Renormalization of spontaneously broken theories.

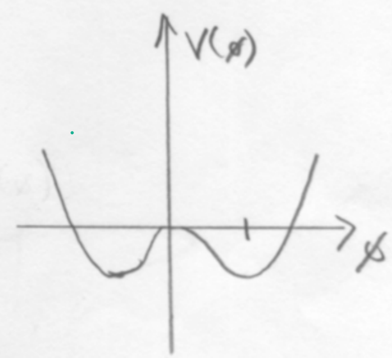
Here the apparent new issue is the proliferation of new couplings and vertices, without any new counter terms. Are these enough? We start by the simple $\lambda\phi^4$ - theory and then move on theories with global symmetries, which introduce the notion of Goldstone bosons. After this we will study the SSB and Higgs mechanism in the case of an $U(1)$ - Abelian Higgs model.

Z_2 - symmetric case ; SSB

Consider theory (with $\mu^2 > 0$)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{1}{4}\lambda\phi^4$$

$$= \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) \qquad (1)$$



Extrema of potential are given by

$$\frac{dV}{d\phi} = 0 \implies \phi = 0 \quad \text{or} \quad \phi = \pm v \qquad (2)$$

where $+\mu^2 = \lambda v^2$ and $V(\pm v) = -\frac{3}{4}\lambda v^2 < 0$.

Now the symmetric minimum is unstable. We can expand theory around the asymmetric minimum by defining

$$\phi \equiv v + \eta \qquad (3)$$

$\uparrow$ new quantum field

In terms of new q-field η theory becomes

$$\mathcal{L}_\eta = \frac{1}{2}(\partial_\mu \eta)^2 - \frac{1}{2}m_\eta^2 \eta^2 - \lambda v \eta^3 - \frac{1}{2}\lambda \eta^4 \qquad (4)$$

where we defined a new positive mass

$$m_\eta^2 \equiv \frac{d^2V}{d\phi^2}\bigg|_{\phi=v} = 3\lambda v^2 - \mu^2 = 2\lambda v^2 \qquad (5)$$

The original Z_2-symmetry is lost in this parametrization. It is not really broken at Lagrangian level. The breakdown is spontaneous, and appears only at the level of states, around the asymmetric minima.

Renormalization

We saw earlier that $\lambda\phi^4$-theory is renormalizable with 3 counter terms. After SSB we have additional superficially divergent 1- and 3-point functions but no new ct's, so the renormalizability is not obvious. Let us define:

$\hookrightarrow$ (at the level of n-pt. functions)

$$\begin{aligned} \phi_0 &= \sqrt{Z_2}\,\phi \\ Z_2\mu_0^2 &\equiv \mu^2 + \delta\mu^2 \qquad (6) \\ Z_2^2\lambda_0 &\equiv \lambda + \delta_\lambda \end{aligned}$$

with $Z_2 \equiv 1 + \delta_2$. Now, the tree-level relation $-\mu^2 = \lambda v^2$ is not automatically preserved by renormalization, and we must start from the broken bare Lagrangian including 1- and 3-point functions, to derive our BPHZ Lagrangian.

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \eta_0)^2 - (-\mu_0^2 v_0 + \lambda_0 v_0^3)\eta_0 - \frac{1}{2}(-\mu_0^2 + 3\lambda v_0^2)\eta_0^2 - \lambda_0 v_0 \eta_0^3 - \frac{1}{4}\lambda_0 \eta_0^4 \qquad (7)$$

Inserting relations (6) we can rewrite this as:

$$\mathcal{L} = \mathcal{L}_\eta + \mathcal{L}_{ct} \qquad (8)$$

where $\mathcal{L}_\eta$ is given by (4) and the ct-lagrangian is $\left[\begin{array}{l} \eta_0 = Z_2 \eta \\ v_0 = Z_2 v \ldots \end{array}\right]$

$$\mathcal{L}_{ct} = \frac{1}{2}\delta_2(\partial_\mu\eta)^2 - \frac{1}{2}(-\delta_\mu^2 + 3\delta_\lambda v^2)\eta^2 - \delta_\lambda v \eta^3 - \frac{1}{4}\delta_\lambda \eta^4$$

$$- \left[-\mu^2 + \lambda v^2 + (-\delta_\mu^2 + \delta_\lambda v^2)\right] v\eta \qquad (9)$$

Note that in the renormalized lagrangian the mass parameter

$$m_\eta^2 \equiv -\mu^2 + 3\lambda v^2 = \frac{d^2 V}{d\phi^2}\Big|_{\phi=v} \qquad (10)$$

but setting $m_\eta^2 = 2\lambda v^2$ is now dependent of retaining the tree-level relation $-\mu^2 + \lambda v^2 = 0$. In an generic renormalization scheme this will not be the case. However, we can always define the parameter v, so this holds (= choice of a scheme). If we denote the 1-loop correction to the 1-point function by $-i\vartheta D$, the new condition for the asymmetric minimum becomes

$\tilde{\Gamma}^{(1)}$

$$\frac{dV}{d\phi}\Big|_{\phi=v} = i\Big($$  $+$  $$\Big) = v\Big(\underbrace{-\mu^2 + \lambda v_R^2}_{\text{tree-level ct}} + \overbrace{\left[D - \underbrace{\delta\mu_R^2 + \delta_\lambda^R v_R^2}_{\text{1-loop. ct}}\right]}^{\equiv 0}\Big) \equiv 0 \qquad (11)$$

$\equiv -iD$

where R refers to the renormalization scheme chosen to define δ_λ^R and $\delta\mu_R^2$. It is thus evident that the loop-corrected minimum v_e is scheme-dependent. We can choose however, that to any order

$$-\delta\mu^2 + \delta_\lambda v^2 \equiv -D_v \qquad (12)$$

as one of our renormalization conditions. Remember now that we can always write around asymmetric point:

$$V_{eff}(\phi) \equiv i \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\Gamma}_v^{(n)}(0)\, (\phi - v)^n \qquad (13)$$

That is

$$i\tilde{\Gamma}^{(n)}(0) = \left.\frac{d^n V}{d\phi^n}\right|_{\phi=v} \qquad (14)$$

And so the condition (11) is equivalent to saying that tadpole vanishes

$$i\tilde{\Gamma}^{(1)} = \left.\frac{dV}{d\phi}\right|_{\phi=v} \equiv 0$$

Together with (12) this then implies that vev does not move:

$$-\mu^2 + \lambda v^2 = 0 \iff v^2 = + \frac{\mu^2}{\lambda} \qquad (15)$$

To all orders. With these conditions we can rewrite the ct-Lagrangian as

$$\mathcal{L}_{ct} = \frac{1}{2}\delta_Z(\partial_\mu \eta)^2 + \frac{1}{2}\delta m^2 \eta^2 - \delta_\lambda v \eta^3 - \frac{1}{4}\delta_\lambda \eta^4 \qquad (16)$$

where

$$\delta m^2 \equiv -\delta\mu^2 + 3\delta_\lambda v^2 = 2\delta_\lambda v^2 - D_v \qquad (17)$$

$$\Longleftrightarrow -D_v = -\delta\mu^2 + \delta_\lambda v^2 = \delta m^2 - 2\delta_\lambda v^2$$

Second, let's observe that the shifted theory 2-point-function at zero momentum coincides with $d^2V/d\phi^2|_{\phi=v}$:

$$\left.\frac{d^2V}{d\phi^2}\right|_{\phi=v} = i\tilde{\Gamma}^{(2)} = \underset{\text{tree}}{m_\eta^2} + \underset{\text{1-loop + ct}}{\Pi(0)} \qquad (18)$$

One possible renormalization condition is to set $\Pi(0) \equiv 0$, in which case $m_\eta^2 = -\mu^2 + 3\lambda v^2 = 2\lambda v^2$ becomes the finite mass parameter of the theory. This turns out to be the $p^2=0$ mass, of course. Eg. the condition

$$\left.\frac{d^2V}{d\phi^2}\right|_{\phi=v} \equiv m_\eta^2 \qquad (19)$$

Rem.: $i\tilde{\Gamma}^{(2)} = i\frac{\delta^2\Gamma}{\delta\phi\delta\phi}_{\text{PI}}$
$- i\left(\frac{(-i)^2\delta^2 W}{\delta J^2}\right)$
$= -i\Delta^{-1}$
$= -i\left(\frac{1}{i}\right)(p^2 - m^2 + \Pi)\Big|_{p^2=0}$
$= \underbrace{m^2 + \Pi(0)}$
$\equiv m^2_{p^2=0}$

is equivalent to setting for the 2-point function $-\Delta^{-1} = i\tilde{\Gamma}^{(2)}$

$$-i\Delta^{-1}(p^2=0) \equiv +m_\eta^2 \qquad (20)$$

We still need a condition for w.f.r.- renormalization. Any consideration of potential will not give this. We shall choose

$$\left.\frac{di\Delta^{-1}}{dp^2}\right|_{p^2=0} = \left.\frac{d\Pi}{dp^2}\right|_{p^2=0} = 0 \Rightarrow \delta_2 = -\left.\frac{d\Pi}{dp^2}\right|_{p^2=0} \qquad (21)$$

1-loop renormalization

The 1-loop tadpole is just  $= 0$

$$vD = i\Big($$  $$\Big) = i(-i\lambda v)\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m_\eta^2} = 3\lambda v i A_0(m_\eta^2) \qquad (15)$$

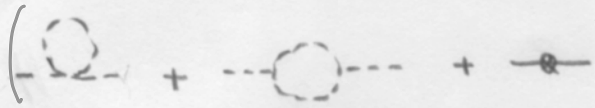
$$\Rightarrow \underline{D_v = 3\lambda i A_0(m_\eta^2)}$$

so that:

$$\overbrace{-\delta\mu^2 + \delta_\lambda v^2}^{= -D_v} = -3i\lambda A_0(m_v^2). \qquad (19)$$

2-point function at 1-loop, broken phase at $p^2=0$:

$$-\delta_2 i\eta\Box\eta \to i\delta_2 p^2$$

$\Pi = i\Big($  $\Big)$

$S = \frac{1}{4}\cdot 4\cdot 3 = 3 \qquad S = \frac{1}{2}\cdot 6\cdot 3\cdot 2$

$$= 3\lambda i \cancel{A_0(m_\eta^2)} + 18\lambda^2 v^2 i B_0(m_\eta^2, m_\eta^2, p^2) + \overbrace{(2\delta_\lambda v^2 - \cancel{D})}^{\delta m^2} + p^2\delta_2$$

$$= 2v^2\left(9\lambda^2 i B_0 + \delta_\lambda\right) + p^2\delta_2 \qquad (20)$$

Note that one complete possible renormalization scheme would be to define m_η^2 as the mass parameter, ie

$$\underline{m^2 \equiv m_\eta^2} = \frac{d^2V}{d\phi^2}\Big|_{\phi=v} = -\mu^2 + 3\lambda v^2 \overset{\text{by (11)}}{=} 2\lambda v^2 \qquad (21)$$

This corresponds to

$$\Pi(0) = 2v^2\left(9\lambda i B_0(m_\eta^2, m_\eta^2, 0) + \delta_\lambda\right) \equiv 0$$

eg:

$$\underline{\delta_\lambda = -9\lambda^2 i B_0(m_\eta^2, m_\eta^2, 0)} \qquad (22)$$

Consistency check. The full 2-point function should be the derivative of the 1-point function ($d^2V/d\phi = d/d\phi\,(dV/d\phi)$)

$$\frac{d}{d\phi}\left[\phi\left(-\mu^2+\lambda\phi^2+(D-\delta\mu^2+\delta_\lambda\phi^2)\right)\right]\Big|_{\phi=v} \qquad ; \; D = 3\lambda i A_0(m_v^2)$$

$$= 0 + 2\lambda v^2 + 2\delta_\lambda v^2 + 3i\lambda v \frac{dA_0}{d\phi}\Big|_{\phi=v} \qquad ; \; \frac{dA}{d\phi} = \frac{dm^2}{d\phi}\frac{dA_0}{dm^2} = 6\lambda\phi B_0(m^2,m^2,0)$$

$$= 2\lambda v^2 + 2v^2\left(9\lambda^2 i B_0(m_\eta^2,m_\eta^2,0) + \delta_\lambda\right)$$

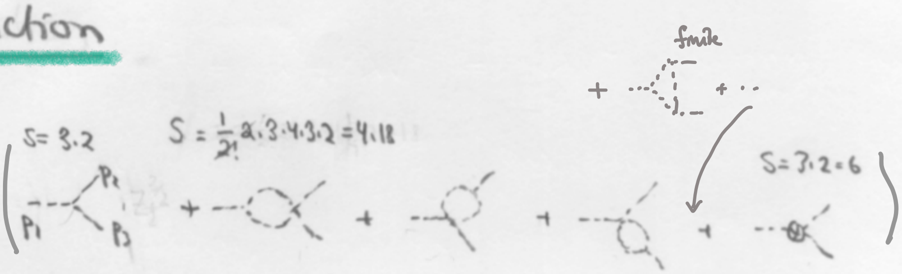
$$= m_\eta^2 + \Pi(0) = \tilde{\Gamma}^{(2)}(0)$$

$$A_0 = \int_k \frac{1}{k^2-m^2}$$

$$B_0(m^2,m^2,0) = \int_k \left(\frac{1}{k^2-m^2}\right)^2$$

3-point function

$\tilde{\Gamma}^{(3)}(p_1,p_2,p_3) = \Big($



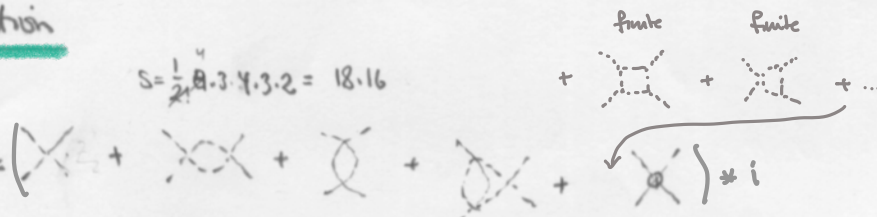
$\Big) \times i$

$$= 6\lambda v + \sum_{i=1}^{3} 18\lambda^2 v\, iB_0(m_\eta^2,m_\eta^2,p_i^2) + 6i\delta_\lambda v + \dots$$

$$= 6\lambda v + \sum_{i=1}^{3} 18\lambda^2 v\left(iB_0(m_\eta^2,m_\eta^2,p_i^2) - iB_0(m_\eta^2,m_\eta^2,0)\right) + \dots \qquad \text{FINITE!}$$

4-point function

$\tilde{\Gamma}^{(4)}(s,t,u) = \Big($



$\Big) * i$

$$= 6\lambda + \sum_{a=s,t,u} 18\lambda^2 iB_0(m_\eta^2,m_\eta^2,a) + 6\delta_\lambda + \dots$$

$$= 6\lambda + \sum_{a=s,t,u} 18\lambda^2\left(iB_0(m_\eta^2,m_\eta^2,a) - iB_0(m_\eta^2,m_\eta^2,0)\right) + \dots \qquad \text{FINITE!}$$

4. Spontaneous symmetry breaking & Higgs mechanism

Global symmetry

Consider a complex scalar field

$$\phi(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right) \qquad ; \ \phi_{1,2} \text{ real.}$$

and

wrong sign, if $\mu^2 > 0$!

$$\mathcal{L}_\phi = (\partial_\mu\phi)^\dagger(\partial_\mu\phi) + \underbrace{\mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2}_{= -V(|\phi|^2).} \qquad (4.1)$$

This theory is invariant under $\phi \to e^{-i\alpha}\phi$; $\alpha \in \mathbb{R}$. The Euler-Lagrange equation for (4.1) leads to equation of motion:

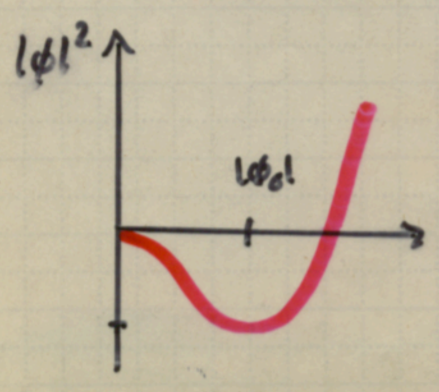
$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^\dagger)}\right) - \frac{\delta\mathcal{L}}{\delta\phi^\dagger} = 0$$

$$\Rightarrow \quad \partial^2\phi + \mu^2\phi = 2\lambda\phi|\phi|^2 \qquad (4.2)$$

The ground state of the system (the vacuum) obeys the e.o.m with $\partial_\mu\phi_0 = 0$, so that

$$\mu^2\phi_0 = 2\lambda\phi_0|\phi_0|^2 \Rightarrow \underline{\phi_0 = 0} \text{ or } \underline{|\phi_0|^2 = \frac{\mu^2}{2\lambda}} = \frac{v^2}{2}$$

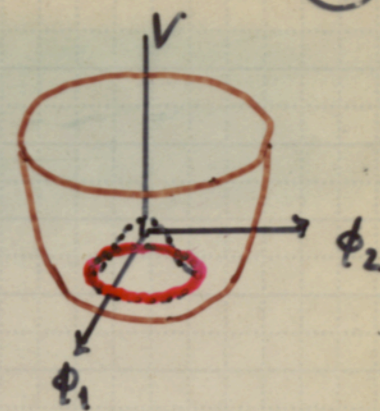
$(\lambda v^2 = \mu^2)$



- $V(0) = 0$
- $V(\phi_0) = -\frac{\mu^4}{4\lambda}$. ← lower minimum when $\mu^2 > 0$

Vacuum is not unique for $\mu^2 > 0$. Instead we have a continuous set of equivalent degenerate vacua $\phi_0 = |\phi_0| e^{i\alpha}$, where

$$\frac{dV}{d|\phi_0|^2} = -\mu^2 + 2\lambda|\phi_0|^2 = 0.$$



When we choose one of these states as the true vacuum, we break the U(1)-symmetry, since for this state

$$e^{i\alpha}|\phi_0\rangle \neq |\phi_0\rangle$$

while of course $|\phi_0|^2$ is invariant. This is the spontaneous symmetry breaking of U(1). It breaks the symmetry in the Hilbert space of states, not in $\mathcal{L}$. !

Let us now take $\theta_0 = 0$ and $|\phi_0| \equiv v$, and rewrite the fields ϕ_1 & ϕ_2 with new fields around the vacuum state:

$$\phi = \frac{1}{\sqrt{2}}(v + \phi_1' + i\phi_2') \qquad (4,3)$$

Inserting this back to (4.1) gives

"right" sign for mass

no mass for ϕ_2

$$\mathcal{L}_\phi = \frac{1}{2}(\partial_\mu\phi_1')^2 - \mu^2\phi_1'^2 + \frac{1}{2}(\partial_\mu\phi_2')^2$$

$$- \underbrace{\frac{1}{2}\lambda v}_{\text{"new" interactions} \approx v.}\phi_1'(\phi_1'^2 + \phi_2'^2) - \frac{1}{4}\lambda(\phi_1'^2 + \phi_2'^2)^2 \qquad (4,4)$$

renormalization 'problems'.

The broken theory thus contains

- one massive state with $m = \mu\sqrt{2}$ (ϕ_1')
- one massless state : goldstone boson (ϕ_2').

There is a general rule for this :

> For every spontaneously broken continuous global symmetry one gets a massless scalar particle. (Goldstone theorem).

Ex. Compute the effective action in this theory. ⇒ Problem in renormalization with $m^2_{p^2=0}$. (with gb:s)

local symmetry

Now consider the case where the symmetry is local, but we yet have the "wrong-sign" mass parameter in (4.1). What is the particle spectrum now?

local symmetry requirement requires the gauge-field A_μ. We thus have

$$\mathcal{L}_\phi = (D_\mu \phi^+)(D_\mu \phi) + \mu^2 \phi^+ \phi - \lambda(\phi^+\phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad (4.5)$$

where

$$D_\mu \equiv \partial_\mu + ie A_\mu \qquad (4.6)$$

This theory is invariant under local gauge-transform (scalar-electrodynamics)

$$\begin{cases} \phi(x) \to e^{-i\alpha(x)} \phi(x) \\ A_\mu(x) \to A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x) \end{cases} \qquad (4.7)$$

Let us now parametrize ϕ as follows:

$$\phi(x) \equiv \frac{\eta(x)+v}{\sqrt{2}} e^{+i\xi/v} \simeq \frac{1}{\sqrt{2}}(\eta + v + i\xi) \qquad (4.8)$$

$\xi \ll v$; small fluctuation

That is $\eta_0 = \langle 0|\eta|0\rangle \equiv 0$, $\xi_0 = \langle 0|\xi|0\rangle = 0$, while $\phi_0 = \langle 0|\phi|0\rangle = v/\sqrt{2}$.

Now use the gauge-invariance (4.7) to bring any configuration of the form (4.8) to a form containing only η-field:

UNITARY GAUGE:

$$\phi(x) \longrightarrow e^{-i\alpha(x)}\phi(x) \equiv \frac{1}{\sqrt{2}}(\eta + v) \qquad (4.9)$$

From (4.8) we see that this transformation is effected by the choice

$$\alpha(x) \equiv + \frac{1}{v}\xi(x) \qquad (4.10)$$

This choice of gauge corresponds to our earlier choice of picking one particular vacuum state. However, we must at the same time transform the A-field

$$A_\mu \to A_\mu' = A_\mu + \frac{1}{ve}\partial_\mu \xi \qquad (4.11)$$

Gauge transf. got rid of ξ in ϕ, but it reappeared in A_μ'.

got "swallowed" by A_μ.

This does not "show up" in the $\mathcal{L}$ however.

What is the spectrum of states in this gauge?

now real : $\frac{1}{\sqrt{2}}(\eta+v)$

$$\mathcal{L}_{\phi', A'} = \left[(\partial_\mu + ieA'_\mu)\phi'\right]^\dagger\left[(\partial^\mu + ieA'^\mu)\phi'\right]$$

$$+ \mu^2|\phi'|^2 - \lambda|\phi'|^4 - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu}$$

$$= \underbrace{(\partial_\mu\phi')^\dagger(\partial^\mu\phi')}_{\frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta)} + e^2|\phi'|^2 A'_\mu A'^\mu + \mu^2|\phi'|^2 - \lambda|\phi'|^4 - \frac{1}{4}F'^2$$

- $|\phi'|^2 = \frac{1}{2}(v^2 + 2v\eta + \eta^2)$
- $V(|\phi'|^2) = -\mu^2\eta^2 - \lambda v\eta^3 - \frac{1}{4}\lambda\eta^4$

$$\Rightarrow \mathcal{L}' = \left(\frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - \mu^2\eta^2\right) + \left(-\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \underbrace{\frac{1}{2}e^2v^2}_{=\frac{1}{2}M_{A'}^2}A'_\mu A'^\mu\right)$$

$$- \lambda v\eta^3 - \frac{1}{4}\lambda\eta^4 + e^2(v\eta + \frac{1}{2}\eta^2)A'^2 \qquad (4.12)$$

So the theory now contains

1. massive scalar field η : $m_\eta^2 = 2\mu^2$: Higgs-field
1. Massive vector field A' ; $M_{A'} = ev$

So what happened? Local gauge-invariance requirement lead to introduction of A_μ. We cannot write a term $\sim M_A^2 A_\mu A^\mu$ into $\mathcal{L}$, because it would break $U(1)$.

However, since $V(|\phi|)$ has an asymmetric minimum, then the ground-state breaks the $U(1)$-symmetry in the state-space. As a result, the gauge-choice (4.9), which effects the SSB by picking a particular vacuum gives rise to a mass to photon!

Unlike in the case of global symmetry, there is no trace of a goldstone mode in (5.12). This "would be" Goldston boson got eaten by the gauge field; it is effectively replaced by the longitudinal mode of A'.

Gauge propagator (unitary gauge)

In the unitary gauge the propagator is the inverse of $(D^{-1})^{\mu\nu}$:

$$i\mathcal{L}_{A^2} = +\frac{i}{2}A_\mu\left(\left(\partial^2+M^2\right)g^{\mu\nu} + \partial^\mu\partial^\nu\right)A_\nu \equiv -\frac{1}{2}A_\mu(D^{-1})^{\mu\nu}A_\nu \quad (*)$$

So given:

$$(D^{-1})^{\mu\nu} = +i\left[(q^2-M^2)g^{\mu\nu} + q^\mu q^\nu\right]$$

$$D_{\mu\nu} \equiv a g_{\mu\nu} + b q_\mu q_\nu \;;\; (D^{-1})_{\mu\nu}D^{\nu\rho} \equiv \delta_\mu{}^\rho$$

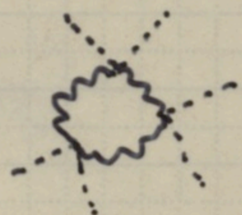
$$\Rightarrow a = \frac{-i}{q^2-M^2} \;;\; b = -\frac{a}{M^2}$$

$$\Rightarrow D_{\mu\nu} = \frac{-i}{q^2-M^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{M^2}\right) \qquad (4.13)$$

"problem"-term

$$\boxed{* \quad M \equiv ev}$$

The $\frac{q^\mu q^\nu}{M^2}$-term spoils the perturbative renormalizability as we can see by power counting. Indeed, for example the diagram contributing to ϕ^6-function



$$\sim \int \frac{d^4k}{k^6} \quad : \text{"without } \frac{k^\mu k^\nu}{M^2}\text{" finite.}$$

$$\sim \int d^4k \, \frac{1}{M^6} \quad \text{"with } \frac{k^\mu k^\nu}{M^2}\text{", diverges.}$$

This is only an apparent problem, due to "unfortunate" choice of gauge (w.r.t. renormalizability).

R_ξ-gauge

Now parametrize ϕ, instead of (4.9) as follows

$$\phi = \frac{1}{\sqrt{2}}(\eta + v + i\pi) \qquad (4.14)$$

Then

$$\overbrace{(\partial_\mu+ieA_\mu)(\eta+v+i\pi)}^{= \partial_\mu\eta + eA_\mu\pi + i(\partial_\mu\pi - eA_\mu(\eta+v))}$$

$$\mathcal{L} = \underbrace{\frac{1}{2}\left|(\partial_\mu + ieA_\mu)(\eta + v + i\pi)\right|^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mu^2|\phi|^2 - \lambda|\phi|^4 \qquad (4.15)$$

$$\frac{1}{2}(\partial_\mu\eta - eA_\mu\pi)^2 + \frac{1}{2}(\partial_\mu\pi + eA_\mu\eta)^2$$

$$+ evA_\mu(\partial_\mu\pi + eA_\mu\eta) + \frac{(ev)^2}{2}A_\mu A^\mu$$

mixing: must be removed by g.f.

A suitable gauge-choice is

$$f(A_\mu,\phi) = \partial^\mu A_\mu - ev\xi\pi = 0$$

with:

mass for π: $-\frac{1}{2}\xi(ev)^2$

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial^\mu A_\mu - \xi e v \pi)^2 \qquad (4.16)$$

$\hookrightarrow$ mixing term $= + ev(\partial_\mu A^\mu)\pi$

$= -ev A^\mu(\partial_\mu \pi)$

Adding the gauge-fixing-term the Lagrangian now becomes:

$$\mathcal{L}_0 = \frac{1}{2}\left[(\partial^\mu\eta)^2 - 2\mu^2\eta^2\right] + \frac{1}{2}\left[(\partial^\mu\pi)^2 - \xi(ev)^2\pi^2\right]$$

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(ev)^2 A_\mu A^\nu - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 + \text{Interactions}$$

$+ \mathcal{L}_{ghost}$ $\leftarrow$ next page (4.17)

Theory now contains η with the same mass as before, but also π, the would be goldstone-mode, that now has a gauge dependent mass $m_\pi^2 = \xi(ev)^2$. Moreover, the gauge propagator now becomes the inverse of

$$D^{-1}_{\mu\nu} = i\left[(q^2 - M^2)g_{\mu\nu} + (1 - \tfrac{1}{\xi})q^\mu q^\nu\right]$$

$$\Rightarrow D_{\mu\nu} = \frac{-i}{q^2 - M^2}\left(g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2 - \xi M^2}\right) \qquad (4.18)$$

R_ξ-gauge

In R_ξ-gauge, for any finite ξ, the gauge-propagator is asymptotically

$$D_{\mu\nu} \sim \frac{1}{q^2} g_{\mu\nu} \quad ; q^2 \to \infty .$$

and the renormalizability by power counting is manifest.
Particular cases:

$\xi = 1$; Feynman gauge $\quad D_{\mu\nu} = -\frac{i g_{\mu\nu}}{q^2 - M^2}$
$\xi = 0$; Landau gauge $\quad \sim D_{\mu\nu} = \frac{i}{q^2 - m^2}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)$
$\xi = \infty$; Unitary gauge. $\quad$ L 4.13

Note that also π-field decouples from the excitation spectrum in the unitarity-limit $\xi \to \infty$, as it should ($m_\pi \to \infty$).

The fact that U-gauge is a limiting case of R_ξ-gauge proves that the spontaneously broken theory is both renormalizable and unitary at the same time.

This is in fact not all. In addition to the renormalizability by power-counting one should prove that all n-point functions are made finite by the counter-terms of the symmetric form of the Lagrangian.

Indeed, starting from the initial Lagrangian:

$$\mathcal{L}_0 = |(\partial_\mu + i e_0 A_{0\mu})\Phi_0|^2 - \frac{1}{4}(F_0^{\mu\nu})^2 + \mu_0^2 |\Phi_0|^2 - \lambda_0 |\Phi_0|^4 \tag{4.14}$$

and redefining

$$\begin{cases} A_{0\mu} = Z_A^{1/2} A_\mu \\ \Phi_0 = Z_\phi^{1/2} \Phi \\ \lambda_0 Z_\phi^2 \equiv Z_\lambda \lambda \equiv \lambda + \delta_\lambda = (\lambda + \delta\lambda) Z_\phi^2 \\ e_0 = e Z_e = e(1+\delta_e) \\ \mu_0^2 Z_\phi \equiv \mu^2 + \delta_\mu = Z_\phi(\mu^2 + \delta\mu^2) \end{cases} \qquad (4.20)$$

One can rewrite the spontaneously broken Lagragian in terms of renormalized fields. One sees that after renormalizing propagators there is a single counter-term δ_λ for the 5 vertices involving λ

$$\delta\mathcal{L}_\lambda = -\lambda v(\eta^3 + \eta\pi^2) - \frac{\lambda}{4}(\eta^2+\pi^2)^2 \qquad (4.21)$$

and similarly only one δ_e for the 4 vertices involving e and A_μ:

$$\delta\mathcal{L}_{\Phi A} = -eA^\mu(\pi\partial_\mu\eta - \eta\partial_\mu\pi) + \frac{1}{2}e^2(\eta^2+\pi^2)A^2 + e^2 v\eta A^2$$

$$\delta\mathcal{L}_{\text{ghost}} = -ie^2 v\,\bar{c}\,\eta\, c \qquad (4.22)$$

That this really works, calls to be proven explicitly. We will sketch the proof at 1-loop level shortly. First we need to write down the full Feynman rules for the theory:

When (4.19) is written in terms of renormalized fields, one ~~finds,~~ one gets

$$\mathcal{L} = \overset{4.17}{\mathcal{L}_0} + \overset{4.21}{\delta\mathcal{L}_\lambda} + \overset{4.22}{\delta\mathcal{L}_{ZA}} + \overset{4.24}{\delta\mathcal{L}_{ct}}, + \mathcal{L}_{ghost} \qquad (4.23)$$

where

$$\boxed{\begin{aligned}\delta\mathcal{L}_{ct} = & -\tfrac{1}{2}(\delta_\mu + \delta_\lambda v^2 + \delta_\pi \xi M^2)\pi^2 - \tfrac{1}{2}(\delta_\mu + 3\delta_\lambda v^2)\eta^2 - \tfrac{1}{4}\delta_A F_{\mu\nu}F^{\mu\nu} \\ & + \tfrac{1}{2}\delta_\phi(\partial\pi)^2 + \tfrac{1}{2}\delta_\phi(\partial\eta)^2 + \tfrac{1}{2}\bar{\delta}_1 M^2 A^2 \\ & - \frac{\delta_\lambda}{4}(\eta^2+\pi^2)^2 - \delta_\lambda v\eta(\eta^2+\pi^2) - (v\delta_\mu + \delta_\lambda v^3)\eta \\ & - e\bar{\delta}_2 A(\pi\partial_\mu\eta - \eta\partial_\mu\pi) + \tfrac{1}{2}e^2\bar{\delta}_1(\eta^2+\pi^2)A^2 + e^2\bar{\delta}_1 v\eta A^2\end{aligned}} \qquad (4.24)$$

where $\bar{\delta}_1 \equiv Z_e^2 Z_\phi Z_A - 1.$ (4.25)

$\bar{\delta}_2 = Z_e Z_\phi Z_A^{1/2} - 1$
$\approx \delta_e + \delta_\phi + \tfrac{1}{2}\delta_A$ etc.

$Z_3 = Z_A$
$Z_2 = Z_\phi$
$Z_1 = Z_e$

$\delta_\pi \equiv Z_e^2 Z_\phi^2 Z_A^2 - 1 \approx \bar{\delta}_1 + \delta_\phi$

We then get by inspection of (4.17), (4.21), (4.22) and (4.24):

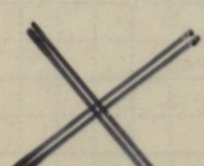
η ═══ $\dfrac{i}{k^2 - 2\mu^2 + i\epsilon}$

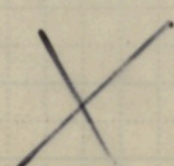
c .... $-\dfrac{i}{k^2 - \xi M^2 + i\epsilon}$

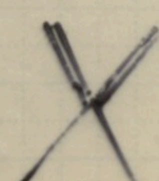
π ——— $\dfrac{i}{k^2 - \xi M^2 + i\epsilon}$

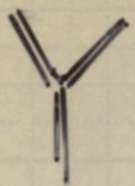
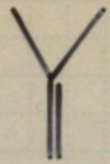
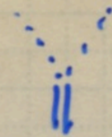
$M \equiv ev$

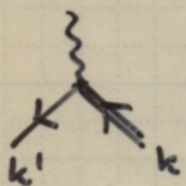
A μ ~~~~~ ν $\dfrac{-i}{k^2 - M^2}\left(g_{\mu\nu} - (1-\xi)\dfrac{k_\mu k_\nu}{k^2 - \xi M^2}\right)$ (4.26)

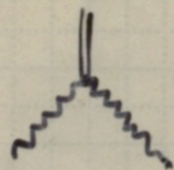
 $\sim -\dfrac{i\lambda}{4}$

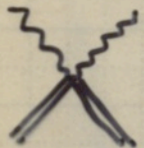
 $-\dfrac{i\lambda}{4}$

 $-\dfrac{i\lambda}{2}$

 $-i\lambda v$  $-i\lambda v$  $-i\xi e^2 v$

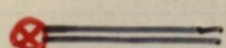


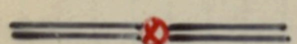
$e(k+k')_\mu$  $ie^2 v g_{\mu\nu}$

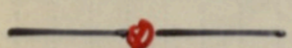
 $\frac{i}{2}e^2 g_{\mu\nu}$  $\frac{i}{2}e^2 g_{\mu\nu}$ (4.27)

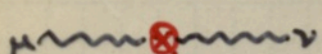
Unlike P&S, I have not included any combinatorics factors in these rules.

In addition, we have the counter-term rules

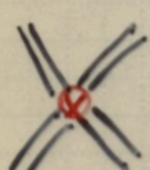
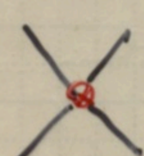
 $-i(v\delta_\mu + \delta_\lambda v^3)$

 $i(p^2\delta_\phi - \delta_\mu - 3\delta_\lambda v^2)$

 $i(p^2\delta_\phi - \delta_\mu - \delta_\lambda v^2 - \delta_\pi \xi M^2)$



$-i(g_{\mu\nu}q^2 - q^\mu q^\nu)\delta_A + iM^2 g_{\mu\nu}\bar{\delta}_1$ (4.28)

 $-\frac{i\delta_\lambda}{4}$  $-\frac{i\delta_\lambda}{4}$  $-\frac{i\delta_\lambda}{2}$

 $-i\delta_\lambda v$  $-i\delta_\lambda v$

 $e\bar{\delta}_2(k+k')_\mu$  $ie^2 v\bar{\delta}_1 g_{\mu\nu}$

 $\frac{i}{2}e^2\bar{\delta}_1 g_{\mu\nu}$  $\frac{i}{2}e^2\bar{\delta}_1 g_{\mu\nu}$

Ghosts

One element still missing from our Feynman rules for our theory in R_ξ-gauge are the ghosts. Interestingly the ghosts now do not couple to A_μ-fields directly, but to η instead. Indeed we had:

$$\boxed{G = \left(\partial_\mu A^\mu - \xi e v \pi\right) = 0} \qquad (4.31)$$

Under an infinitesimal gauge-transform $\phi \to (1 - i\alpha)\phi \Rightarrow$

$$= (1 - i\alpha)(v + \eta + i\pi]$$

$$\begin{aligned} \delta\pi &= -(v+\eta)\alpha \\ \delta\eta &= +\alpha\pi \\ \delta A_\mu &= +\frac{1}{e}\partial_\mu\alpha \end{aligned} \quad ;$$

$$\Rightarrow \eta \to \eta + \alpha\pi$$
$$\pi \to \pi - \alpha(\eta + v)$$
$$A \to A + \frac{i}{e}\partial_\mu\alpha \qquad (4.32)$$

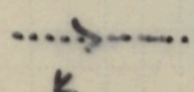
$$\Rightarrow \delta G = +\frac{1}{e}\partial^2\alpha + \xi e v (v+\eta)\alpha$$

$$\Rightarrow \frac{\delta G}{\delta\alpha} \sim +\partial^2 + \xi M^2\left(1 + \frac{\eta}{v}\right) \qquad (4.33)$$

This introduces a ghost-term:

$$\det\left(e\frac{\partial G}{\partial\alpha}\right) = \int \mathcal{D}c\mathcal{D}\bar{c}\; e^{\overbrace{-i\int d^4x\, \bar{c}\left[+\partial^2 + \xi M^2 + \xi e^2 v \eta\right]c}^{= i\int d^4x\, \mathcal{L}_{ghost}}} \qquad (4.34)$$

I_5. there are new rules

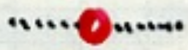


$$\frac{i}{k^2 - \xi M^2} \quad ;$$

$$\sim -i\xi e^2 v \qquad (4.35)$$

Obs.:

and new counter-terms:



$$+i\left(\delta_g p^2 - \bar\delta_g\, \xi M^2\right) \qquad\qquad -i\,\bar\delta_g\, \xi e^2 v$$

with $\bar\delta_g + 1 \equiv Z_g Z_\varphi Z_e^2 Z_A$

These come from:

$$-i\,\bar c_0\left(+\partial^2 + \xi_0 e_0^2 v_0^2 + \xi_0 e_0^2 v_0 \eta_0\right) c_0$$

$$\to -i\,\bar c_1\left(+\overset{c\text{'s wfr}}{Z_g}\,\partial^2 + \underbrace{\overset{c\text{'s}}{Z_g}\,\overset{v,\eta}{Z_\varphi}\,\overset{e\text{'s}}{Z_e^2}\,\overset{\xi}{Z_A}}\left(\xi e^2 v^2 + \xi e^2 v\eta\right)\right)$$

$$\equiv 1 + \bar\delta_g \equiv Z_g(1+\bar\delta_1) \simeq 1 + \delta_g + \bar\delta_1$$

not independent

$\approx \delta_\varphi + \delta_A + 2\delta_e$

Note: we get $\xi_0 \equiv Z_A \xi$ from requirement that

$$\frac{1}{2\xi_0}(\partial_\mu A_0)^2 = \frac{1}{2Z_\xi \xi} Z_A(\partial_\mu A)^2 \equiv \frac{1}{2\xi}(\partial_\mu A)^2$$

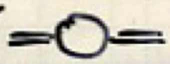
Only 1 new ct. δ_g.

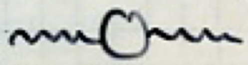
1- Loop renormalizability : Sketch

Renormalization conditions. Most convinient choice is to sacrifice on-shell mass fitting for cancellation of 1-point functions

 $= 0$

With this choice mass becomes a prediction. Other conditions

$\frac{d}{dp^2}\Big($  $\Big) = 0$

" $\frac{d}{dp^2}\Big($  $\Big) = 0$ " ← residue of $g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$ - term to zero at pole

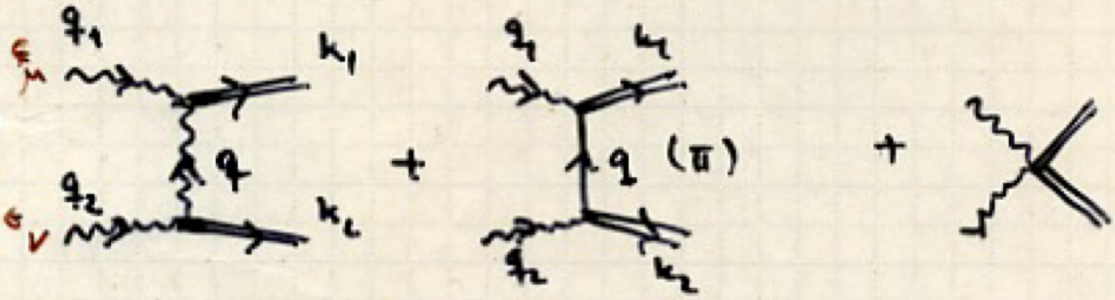
 $= 6i\lambda$

 $= ie^2 v\, g_{\mu\nu}$ (for example)

From these derive ct's and then prove that <u>all</u> one-loop corrections became finite. Long calculation ;).

Cancellation of unphysical poles in R_ξ; example

Consider scattering $A+A \to \eta+\eta$: In R_ξ-gauge we get two diagrams:



$S = \frac{1}{2}\cdot 2\cdot 2\cdot 2 = 4$ $\qquad$ $S = \frac{1}{2}\cdot 2 = 1$

$$iM_{13} = 4\cdot(ie^2 v)^2\, \epsilon^\mu(q_1)\,\epsilon^\nu(q_2)\,(-i)\left(\frac{g_{\mu\nu}-\frac{q_\mu q_\nu}{M^2}}{q^2-M^2} + \frac{q_\mu q_\nu/M^2}{q^2-\xi M^2}\right) + iM_3$$

$$= iM_{U\text{-gauge}} + \underbrace{\frac{4ie^4v^2}{M^2}\,\frac{\epsilon_1\cdot q\;\epsilon_2\cdot q}{q^2-\xi M^2}} \qquad ; \; q = q_2 - k_2 = k_1 - q_1$$

$$= -4ie^2\,\frac{\epsilon_1\cdot k_1\;\epsilon_2\cdot k_2}{q^2-\xi M^2} \qquad ; \; \epsilon_i\cdot q_i = 0$$

and

$$iM_2 = e^2\,\epsilon_1\cdot(-k_1-q)\,\epsilon_2\cdot(-k_2+q)\,\frac{i}{q^2-\xi M^2}$$

$$= 4ie^2\,\frac{\epsilon_1\cdot k_1\;\epsilon_2\cdot k_1}{q^2-\xi M^2}$$

so clearly spurious ξ-dependent part cancels, and one recovers the U-gauge result.

Full spontaneously broken local U(1)-th. Lagrangian in R_ξ-gauge

$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_{ct}, \qquad \left(\begin{array}{ll} A_0 \equiv Z_3^{1/2} A & c_0 \equiv Z_{2g}^{1/2} c \\ \phi_0 \equiv Z_2 \phi & e_0 = e Z_1 \equiv e(1+\delta_1) \\ \lambda_0 Z_2^2 \equiv Z_\lambda \lambda = \lambda + \delta_\lambda & \mu_0^2 Z_2 = \mu^2 - \delta_\mu \end{array} \right)$$

where

$$\mathcal{L}_R = \frac{1}{2}\left[(\partial_\mu \eta)^2 - 2\mu^2\eta^2\right] + \frac{1}{2}\left[(\partial_\mu \pi)^2 - \xi(ev)^2\pi^2\right]$$

$$- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(ev)^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \quad - \bar{c}\left(\partial^2 + \xi(ev)^2\right)c$$

$$- \lambda v(\eta^3 + \eta\pi^2) - \frac{\lambda}{4}(\eta^2+\pi^2)^2$$

$$- eA^\mu(\pi\partial_\mu\eta - \eta\partial_\mu\pi) + \frac{1}{2}e^2(\eta^2+\pi^2)A^2 + e^2 v\eta A^2 - ie^2 v\,\bar{c}\eta c$$

and

$$\mathcal{L}_{ct} = -\frac{1}{2}(\delta_\mu + 3\delta_\lambda v^2)\eta^2 - \frac{1}{2}(\delta_\mu + \delta_\lambda v^2 + \delta_\pi \xi(ev)^2)\pi^2 + \frac{1}{2}\bar{\delta}_1 (ev)^2 A^2$$

$$+ \frac{1}{2}\delta_2(\partial_\mu\eta)^2 + \frac{1}{2}\delta_2(\partial_\mu\pi)^2 - \frac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu}$$

$$- \frac{\delta_\lambda}{4}(\eta^2+\pi^2)^2 - \delta_\lambda v\eta(\eta^2+\pi^2) - (v\delta_\mu + \delta_\lambda v^3 \overbrace{- v\mu^2 + \lambda v^3}^{=0 \text{ if set } \mu^2 \equiv \lambda v^2})\eta$$

$$- e\bar{\delta} A^\mu(\pi\partial_\mu\eta - \eta\partial_\mu\pi) + \frac{1}{2}e^2\bar{\delta}_1(\eta^2+\pi^2)A^2 + e^2\bar{\delta}_1 v\eta A^2$$

$$- \delta_{2g}\,\bar{c}\partial^2 c - \bar{\delta}_{1g}\,\xi(ev)^2\bar{c}c$$

With: $\delta_2 \equiv Z_2 - 1$; $\delta_3 \equiv Z_3 - 1$; $\delta_\lambda = \lambda(Z_\lambda - 1)$; $(\lambda_0 Z_2^2 \equiv Z_\lambda \lambda)$

$\delta_\mu = \mu^2 - \mu_0^2 Z_2$; $\delta_\pi = Z_1^2 Z_2^2 Z_3 - 1$; $\delta_{2g} = Z_{2g} - 1$

$\bar{\delta}_1 = Z_1^2 Z_2 Z_3 - 1$; $\bar{\delta} = Z_1 Z_2 Z_3^{1/2} - 1$; $\bar{\delta}_{1g} = Z_{2g} Z_1^2 Z_2 Z_3 - 1$