_{fl}}{3\pi f_\pi}\right)^2 \underset{\substack{\uparrow\\ \text{symm. factor}}}{\frac{1}{2}} \sum_{\text{pol}}(\epsilon_1^\mu\epsilon_1^{\mu'})\overbrace{(\epsilon_2^\nu\epsilon_2^{\nu'})}^{-g^{\nu\nu'}+\text{ term giving zero}} k_1^\alpha k_2^\beta k_1^{\alpha'} k_2^{\beta'} \epsilon_{\mu\nu\alpha\beta}\epsilon_{\mu'\nu'\alpha'\beta'} \cdot \frac{\overbrace{(m_\pi/2)}^{P_{cm}}}{8\pi m_\pi^2}$$

$$= \left(\frac{\alpha n_{fl}}{3\pi f_\pi}\right)^2 \frac{1}{2}\Big(\underbrace{\epsilon_{\mu\nu\alpha'\beta'}\epsilon^{\mu\nu}{}_{\alpha\beta}}_{2(g_{\alpha'\beta}g_{\alpha\beta'}-\cancel{g_{\alpha\alpha'}g_{\beta\beta'}}^{0})}\Big) k_1^\alpha k_2^\beta k_1^{\alpha'} k_1^{\beta'} = \left(\frac{\alpha n_{fl}}{3\pi f_\pi}\right)^2 (\underbrace{k_1\cdot k_2}_{\approx (\frac{m_\pi^2}{2})})^2$$

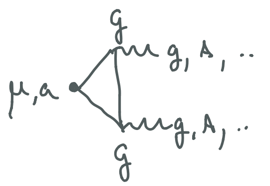
$$= \frac{\alpha^2}{64\pi^3}\left(\frac{n_{fl}}{3}\right)^2 \frac{m_{\pi^0}^3}{f_\pi^2}$$

Now, that f_π was already measured from $\pi^\pm$-decay, this is a firm prediction. Putting in the numbers: $f_\pi = 93$ MeV, $m_{\pi_0} = 135$ MeV, $\alpha = 1/137$ one gets

$$\Gamma_{\pi^0\to\gamma\gamma} \approx 7{,}64\cdot 10^{-7}\,\text{MeV} \Rightarrow \tau_{\pi^0\to\gamma\gamma} \simeq 8{,}6\cdot 10^{-17}\,\text{sek} \quad \left(\begin{matrix}\text{PDG:}\\ \tau_\pi = 8{,}52\cdot 10^{-17}\,\text{sek}\end{matrix}\right)$$

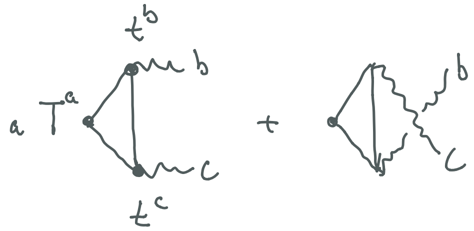
Anomaly cancellation in the SM (Adler–Bell–Jackiw – anomalies)

Chirality of $SU(2)_L$ theory poses the additional problem with anomalies. The whole structure of a local gauge theory is based on exact gauge-invariance. Violation of the current conservation violates this structure. Diagrams of form



$$+ \cdot\cdot \Rightarrow j^{\mu a} = \bar{\psi}\gamma^\mu P_L t^a \psi \quad \text{anomalous}$$

destroy Ward identities and hence renormalizability. Also this leads to non-cancellation of unphysical states and hence loss of unitarity. The only way to avoid this is to have the diagram vanishing. How does this happen? More general calculation with generators included gives



$$\sim A^{abc} = \mathrm{tr}\left[T^a, \{t^b, t^c\}\right]$$

For example if $t^b, t^c \in SU(2) \Rightarrow \{t^b, t^c\} = \frac{1}{2}\delta^{bc} \Rightarrow A^{abc} = \frac{1}{2}\delta^{bc}\,\mathrm{Tr}(T^a)$

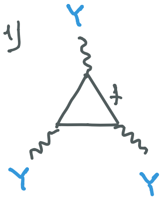
This vanishes whenever T^a is non-Abelian generator, However, if $T^a = Q$, we get

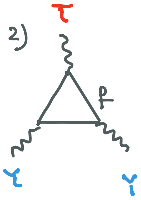
$$A^{Qbc}_{SM} = \frac{1}{2}\delta^{bc}\,\mathrm{Tr}(Q) = \frac{1}{2}\delta^{bc}\Big(\underbrace{N_c\left(\tfrac{2}{3}-\tfrac{1}{3}\right)}_{\text{quarks}} + \underbrace{0-1}_{\text{leptons}}\Big) = \frac{N_c}{3} - 1 \overset{N_c \equiv 3}{=} 0$$

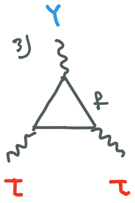
(both in fundamental representation

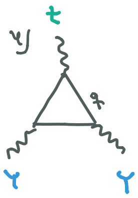
That is SM is anomaly-free (renormalizable, etc) only if there are 3 families of quarks.

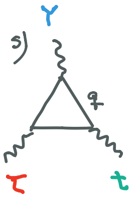
There are many more potentially anomalous diagrams in the SM:

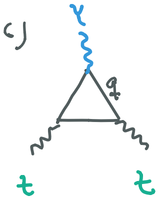
1)


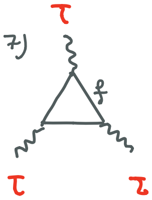
2)


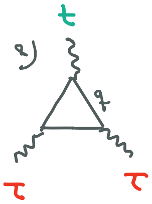
3)


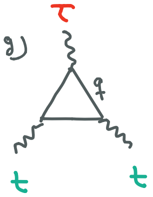
4)


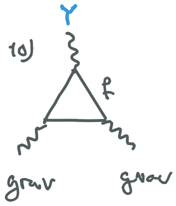
5)


6)


7)


8)


9)


10)


Here τ refers to SU(2), Y to hypercharge, and t to SU(3).

- $tr_f(\tau^a) = tr_q(t^a) = 0 \Rightarrow \mathcal{A}^{2,4,5,9} \to 0$

- $\mathcal{A}^3 \propto \frac{1}{2}\delta^{bc}\,\mathrm{Tr}_f Y$; $\mathrm{Tr}_f Y = -2\overset{e}{\underset{\uparrow}{\left(-\frac{1}{2}\right)}} \underset{\uparrow}{+} \overset{e_R}{(-1)} - 3\left(\overset{L}{2\left(\frac{1}{6}\right)} - \overset{u_R}{\frac{2}{3}} - \overset{d_R}{\left(-\frac{1}{3}\right)}\right) = 0$

  $P_L = \frac{1}{2}(1-\gamma_5)$ $P_R = \frac{1}{2}(1+\gamma_5)$

- $\mathcal{A}^{10} \propto \mathrm{Tr}_f Y$

  $\mathcal{A}^6 \propto \mathrm{Tr}Y \cdot \mathrm{Tr}\{t^b, t^c\}$

$$\Rightarrow \mathcal{A}^{3,6,10} = 0$$

- $\mathrm{Tr}(\tau^a, \{\tau^b, \tau^c\}) = \frac{1}{2}\delta^{bc}\mathrm{Tr}(\tau^a) = 0 \quad \Rightarrow \mathcal{A}^7 = 0 \quad (\mathcal{A}^8 = 0)$

- $\mathrm{Tr}_f Y^3 = -2\left(-\frac{1}{2}\right)^3 + (-1)^3 - 3\left[2\left(\frac{1}{6}\right)^3 - \left(\frac{2}{3}\right)^3 - \left(\frac{1}{3}\right)^3\right] = 0 \Rightarrow \mathcal{A}^1 = 0$

⇒ All anomalies cancel. SM. OK.

III) B-violation in SU(2)×U(1)-theory

⊛ **Baryons** : p, n, … three-quark bound states

(qqq)

One sets baryon numbers

$$B(q) \equiv \frac{1}{3} \quad ; \quad B(\bar{q}) = -\frac{1}{3}$$

for quarks, so that

$$B(p) = B(n) = 1, \ldots \qquad B(\bar{p}) = -1 \ldots$$

Similarly, one can define lepton number

$$L(e^-) = 1 \quad , \quad L(e^+) = -1 \quad \text{etc.}$$

(at classical level)

In standard model both B and L are conserved. This can be attributed to global nongauged symmetries in the lagrangean. (*later we do differently)
For example, we can make the transform

$$\psi_q \longrightarrow e^{i\overset{\text{constant}}{\overset{\downarrow}{\alpha_B}}}\psi_q \simeq \psi_q + i\alpha_B\psi_q \tag{31}$$

for all quark-fields in the lagrangian. Then obviously

$$\delta\mathcal{L} = \delta\left(\bar{\psi}_q \not\partial \psi_q - \bar{\psi}_q M \psi_q\right) = 0$$

But then, (Noethers theorem)

Euler-Lagrange $\frac{\delta\mathcal{L}}{\delta\psi} = \partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}$

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\psi_q}\delta\psi_q + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi_q)}\delta(\partial_\mu\psi_q) \overset{\downarrow}{=} \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi_q)}\delta\psi_q\right) = 0$$

Now remembering that our $\delta\psi_q = i\overset{\text{constant}}{\alpha_B}\psi_q$, we get

$$\partial_\mu j^\mu_B = \partial_\mu(\underset{\substack{\uparrow \\ \text{conserved "baryonic" current.}}}{\bar{\psi}_q\gamma^\mu\psi_q}) = 0 \qquad (32)$$

In fact, because of quark mixing only the total B-current is conserved. For leptons there is no mixing and each lepton flavour is separately conserved.

One can also similarly show that (in a massless theory the) axial current is conserved at classical level:

$$\psi_q \to e^{i\gamma_5\theta}\psi_q \qquad (33)$$

$$\delta\mathcal{L} = 0 \quad \Rightarrow \quad \partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi) = \underline{\partial_\mu j^\mu_5 = 0}$$

Combining we get

$$\begin{aligned} \partial_\mu j^\mu_L &= 0 \qquad \text{l-chiral current} \\ \partial_\mu j^\mu_R &= 0 \qquad \text{r-chiral current} \end{aligned} \qquad (34)$$

where

$$\begin{aligned} j^\mu_L &\equiv \bar{\psi}_{qL}\gamma^\mu\psi_{qL} = \frac{1}{2}\bar{\psi}_q\gamma^\mu(1-\gamma_5)\psi_q \\ j^\mu_R &\equiv \bar{\psi}_{qR}\gamma^\mu\psi_{qR} = \frac{1}{2}\bar{\psi}_q\gamma^\mu(1+\gamma_5)\psi_q \end{aligned}$$

Chiral fields

Given current conservation $\partial_\mu j^\mu = 0$ it is easy to see that the total charge

$$Q \equiv \int d^3x \, J^0 \qquad (35)$$

is also conserved:

$$\frac{dQ}{dt} = \int d^3x \, \partial_0 J^0 = \int d^3x \, \partial_\mu J^\mu = 0, \qquad (36)$$

So, from (23) we get that, apparently, in MSM total L- and R-chiral baryon numbers are <u>conserved</u>.

$$\frac{dQ_L}{dt} = 0$$

$$\frac{dQ_R}{dt} = 0 \, .$$

This is not the whole story, however, because the axial currents are <u>anomalous</u> in gauge theories.

As a result one gets instead

$$\boxed{\partial_\mu j^\mu_{B_L} = \underset{\substack{\uparrow \\ \text{\# of flavours}}}{N_{fl}} \frac{-g^2}{32\pi^2} \underset{\substack{\uparrow \\ \tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}}}{\tilde{F}_{\mu\nu}} F^{\mu\nu}} + U(1)\text{-parts} \qquad (37)$$

$$\Rightarrow \quad \frac{dQ_{B_L}}{dt} \neq 0 \; !$$

Buckle up, we now derive this result!

3.1) Axial anomaly, Fujikawas method

A more fundamental way to derive current conservation laws is do check for invariance of the path integral (which defines the whole theory; also global topological properties, which will be missed in local lagrangean approach) under field redefinitions (transformations of variable)!

Consider a chiral theory with action:

Field strength tensor

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

$$S = \int d^4x \left(\bar\Psi i \not{D} \Psi - \frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} \right) \qquad (38)$$

gauge-field

$$D_\mu \equiv \partial_\mu - ig\tau \cdot A_\mu$$

group generator ; $\begin{cases} U(1) : \tau = 1. \\ SU(2) : \tau_i = \frac{1}{2}\sigma_i \end{cases}$

The path integral is

$$\boxed{\int \mathcal{D}\bar\Psi \mathcal{D}\Psi \, e^{iS},} \qquad (29)$$

where

$$\begin{aligned} \mathcal{D}\bar\Psi &= \prod_{\vec x} d\bar\Psi(\vec x) \\ \mathcal{D}\Psi &= \prod_{\vec x'} d\Psi(\vec x) \end{aligned} \qquad (40)$$

Perform a local field-redefinition (transform of variables)

$$\begin{cases} \Psi(x) \rightarrow e^{i\theta(x)\gamma_5} \Psi(x) \equiv \Psi' \\ \bar\Psi \rightarrow \bar\Psi e^{i\theta(x)\gamma_5} \equiv \bar\Psi' \end{cases} \qquad (41)$$

Then

$$S \rightarrow S + \int d^4x \, \bar\Psi e^{i\theta\gamma_5} \overbrace{i\gamma^\mu i\partial_\mu\theta\gamma_5}^{-\partial_\mu\theta\gamma^\mu\gamma_5} e^{i\theta\gamma_5} \Psi$$

$$= \underline{S + \int d^4x \, \theta(x) \partial_\mu J^\mu_5} \qquad (42)$$

(our old result!)

So action is conserved if $\partial_\mu J_5^\mu = 0$. However, the measure changes

$$\mathcal{D}\bar\Psi\,\mathcal{D}\Psi \to \det\left(e^{i\theta(x)\gamma_5}\right)^2 \mathcal{D}\psi\,\mathcal{D}\psi \qquad (43)$$

Because of infinities involved with the measure (in fact ill-defined) one must compute this det( ) with care. It will turn out to be finite and nonzero. (anomaly)

It will be convenient to define a complete set of eigenstates of $\not{D}$:

$$\not{D}\phi_n = \lambda_n \phi_n \qquad (44)$$

such that

(over spin and group indices)

$$\mathrm{Tr}\int d^4x\, \phi_n^\dagger(x)\,\phi_m(x) = \delta_{nm} \qquad (45)$$

Then any field configuration ψ and $\bar\psi$ can be written as

$$\psi(x) = \sum_n \overset{\text{grassmann}}{a_n}\, \underset{\text{real scalar}}{\phi_n(x)} \;;\quad \bar\psi(x) = \sum_n \bar b_n\, \phi_n^\dagger(x) \qquad (46)$$

and therefore the measure may be rewritten as

$$\mathcal{D}\bar\psi\,\mathcal{D}\psi \longrightarrow \prod_n d\bar b_n \prod_m da_m . \qquad (47)$$

(both lead to integration over all possible field configurations)

Now, in the chiral transform

$$\psi' = \sum_n a'_n \phi_n(x) \equiv e^{i\theta\gamma_5}\psi \equiv \sum_n a_n e^{i\theta\gamma_5}\phi_n(x)$$

$$\Rightarrow \quad a'_n = \sum_m a_m \mathrm{Tr}\underbrace{\int d^4x\, \phi_n^{\dagger}(x)\, e^{i\theta\gamma_5}\, \phi_m(x)}_{\equiv C_{nm}} \qquad (48)$$

*

Box: Grassman variables.

Our numbers a_n and $\bar{b}_n$ are Grassman variables, which anti-commute

$$\{a_i, a_j\} = 0 \qquad (49)$$

It then follows that only nontrivial N-dimensional G-integral is

$$\int da_1 \ldots da_N \; a_1 \ldots a_N \qquad (49a)$$

Proof. follows from requirement of translational invariance $\int d\theta\, \xi(\theta) \equiv \int d\theta\, \xi(\theta + c)$. Exercise!

i.e. in fact $\int da_i = \frac{\partial}{\partial a_i}$! (Exercise)

then, since under a change of variables

$$a_i \rightarrow a'_i = \sum_j^N C_{ij} a_j$$

$$\Rightarrow \prod_i^N a_i \rightarrow \prod_i^N a'_i = \prod_{i=1}^N \sum_{j=1}^N C_{ij} a_j$$

$$= \sum_{\text{perm}} \epsilon^{i_1 \ldots i_N} C_{i_1 1} \cdots C_{i_N N} \prod_{i=1}^N a_j = \det C \prod_{i=1}^N a_j$$

f.ex.: $(C_{11}a_1 + C_{12}a_2)(C_{21}a_1 + C_{22}a_2)$

$$= C_{11}a_1C_{21}a_2 + C_{12}\underbrace{a_2C_{21}a_1}_{-a_2a_1} + 0.$$

$$= (C_{11}C_{22} - C_{12}C_{21})\, a_1a_2$$

$$= \det C\; a_1a_2$$

Thus the invariance of (49a) under change of variables requires

$$\prod_{i=1}^{N} da_i' = (\det C)^{-1} \prod_{i=1}^{N} da_i \tag{50}$$

| *

In (48) we have infinite product, but that's ok, and (50) holds with $\lim_{N\to\infty}$.

Next consider an infinitesimal chiral transform

$$e^{i\theta\gamma_5} \simeq 1 + i\theta\gamma_5 \tag{51}$$

Then

$$(\det C)^{-1} \simeq \det\left(\delta_{nm} + i\,\mathrm{Tr}\int d^4x\,\theta(x)\,\overbrace{\phi_n^{\dagger}(x)\gamma^5\phi_m(x)}^{\equiv A_{nm}}\right)^{-1}$$

$$\simeq \det\left(\delta_{nm} - i\,\mathrm{Tr}\int d^4x\,\theta(x)\,A_{nm}\right)$$

$$\simeq \exp\Big(-i\underbrace{\sum_n \mathrm{Tr}\int d^4x\,\theta(x)\,A_{nn}}_{\equiv \int d^4x\,A(x)\,\theta(x)}\Big),$$

where in the last step I used the identity

$$\boxed{\det C = e^{\mathrm{Tr}\ln C}.} \quad (C \text{ hermitian}) \tag{52}$$

It remains to compute A(x). This is divergent, and we must regularize it in a gauge invariant way. Because we are explicitly using eigenstates of $\not{D}$ (a gauge invariant operator) we can do this easily:

$$A(x) = \lim_{M\to\infty} \mathrm{Tr} \sum_n \phi_n^\dagger(x)\, \gamma_5\, e^{-(\lambda_n/M)^2} \phi_n(x)$$

$$= \lim_{M\to\infty} \mathrm{Tr} \sum_n \phi_n^\dagger(x)\, \gamma_5\, e^{-(\not{D}/M)^2} \phi_n(x)$$

Now, to re-evaluate the effet of operator $\not{D}$, switch to k-representation;

$$\phi_n(x) = \langle x | n \rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \langle k | n \rangle .$$

$$\Rightarrow A(x) = \lim_{M\to\infty} \mathrm{Tr} \sum_n \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} e^{ik'\cdot x} \langle n | k' \rangle \gamma^5 e^{-(\not{D}/M)^2}$$

$$\times e^{-ik\cdot x} \langle k | n \rangle$$

using completeness of $\{n\}$ this becomes

$$= \lim_{M\to\infty} \mathrm{Tr} \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} \gamma_5 \, \underbrace{e^{-(\not{D}/M)^2} \; e^{-ik\cdot x}}$$

(this is an operator acting on x)

Now observe that

$$\not{D}^2 = D_\mu D_\nu \gamma^\mu \gamma^\nu = \frac{1}{2} D_\mu D_\nu \Big(\underbrace{\{\gamma^\mu, \gamma^\nu\}}_{= 2g_{\mu\nu}} + \underbrace{[\gamma^\mu, \gamma^\nu]}_{\text{antisymm.}} \Big)$$

$$= D^\mu D_\nu + \frac{1}{4} \underbrace{[D_\mu, D_\nu]} [\gamma^\mu, \gamma^\nu]$$

$$= [(\partial_\mu - ig\tau\cdot A_\mu), (\partial_\nu - ig\tau\cdot A_\nu)] = -ig\,\tau\cdot F_{\mu\nu}$$

$$= (\overset{\rightarrow -ik_\mu}{\partial_\mu} - ig\tau\cdot A_\mu)^2 - \frac{ig}{4} \tau\cdot F_{\mu\nu} [\gamma^\mu, \gamma^\nu]$$

↑ all derivs here act on A's only

So we have

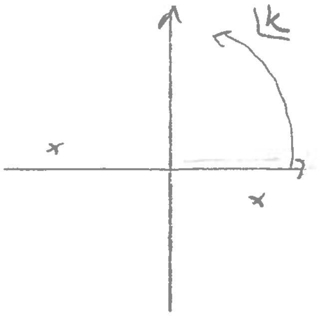
$$A(x) = \lim_{M\to\infty} \mathrm{Tr} \int \frac{d^4k_M}{(2\pi)^4}\, \gamma_5\, e^{-\frac{1}{M^2}\left(\overbrace{-(k_\mu+d_\mu)^2}^{+\frac{k_M^2}{M^2} = \frac{k_0^2-k^2}{M^2}} + \underbrace{\frac{ig}{4}\tau\cdot F_{\mu\nu}\,[\gamma^\mu,\gamma^\nu]}_{\text{treat as perturbation (does not depend on } k)}\right)}$$

↑

Minkowskian, continue to Euclidian:

$$d^4k_M \to i\,d^4k_E$$

$$k_M^2 \to -k_E^2$$



$$\Rightarrow A(x) = \lim_{M\to\infty} \underbrace{i\int \frac{d^4k_E}{(2\pi)^4}\, e^{-k_E^2/M^2}}\cdot \mathrm{Tr}\,\gamma_5 \sum_{p=0}^{\infty} \frac{(-1)^p}{p!}\left(\frac{-ig}{4M^2}\tau\cdot F_{\mu\nu}[\gamma^\mu,\gamma^\nu]\right)^p$$

$$= \frac{2\pi^2}{(2\pi)^4}\int dk_E\, k_E^3\, e^{-k_E^2/M^2} = \frac{M^4}{8\pi^2}\cdot\frac{1}{2}\int_0^\infty dx\, x e^{-x} = \underline{\underline{\frac{M^4}{16\pi^2}}}$$

Obviously all terms beyond order $p=2$ vanish as $M\to\infty$.

$$p=0 \quad \sim \quad \mathrm{Tr}\,\gamma_5 = 0 \qquad !$$

$$p=1 \quad \sim \quad \mathrm{Tr}\,\gamma_5[\gamma^\mu,\gamma^\nu] = 0 \qquad !$$

$$p=2 \quad \sim \quad \mathrm{Tr}\,\gamma_5[\gamma^\mu,\gamma^\nu][\gamma^\alpha,\gamma^\beta]$$

$$= 4\,\mathrm{Tr}\,\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = -16i\epsilon^{\mu\nu\alpha\beta}$$

So we find

$$A(x) = \frac{i}{16\pi^2}\,\frac{1}{2!}\,\frac{g^2}{16}\cdot(+16i\epsilon^{\mu\nu\alpha\beta})\,\underset{\substack{\uparrow \\ \text{group. indices.}}}{\mathrm{Tr}}\left(\tau\cdot F_{\mu\nu}\,\tau\cdot F_{\alpha\beta}\right)$$

The last trace depends on the group:

$$U(1) \quad : \tau = 1 \quad \Rightarrow \quad \mathrm{Tr} = F_{\alpha\beta} F_{\mu\nu}$$

$$SU(2) \quad ; \tau_i = \frac{1}{2}\sigma_i \quad \Rightarrow \quad \mathrm{Tr} = \frac{1}{4} \underbrace{\mathrm{Tr}\,\sigma_a \sigma_b}_{=2\delta_{ab}} F^a_{\alpha\beta} F^b_{\mu\nu}$$

$$= \frac{1}{2} F^a_{\alpha\beta} F^a_{\mu\nu}$$

So, for SU(2)-group (msm) we get

$$A(x) = -\frac{g^2}{32\pi^2} \Big[\overbrace{\frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F^a_{\alpha\beta}}^{=\tilde{F}^{\mu\nu}} \Big] F^a_{\mu\nu} = -\frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \qquad (53)$$

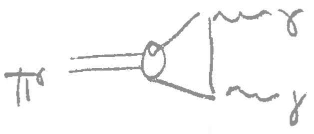
Finally note that both $\mathcal{D}\psi$ and $\mathcal{D}\bar{\Psi}$ give rise to the same factor A, and hence altogether

$$\mathcal{D}\bar{\Psi}\mathcal{D}\Psi\, e^{iS} \longrightarrow$$

$$= \mathrm{Old} \times \underbrace{e^{i\int \theta(x)\left(\partial_\mu J_5^\mu + \frac{g^2}{16\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu}\right)}}$$

$$\equiv 1 \quad \Rightarrow \quad \boxed{\partial_\mu J_5^\mu = \cancel{-}\frac{g^2}{16\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu}} \qquad (54)$$

U(1): PCAC



This is the celebrated axial anomaly. It is not fiction. It is well known that pion decays dominantly via this anomalous coupling.

Note that under the vector-like transform $\psi \to e^{i\theta}\psi$, $\mathcal{D}\bar{\Psi}$ gives rise to the <u>opposite, cancelling contribution</u> to that from $\mathcal{D}\psi$ hence $\partial_\mu j^\mu$ is not anomalous.

(infinite)

We still have to connect these results with <u>B-violation</u>. This is now fairly easy.

In MSM only L-chiral fields couple to SU(2)-fields. So the lagrangian is:

$$\sum_i \bar{\Psi}_{iL} i \not{D}^i_L \psi_{iL} + \sum_i \bar{\Psi}_{iR} i \not{D}^i_L \Psi_{iR} \qquad (55)$$

with

$$\begin{aligned} D^i_{L\mu} &= \partial_\mu - ig\tau\cdot A_\mu - ig'\tfrac{1}{2}\underset{\uparrow \text{ hypercharges}}{Y_{iL}} \\ D^i_{R\mu} &= \partial_\mu - ig'\tfrac{1}{2}Y_{iR} \end{aligned} \qquad (56)$$

and the sum running over all fields $(e,\mu,\tau, \nu_e,\nu_\mu,\nu_\tau, u,d,s,c,b,t)$.

* The hypercharges are assigned according to Klein-Nishina relation

$$Y = 2(Q - \overset{\substack{\text{isospin}\\ \downarrow}}{T_3}) \qquad \begin{pmatrix} u \\ d \end{pmatrix} \begin{matrix} \leftarrow T_3 = +\frac{1}{2} \\ \leftarrow T_3 = -\frac{1}{2} \end{matrix}$$

i.e.

$$\overset{\text{left lepton}}{Y_{\ell_L}} = -1 \;;\; \overset{\text{right lepton}}{Y_{eR}} = -2 \;;\; \overset{\substack{\text{left quark}\\ \downarrow}}{Y_{q_L}} = \frac{1}{3} \;;\; \overset{\substack{\text{right up-quarks}\\ u,c,t}}{Y_{u_R}} = \frac{4}{3} \text{ and } \overset{\substack{\text{right down-quarks}\\ d,s,b}}{Y_{d_R}} = -\frac{2}{3}$$

*

While the result is in fact obvious, let us go through the steps explicitly. Now it is important to keep fields <u>ψ_L and ψ_R independent</u>.

To this end we perform the transform for all quark fields

$$\psi_{Q_i} \longrightarrow e^{i\theta L}\psi_{Q_i} \qquad (57)$$

under which

$$\begin{aligned} \psi_{Q_iL} &\to e^{i\theta L}\psi_{Q_iL} \ ; \quad \bar{\psi}_{Q_iL} \to \bar{\psi}_{Q_iL} e^{-i\theta R} \\ \psi_{Q_iR} &\to \psi_{Q_iR} \qquad\ \ ; \quad \bar{\psi}_{Q_iR} \to \bar{\psi}_{Q_iR} \end{aligned} \qquad (58)$$

Then obviously

$$\delta S = \int d^4x \ \theta \ \partial_\mu J^\mu_{Q_L}$$

colour = 3

$$= \sum_i \bar{\psi}_{Q_iL}\gamma^\mu\psi_{Q_iL} = \overset{\downarrow}{N_c} \sum_{\text{flavour}} \bar{\psi}_{Q_fL}\gamma^\mu\psi_{Q_f}$$

Moreover, the measure becomes

sign!, Important.

$$\mathcal{D}\bar{\Psi}\mathcal{D}\Psi \longrightarrow \mathrm{Det}(e^{-i\theta R})\,\mathrm{Det}(e^{i\theta L})\,\mathcal{D}\bar{\Psi}\mathcal{D}\Psi$$

$$= \mathrm{Det}(e^{-i\frac{\theta}{2}})\,\mathrm{Det}(e^{\ominus\frac{i}{2}\theta\gamma_5})\,\mathrm{Det}(e^{+i\frac{\theta}{2}})\,\mathrm{Det}(e^{\ominus\frac{i}{2}\theta\gamma_5})\,\mathcal{D}\bar{\Psi}\mathcal{D}\Psi$$

determinants corresp. to "vector currents" cancel

these are just $\boxed{\frac{1}{2}}A(x)$ in eq. 19

with $(\not{D}/M)^2 \to \sum_i (\not{D}_i/M)^2$

# of flavours = 3.

$$= N_c \int d^4x\,\theta(x) \left\{ - \overset{\downarrow}{N_f} \frac{g^2}{32\pi^2} \tilde{F}^a_{\mu\nu} F^{a\mu\nu} - \sum_{\substack{\text{flav.} \\ \text{I-spin}}} \left(\frac{Y_{Q_iL}}{2}\right)^2 \frac{g'^2}{16\pi^2} \tilde{f}_{\mu\nu} f^{\mu\nu} \right\} \mathcal{D}\bar{\Psi}\mathcal{D}\Psi$$

Hyper-field tensor

$\partial_\mu B_\nu - \partial_\nu B_\mu$

This implies then

$$\equiv \sum \left(\frac{Y_{QiL}}{2}\right)^2 = \underset{2}{\frac{1}{4}} \cdot \frac{2}{9} \cdot \overset{\text{flav}}{3} = \frac{1}{6}$$

$$\boxed{\partial_\mu J^\mu_{Q_L} = N_c N_f \frac{g^2}{32\pi^2} F\tilde{F}} + N_c Y^2_{QL} \frac{g'^2}{16\pi^2} f\tilde{f} \qquad (59)$$

Similarly one finds

$$\sum_i \frac{Y_{Qi}}{2}$$

$$\partial_\mu J^\mu_{Q_R} = \textcircled{-} N_c Y^2_{QL} \frac{g'^2}{16\pi^2} f\tilde{f} \qquad (60)$$

and for leptons (for which each current is separately conserved also)

$$\partial_\mu J^\mu_{\ell_L} = N_f \frac{g^2}{32\pi^2} F\tilde{F} + Y^2_{\ell_L} \frac{g'^2}{16\pi^2} f\tilde{f} \qquad (61)$$

and

$$\partial_\mu J^\mu_{\ell_R} = - Y^2_{\ell_L} \frac{g'^2}{16\pi^2} f\tilde{f} \qquad (62)$$

From these one can compute immediately

$$\partial_\mu J^\mu_B \equiv \textcircled{\tfrac{1}{3}} \partial_\mu J^\mu_{Q_L} = N_{f\ell} \frac{g^2}{32\pi^2} F\tilde{F} + \underbrace{\overset{-3/2}{\left(Y^2_{Q_L} - Y^2_{Q_R}\right)}} \frac{g'^2}{16\pi^2} f\tilde{f}$$

$$= \frac{1}{4} N_{f\ell}' \left(\frac{1}{9} + \frac{1}{9} - \frac{16}{9} - \frac{4}{9}\right) = -\frac{3}{2}.$$

and

$$\partial_\mu J^\mu_\ell = N_f \frac{g^2}{32\pi^2} F\tilde{F} + \underbrace{\overbrace{\left(Y^2_{\ell} - Y^2_{\ell R}\right)}} \frac{g'^2}{16\pi^2} f\tilde{f}$$

$$= \frac{1}{4} \cdot N_{fl} \left(1 + 1 - 2 - 2\right) = -\frac{N_{fl}}{2}$$

I.e. finally the total current divergences are (result often cited in literature):

$$\boxed{\partial_\mu J^\mu_B = \partial_\mu J^\mu_\ell = N_{f\ell}\left(\frac{g^2}{32\pi^2} F\tilde{F} - \frac{g'^2}{32\pi^2} f\tilde{f}\right)} \qquad (63)$$

So electroweak anomaly breaks

Breaks: $B+L$ (64)

conserves: $B-L$! (65)

We do not know if $B-L=0$ in the universe, because the possible asymmetry in the primordial neutrino background radiation cannot (ever) be measured.

Above, I included the U(1)-part of the anomaly for completeness. This part is however, not interesting as it will not produce any physically significant B-violation rate. This is why I restricted the attention to $F\tilde{F}$-part in eqn. (37).

From now on we shall drop U(1)-part and concentrate on the form (37):

$$\boxed{\partial_\mu J^\mu_{B_L} = N_{f\ell}\frac{g^2}{32\pi^2} F^a_{\mu\nu}\tilde{F}^{a,\mu\nu}} \qquad (66)$$

Of course, in fact B and L change simultaneously, so we have

$$\partial_\mu J^\mu_{B+L} = 2N_{f\ell}\frac{g^2}{32\pi^2} F\tilde{F} \quad ; \quad \partial_\mu J^\mu_{B-L} = 0 .$$

Chern Simons current

One can rewrite $F\tilde{F}$ as a divergence

$$\frac{g^2}{32\pi^2} F\tilde{F} = \partial_\mu K^\mu_{cs} , \qquad (67)$$

where

$$K^\mu_{cs} = \frac{g^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}\left\{F^a_{\nu\alpha}A^a_\beta - \frac{1}{3} g\, \epsilon^{abc} A^a_\nu A^b_\alpha A^c_\beta\right\} \qquad (68)$$

is a (gauge-variant) Chern-Simons current.

So we can formally integrate (66) ofer 3-space and use

$$\underset{\substack{\uparrow \\ \text{no currents at } \infty}}{\int d^3x\, \nabla\cdot j_B} = 0$$

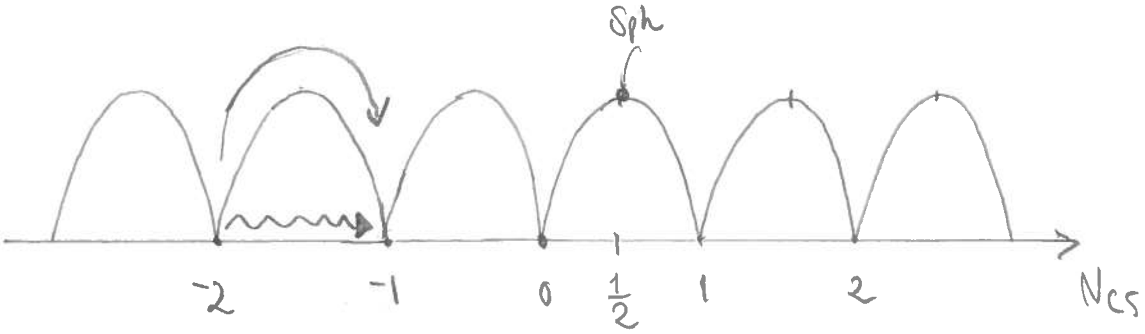
which gives

$$\overset{= \Delta N_L(t)}{\Delta N_B(t)} = \int_{-\infty}^{t} dt\, d^3x\, \partial_t j^\mu_B = \int_{-\infty}^{t} dt\, d^3x\, \partial_\mu j^\mu_B$$

$$= N_{fl}\int_{-\infty}^{t} dt\, d^3x\, \partial_\mu K^\mu_{cs} = N_{fl}\int_{-\infty}^{t} dt\, \partial_t K^0_{cs}$$

$$= N_{fl}\, \underset{\substack{\uparrow \\ \text{Chern-Simons number} \\ \text{(winding)}}}{\Delta N_{cs}(t)} \qquad \text{(changes are gauge invariant)}$$

What makes this interesting, is that the Gauge vacuum of a nonabelian theory is highly nontrivial. In fact there are an infinite set of vacuums, labeled by integer Chern-Simons number:



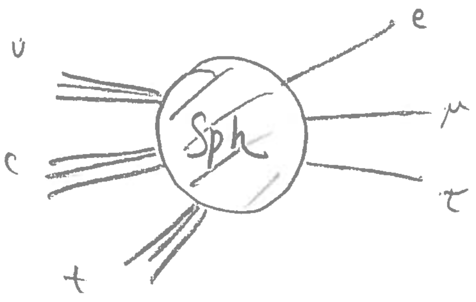
$$A_\mu = -\frac{i}{g} U(\partial U^{-1})$$

All these vacuums are pure gauge and entirely equally good ground states. Vacuum is degenerate.

We will not have time for the very interesting topological properties of nonabelian theories that lead to this phenomenon. Instead, we will proceed to see the consequences.

Obviously, when gauge fields move from one vacuum to another, the Baryon number will change by a factor of $N_{fl} = 3$.

$$\boxed{\Delta N_{CS} = 1 \quad \Rightarrow \quad \Delta B = 3} \quad (\text{and } \Delta L = 3)$$



The barrier between the vacuum configurations is very high however.

Estimate of the rate at T=0

At T=0 one may tunnel from one vacuum to another Such rate is proportional to

$$\Gamma \sim e^{-S_{inst}}, \qquad (69)$$

where S is the Euclidian action for an instanton connecting the two vacua. We do not need to construct the instanton here, however. It suffices to note ('t Hooft 76) that for any configuration:

$$\int d^4x \, (F^a_{\mu\nu} - \tilde{F}^a_{\mu\nu})^2 \geqslant 0$$

$$\Rightarrow \int d^4x_E \left(FF + \tilde{F}\tilde{F} - 2F\tilde{F} \right)$$

$$= 8 S_E - 2 \cdot \left(\frac{32\pi^2}{g^2} \right) \Delta N_{CS} \geqslant 0 \qquad (70)$$

So that even for $\Delta N_{CS} = 1$ we have the bound

$$\boxed{S_E \geqslant \frac{8\pi^2}{g^2}} \qquad (71)$$

and hence

$$\Gamma < e^{-8\pi^2/g^2} \sim e^{-180}.$$

Pretty small

$\Rightarrow$

Proton is Stable !

At high T the transitions can occur faster because of thermal activation. [Kuzmin, Rubakov & Shaposhnikov -86]. In this case one expects

$$\boxed{\Gamma_{sph} \sim e^{-E_{sph}(T)/T}} \qquad (72)$$

where E_{sph} is the energy of the configuration on the top of the barrier.

This configuration was found by Klinkhamer and Manton -8? .

$$\begin{cases} A_i = \dfrac{2}{gr} f(r)\, \vec{r} \times \vec{\sigma} \\[2ex] \phi = \dfrac{\phi_c}{\sqrt{2}} h(r)\, \vec{\sigma} \cdot \hat{r} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

With this ansaz the Hamiltonian of the system can be put to form

$$E_{sph} = \frac{4\pi}{g} \phi \cdot \underset{\uparrow}{\mathcal{E}}\left[f, h ; \frac{\lambda}{g^2} \right]$$

scaled some functional integral over functions f and h depending on λ (cf. eqn) and gauge coupling g.

minimizing $\mathcal{E}$ w.r.t h and g gives

$$\mathcal{E}_{min}[\ \ ] = B\left[\frac{\lambda}{g^2}\right] = \underset{\underset{\frac{\lambda}{g^2} = 0}{\uparrow}}{1{,}52} - \underset{\underset{\frac{\lambda}{g^2} \to \infty}{\uparrow}}{2{,}7}$$

So a fairly good estimate is that ($g = 2/3$, $B(\frac{\lambda}{g^2}) \sim 2$)

$$\boxed{\frac{E_{sph}}{T} \simeq 40 \, \frac{\phi}{T}} \qquad (73)$$

Obviously with reasonably small ϕ/T this can become very large.

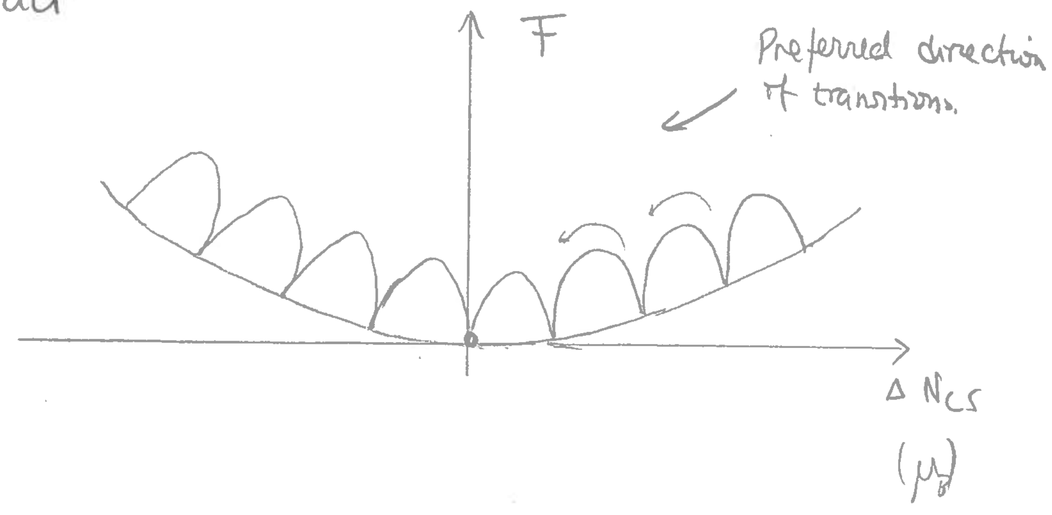
So at high T we expect that B is violated by gauge field transitions between different vacuum states, and we can expect this rate to be large.

Coupling to μ. Sphaleron wash out

Particles created by anomaly thermalize and increase the free energy.

$$F \propto \mu_B N_B \sim \mu_B^2 \sim N_{CS}^2 \qquad (74)$$

So in fact



So, instead of creating anything, sphaleron interactions tend to <u>destroy</u> any pre-existing baryon number!

Indeed, if in equilibrium, we get

$$B + L = 0 \tag{75}$$

So, If originally $B_{in} - L_{in} = 0$, then finally

$$B_f = L_f = 0 . \tag{76}$$

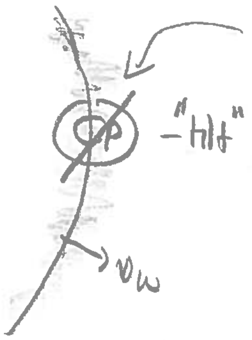
This is the well known <u>sphaleron wash-out problem</u>.

How do we get any baryons then?

> In the presence of <u>CP-violating effects</u> the picture gets modified.

Effectively in CP violating <u>background</u> the free-energy potential gets <u>tilted</u>, favouring some baryon production.

> This "tilt" will be provided by the bubble wall, which also produced the required out-of-eq. conditions.



-"tilt" ⇒ possibility to induce $\mu_{B+L}(x) \neq 0$

However, a problem remains that we are generating B+L via anomaly so we do have just the case B−L = 0 and hence:

Sphaleron rate must turn off inside the bubble!

Putting in the pre-factor and comparing with H (roughly $\Gamma_{sph}/H < 1$), one finds

$$\boxed{\left(\frac{\phi}{T}\right)_c \gtrsim 1} \tag{77}$$

This is the bound I already referred to in the case of MSM. It cannot be satisfied there and therefore we may conclude

> Matter exists $\Longrightarrow$ Must exist new physics
> beyond MSM!

Yet we need Γ_{sph} large in the vicinity of the wall in order to be able to create anything while CP is providing the "tilt."

Eqn. (73) has the answer. When $\phi \to 0$, as it does in the unbroken phase, the barrier suppression vanishes! (and the approximation leading to 73 breaks down.)

In the unbroken phase the rate must be found by numerical methods (cf. Rummukainen). We can, however, make a simple analytical estimate: