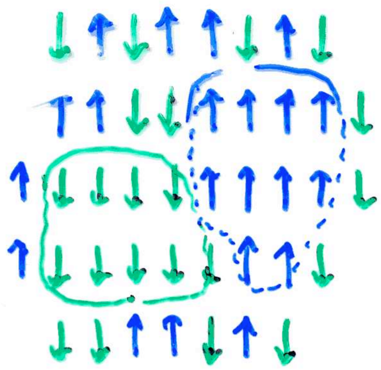
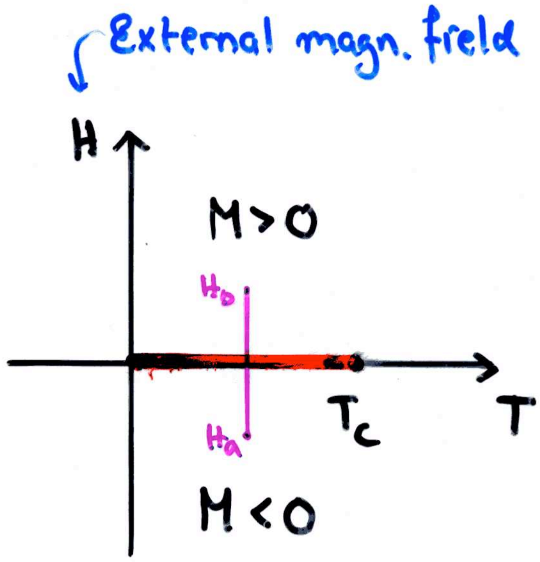
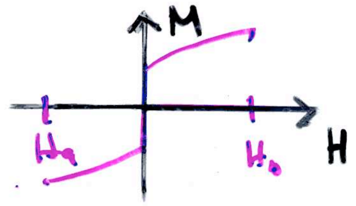
Landau theory (for a ferromagnet)

Consider a simple ferromagnetic system with a symmetry axis ($N=1$ -system)



- First order phase transition
- Order parameter magnetization M





magnetization changes discontinuously over $H=0$, when $T<T_c$.

Problem. When $T<T_c$ the macroscopic state of the system displays long (∞) range order. No order exists for $T>T_c$. How does this order arise quantitatively when $T\to T_c$ from above? Critical scaling laws. Universality.

Landau: Take $H=0$, and consider $T\sim T_c$
There M should be small $\Rightarrow$ the system can be described by truncated expansion Gibbs energy:

$$G(M,T) \simeq A(T) + B(T)M^2 + \underbrace{C(T)M^4 + ..}_{\text{small!}} \qquad (2.113)$$

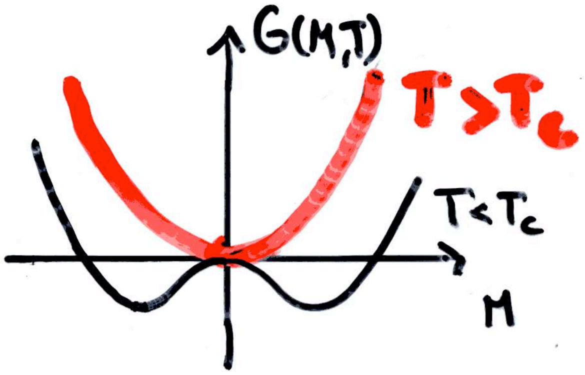
{ M^{2n+1} excluded by symmetry: $G(-M)=G(M)$ }

In absence of external field, minimizing G w.r.t M gives the preferred value of magnetization:

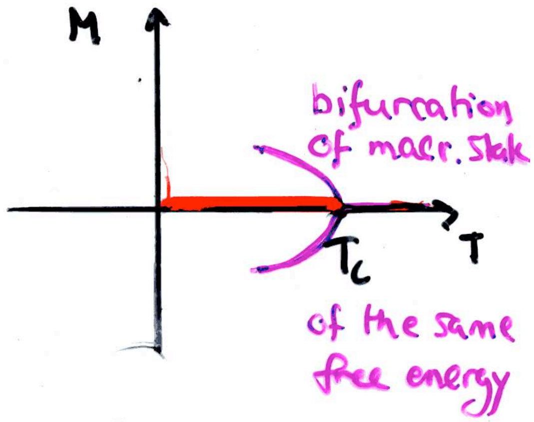
$$0 = \left(\frac{\partial G}{\partial M}\right)_T = 2M\left(B(T) + 2C(T)M^2\right) = 0 \qquad (2.114)$$

$$\Leftrightarrow \quad M = 0 \quad \text{or} \quad M = \pm\sqrt{\frac{-B(T)}{2C(T)}} \qquad (2.115)$$

Thus, choosing the phenomenological parameters suitably, we can arrange that $M=0$ is the minimum for $T>T_c$ and $M\neq 0$ when $T<T_c$. The simplest choice, made by Landau is



$$\begin{cases} B(T) \equiv b(T-T_c) \\ C(T) \equiv c \end{cases} \qquad (2.116)$$

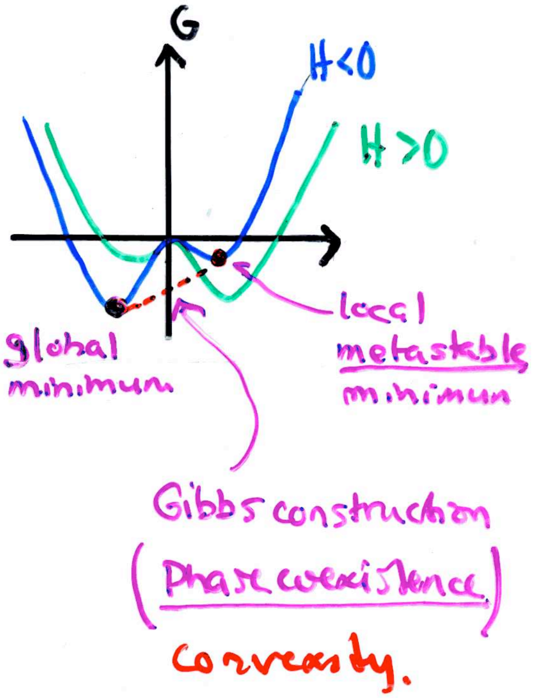


$$\Rightarrow \quad \underline{M = \pm\sqrt{\frac{b}{2c}}\,(T_c - T)^{1/2}} \qquad (2.117)$$

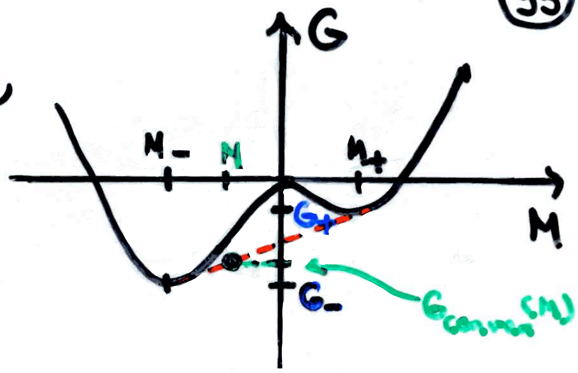
in the "broken" phase $T<T_c$, while $M=0$ for $T>T_c$.

With an external field included, we have

$$G(M,T,H) = A(T) + B(T)M^2 + C(T)M^4 - HM \qquad (2.118)$$



Now we have global and local minima in an asymmetric potential. These minima are physically meaningful despite the fact that the true thermodynamical $G(M,H,T)$ must be convex. (See fig.) The convexity requires taking into account possibility of phase coexistence. When this is allowed, the convex construction comes from



$$G_{convex} = (1-x)G_- + x\,G_+ \qquad (2.119)$$

where a fraction x of the system is in phase $M>0$ and fraction $1-x$ in state $M<0$, with $x = (M-M_-)/(M_+-M_-)$. (see fig.).

Fluctuations

In order to study fluctuations, generalize the purely macroscopic Landau theory to a field theory with local spin-field $s(\vec{x})$:

$$\boxed{G[s(x),T] \equiv \int d^3x \left[\frac{1}{2}(\nabla s)^2 + b(T-T_c)\,s^2 + c\,s^4 - H(x)\,s(x)\right]} \qquad (2.120)$$

$$\boxed{M \equiv \int d^3x\; s(\vec{x})}$$

keep this from macroscopic theory. Induces phase transition

This is of course nothing but an effective ($\lambda\phi^4$-) Euclidean QFT in 3 dimensions, so all of our RGE-analysis in d-dimensions will apply. This is still not a fundamental theory for the spin system, but it should be a good description for it near T_c.

Minimize (2.120):

if $M\approx 0$, $T\sim T_c$

≈ 0

local ext. field.

$$0 = \delta G[s] = -\nabla^2 s + 2b(T-T_c)\,s + 4s^3 - H(\vec{x}) \qquad (2.121)$$

Close to the critical point this gives

$$\left[-\nabla^2 + 2b(T-T_c)\right] s(\vec{x}) = H(\vec{x}) \qquad (2.122)$$

("Klein-Gordon" eq in 3-d Eucl. th.)

If we use $H(x) \equiv H_0\,\delta(\vec{x})$, then $s(\vec{x})$ describes the response of the spin field at $\vec{x}$ for aligning the spin at $\vec{x}=0$ along the external field. The solution thus measures long distance correlation. This solution also corresponds to the spin-spin correlation function

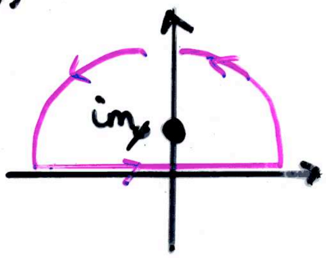
$$D(x) = \langle s(\vec{x})\, s(0)\rangle, \qquad (2.123)$$

which is the Greens function for (2.122) with a generic $H(\vec{x})$. Fourier transforming we get

$$D(x) = \int \frac{d^3k}{(2\pi)^3} \frac{H_0\, e^{i\vec{k}\cdot\vec{x}}}{|\vec{k}|^2 + \underbrace{2b(T-T_c)}_{\equiv m_x^2}} = \frac{H_0}{4\pi^2}\int_0^\infty dk\, k^2\, \frac{e^{ikr}-e^{-ikr}}{ik\,(k^2+m_x^2)}$$

$$= \frac{H_0}{4\pi^2 r i}\int_{-\infty}^{\infty} dq\, \frac{q\, e^{iqr}}{q^2+m_x^2} = \frac{H_0}{4\pi r}\, e^{-r m_x}$$

$$= \frac{H_0}{4\pi r}\, e^{-r/\xi}$$



So, the correlation length for fluctuations is

$$\xi = \left[2b(T-T_c)\right]^{-1/2} \qquad (2.124)$$

This result corresponds to the tree-level QFT-prediction from (2.111), with $y_n \equiv 0$. We already know what is the 1-loop correction from the fact that (2.120) is described by the WF-fixed point at $d=3$, whereby the Landau-th prediction

$$\nu = \frac{1}{2} \xrightarrow[\text{QFT}]{\text{1-loop}} \nu = 0.6 \qquad (2.125)$$

Experimentally $\nu \sim 0.63$ for a variety of systems

Some comments.

* parameters b & c depend on the physics at atomic scale, (but they can be measured from critical scaling exp.)

* Dependence of M & ξ on $T-T_c$ including the critical exponents $M \sim (T-T_c)^{1/2}$ and $\xi \sim (T-T_c)^{-1/2}$ are universal, independent of the microscopic system. Only an order parameter and symmetry $G(-M) = G(M)$ were needed.

We already saw how QFT-loop corrections improved the computation of ν. We can similarly find corrections to many other critical scaling laws, including magnetization $M \sim t^{\beta}$, $M \sim H^{1/\delta}$ anomalous scaling of spin-spin correlations $\langle s(x) s(0) \rangle \sim |x|^{2-d-\eta}$, and magnetic susceptibility $\chi \sim t^{-\gamma}$, where $t \equiv (T-T_c)/T_c$. We will compute all these systematically from RGE near WF-fixed point. Before that we pause to re-state the analog between $\lambda\phi^4$-QFT & theory of magnetization.

We also need to learn how to RGE-improve the Gibbs free energy itself!

QFT - ferromagnetism analog - effective action in QFT

We have already seen that

Stat. system	QFT
$s(x)$ (effective!)	ϕ (fundamental?)
$s_{a}(x)$; M	$\phi_{cl}(x)$; ϕ_{cl}
local global mag.	class. field global, const. c.f.
$F[h(x)]$	$W[J]$
$G[s(x)]$; $G[M]$	$\Gamma[\phi_{cl}(x)]$, $V_{eff}(\phi_c)$

Indeed, let us remember that we defined the effective action $\Gamma[\phi_{cl}]$ as the Legendre transformation

$$\underline{\Gamma[\phi_{cl}] \equiv W[J] - \int d^4x\, \phi_{cl} J} \qquad (2.126)$$

with

$$\phi_{cl} = \frac{\partial W}{\partial J} = {}_J\langle \Omega | \phi | \Omega \rangle_J \qquad (2.127)$$

expectation value of the qm. field given source J

$$\frac{\partial \Gamma[\phi_{cl}]}{\partial \phi_{cl}} = J(x) \xrightarrow{J\to 0} 0 \qquad (2.128)$$

true minimum of the system including quantum corrections

These are clearly just the analogs of statistical relations

$$G[s_a] = F[h] - \int d^4x\, s_a(x) h(x)$$

$$s_a(x) = \frac{\partial F}{\partial h} \quad ; \quad \frac{\partial G}{\partial s_a} = h(x) \qquad (2.129)$$

local magnetization in field $h(x)$

gives the local conf. of s when h=0.

At tree level of course $S(\vec{x}) = S_{cl}(\vec{x})$, and so we did not need to make the distinction.

$\Gamma[\phi_{cl}]$ can in general be expanded as (expansion in momentum)

$$\Gamma[\phi_{cl}] = \int d^4x \left[Z[\phi_{cl}] (\partial \phi_{cl})^2 - V[\phi_{cl}] + \dots \right] \qquad (2.130)$$

where $\phi_{cl} = \phi_{cl}(x)$. The minimum energy configuration in absence of external field is given by $\partial \Gamma / \partial \phi_{cl} = 0$ (cf. $\partial G / \partial s = 0$).

The absolute minimum configuration must be homogenous ϕ_{cl} = const. For such configurations all derivative terms drop in (2.130) and $\Gamma[\phi_{cl}]$ reduces to effective potential

$$\underset{\text{functional}}{\Gamma[\phi_{cl}]} \xrightarrow{\phi_{cl} = \text{const}} \underset{\text{ordinary function.}}{-V_{eff}(\phi_{cl})} \qquad (2.131)$$

The analog of this situation is moving backwards from theory (2.120) for local spin field to Landau theory (2.113, 2,116) for global constant magnetization: $G[s] \longrightarrow G(M)$.

Quantum corrections to the tree level potential, eg.

$$V(\phi) = +\frac{\lambda}{4!}\phi^4 \longrightarrow V_{eff}(\phi_{cl}) \qquad (2.132)$$

play important role in QFT, both at zero- and at finite temperatures, in connection with vacuum stability analysis, and in the theory of phase transitions. We will perform explicit calculations of V_{eff} shortly.

Homogeneity, made to get

- Note that the assumption (2.131) leads to the demise of phase coexistence and therefore of convexity of $V_{eff}(\phi_{cl})$. (And similarly for the ferromagnet in going from $G[s]$ to $G[M]$.)

Effective action was also found to be the generator of all connected irreducible n-point functions in the theory:

$$\Gamma^{(n)}(x_1,\dots,x_n) = i\,\frac{\delta^n \Gamma[\phi_{cl}]}{\delta\phi_{cl}(x_1)\cdots\delta\phi_{cl}(x_n)} \qquad (2.133)$$

Thus we can write ($\Gamma^{(1)}=0$ for $\lambda\phi^4$-theory with symmetry $\phi\to-\phi$)

$$i\Gamma[\phi_c] = \sum_{n=2}^{\infty}\frac{1}{n!}\int d^4x_1\dots d^4x_n\,\phi_{cl}(x_1)\cdots\phi_{cl}(x_n)\,\Gamma^{(n)}(x_1,\dots,x_n) \qquad (2.134)$$

Now, given the CS-equation for $\Gamma^{(n)}$:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+\beta_m\rho_m\frac{\partial}{\partial\rho_m}+n\gamma\right)\Gamma^{(n)} = 0 \qquad (2.135)$$

We find (sum over all n-point functions, integrating with $\int\prod_{i=1}^{n} d^4x_i\,\phi_{cl}(x_i)$)

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+\beta_m\rho_m\frac{\partial}{\partial\rho_m}\right)\Gamma[\phi_{cl}]$$

$$+\gamma\underbrace{\sum\int d^4x_1\dots d^4x_n\frac{1}{(n-1)!}\,\phi_1\cdots\phi_n\,\Gamma^{(n)}}_{=\int dx\,\phi_{cl}(x)\frac{\delta}{\delta\phi_{cl}(x)}\Gamma[\phi_{cl}]} = 0$$

Ie, we have the CS-RGE for $\Gamma[\phi_{cl}]$:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+\tilde{\beta}_m\rho_m\frac{\partial}{\partial\rho_m}+\gamma\int d^4x\,\phi_{cl}\frac{\delta}{\delta\phi_{cl}}\right]\Gamma[\phi_{cl}]=0 \quad (2.136)$$

In particular for a constant field case this becomes an CS-eqn. for V_{eff}:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+\tilde{\beta}_m\rho_m\frac{\partial}{\partial\rho_m}+\gamma\phi_{cl}\frac{\partial}{\partial\phi_{cl}}\right]V_{eff}(\phi_{cl})=0 \quad (2.137)$$

We shall put these equations to good use shortly. For now we stop the development of 4-d QFT and return to spin-system and critical phenomena. The analog of (2.137) for spin-system is the CS-equation for G in M:

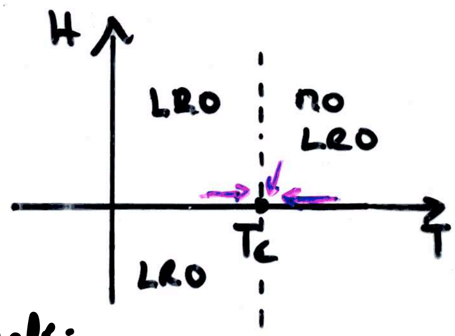
$$\left(\mu\frac{\partial}{\partial\mu}+\sum_i\beta_i\frac{\partial}{\partial\rho_i}+\gamma M\frac{\partial}{\partial M}\right)G(M,\mu,\{\rho_i\})=0 \quad (2.138)$$

this contains $\beta(\lambda)\frac{\partial}{\partial\lambda}$; $\tilde{\beta}_m\rho_m\frac{\partial}{\partial\rho_m}$

and possibly many other (irrelevant) operators which are removed in the end by RGE flow near WF-fixed point.

(assumption)

Critical exponents ; systematics



As noted before, the onset of macroscopic long range order when one approaches the critical point leads to characteristic power law scaling of many physical quantities. For example:

$$M \sim t^{\beta} \quad \text{and} \quad M \sim H^{1/\delta} \quad (T \gtrsim T_c) \qquad (2.139)$$

$t = \bar{p}_m$

where $t \equiv (T-T_c)/T_c$. These are positive powers, since M vanishes for $T \gtrsim T_c$ when $H=0$. Moreover, the specific heat should diverge at T_c. Indeed, from $\Delta Q = n c_v \Delta T$, and $\Delta Q = T\Delta S + \ldots$ we see that $c \sim \Delta S/\Delta T$. Due to onset of LRO infinitesimal change in T brings a macroscopic change in entropy: One defines

$$C_H = T\left(\frac{dS}{dT}\right)_H$$

$$\propto \frac{\partial^2 G}{\partial t^2} \cong t^{-\alpha} \qquad (2.140)$$

Moreover, the magnetic susceptibility $\chi_{ij} = \partial M_i/\partial H_j$ should diverge. Here χ is a scalar:

$$\chi = \frac{\partial M}{\partial H} \sim t^{-\gamma} \qquad (2.141)$$

Finally, the functional form of the spin-spin correlation function is expected to change:

correlation length

$$D(x) \sim e^{-|x|/\xi} \frac{1}{|x|^{d-2+\eta}} \qquad (2.142)$$

compare with (DG)

anomalous scaling due to interactions

and we have already computed the scaling

$$\xi \sim t^{-\nu} \qquad (2.143)$$

to 1-loop in N=1 symmetric ($G(M)=G(-M)$) model. All these exponents $\beta, \delta, \alpha, \gamma, \eta$ and ν will be computed in terms of the anomalous dimensions $\gamma(g^*)$ and $\gamma_m(g^*)$ at WF fixed point, set by $\beta(g)$.

① Two-point function: exponents ν & η.

consider the spin-spin correlation function, now promoted to "quantum level". It obeys

dimensionless.

$$\left[\mu\frac{\partial}{\partial\mu} + \sum_i \beta_i \rho_i \frac{\partial}{\partial \rho_i} - 2\gamma\right] D(x; \mu, \{\rho_i\}) = 0 \qquad (2.144)$$

We can equally well perform the analysis in direct space. Then by dimensional analysis ($\mu|\vec{x}|$ is dimensionless)

$$D(\vec{x}, \mu, \{\rho_i\}) = \frac{1}{|\vec{x}|^{d-2}} d(\mu|\vec{x}|, \{\rho_i\}) \qquad (2.145)$$

and, as usual, the solution becomes

$$D(\vec{x}, \mu, \{\bar{\rho}_i\}) = \frac{1}{|\vec{x}|^{d-2}} h(\{\bar{\rho}_i\})\, e^{+2\int_0^t \gamma(\{\bar{\rho}_i\})\,dt'} \qquad (2.146)$$

where

$$\frac{\mu|\vec{x}|}{\bar{\rho}_i}\frac{d\bar{\rho}_i}{\partial \mu|\vec{x}|} = \beta_i(\{\bar{\rho}_j\}), \qquad (2.147)$$

$\beta_m = 2 - 2\gamma_m$

$\beta_i < 0 \quad i \neq m.$ irrelevant, neglect.

If we are close to a nontrivial fixed point, then

$$\bar{\rho}_m = \rho_m (\mu|\vec{x}|)^{2-2\gamma_m^*} = \rho_m (\mu|\vec{x}|)^{\tilde{\beta}_m} \qquad (2.147)$$

$$\rho_i = \rho_i (\mu|\vec{x}|)^{-A_i} \qquad A_i > 0 \; ; \; i \neq m$$

Similarly, the anomalous exponent becomes a power in $\mu|\vec{x}|$, and altogether, one expects that

$$D(\vec{x}) \sim \frac{1}{|\vec{x}|^{d-2}} (\mu|x|)^{2\gamma^*} f(\bar{\rho}_m)$$

$$\sim \frac{1}{|\vec{x}|^{d-2-2\gamma^*}} \cdot f\left(t (\mu|\vec{x}|)^{2-2\gamma_m^*}\right) \qquad (2.148)$$

where in the last step we took $\rho_m \sim t$ to allow adjust mass to the critical point.

First, we immediately see that

$$\eta = -2\gamma^* \qquad (2.149)$$

This is still zero at 1-loop, because we do not have w.f.r.-coefficient at 1-loop. The 2-loop result is (ex. 1/7):

$$\eta = -2\gamma^* = \frac{2\lambda_*^2}{12(16\pi^2)^2} = \left(\frac{16\pi^2}{3}\epsilon\right)^2 \frac{1}{6(16\pi^2)^2}$$

$$= \frac{1}{54}\epsilon^2 \rightarrow \frac{1}{54} \approx 0{,}0185 \qquad (2.150)$$

Moreover, comparing the form (2.148) to the expected form (2.142) we get

$$\exp(-|\vec{x}|/\xi) \sim f(\bar{\rho}_m) \sim f(t(\mu|\vec{x}|)^{\tilde{\beta}_m})$$

$$\sim \hat{f}(|\vec{x}|(\mu t^{1/\tilde{\beta}_m})) \qquad (2.151)$$

$$\Rightarrow \xi \sim t^{-\frac{1}{\tilde{\beta}_m}} \equiv t^{-\nu} \qquad (2.152)$$

i.e. we get our old result (2.111) for the exponent ν:

$$\underline{\nu = \frac{1}{2-2\gamma_m^*}} . \quad \left(= \frac{1}{\tilde{\beta}_m^*}\right) \qquad (2.153)$$

$\gamma_m = \frac{1}{6} \Rightarrow \nu = \frac{1}{2-\frac{1}{3}} = \frac{3}{5} = 0.6$

(II) Gibbs free energy

The other exponents β, δ, α and γ can be found from the scaling properties of the Gibbs free energy. To find these (in terms of t and H), we must solve eqn. (2.138). Assuming already that only ρ_m survives the scaling near λ_* we have

$\tilde{\beta}_m = 2-2\gamma_m$ (2.105)

$$\underline{\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + \tilde{\beta}_m\rho_m\frac{\partial}{\partial\rho_m} + \gamma H\frac{\partial}{\partial H}\right) G(\mu,\lambda,\rho_m,H) = 0} \qquad (2.154)$$

Where again $\beta(\lambda) = \mu\frac{\partial\lambda}{\partial\mu}$, $\tilde{\beta}_m = \frac{\mu}{\rho_m}\frac{\partial\rho_m}{\partial\mu}$ etc.

Consider being very close to $\lambda = \lambda_*$ (WF), such that also $\partial G/\partial\lambda \approx 0$. (This is the most slowly vanishing correlation because λ is assumed to be near fixed point. See P&S, problem 13.1 for further insight.) Furthermore, by dimensional grounds we see that

$$[G] = L^{-d} \quad \text{and} \quad [M] = [s] = L^{\frac{2-d}{2}}, \qquad (2.155)$$

so that $\quad [L] = M^{\frac{2}{2-d}} \Rightarrow [G] = M^{2d/(d-2)}$

↑ magnetization

$$\underline{G(\mu, \lambda, \rho_m, M)} \approx G(\lambda_*, \rho_m, M\mu^{\frac{2-d}{2}})$$

$$= \underline{M^{\frac{2d}{d-2}} \hat{g}_*(M\mu^{\frac{2-d}{2}}, \rho_m)} \qquad (2.156)$$

Here the dimensionless $\hat{g}_*$ satisfies the equation

$$\left[\left(\frac{2-d}{2} + \gamma^*\right) M \frac{\partial}{\partial M} + \tilde{\beta}_m \rho_m \frac{\partial}{\partial \rho_m} + \frac{2d}{d-2}\gamma^*\right] \hat{g}_* = 0 \qquad (2.157)$$

$$\Leftrightarrow \quad \underline{\left[M \frac{\partial}{\partial M} + \hat{\beta}_m \bar{\rho}_m \frac{\partial}{\partial \bar{\rho}_m} + \hat{\gamma}_*\right] \hat{g}_* = 0} \qquad (2.158)$$

where

$$\underline{\hat{\beta}_m \equiv \frac{-2\tilde{\beta}_m}{d-2-2\gamma^*}} \; ; \quad \underline{\hat{\gamma} = \frac{-4d\gamma^*}{(d-2-2\gamma^*)(d-2)}} \qquad (2.159)$$

with $\tilde{\beta}_m \equiv 2 - 2\gamma_m$, so both $\hat{\beta}_m$ and $\hat{\gamma}$ are known to us at $\lambda = \lambda_*$. (2.158) is solved normally:

$$\underline{G(M) = M^{\frac{2d}{d-2}} \hat{h}_*(\bar{\rho}_m) \exp\left[-\int_0^{\hat{t}} d\hat{t}' \, \hat{\gamma}(\{\bar{\rho}_i\})\right]} \qquad (2.160)$$

↑ in principle. Here $\hat{\gamma}_* = \hat{\gamma}_* = \underline{\text{const}}$.

with $\frac{1}{\bar{\rho}_m}\frac{\partial \bar{\rho}_m}{\partial \hat{t}} = \hat{\beta}_m$.

In our approximation $\hat{\beta}_m$ and $\hat{\gamma}$ are constants, defined by WF fixed point (IR limit), so that (here $\hat{t}$ is just the scaling parameter $\exp(\hat{t}-\hat{t}_0) = M/M_0$ do not confuse with $t \equiv (T-T_c)/T_c$)

$$\bar{\rho}_m = \rho_m e^{\hat{\beta}_m \hat{t}}$$

and

$$G(M,t) \sim M^{\frac{2d}{d-2}} e^{-\hat{\gamma}\hat{t}} \hat{h}_*\left(\rho_m e^{\hat{\beta}_m \hat{t}}\right) \qquad (2.162)$$

Taking $\hat{t} \equiv \log M\mu^{\frac{2-d}{2}}$ to get the appropriate scaling in M, we get

$$\bar{\rho}_m = \rho_m \left(M\mu^{\frac{2-d}{2}}\right)^{\hat{\beta}_m} \sim t M^{\hat{\beta}_m} \qquad (2.163)$$

and

$$G(M,t) = M^{\frac{2d}{d-2}} \left(M\mu^{\frac{2-d}{2}}\right)^{-\hat{\gamma}} \hat{h}_*\left(t\left(M\mu^{\frac{2-d}{2}}\right)^{\hat{\beta}_m}\right)$$

$$\cong M^{1+\delta} \hat{h}_*\left(tM^{-1/\beta}\right) \qquad (2.164)$$

Where I used at last $\rho_m \sim t \equiv (T-T_c)/T_c$. The exponents δ and β are given by

$$1+\delta = \frac{2d}{d-2} - \hat{\gamma} = \frac{2d}{d-2}\left(1 + \frac{2\gamma_*}{d-2-2\gamma_*}\right) = \frac{2d}{d-2-2\gamma_*}$$

$$\Leftrightarrow \quad \delta = \frac{d+2+2\gamma_*}{d-2-2\gamma_*} \qquad (2.165)$$

$d=3$, $2\gamma_* = -\frac{1}{54}$ $\Rightarrow \delta = \frac{5-\frac{1}{54}}{1+\frac{1}{54}} = \frac{269}{55} \sim 4{,}89$

$\gamma_m^* = \frac{1}{6}$

And

$$\beta \equiv -\frac{1}{\hat{\beta}_m} = \frac{d-2-2\gamma_*}{4(1-\gamma_m^*)} \qquad (2.166)$$

$\beta = \frac{1+\frac{1}{54}}{4\left(1-\frac{1}{6}\right)} = \frac{55}{54}\,\frac{3}{10} = \frac{55}{180} \sim 0{,}306$
$= \frac{11}{36}$

The critical exponents needed are found from (2.164) and

(2.165 - 2.166). First note that along the line $t=0$ (or $t\approx 0$)

$$H = \frac{\partial G}{\partial M} = \hat{h}_*(0)\, M^{\delta}, \quad \Rightarrow \quad \underline{M \sim H^{1/\delta}} \qquad (2.167)$$

so that δ really is the exponent giving Magnetization as a function of H. Second, rewrite (2.164) slightly to form

$$\underline{G(M,t) = t^{\beta(1+\delta)}\, \hat{u}_*(M t^{-\beta})} \qquad (2.168)$$

Below T_c, as we have seen, magnetization is found by minimizing G with respect to M. In the scaling region this minimum occurs at some m_0 minimizing $\hat{u}_*$. That is

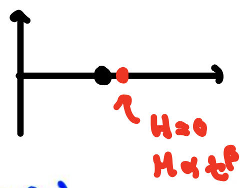
$$m_0 = M t^{-\beta} = \text{const} \quad \Rightarrow \quad \underline{M \sim t^{+\beta}} \qquad (2.169)$$

So β was also just the critical exponent for magnetization as a function of t near T_c. An alternative way to see (2.169) is by use of $\bar{p}_m$-scaling relation (2.163). Correlations in M are relevant when $\bar{p}_m \lesssim 1$, so that on border-line

$$\bar{p}_m \sim t M^{\hat{\beta}_m} \sim 1 \quad \Rightarrow \quad M \sim t^{-1/\hat{\beta}_m} \equiv t^{\beta}. \qquad (2.170)$$

We still need the exponents for specific heat and magnetic susceptibility. From (2.169) one sees that above T_c in $H=0$,

$$G(t) \sim t^{\beta(1+\delta)} \qquad (2.171)$$



Then

$$C_H \sim \frac{\partial^2 G}{\partial t^2} \sim t^{\beta(1+\delta)-2} \cong t^{-\alpha} \qquad (2.172)$$

So that by use of (2.165 - 2.166):

$$2-\alpha = \beta(1+\delta) = \frac{d}{2-2\gamma_m^*} \Leftrightarrow \alpha = 2 - \frac{d}{2-2\gamma_m^*} \quad (2.173)$$

Finally magnetic susceptibility is $\chi = \partial M / \partial H$. Now at nonzero t we have from (2.168) that

$$H = \frac{\partial G}{\partial M} = t^{\beta\delta}\, \hat{u}_*'(Mt^{-\beta}) \quad (2.174)$$

Inverting this relation gives M

$$M = t^{\beta}\, \hat{c}(Ht^{-\beta\delta}) \quad (2.175)$$

whereby, close to $H \approx 0$

$$\frac{\partial M}{\partial H} = \chi = t^{(1-\delta)\beta}\, \hat{c}'(0) \cong t^{-\gamma} \Rightarrow \quad (2.176)$$

and so eventually

$$\gamma = (\delta-1)\beta = \frac{4(1-\gamma^*)}{d-2+2\gamma^*}\,\frac{d-2+2\gamma^*}{4(1-\gamma_m^*)} = \frac{1+\gamma^*}{1-\gamma_m^*} \quad (2.177)$$

$$= \frac{1-\frac{1}{108}}{1-\frac{1}{6}} = \frac{107}{108}\,\frac{6}{5} = \frac{107}{90}$$

This was the last exponent we needed. Computing to order ϵ^2, we have

$\epsilon = 1$

$$\begin{aligned} \gamma_m^* &= \frac{1}{6}\epsilon + \mathcal{O}(\epsilon^2) \approx \frac{1}{6} \\ -\gamma^* &= \frac{1}{108}\epsilon^2 + \mathcal{O}(\epsilon^3) \approx \frac{1}{108} \end{aligned} \quad (2.178)$$

Using these as our best knowledge (best expansion on γ_m and γ^*, which are the field th. parameters we have), we get:

	$D \sim \frac{1}{\lvert x \rvert^{1+\eta}}$	$\xi \sim t^{-\nu}$	$M \propto t^{\beta}$	$M \propto H^{1/\delta}$	$C_H \sim t^{-\alpha}$	$\chi \sim t^{-\gamma}$
Exponent	η	ν	β	δ	α	γ
Expression in $d=3$	$-2\gamma^*$	$\frac{1}{2(1-\gamma_m^*)}$	$\frac{1-2\gamma^*}{4(1-\gamma_m^*)}$	$\frac{5+2\gamma^*}{1-2\gamma^*}$	$2-\frac{3}{2-2\gamma_m^*}$	$\frac{1+\gamma^*}{1-\gamma_m^*}$
(1-loop γ_m, 2-loop γ)	$\frac{1}{54}=0{,}019$	$\frac{3}{5}=0{,}6$	$\frac{11}{36}\approx 0{,}306$	$\frac{269}{55}\approx 4{,}89$	$\frac{1}{5}=0{,}2$	$\frac{107}{90}\approx 1.19$
best ϵ-exp	0,032	0,63	0,325		0,110	1,241
experiment	0,016	0.625	0.325		0,113	1,240
	0,04	0,65	0,34		0,12	1,22
Landau		0,5	0,5			

Observational values actually come from binary liquid (green) and from binary alloy (red) systems!

Clearly already our simple 1- to 2-loop CS-analysis results are in pretty good agreement with observations!

In exercise 2/7 you will generalize these results to $O(N)$-symmetric systems.

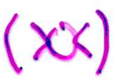
CRITICAL BEHAVIOUR IN 4D. RGE improved effective potential.

As was discussed earlier, the effective action provides the way to compute the minimum energy configuration of the system including the effect of interactions. In particular, the global minimum will be given by V_{eff}. Comparing (2.134) and (2.130) we see that

$$-V_{eff}(\phi_{cl}) = i\sum_{n=2}^{\infty} \frac{1}{n!} \Gamma^{(n)}(p_i=0)\, \phi_{cl}^n \qquad (2.179)$$

where $\Gamma^{(n)}(p_i=0)$ is the n-point 1PI-diagram computed at zero external momentum.

Equation (2.179) can, and have been used to compute V_{eff} directly in loop expansion:

$$-V_{eff} \sim \frac{i}{2}\Big($$  $$\Big)\phi_{cl}^2 + \frac{i}{4!}\Big($$  $$\Big)\phi_{cl}^4 + \frac{1}{3!}\Big($$  $$\Big)\phi_{cl}^6 + \dots \qquad (2.180)$$

However, there are also easier, more direct ways to get V_{eff}. Indeed, consider expanding the theory around some $\phi = \eta \neq 0$. Then:

$$-V_{eff}(\phi_{cl}) = i\sum_{n=2}^{\infty} \frac{1}{n!} \tilde{\Gamma}_{\eta}^{(n)}(p_i=0)\, (\phi_{cl}-\eta)^n . \qquad (2.181)$$

Here $\tilde{\Gamma}^{(n)}$ must be computed in a shifted theory, expanded around the background field η. This causes some trouble, but the reward is that quite simply

$$\boxed{\left.\frac{\partial V_{eff}}{\partial \phi_{cl}}\right|_{\phi_{cl}=\eta} = i\tilde{\Gamma}^{(1)}_{\phi_{cl}}(p_i=0)} \qquad (2.182)$$

That is, V_{eff} is given by the 1-point function (Tadpole) in the shifted theory. For $\lambda\phi^4$-theory (here $\phi = \phi_q$ and η is classical field)

$$\mathcal{L}(\phi+\eta) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}(\phi+\eta)^2 - \frac{\lambda}{4!}(\phi+\eta)^4$$
$$+ \frac{\delta_Z}{2}(\partial\phi)^2 - \frac{\delta_m}{2}(\phi+\eta)^2 - \frac{\delta_\lambda}{4!}(\phi+\eta)^4 \qquad (2,183)$$
$$= \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\underbrace{(m^2+\tfrac{1}{2}\lambda\eta^2)}_{\text{new } m^2_{eff}(\eta)}\phi^2 - \underbrace{\left[(m^2+\delta_m)\eta + \frac{\lambda+\delta_\lambda}{6}\eta^3\right]}_{0^{th}\text{ order corr.}}\phi - \underbrace{\frac{\lambda}{6}\eta}\phi^3 + \dots$$

$\frac{\lambda\eta}{6}$

Where we have shown only terms relevant for computation to 1-loop order. Obviously

$$+i\Gamma^{(1)}_{0\text{-th order}} = (m^2+\delta_m)\eta + \frac{\lambda+\delta_\lambda}{6}\eta^3 \qquad (2,184)$$

So that

$$V_{eff}^{0th\text{-order}}(\phi_{cl}) = +\int_0^{\phi_c} d\eta\, i\Gamma^{(1)}_{0th\text{-ord}} = \frac{1}{2}(m^2+\delta_m)\phi_{cl}^2 + \frac{1}{4!}(\lambda+\delta_\lambda)\phi_{cl}^4 \qquad (2,185)$$

As one would expect. Note that δ_m and δ_λ are all that one needs to renormalize $V_{eff}(\phi_{cl})$. All other terms but $n=2$ & $n=4$ in expansion (2.180) are finite!

The first order correction is

$$+i\Gamma^{(1)}_{1st\text{ order}} = i$$  $$= i\,\frac{-i\lambda\eta}{6}\cdot 3\cdot\mu^\epsilon \overbrace{\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2_{eff}(\eta)}}^{iA_0(m^2_{eff}(\eta))} \qquad (2,185b)$$
$$= +\frac{\lambda i}{2}\eta\,(4\pi\mu^2)^{\epsilon/2}\cdot\frac{-i}{16\pi^2}\,\Gamma\!\left(1-\frac{\epsilon}{2}\right)\left[m^2_{eff}(\eta)\right]^{1-\frac{\epsilon}{2}}$$

$$= + \frac{\lambda \eta}{32\pi^2}\left(-\frac{2}{\epsilon} - 1 - \gamma_E\right)\left(4\pi \frac{\mu^2}{m_{\text{eff}}^2(\eta)}\right)^{\frac{\epsilon}{2}} m_{\text{eff}}^2(\eta)$$

$$= \frac{\lambda \eta}{32\pi^2} m_{\text{eff}}^2(\eta)\left(B_\epsilon + \log \frac{m_{\text{eff}}^2}{\mu^2}\right) \qquad (2.186)$$

That is, to 1-loop order (evt. $\eta \to \phi_{cl}$)

$$\frac{dV_{\text{eff}}}{d\eta} = (m^2 + \delta_m)\eta + \frac{1}{6}(\lambda + \delta_\lambda)\eta^3 + \frac{\lambda\eta}{32\pi^2} m_{\text{eff}}^2(\eta)\left(B_\epsilon + \log\frac{m_{\text{eff}}^2}{\mu^2}\right). \qquad (2.187)$$

Renormalization of V_{eff}.

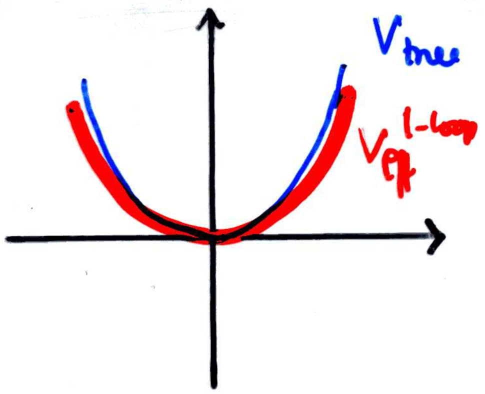
From original definition (2.179) we see that

$$\left.\frac{\partial^2 V_{\text{eff}}}{\partial \phi_{cl}^2}\right|_{\phi_{cl}=0} = i\Gamma^{(2)}(p=0) \qquad (= m^2)$$

$$\left.\frac{\partial^4 V_{\text{eff}}}{\partial \phi_{cl}^4}\right|_{\phi_{cl}=0} = i\Gamma^{(4)}(p=0) \qquad (= \lambda) \qquad (2.188)$$

So, if $\phi_{cl}=0$ is still the minimum of V_{eff}; $\left(\partial V/\partial \phi_{cl}\big|_{\phi_{cl}=0} = 0\right)$ and V_{eff} is regular, we can identify $i\Gamma^{(2)} \equiv m^2$ and $i\Gamma^{(4)} \equiv i\lambda$ as the renormalized mass and coupling of the theory.

$\Rightarrow V_{\text{eff}}(m, \lambda, \phi_{cl})$: finite.



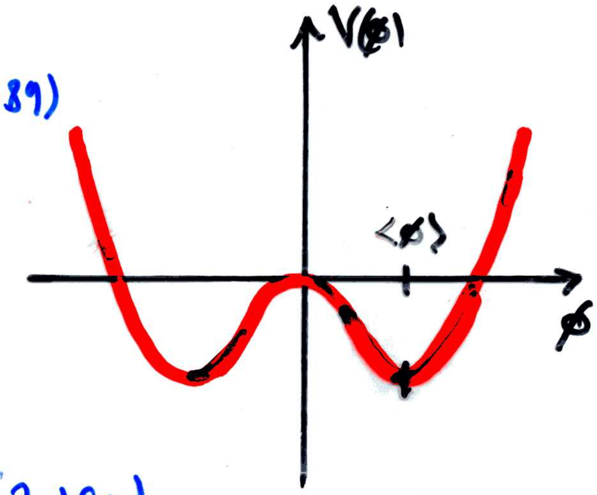
This approach is not always possible however. For example if the mass parameter in Lagrangean is negative $m^2<0$, it cannot be identified as the physical mass. In this case the symmetry is spontaneously broken.

Indeed consider

$$V_{tree} = -\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \qquad (2.189)$$

$$0 = \frac{\partial V_{tree}}{\partial\phi} = -\phi\left(m^2 - \frac{\lambda}{6}\phi^2\right)$$

$$\Rightarrow \phi = 0 \quad \vee \quad \phi = \sqrt{\frac{6}{\lambda}}\,m \equiv \langle\phi\rangle$$



(2.190)

In such case the theory will be quantized around the shifted field $\phi \equiv \langle\phi\rangle + \eta$, after which

$$\mathcal{L}(\eta) = \frac{1}{2}(\partial_\mu\eta)^2 + \frac{1}{2}\left(-m^2 + \frac{\lambda}{2}\langle v\rangle^2\right)\eta^2 + \dots$$

where

$$M^2 = -m^2 + \frac{\lambda}{2}\langle v\rangle^2 = \left.\frac{\partial^2 V}{\partial\phi^2}\right|_{\phi=\langle v\rangle} \overset{\text{tree level}}{=} +2m^2 > 0$$

(2.191)

This definition can be extended to $V \to V_{eff}$, as the definition of mass. There are other issues with renormalization of spontaneously broken theory, which we will leave aside for now. (Will return to it later.)

Let us consider two specific examples here. I) the case with $m^2>0$ and II) case with $m^2=0$ (Coleman-Weinberg).

I Case with $m^2>0$ (no SSB)

Choose the conditions (2.188). Obviously they will set

$$\delta_m \equiv -\frac{\partial^2}{\partial\eta^2}\delta V^{(1)}\Big|_{\eta=0}$$

$$\delta_\lambda = -\frac{\partial^4}{\partial\eta^4}\delta V^{(1)}\Big|_{\eta=0} \qquad (2.192)$$

Now write $\frac{\partial}{\partial\eta}\delta V^{(1)} \equiv \lambda\eta f(m_{eff}^2)$ with $f(x)=\frac{1}{32\pi^2}x\left(B_\epsilon+\log\frac{x}{\mu^2}\right)$ and note that $m_{eff}^2(0)=m^2$ and $\frac{\partial m_{eff}^2}{\partial\eta}=\lambda\eta$. Then

$$\frac{\partial^2}{\partial\eta^2}\delta V^{(1)} = \lambda f(m_{eff}^2)+(\lambda\eta)^2 f'(m_{eff}^2)$$

$$\Rightarrow \quad \delta_m = -\lambda f(m^2) = -\frac{\lambda}{32\pi^2}m^2\left(B_\epsilon+\log\frac{m^2}{\mu^2}\right). \qquad (2.193)$$

moreover.

$$\frac{\partial^4}{\partial\eta^4}\delta V^{(1)} = \frac{\partial}{\partial\eta}\left(3\lambda^2\eta\, f'(m_{eff}^2)+(\lambda\eta)^3 f''(m_{eff}^2)\right)$$

$$= 3\lambda^2 f'(m_{eff}^2) + \text{terms that vanish at } \eta=0$$

$$\Rightarrow \quad \delta_\lambda = -3\lambda^2 f'(m^2) = -\frac{3\lambda^2}{32\pi^2}\left(B_\epsilon+1+\log\frac{m^2}{\mu^2}\right) \qquad (2.194)$$

Inserting these expressions back to (2.187) gives

$$\frac{\partial V_{eff}}{\partial \eta} = m^2\eta + \frac{1}{6}\lambda\eta^3 + \frac{\lambda\eta}{32\pi^2} m_{eff}^2 \log m_{eff}^2$$

$$+ \frac{\lambda\eta}{32\pi^2}\Big((m^2+\tfrac{1}{2}\lambda\eta^2)(B_\epsilon - \log\mu^2)$$

$$- m^2(B_\epsilon - \log\frac{\mu^2}{m^2}) - \frac{1}{2}\lambda\eta^2(B_\epsilon - \log\frac{\mu^2}{m^2}+1)\Big)$$

$$= m^2\eta + \frac{1}{2}\lambda\eta^3 + \frac{\lambda\eta}{32\pi^2}\left(m_{eff}^2 \log\frac{m_{eff}^2}{m^2} - \frac{1}{2}\lambda\eta^2\right) \qquad (2.195)$$

Integrating this out gives:

$$V_{eff}(\phi_{cl}) = \int_0^{\phi_{cl}} d\phi_{cl}\frac{\partial V}{\partial\eta} = V_{tree}(\phi_{cl}) - \frac{1}{4}\frac{\lambda^2\phi_{cl}^4}{64\pi^2} + \frac{\lambda}{32\pi^2}\int_{m^2}^{m_{eff}^2} dx\, x\log x$$

$$= V_{tree} - \frac{\lambda^2\phi_{cl}^4}{4\cdot 32\pi^2}(m^2+\tfrac{3}{2}\lambda\phi_{cl}^2) + \frac{m_{eff}^4}{64\pi^2}\log\frac{m_{eff}^2}{m^2}. \qquad (2.196)$$

This result is slightly different from the $N\to 1$ limit of P&S on p. 372 due to different choice of renormalization condition.

(II) Case with $m^2=0$

Obviously our previous renormalization prescription cannot be used here! This can be traced to ill behaviour of $\partial^4 V/\partial\phi^4$ at $\phi=0$ (logarithmic divergence there). That is, we cannot define the coupling at $\phi=0$. We can still define the mass to be zero at $\phi=0$. So we take now:

$$\left.\frac{\partial^2 V}{\partial \phi^2}\right|_{\phi=0} = 0$$
$$\left.\frac{\partial^4 V}{\partial \phi^4}\right|_{\phi=M} = \lambda_M \qquad (2,197)$$

Where M is some arbitrary scale. We can perform analysis as before, only now $m_{eff}^2 \equiv \frac{1}{2}\lambda\eta^2$. so

$$\frac{\partial V}{\partial \eta} = \delta_m \eta + \frac{1}{6}(\lambda+\delta_\lambda)\eta^3 + \frac{\lambda^2\eta^3}{64\pi^2}\left(B_\epsilon + \log\frac{\lambda\eta^2}{2\mu^2}\right) \qquad (2,198)$$

One has:

$$\frac{\partial^2 V}{\partial \eta^2} = \delta_m + \frac{1}{2}(\lambda+\delta_\lambda)\eta^2 + \frac{\lambda^2\eta^2}{32\pi^2}\left(1+\frac{3}{2}\left(B+\log\frac{\lambda\eta^2}{2\mu^2}\right)\right) \overset{\eta=0}{=} \delta_m$$

$$\frac{\partial^4 V}{\partial \eta^4} = \lambda+\delta_\lambda + \frac{3\lambda^2}{32\pi^2}\left(B+\frac{11}{3}+\log\frac{\lambda\eta^2}{2\mu^2}\right) \qquad (2,199)$$

Thus, from (2,197), one has

$$\delta_m = 0$$
$$\delta_\lambda = -\frac{3\lambda_M^2}{32\pi^2}\left(B_\epsilon+\frac{11}{3}+\log\frac{\lambda M^2}{2\mu^2}\right) \qquad (2,200)$$

Whereby:

$$\frac{\partial V}{\partial \eta} = \frac{1}{6}\lambda_M\eta^3 + \frac{\lambda_M^2\eta^3}{64\pi^2}\left(\log\frac{\eta^2}{M^2}-\frac{11}{3}\right) \qquad (2,201)$$

Integrating this out immediately gives:

$$\boxed{V_{eff}(\phi_{cl}) = \frac{1}{4!}\lambda_M \phi_{cl}^4 + \frac{\lambda_M^2 \phi_{cl}^4}{256\pi^2}\left(\log\frac{\phi_{cl}^2}{M^2} - \frac{25}{6}\right)} \qquad (2.202)$$

The final expression is again finite and unique, given the definition of the coupling λ_M at $\phi_{cl} = M$. Loop correction is proportional to λ^2, since the coupling constant depence of the term inside log vanished due to ren. condition.

Obviously, a different choice of $M \to M'$ would not affect the form of V_{eff}, to order λ^2. Indeed, if we pick another M', then from

$$V_{eff}(\lambda_M, M, \phi_{cl}) - V_{eff}(\lambda_M', M', \varphi_{cl})$$

$$= \frac{1}{4!}(\lambda_M - \lambda_{M'})\phi_{cl}^4 + \frac{\lambda_M^2 \phi_{cl}^4}{256\pi^2}\log\frac{M'^2}{M^2} + O(\lambda^3)$$

That is, if we relate

$$\lambda_{M'} \equiv \left.\frac{\partial^4 V}{\partial\phi^4}\right|_{M'} \equiv \lambda_M + \frac{3\lambda_M^2}{32\pi^2}\log\frac{M'^2}{M^2} \qquad (2.203)$$

we see that V_{eff} remains same to order λ^3, i.e., the shift

$$V(\lambda_M, M, \phi_{cl}) \to V(\lambda_{M'}, M', \phi_{cl}) \qquad (2.204)$$

is but a reparametrization of the same function. This, as we shall see shortly, defines RGE for V_{eff}.

A proper analysis of vacuum structure in this theory in fact requires RGE-techniques.

Indeed, consider the true vacuum of the effective action (2,202). At tree level of course

$$\frac{\partial V_{tree}}{\partial \phi} = \frac{1}{6}\lambda\phi^3 = 0 \Rightarrow \phi = 0$$

From (2,202) however:

$$0 = \frac{\partial V_{eff}}{\partial \phi} = \frac{1}{6}\lambda_M \phi^3 \left(1 - \frac{\lambda_M}{256\pi^2}\left(88 + 24\log\frac{\phi^2}{M^2}\right)\right) \qquad (2,205)$$

This develops a new minimum at $\phi = \langle\phi\rangle$, where

$$\frac{3}{32\pi^2}\lambda_M \log\frac{\langle\phi\rangle^2}{M^2} \approx 1. \qquad (2,206)$$

Moreover, in this minimum V_{eff} is obviously negative, so it seems that we are seeing a spontaneous symmetry breaking induced by radiative corrections. Whether this is really the case remains unsure because the expansion parameter is of order one at $\phi \sim \langle\phi\rangle$. We need RGE-improvement.

Well, we know the RGE for V_{eff} already:

$$\left(M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + \gamma\phi_{cl}\frac{\partial}{\partial\phi_{cl}}\right)V_{eff}(M,\lambda,\phi_{cl}) = 0 \qquad (2,207)$$

Using dimensional analysis we can set

$$\boxed{V_{eff}(M,\lambda,\phi_{cl}) = \phi_{cl}^4\, v\left(\frac{\phi_{cl}}{M},\lambda\right)} \qquad (2,208)$$

where v obeys:

$$\left(x\frac{\partial}{\partial x} - \frac{\beta}{1-\gamma}\frac{\partial}{\partial \lambda} - \frac{4\gamma}{1-\gamma} \right) v(x,\lambda) = 0 \qquad (2.209)$$

where $x \equiv \phi_{cl}/M$. The solution is standard:

$$\boxed{v\left(\frac{\phi_{cl}}{M},\lambda\right) = v_0(\bar{\lambda})\, e^{+\int_1^{\phi_{cl}/M} d\log x \frac{4\gamma}{1-\gamma}}} \qquad (2.210)$$

and

$$\boxed{\frac{d\bar{\lambda}}{d\log x} = \frac{\beta(\bar{\lambda})}{1-\gamma(\bar{\lambda})}.} \qquad (2.211)$$

However, at one loop order we have $\gamma = 0$, whereby to this approximation

$$V_{eff}(\phi_{cl},M,\lambda) = \phi_{cl}^4\, v_0(\bar{\lambda}), \qquad (2.212)$$

where

$$\frac{\partial\bar{\lambda}}{\partial \log\frac{\phi_{cl}}{M}} = \beta(\bar{\lambda}) \overset{(3.63)}{=} \frac{3\bar{\lambda}^2}{16\pi^2} \qquad (2.213) \qquad \Rightarrow \qquad \bar{\lambda}(\phi_{cl}) = \frac{\lambda_M}{1-\frac{3\lambda_M}{16\pi^2}\log\frac{\phi_{cl}}{M}} \qquad (2.214)$$

As usual, CZ-analysis does not tell the precise form of the function $v_0(\bar{\lambda})$. However, we can find an expansion expression for $v_0(\bar{\lambda})$, valid at small $\bar{\lambda}$, by comparing (2.212) with (2.202). The correct identification is

$$V_{eff}(\bar{\lambda}) = \frac{1}{4!}\phi_{cl}^4\bar{\lambda} \qquad (2.215)$$

Exact match with 1-loop potential can be gotten by small scale change $\log\frac{\phi_{cl}^2}{M^2} = \log\frac{\phi_{cl}^2}{M'^2} - \frac{25}{6}$ → $M^2 = M'^2 e^{25/6}$.

This function no more shows the additional minimum on scale $\frac{3}{32\pi^2}\lambda_M \log\frac{\phi}{M} \simeq 1$. The coupling becomes small for $\phi_c \to 0$.

EFFECTIVE ACTION; PATH INTEGRAL – Third way to get V_{eff}.

Tadpole method provides a quick way to V_{eff}. Almost as easy and more intuitive way is to impose the shifted field method already to path integral.

We know already that

$$-i\log Z[J] = W[J] = \Gamma[\phi_{cl}] - \int d^4x\, J\phi_{cl} \qquad (2.216)$$

where

$$\phi_{cl} = \frac{\partial W}{\partial J} = {}_J\langle\Omega|\phi|\Omega\rangle_J \qquad (2.217)$$

and

$$\Gamma_{tree}[\phi_{cl}] = \int d^4x\, \mathcal{L}[\phi_{cl}] \qquad (2.218)$$

Effective action is the 'free energy functional' that describes the dynamics of the local expectation value of the field ϕ under the influence of the external source $J(x)$ (remember the magnetization analogue). Bearing in mind the result (2.218), it seems to be a good idea to write

$$\phi \equiv \phi_{cl} + \overset{\text{arbitr. q-field}}{\eta} \qquad (2.219)$$

Then divide

$$\mathcal{L} = \mathcal{L}_1 + \underset{\uparrow\ \text{counter terms}}{\delta\mathcal{L}} \tag{2.220}$$

and

$$J = J_1 + \underset{\uparrow\ \text{counter source}}{\delta J} \tag{2.221}$$

Such that

$$\frac{\delta\mathcal{L}_1}{\delta\phi_{cl}} \equiv J_1 \tag{2.222}$$

Then

$$\mathcal{L}(\phi_{cl}+\eta) + J_1(\phi_{cl}+\eta) = \mathcal{L}(\phi_{cl}) + \overbrace{\left(\frac{\delta\mathcal{L}_1}{\delta\phi_{cl}} + J_1\right)}^{=0}\eta + \mathcal{L}_{1eff}(\eta) + \eta J_1 \tag{2.223}$$

where

$$\mathcal{L}_{1eff}(\eta) = \frac{1}{2}(\partial_\mu\eta)^2 - \frac{1}{2}\underbrace{\left(m^2+\frac{1}{2}\lambda\phi_{cl}^2\right)}_{m^2_{eff}(\eta)!}\eta^2 - \underbrace{\frac{\lambda}{6}\phi_{cl}}_{\text{new coupling}}\eta^3 - \frac{\lambda}{4!}\eta^4 \tag{2.224}$$

Similarly, if we set

$$\frac{\delta\,\delta\mathcal{L}}{\delta\phi}\bigg|_{\phi=\phi_{cl}} = -\delta_m\phi_{cl} - \frac{1}{6}\delta_\lambda\phi_{cl}^3 \equiv -\delta J_1 \tag{2.225}$$

we get

$$\delta\mathcal{L}_{eff}(\eta) = \frac{\delta_Z}{2}(\partial_\mu\eta)^2 + \frac{1}{2}\left(\delta_m + \frac{1}{2}\delta_\lambda\phi_{cl}^2\right)\eta^2 + \frac{\delta_\lambda}{6}\phi_{cl}\eta^3 + \frac{\delta_\lambda}{4!}\eta^4 + \widetilde{\delta J}_1\eta \tag{2.226}$$

only one c.t. for all these!

where $\widetilde{\delta J}_1 \equiv \delta J - \delta J_1$. This is still arbitrary c.t., and it can be used to exactly cancel the loop-tadpole

 $= 0 \Rightarrow$ no linear term in $\mathcal{L}_{eff}$.

$\sim \widetilde{\delta J}_1$

$$Z[J] = e^{i\int d^4x \left(\mathcal{L}[\phi_{cl}] + J\phi_{cl}\right)} \times$$

$$\times \int \mathcal{D}\eta \, e^{i\int d^4x \, \mathcal{L}_{1eff}(\eta, \phi_{cl}) + \delta\mathcal{L}_{1eff}(\phi_{cl})} \qquad (2.227)$$

Comparing to (2.216) then gives:

$$\Gamma[\phi_{cl}] = \int d^4x \, \mathcal{L}[\phi_{cl}] - i \log\left(\int \mathcal{D}\eta \, e^{i\int d^4x \left[\mathcal{L}_{1eff}(\eta,\phi_{cl}) + \delta\mathcal{L}_{1eff}\right]} \right) \qquad (2.228)$$

The correction logarithm corresponds to the sum of all connected vacuum graphs in the shifted theory. To the lowest order this is just the gaussian:

$$i \cdot \bigcirc = \log\left(\int \mathcal{D}\eta \, e^{-i\int d^4x \, \frac{1}{2}\eta\left(\partial^2 + m_{eff}^2(\phi_{cl})\right)\eta} \right) \qquad \begin{array}{l} d^4x \to -i d^4x_E \\ \partial^2 \to -\partial_E^2 \end{array}$$

$$= \log\left(\mathrm{Det}\left[\tfrac{1}{2}\overbrace{(-\partial_E^2 + m_{eff}^2)}^{=G}\right]\right) = \log \prod \lambda_G$$

$$= \mathrm{Tr}\log\lambda_G = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \log\left(k_E^2 + m_{eff}^2\right)$$

$$= \frac{1}{2}\int^{m_{eff}^2} dm^2 \int \frac{d^dk}{(2\pi)^d} \frac{\mu^\epsilon}{k^2 + m^2} + \text{irrelevant constant}$$

$$= \int^{\phi_{cl}} d\eta \; \underbrace{\frac{\lambda\eta}{2}\int \frac{d^dk_E}{(2\pi)^d} \frac{\mu^\epsilon}{k_E^2 + m_{eff}^2}} = \int^{\phi_{cl}} d\eta \, \frac{dV_{1-loop}}{d\eta} \qquad (2.229)$$

= expression (2.185 b) for the tadpole correction to $\frac{dV}{d\eta}$.

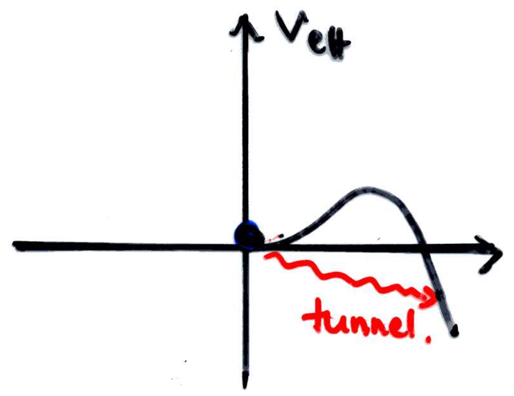
We do not need to go further since this result obviously agrees (up to an irrelevant constant) with our previous calculation for $\partial V/\partial m$.

Higher loop corrections here look as follows

+ ... + counter terms.

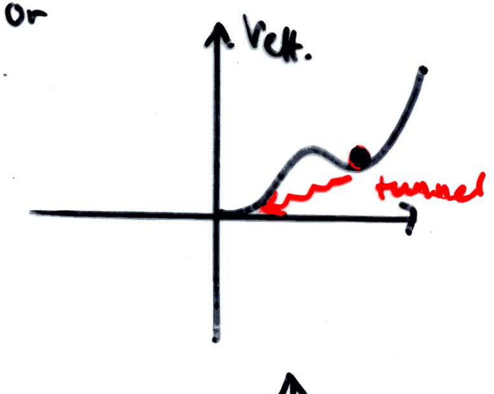
Let us conclude by noting some applications of V_{eff}-calculations.

1] Vacuum stability.



Consider the possibility depicted in the figure 1. Question: If this was the form of V_{eff}, how long would the metastable state live?

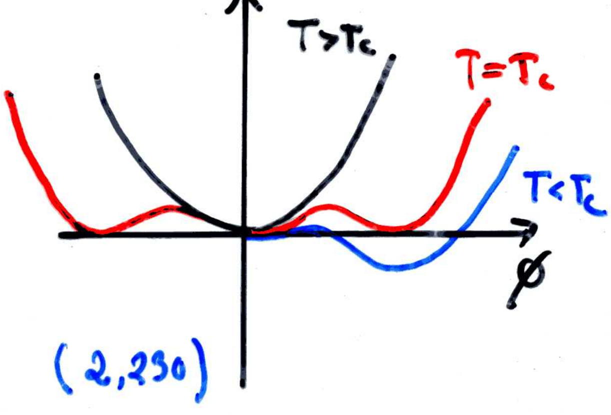
or



2] Phase transitions ; finite T.

Consider case where the mass parameter is negative. In finite T thermal corrections are just dimensionally of form $\sim cT^2\phi^2 + \ldots$ Here

$$\phi^2 m^2 \longrightarrow m_{eff}^2(T)\phi^2 = (-m^2 + cT^2)\phi^2$$



(2,230)