2. THE RENORMALIZATION GROUP

We now have learned to obtain finite results for physical observables from renormalizable theories. We also understand that renormalizable theories are at best effective theories valid at momenta small compared with some fundamental cut-off Λ. But why is this possible? How come the unknown details of the full theory are erased at the large distances? Why do the observed couplings depend on the scale? To answer these questions we now try to understand better the short distance limit of QFT.
(P&S. chapter 12).

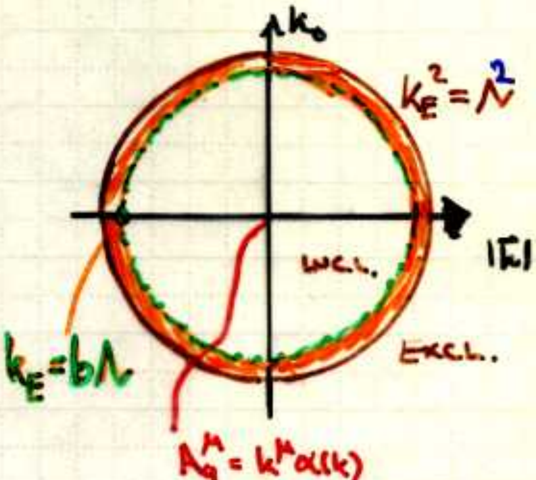
whatever that might mean!

2.1 Wilsons renormalization theory.

To isolate the short distance behaviour we use cut-off regulator. That is, for $\lambda\phi^4$-theory

$$Z_\Lambda[0] = \int \mathcal{D}\phi \, e^{i\int d^4x \mathcal{L}}$$

$$\equiv \int_{|k_E|<\Lambda} \prod d\phi(k) \, e^{-\int d^4x_E \mathcal{L}_E^{(0)}} \qquad (2.1)$$



(cut-off breaks gauge-invariance at large k^μ!)

where

$$\mathcal{L}_E^{(0)} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \qquad (2.2)$$

To extract the effect of highest modes on Z perform explicitly the integrals over the modes $b\Lambda < |k_E| < \Lambda$

in (2.1). To this end introduce variables

$$\phi^{>}(k) = \begin{cases} \phi(k) & ; \; b\Lambda < |k| < \Lambda \\ 0 & ; \; \text{otherwise} \end{cases} \qquad b \in [0,1]$$

$$\phi^{<}(k) = \begin{cases} 0 & ; \; |k| > b\Lambda \\ \phi(k) & ; \; \text{otherwise} \end{cases} \qquad (2.3)$$

Then

$$Z_\Lambda = \int \Pi d\phi^{<} \, \Pi d\phi^{>} \, e^{-\int d^dx \, \mathcal{L}(\phi^{<} + \phi^{>})}$$

$$= \int \Pi d\phi^{<} \, e^{-\int d^dx \, \mathcal{L}^{(0)}(\phi^{<})} \underbrace{\int \Pi d\phi^{>} \, e^{-\int d^dx \, \tilde{\mathcal{L}}(\phi^{>}, \phi^{<})}}_{\equiv I}$$

integrate and exponentiate in $\phi^{>}$ $\Rightarrow e^{-\int \delta\mathcal{L}_{eff}(\phi^{<})}$.

$$\equiv \int \prod_{|k|<b\Lambda} d\phi \; e^{-\int d^dx \, \mathcal{L}_{eff}(\phi^{<})} \qquad (2.4)$$

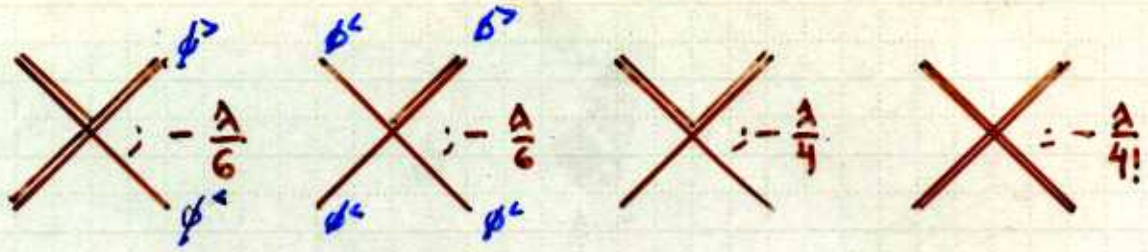
When evaluating I we will assume that λ is small, so that we can treat the quartic part as a perturbation:

$$\tilde{\mathcal{L}} = \underbrace{\tfrac{1}{2}(\partial_\mu \phi^{>})^2 + \tfrac{1}{2} m^2 \phi^{>2}}_{\tilde{\mathcal{L}}_0^{>}} + \underbrace{\lambda \left[\tfrac{1}{6} \phi^{<} \phi^{>3} + \tfrac{1}{4} (\phi^{<}\phi^{>})^2 + \tfrac{1}{6} \phi^{<3} \phi^{>} + \tfrac{1}{4!} \phi^{>4} \right]}_{\text{perturbation } \delta\tilde{\mathcal{L}}_I} \qquad (2.5)$$

Integral can then be done using Feynman diagrammatic methods given the Feynman rules: (in d-dimensions)

$$=\!=\!= \;\; \hat{=} \;\; \frac{1}{k^2+m^2} \Theta(k) \; ; \quad \Theta(k) = \begin{cases} 1 & ; \; b\Lambda < |k| < \Lambda \\ 0 & ; \; \text{otherwise} \end{cases} \qquad (2.6)$$

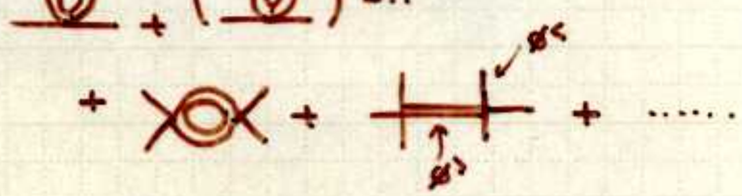
and



We then observe that

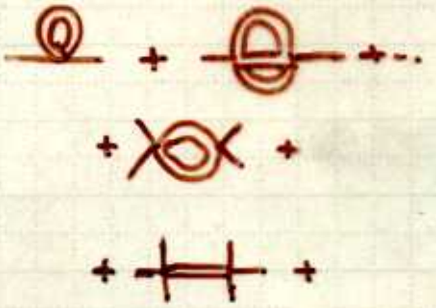
$$I = \int \Pi d\phi^> \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \prod_{i=1}^{N} \left[\int d^4x_i \tilde{\mathcal{L}}_I\right]^n e^{-\int d^4x \tilde{\mathcal{L}}_0}$$

$$= \underbrace{I(\lambda=0)}_{= e^{\{\circledcirc + ..\}}} \times \underbrace{\left\{ \text{Sum of all } \phi^<\phi^<..\phi^< \text{- diagrams} \right\}} \qquad (2.7)$$



$$= I(0) \times \exp\left\{ \text{Sum of all connected diagrams} \right\} \qquad (2.8)$$

(all external legs are eventually coupled to same point x in these diagrams)



$\Rightarrow \phi^{<2}, (\partial\phi^<)^2$

$\Rightarrow \phi^{<4}, \phi^{<2}(\partial\phi^<)^2 + ..$

$\Rightarrow \phi^{<6}, \ldots$

(LOTS!) New terms in the effective action!

Examples of new terms introduced to $\mathcal{L}_{eff}$ include mass & coupling renormalizations:

-  $= -\frac{\lambda}{4}\int d^4x\,(\phi^<(x))^2 \underbrace{\int \mathcal{D}\phi^> [\phi^>(x)]^2 e^{-\int \mathcal{L}_0}}\cdot\frac{1}{I(0)}$

$$= \int_{q,q'} \underbrace{\int \mathcal{D}\phi^> \phi^>(q)\phi^>(q') e^{-\int\mathcal{L}_0}} \cdot e^{i(q-q')\cdot x}$$

$$= (2\pi)^4 \delta^4(q-q')\,\Theta(q) \underbrace{\int \mathcal{D}\phi^> \phi_q^{>2} e^{-\int\mathcal{L}_0}}_{= I(0)\cdot\frac{1}{q^2+m^2}}$$

$$= -\int d^4x\,[\phi^<(x)]^2 \left[\frac{\lambda}{4}\int_{b\Lambda<|q|<\Lambda} \frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2}\right]$$

[just what PT would have given directly.]

$$= -\int d^4x\,\frac{1}{2}\mu\,\phi^{<2}(x) \qquad (2.9)$$

where

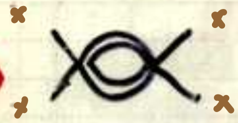
$$\mu = \frac{\lambda}{2}\cdot\frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2)}\int_{b\Lambda}^{\Lambda} dq\,\frac{\overbrace{q^{d-1}}^{\sim q^{d-3}}}{q^2+m^2} \approx \frac{\lambda}{(4\pi)^{d/2}}\frac{1-b^{d-2}}{d-2}\Lambda^{d-2} \qquad (2.10)$$

$$\xrightarrow{d\to 4} \frac{\lambda(1-b^2)}{32\pi^2}\Lambda^2$$

↑ quadratic in Λ !

is clearly a mass renormalization to the original mass of m^2 of the effective field $\phi^<$.

Similarly, when external momenta of $\phi^<$-fields are zero:

-  $= -\frac{1}{4!}\int d^4x\,\zeta\,\phi^<(x)^4 \qquad (2.11)$

with

only combinatoric factor

$$\zeta = -4!\cdot\frac{2}{2!}\cdot\left(\frac{\lambda}{4}\right)^2 \int_{b\Lambda<|q|<\Lambda} \frac{d^dq}{(2\pi)^d}\frac{1}{(k^2+m^2)^2} \approx -\frac{3\lambda^2}{(4\pi)^{d/2}\Gamma(\frac{d}{2})}\frac{(1-b^{4-d})}{d-4}\Lambda^{d-4} \qquad (2.12)$$

In 4d this becomes

$$\boxed{\zeta \longrightarrow -\frac{3\lambda^2}{16\pi^2}\log\frac{1}{b}} \qquad (2,13)$$

So clearly ζ-term gives rise to the coupling-constant renormalization for the effective field $\phi^<$.

Note that unlike was the case with mass renormalization, coupling constant correction (2,13) is independent of the cut-off Λ. (FINITE)

Each log interval in $|q|$ gives rise to equally large correction to λ between the interval $m^2 < q^2 < \Lambda^2$. No pathology.

In addition to these renormalizations the effective action will contain an infinite set of nonrenormalizable operators such as

$$\phi^6, \quad (\partial\phi)^4, \quad \phi^2(\partial\phi)^2, \quad \text{etc} \qquad (2,14)$$

The role of these terms (together with renormalizations), is to emulate the effect of the high q-fluctuations $\phi^>$ in the original expression for Z. (Obviously we ~~need~~ would like to! to get rid of these somehow....)

Schematically, we then have

$$\int d^dx\, \mathcal{L}_{eff}(\phi^<) = \int \overset{\to b^{-d}d^dx'!}{d^dx} \Big[\tfrac{1}{2}(1+\Delta Z)\overset{\to b^2\partial'^2}{(\partial_\mu\phi^<)^2} + \tfrac{1}{2}(m^2+\Delta m^2)\phi^{<2}$$

$$+ \frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 + \Delta C\underset{\to b^4\partial'^4}{(\partial_\mu\phi)^4} + \Delta D\phi^6 + \ldots \Big] \qquad (2.15)$$

small $\sim\lambda$

LOTS of ROSS.

Now rescale the variables & the field

$$k \to k' = k/b$$
$$x \to x' = bx$$

designed to return to canonical kinetic term $\frac{1}{2}(\partial\phi')^2$.

$$\phi^< \to \phi' = \left[b^{2-d}(1+\Delta Z)\right]^{1/2}\phi^< \qquad (2.16)$$

In terms of these new variables

$$\boxed{Z_{b\Lambda} = \int \prod_{\underbrace{k'<\Lambda}_{\text{same range}}} e^{-\int d^dx' \,\mathcal{L}_{eff}(\phi')}} \times \text{const.} \qquad (2.17)$$

where

$$\mathcal{L}_{eff}(\phi') = \overbrace{\tfrac{1}{2}(\partial'_\mu\phi')^2 + \tfrac{1}{2}m'^2\phi'^2 + \tfrac{1}{4}\lambda'\phi'^4}^{\text{old form}}$$
$$+ \underbrace{C'(\partial'_\mu\phi')^4 + D'\phi'^6}_{\text{"New" terms}} + \ldots\ldots \qquad (2.18)$$

and:

$$m'^2 \equiv (m^2+\Delta m^2)(1+\Delta Z)^{-1}\,b^{-2} \approx m^2 b^{-2}$$
$$\lambda' \equiv (\lambda+\Delta\lambda)(1+\Delta Z)^{-2}\,b^{d-4} \approx \lambda b^{d-4}$$
$$C' \equiv (C+\Delta C)(1+\Delta Z)^{-2}\,b^{d} \approx C b^{d}$$
$$D' \equiv (D+\Delta D)(1+\Delta Z)^{-3}\,b^{2d-6} \approx D b^{2d-6} \qquad (2.19)$$
$$\vdots$$

when $\Delta Z, \Delta\lambda$ etc can be neglected..

We can actually allow these terms in the original lagrangian!
(admitting we do not know $\mathcal{L}_\Lambda$)

The combined effect of integrating out the shell $b\Lambda < |q| < \Lambda$ and rescaling can be viewed as a shift of the original theory in the (infinite dimensional) space of possible Lagrangians.

One can make successive transformations and even take the limit $b \to 1$ in each step, which makes the transformation a continuous one. This set of shifts is what is called the renormalization group.*

Renormalization group flow.

Now comes the key result that makes all the subsequent work worthwhile.

Consider computing a correlation function of fields with $p_i \ll \Lambda$. In our old way, starting from bare theory large corrections $\sim \Lambda^2$ & $\log \Lambda$ appear suddenly in loop calculations. Alternatively, in Wilsons approach one first integrates out all momentum shells from Λ down to $\sim p_i$ and computes the correlation function from the effective theory. $\Rightarrow$ No big corrections. However, the tradeoff seems to be a very messy Lagrangian $\mathcal{L}_{eff}$!

The point is of course that $\mathcal{L}_{eff}$ is of course not at all messy, quite the contrary!

---

* Obviously this set of operations does not contain an inverse (at least in any finite-d space of theories), so RG is not really a group.

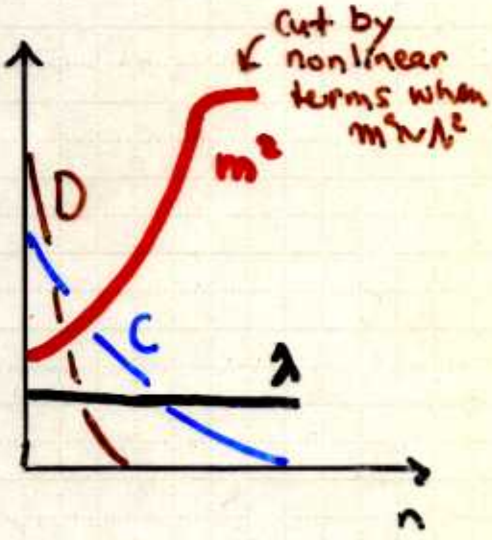
Indeed consider the flow (2.19) in the vicinity of point $m^2 = \lambda = C = D = \ldots\ldots = 0$ in the theory space, so that

$$\mathcal{L}^{(0)} = \frac{1}{2}(\partial_\mu \phi)^2 \qquad (2.20)$$

- If we start exactly from this point then all corrections vanish and each successive iteration leaves $\mathcal{L}$ invariant. We say that (2.20) is a fixed point.

- When not quite at (2.20) but very close to it, all nonlinear corrections Δm^2, $\Delta\lambda$, ΔC,... are small and can be neglected, resulting in the simple scaling laws $m'^2 \approx m^2 b^{-2}$; $\lambda' = \lambda^{d-4}$ etc.

Now imagine performing successive integrations of the momentum shells near (2.20). Obviously after n integrations in momenta we get



$b < 1$

relevant $\left\{ \; m'^2 \approx b^{-2n} m^2 \xrightarrow{n\to\infty} \infty \right.$

marginal $\left\{ \; \lambda' \approx (b)^{n(d-4)} \lambda \longrightarrow \lambda \right.$

irrelevant $\left\{ \begin{array}{l} C' \approx b^{dn} C \longrightarrow 0 \\ D' \approx b^{(2d-6)n} D \longrightarrow 0 \end{array} \right.$

Since $b < 1$ all terms with positive scaling power die out and are removed from $\mathcal{L}_{eff}$ by successive iterations!

This leaves only operators with negative or zero scaling power to $\mathcal{L}_{eff}$. $\Rightarrow$ $\mathcal{L}_{eff}$ is simple ($\mathcal{L}^0$ is the messy one!)

We can make this statement a very precise one. A generic term in $\mathcal{L}_{eff}$ contains N powers of fields and M derivatives, for example

$$\Delta\mathcal{L}_{NM} = c_{NM}\,\phi^n(\partial\phi)^M \quad ; \; n+M \equiv N \qquad (2.21)$$

Such operators flow near the fixed point (2.20) as

$$c'_{NM} = c_{NM}\, b^{-d+M+N\left(\frac{d}{2}-1\right)}$$

$$= \underline{c_{NM}\, b^{+(d_{NM}-d)}} \; ; \quad \underline{d_{NM} \equiv M + N\left(\frac{d}{2}-1\right)} \qquad (2.21)$$

So we see that all operators with $d_{NM} > d$ are removed from the low energy effective theory. Such operators are called irrelevant. If $d_{NM} < d$ on the other hand the significance of the operator grows during the flow and the operator is called relevant. If $d_{NM} = d$ the fate of the operator cannot be judged from the scaling flow, but is determined by the nonlinear terms in (2.19). Such operators are called marginal.

So we see that at small momentum scales

$$\mathcal{L}_{eff} \approx \mathcal{L}\,(\text{relevant \& marginal operators}) \qquad (2,22)$$

Now observe that $d_{NM}-d$ also contains the renormalizability of diagrams. The operator of form $\phi^n(\partial\phi)^M$ comes from extracting from an N-point function the part $\propto p^M$. The superficial degree of divergence of such operator

is
$$D = dL - 2P - M$$
which after some simple algebra (with N external legs, V vertices & P propagators constrained by $4V = 2P + N$ and $L = P - V + 1$) gives

$$D = d + [\underbrace{d-4}_{=0 \text{ to ensure renormalizability}}]V - \left(\frac{d-2}{2}N + M\right)$$

$d(P-V+1)-2P-M$
$= (d-2)P - dV + d - M$
$= (\frac{1}{2}(d-2)-d)V + d - (\frac{d}{2}-1)N - M$
$= (d-4)V + d - ((\frac{d-2}{2})N + M)$

$$= d - d_{MN}. \qquad (2.23)$$

That is, the criterion for relevancy of operators coincides with the criterion of renormalizability.

relevant operators are super renormalizable

marginal operators are renormalizable

irrelevant operators are nonrenormalizable

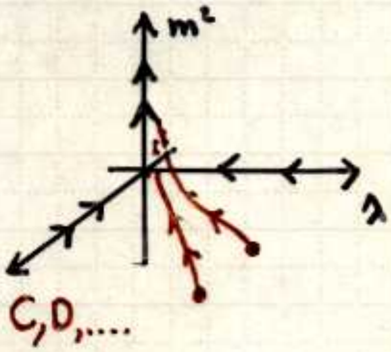
Wilsons procedure thus explains why nature at large scales is described by simple renormalizable theories. No matter how mesy $\mathcal{L}_0$ one has at the cut-off. Indeed we can now continue (2.22) to read

$$\mathcal{L}_{eff} = \mathcal{L}(\text{(super) renormalizable operators}) \qquad (2.24)$$

Let us take a more closer look onto the RG-flow. Consider $\lambda\phi^4$-theory near the fixed point (2.20), where $\mathcal{L} \approx \frac{1}{2}(\partial\phi)^2$.

- First note that mass operator $m^2\phi^2$ is relevant for all dimensions $d > 2$. (It becomes marginal in $d=2$).

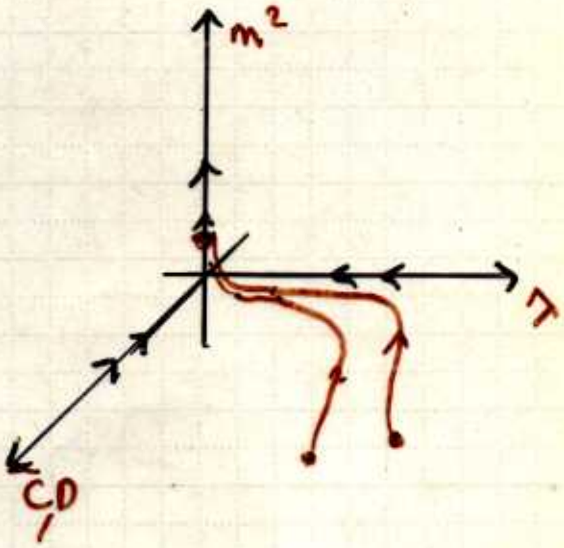
- $d > 4$. $\lambda\phi^4$-term is irrelevant. only $m^2\phi^2$ term relevant.



- $d = 4$ λ is now marginal and does not evolve in scaling. It does evolve due to nonlinear term $\Delta\lambda$ however. From (2.13):

$$\lambda' = \lambda - \frac{3\lambda^2}{16\pi^2} \log \frac{1}{b} \qquad (2.25)$$

$\Rightarrow$ λ decreases slowly



Although slow, this decrease implies that $\lambda \to 0$ as $\Lambda \to \infty$ (∞ # of iterations needed to get out from the cutoff). $\Rightarrow$ $\lambda\phi^4$ theory in $d=4$ does not exist in the limit $\Lambda \to \infty$.

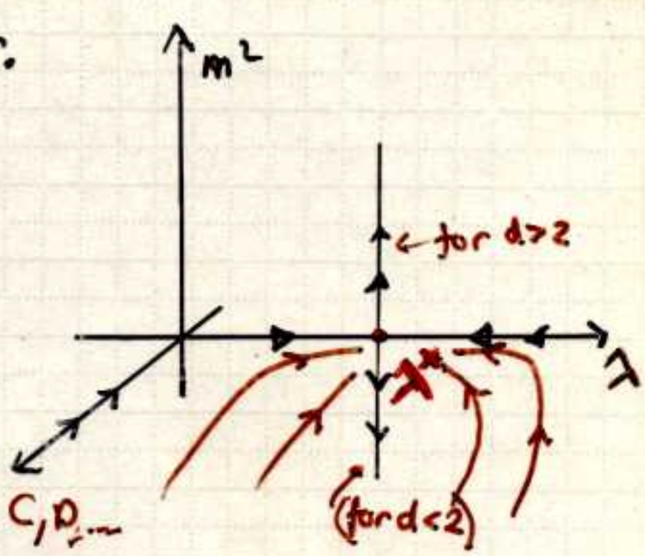
* Note added. For m^2-running Δm^2 is not necessarily smaller than correction due to scaling.

$$m'^2 - m^2 = (m+\Delta m)b^{-2} - m^2 \approx m^2(1-b^2) + \Delta m^2 \approx \left(m^2 + \frac{\lambda}{32\pi^2}\Lambda^2\right)(1-b^2)$$

eg one needs extra condition $\frac{\lambda}{32\pi^2}\Lambda^2 < m^2 \ll \Lambda^2$ to ensure that m^2 remains small in the process.

- **d<4** λ is now relevant operator.

The transformation rule now is to one loop order ($\Delta Z_{1\text{-loop}}=0$) given by (2.19) & (2.12)



$$\lambda' = \underbrace{\left[\lambda - \frac{3\lambda^2}{(4\pi)^{d/2}\Gamma(\frac{d}{2})}\,\frac{b^{d-4}-1}{4-d}\Lambda^{d-4}\right]}_{\text{If this} \to 0 \text{ due to nonlinearity} \Rightarrow \text{it will not evolve in further scalings}} b^{d-4} \qquad (2.26)$$

$\Rightarrow$ $\boxed{\text{New fixed point } \lambda = \lambda_* \neq 0}$

(Wilson-Fisher f.p.)

In $4-\epsilon$ dim: $\lambda_* = \frac{16\pi^2}{3}\epsilon$

For $d>2$ mass term grows near λ_* and the flow becomes similar to the case with $d=4$ near $\lambda=0$.

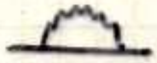
Can have $\lambda \neq 0$ at $d=4-\epsilon$.

- All known QFT:s are controlled by either free field fixed points, or ones that approach free field fixed point in some limit ($\epsilon\to 0$).

- Other types of fixed points could exist (strong coupling...)

- Scalar mass term is special, since for m^2 the nonlinear correction 'sees' the cut-off scale directly: $\Delta m^2 \sim \Lambda^2$. For all other terms in all known QFT's (including fermion masses the corrections in Wilson flow are independent of Λ, as in (8.13). $[m_f \to m_f (1 + Cg^2 \log \frac{1}{b})]$. It is this qualitative difference that makes fundamental scalar fields unattractive from the renormalization point of view and warrants talking about hierarchy problem with scalar masses.

 $\sim \Lambda^2 \triangleq$ hierarchy problem

 $\sim \log \Lambda$ no problem.

Wilsons method provides a very clear understanding of why the nature is described by renormalizable theories. It is clumsy to use to derive scaling properties of the correlation functions however, and the cut-off is also known to be problematic with gauge-theories.

Now let us drop all irrelevant (nonrenormalizable) operators and concentrate on renormalizable theories.

In this restricted sense RG becomes a way to study the scaling properties of greens functions with respect to choice of renormalization points.

$\Rightarrow$ (for example) asymptotic freedom.

2.2 Renormalization group.

Preliminaries

Let us first remind ourselves about the arbitrariness related to the renormalization procedure, namely the scheme dependence.

For example we can define different subtraction schemes wrt. different renormalization points in $\lambda\phi^4$-theory. A physically measurable coupling

$$\lambda \equiv \Gamma^{(4)}(s=4m^2, t=u=0) \qquad (2.27)$$

can be related to a coupling defined in the unphysical region

$$\lambda' \equiv \Gamma^{(4)}(0,0,0) \qquad (2.28)$$

by a finite renormalization. If in the physical scheme the renormalized 4-point function is

$$\Gamma^{(4)}(s,t,u) = \lambda + \sum_{n=1}^{\infty} (a_n(s,t,u) - a(4m^2,0,0))\lambda^n \qquad (2.29)$$

we have

$$\lambda' = \lambda + \sum_{n=1}^{\infty} \overbrace{(a_n(0,0,0) - a_n(4m^2,0,0))}^{\text{finite}}\lambda^n \qquad (2.30)$$

More generally, we can imagine two different renormalization procedures R and R', both starting from the same bare Lagrangian:

$$\mathcal{L} = \mathcal{L}_R(R\text{-parameters}) = \mathcal{L}_{R'}(R'\text{-parameters}) \qquad (2.31)$$

Then obviously

$$\phi_R = Z_\phi^{-1/2}(R)\,\phi_0 \quad \text{and} \quad \phi_{R'} = Z_\phi^{-1/2}(R')\,\phi_0$$

$$\Rightarrow \phi_{R'} \equiv \underbrace{Z_\phi^{-1/2}(R,R')}_{\text{finite !}}\,\phi_R = \left(\frac{Z_\phi(R')}{Z_\phi(R)}\right)^{-1/2}\phi_R \qquad (2.32)$$

Similarly ($\lambda_0 = (Z_\lambda/Z_\phi^2)\lambda_r = Z_1\lambda_r$; see p.)

$$\lambda_{R'} = \frac{Z_\phi^2(R,R')}{Z_\lambda(R,R')}\,\lambda_R \equiv Z_1(R,R')\,\lambda_R$$

$$m_{R'}^2 = m_R^2 + \Delta m^2(R',R) \qquad (2.34)$$

where

$$\left.\begin{aligned} Z_\lambda(R',R) &= Z_\lambda(R')/Z_\lambda(R) \\ \Delta m^2(R',R) &= \Delta m^2(R') - \Delta m^2(R) \end{aligned}\right\}\ \text{finite!} \qquad (2.35)$$

An operation that takes the parameters in scheme R to those in scheme R' constitutes a transformation in the space of Lagrangians. The set of all such transformations is what we understand with renormalization group here. (This is clearly related in spirit to the Wilsons method.)

MS and $\overline{\text{MS}}$ schemes

Until now we have always used renormalization schemes corresponding to on-shell subtraction or to subtraction on some off shell position in the phase space. In these schemes the arbitrary μ-dependence introduced by our scaling of $4-\epsilon$-dimensional integrals was always replaced by dependence on the renormalization point.

The renormalization scheme does not need to be of this type however. We can also decide to define finite parameters by simply subtracting the pole terms $\sim \frac{2}{\epsilon}$ (Minimal subtraction, MS) or the combination $\frac{2}{\epsilon} - \gamma + \ln 4\pi$ ($\overline{MS}$ - scheme) from the divergent quantities.

Indeed, if we want to compare λ' of eqn. (2.28) to $\lambda_{\overline{MS}}$, we find at 1-loop level:

$$\lambda' = \lambda_{\overline{MS}} + i \frac{3\lambda_{\overline{MS}}^2}{2} i\mu^\epsilon \int \frac{d^dq}{(2\pi)^d} \frac{1}{[q^2-m^2]^2} + \delta_{\lambda_{\overline{MS}}}$$

$$= \lambda_{\overline{MS}} - \frac{3\lambda_{\overline{MS}}^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma_E + \ln 4\pi + \log\left(\frac{\mu^2}{m^2}\right)\right) + \delta_{\lambda_{\overline{MS}}} \quad (2.36)$$

Since we are defining

$$\delta_{\lambda_{\overline{MS}}} \equiv \frac{3\lambda_{\overline{MS}}}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma_E + \ln 4\pi\right) \equiv \frac{3\lambda_{\overline{MS}}}{16\pi^2}\frac{1}{\epsilon_{\overline{MS}}} \quad (2.37)$$

we get

$$\lambda' = \lambda_{\overline{MS}}\left(1 - \frac{3\lambda_{\overline{MS}}}{32\pi^2}\log\frac{\mu^2}{m^2}\right) \quad (2.38)$$

μ^2-dependent connection.

Let us further remember that in $\lambda\phi^4$-theory there is no w.f.r-correction at one loop, so that

$$\begin{aligned} Z_1^{\overline{MS}} &\simeq 1 + \frac{3\lambda_{\overline{MS}}}{16\pi^2}\frac{1}{\epsilon_{\overline{MS}}} \\ Z_\phi^{\overline{MS}} &\simeq 1 \end{aligned} \qquad (2.39)$$

Furthermore for completeness observe that (for $m \neq 0$)

$$\begin{aligned} m^2_{\overline{MS}} &= m^2\left(1 - \frac{\lambda_{\overline{MS}}}{32\pi^2}\left(1 + \log\frac{\mu^2}{m^2}\right)\right) \\ \delta m_Z^2 &= + \frac{\lambda_{\overline{MS}}}{16\pi^2}\frac{1}{\epsilon_{\overline{MS}}} \end{aligned} \qquad (2.40)$$

(To this order m^2 is the physical pole-mass.

Renormalization group (space-like momentum subtraction scheme)

Peskin & Schröder introduces the RGE for an off-shell momentum subtraction scheme for $\lambda\phi^4$-theory. Treating mass-terms is a bit delicate in this scheme and one therefore first looks at a massless theory.

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi_0)^2 - \frac{\lambda}{4!}\phi_0^4 \qquad (2.41)$$

From the outset this looks like a scale-free theory, but as we know, the hidden scale arises from renormalization conditions. This theory cannot be renormalized on-shell, because then the counter-terms would be infrared divergent $\sim \log m^2$.

One can instead expand Π at some unphysical, space-like momentum $-p^2 = M^2$:

$$\Pi_{full} = -p^2\delta_\phi + \delta_m + \Pi(-M^2) + \frac{d}{dp^2}\Pi\Big|_{-M^2}(p^2+M^2) + \tilde{\Pi}_M$$

Given that

$$\delta_\phi = \Pi'(-M^2) \quad \Leftrightarrow \quad \underline{Z_\phi \equiv 1 + \Pi'(-M^2)} \qquad (2.42)$$

$$\underline{\delta_m \equiv -\Pi(-M^2) - M^2\delta_\phi} \qquad (2.43)$$

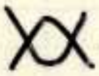
one has

$$i\Delta_M^{-1}(p^2) = p^2 + \tilde{\Pi}(p^2) \xrightarrow{p^2\to -M^2} -M^2 \qquad (2.44)$$

which is finite and independent of $\tilde{\Pi}$ at $p^2 = -M^2$. *

Similarly one can define the renormalized coupling λ_M at $s = t = u = -M^2$:

$$\underline{-i\lambda_M \equiv \Gamma^{(4)}(-M^2, -M^2, -M^2)} \qquad (2.45)$$

At one loop  +  + crossed + 

$$\Gamma^4(s,t,u) = -i\lambda_M - i\left(\frac{\lambda_M^2}{32\pi^2}\Gamma\left(\frac{\epsilon}{2}\right)(4\pi\mu^2)^{\frac{\epsilon}{2}}\left(\Delta_s^{-\epsilon/2} + \Delta_t^{-\epsilon/2} + \Delta_u^{-\epsilon/2}\right) - \delta_{\lambda_M}\right) \qquad (2.46)$$

where $\Delta_i = -x(1-x)p_i^2$. Keeping $\lambda_R = \lambda_M$ fixed at $s=t=u=-M^2$

This fixes $Z_{1M} = Z_\lambda^M / Z_\phi^{M\,2} \approx Z_\lambda^M = 1 - \frac{1}{\lambda}\delta_\lambda$ (see p.)

$$Z_{\lambda M} = 1 + \frac{3\lambda_M}{16\pi^2}\left(\frac{1}{\epsilon_{\overline{MS}}} - \frac{1}{2}\int_0^1 dx \log \frac{x(1-x)M^2}{\mu^2}\right)$$

$$\underline{1 + \frac{3\lambda_M}{32\pi^2}\left(\frac{2}{\epsilon_{\overline{MS}}} - \log M^2 + \text{finite terms}\right)} \qquad (2.47)$$

Compare this with $\overline{MS}$-result (2.39).

We are finally ready to write down the RGE for an n-point function. Start from equality:

$$\boxed{\begin{aligned} G_0^{(n)} &\equiv \langle\Omega|T(\phi_0(x_1)\ldots\phi_0(x_n))|\Omega\rangle_c = \\ &= Z_M^{n/2}\langle\Omega|T(\phi_M(x_1)\ldots\phi_M(x_n))|\Omega\rangle \equiv Z_M^{n/2} G_M^n \end{aligned}} \qquad (2.48)$$

where the quantities on the R.H.S. (it is a full Greens function to be evaluated perturbatively) depend on the scale M. Now consider shifting $M \to M + \delta M$. This now corresponds to a RG-transformation in the sense described on p. 66. We thus have

Finite transf.

$$\left\{\begin{aligned} M &\to M' = M + \delta M \\ \lambda_M &\to \lambda_{M'} = (Z_{1M'}/Z_{1M})^{-1}\lambda_M \equiv \lambda_M + \delta\lambda_M \\ \phi_M &\to \phi_{M'} = (Z_{\phi_{M'}}/Z_{\phi_M})^{-1/2}\phi_M \equiv (1+\delta\eta)\phi_M^{\times} \end{aligned}\right. \qquad (2.49)$$

Now, since l.h.s. of eqn (2.48) is invariant in such redefinition we get

$^\times$ eg $\delta\eta = \left(\frac{Z_M}{Z_{M'}}\right)^{1/2} - 1$

First observe that under a shift $\phi \to \phi + \delta\phi = (1+\delta\eta)\phi$

$$\delta G_M^{(n)} = n\delta\eta\, G_M^{(n)} = \frac{\partial G}{\partial M}\delta M + \frac{\delta G}{\partial \lambda}\delta\lambda \qquad (2.50)$$

where the second equality follows from the dependence

$$G_M^{(n)} = G_M^{(n)}(M;\lambda). \qquad (2.51)$$

Multiplying (2.51) by $M/\delta M$ we find

$$\boxed{\left(M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda_M} - n\gamma\right) G_M^{(n)} = 0} \qquad (2.52)$$

where

$$\gamma \equiv + \frac{\delta\eta}{\delta M}M = \frac{M}{\delta M}\left(-1 + \left[\frac{Z_{\phi_M}}{Z_{\phi_{M+\delta M}}}\right]^{1/2}\right) \xrightarrow{\delta M\to 0} -\frac{M}{2Z_{\phi_M}}\frac{\partial Z_{\phi_M}}{\partial M}\bigg|_{\lambda_0,\epsilon} \qquad (2.53)$$

$$= -M\frac{\partial}{M}\log\sqrt{Z_M}$$

$$\beta(\lambda) \equiv M\frac{\delta\lambda}{\delta M} \xrightarrow{\delta M\to 0} M\frac{\partial\lambda}{\partial M}\bigg|_{\lambda_0,\epsilon} \qquad (2.54)$$

(2.52) is the Callan-Symanzik equation for the n-point Green function. The anomalous dimension γ and the β-function $\beta(\lambda)$ are finite functions (independent of the cutoff Λ (ϵ)) as is directly seen from definitions (2.49).

The physical interpretation of β and γ, on the basis of (2.53) and (2.54) is to express the rate of change of the coupling λ and of the (log of) the w.f.r-factor as a function of M for fixed bare parameters (λ_0, ϵ).

Examples.

2-point function, on dimensional grounds is:

$$G^{(2)}(p) = \frac{i}{p^2}\, g_2\left(-\frac{p^2}{M^2}, \lambda\right) \qquad (2.55)$$

Thus trading $M\frac{\partial}{\partial M} \to -P\frac{\partial}{\partial P}$ one has ($x \equiv P/M$)

$$\left[x\frac{\partial}{\partial x} - \beta(\lambda)\frac{\partial}{\partial \lambda} + 2\gamma(\lambda)\right] g_2(x,\lambda) = 0 \qquad (2.56)$$

One can immediately see that this eqn. has the solution

$$g(x,\lambda) = \tilde{g}_2\left(\bar{\lambda}\left(\tfrac{P}{M},\lambda\right)\right) \exp\left[+2\int_1^{P/M} d\log x\; \gamma(\bar{\lambda}(x,\lambda))\right] \qquad (2.57)$$

where

$$x\frac{\partial \bar{\lambda}}{\partial x} = \boxed{\frac{d\bar{\lambda}}{d\log\frac{P}{M}} = \beta(\bar{\lambda})} \qquad (2.58)$$

with $\bar{\lambda}(M,\lambda) \equiv \lambda$

RG-Eqn. for the running coupling $\bar{\lambda}$.

Similarly, for the 4-point function

$$G^{(4)}(p) \equiv \frac{i^4}{p^8}\, g_4\left(\frac{-p^2}{M^2}, \lambda\right) \qquad (2.59)$$

g_4 obeys an equation like (2.56) with $2 \to 4$, and has the

Solution.

$$g_4(x,\lambda) = \tilde{g}_4\left(\bar{\lambda}\left(\tfrac{p}{M},\lambda\right)\right) \cdot \left[\exp\left(+\int_1^{p/M} d\log x' \, \gamma(\bar{\lambda})\right)\right]^{+4} \qquad (2.60)$$

where $\bar{\lambda}$ is again given by (2.58). The functions $\tilde{g}_2$ and $\tilde{g}_4$ are undefined by this procedure, but they can be matched from the perturbative calculation, according to which

$$\tilde{g}_2 = 1 + \mathcal{O}(\lambda^2)$$

$$\tilde{g}_4 = -i\bar{\lambda} + \mathcal{O}(\lambda^2) \qquad (2.61)$$

= renormalized, running coupling at scale p.

This analysis generalizes to an arbitrary n-point function $G^{(n)} \equiv (i/p^2)^n g_n(-\frac{p^2}{M^2},\lambda)$. One finds that the separate dependence of G^n on p^2/M^2 and λ reorganizes itself to a dependence on the running coupling $\bar{\lambda}$ of (2.58) and an exponential factor for each external leg:

$$g_n\left(-\frac{p^2}{M^2},\lambda\right) = \tilde{g}_n(\bar{\lambda}) \left[\exp\left(+\int_1^{p/M} d\log x' \, \gamma(\bar{\lambda})\right)\right]^{+n} \qquad (2.62)$$

without RGE an expansion in $(\lambda \log \frac{p}{M})^n$

running coupling resums these to ∞ order.

$\Rightarrow$ scale dependence on $\bar{\lambda}$.

cumulative wfr from getting from ref scale M to the obs. scale p.

One loop running of λ

From (2.54), (2.49) and (2.47) with $Z_\phi \approx 1 + \mathcal{O}(\lambda^2)$ one gets

$$\underline{\beta(\lambda)} \overset{2.49}{=} \lim_{\delta M \to 0} \frac{\lambda M}{\delta M}\left(\frac{Z_\lambda}{Z_\lambda'}\frac{Z_{\phi'}^2}{Z_\phi^2} - 1\right) \approx \lim_{\delta M \to 0} \frac{\lambda M}{\delta M}\frac{Z_\lambda - Z_\lambda'}{Z_\lambda}$$

$$= -\frac{\lambda M}{Z_\lambda}\frac{\partial Z_\lambda}{\partial M} \approx -\frac{2\lambda}{\underbrace{Z_\lambda}_{\to 1 \text{ to order } \lambda}}\frac{d Z_\lambda}{d\log M^2} \overset{2.47}{=} \underline{\frac{3\lambda^2}{16\pi^2}} + \mathcal{O}(\lambda^3) \quad (2.63)$$

Thus from (2.58)

$$\underline{\frac{d\bar\lambda}{d\log\frac{p}{M}} = \frac{3\bar\lambda^2}{16\pi^2}}, \quad \text{with} \quad \bar\lambda(M,\lambda) = \lambda$$

$$\Rightarrow \quad \frac{1}{\lambda} - \frac{1}{\lambda(p)} = \frac{3}{16\pi^2}\log\frac{p}{M} \iff \underline{\lambda(p) = \frac{\lambda}{1 - \frac{3\lambda}{16\pi^2}\log\frac{p}{M}}} \quad (2.64)$$

Thus RGE improved PT uses the coupling (2.64) in favour of $\lambda(M)$ at the reference scale M. $\triangleq$ resumming leading logs. This becomes evident if one writes

$$\lambda(p) = \lambda \sum_{n=0}^{\infty}\left(\frac{3\lambda}{16\pi^2}\log\frac{p}{M}\right)^n. \quad (2.65)$$

Dimensional transmutation

Now observe that $\lambda(p)$ becomes infinite at scale p_{M_∞} where $1 - \frac{3\lambda_M}{16\pi^2}\log\frac{p_{M_\infty}}{M} = 0$, i.e.

$$p_{M_\infty} = M \cdot e^{+\frac{16\pi^2}{3\lambda(M)}} \qquad (2.66)$$

Of course the 1-loop expression breaks down much before the scale p_{M_∞}. We can still use p_{M_∞} to rewrite $\lambda(p)$ without a reference coupling λ_M:

$$\lambda(p) = \frac{16\pi^2}{3\log\left(\frac{p_{M_\infty}}{p}\right)} \qquad (2.67)$$

Here the dimensionless coupling in the Lagrangian has been traded to a dimensionful intrinsic scale in the theory.

RGE for non-marginal operators

Space-like momentum subtraction works well when $m=0$. If $m \neq 0$ there is the additional problem that counter-terms depend also on the ratio m^2/M^2 (and not only p^2/M^2), whence CS-equation becomes more complicated. One solves this problem by treating mass as a perturbation (see P&S chapters 12.4 and 12.5). This formalism can be generalized to arbitrary operators with positive and negative mass dimensions. An important result coming from this is that the coupling of an operator with dimension d_i obeys

$$\frac{d}{d\log\frac{p}{M}} \rho_i = [\underbrace{d_i - 4 + \dots}_{\text{loop. corr. } \sim g, \lambda}]\, \rho_i \;\Rightarrow\; \bar{\rho}_i = \rho_i \underbrace{\left(\frac{p}{M}\right)^{d_i-4}}_{\to 0 \text{ with } p\to 0 \text{ for } d_i > 4} \qquad (2.68)$$

That is, operators with $d_i > 4$ (non-renormalizable operators) become irrelevant at $p \ll M$. This result recovers Wilsons RGE-result in the Callan-Symanzik approach.

RGE for the $\overline{MS}$-scheme.

RGE can also be defined with no reference a particular subtraction point. Indeed consider the bare Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{\lambda_0}{4!}\phi_0^4 \qquad (2.69)$$

and rewrite it as a regulated theory in renormalized BPHZ-scheme:

$$\mathcal{L}_\epsilon = \frac{1}{2}(\partial_\mu \phi_r)^2 - \frac{\lambda_r \mu^\epsilon}{4!}\phi_r^4 + \text{c.t} \qquad (2.70)$$

hence λ_0 is dimensionful!

This Lagrangian is to be understood as having been defined in "$4-\epsilon$-dimensions" and hence, to keep λ_r dimensionless, an explicit μ^ϵ-factor has been introduced $\Rightarrow$ scale enters $\mathcal{L}_\epsilon$.
n-point 1PI-functions obey (*)

$$\Gamma_r^{(n)}(\lambda_r, \mu) = Z_\phi^{n/2}(\lambda_0 \mu^{-\epsilon}, \epsilon)\, \Gamma_0^{(n)}(\lambda_0, \epsilon) \qquad (2.71)$$

In this approach RGE follows from μ-independence of $\Gamma_0^{(n)}$:

$$\left(\mu\frac{\partial}{\partial\mu} + \underbrace{\beta(\lambda_r)}_{=\mu\frac{\partial}{\partial\mu}\lambda_r|_{\text{bare}}}\frac{\partial}{\partial\lambda_r} + n\gamma\right)\Gamma_r^{(n)}(\lambda_r, \mu) = 0 \qquad (2.72)$$

---

* you can think of Z_ϕ being computed in bare expansion, whence it must be a function of $\underbrace{\lambda_0 \mu^{-\epsilon}}_{\text{dimensionless}}$, (see. p. 82)

Now define an invariant charge

$$G(p_i;\lambda,\mu) \equiv \Gamma^{(4)}(\{p_i\}) \prod_{i=1}^{4} \frac{[(p_i)^2]^{1/2}}{(\Gamma^{(2)}(p_i))^{1/2}} . \qquad (2.73)$$

This obviously obeys the homogenous RGE:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda}\right) G(p_i;\lambda,\mu) = 0 \qquad (2.74)$$

Define now RGE-transformation

$$\boxed{\mu \to \bar{\mu}(t) = e^t\mu} \qquad (2.75)$$

We find solutions to $G(\lambda,\mu)$ which obey $G(\lambda,\mu) = G(\bar{\lambda}(t),\bar{\mu}(t))$, i.e.

$$0 = \frac{\partial}{\partial t} G(\bar{\lambda}(t),\bar{\mu}(t)) = \left(\frac{\partial\bar{\lambda}}{\partial t}\frac{\partial}{\partial\bar{\lambda}} + \underbrace{\frac{\partial\bar{\mu}}{\partial t}}^{=\bar{\mu}}\frac{\partial}{\partial\bar{\mu}}\right) G(\bar{\lambda},\bar{\mu}) \qquad (2.76)$$

Comparing with (2.74) this implies

$$\boxed{\frac{\partial\bar{\lambda}(t)}{\partial t} = \beta(\bar{\lambda}(t))} \quad ; \quad \bar{\lambda}(0) = \lambda. \qquad (2.77)$$

How do we relate this to physical charge in this scheme? First note that G is actually $\sim$ physical coupling on shell. So we can define λ by some reference point as follows.

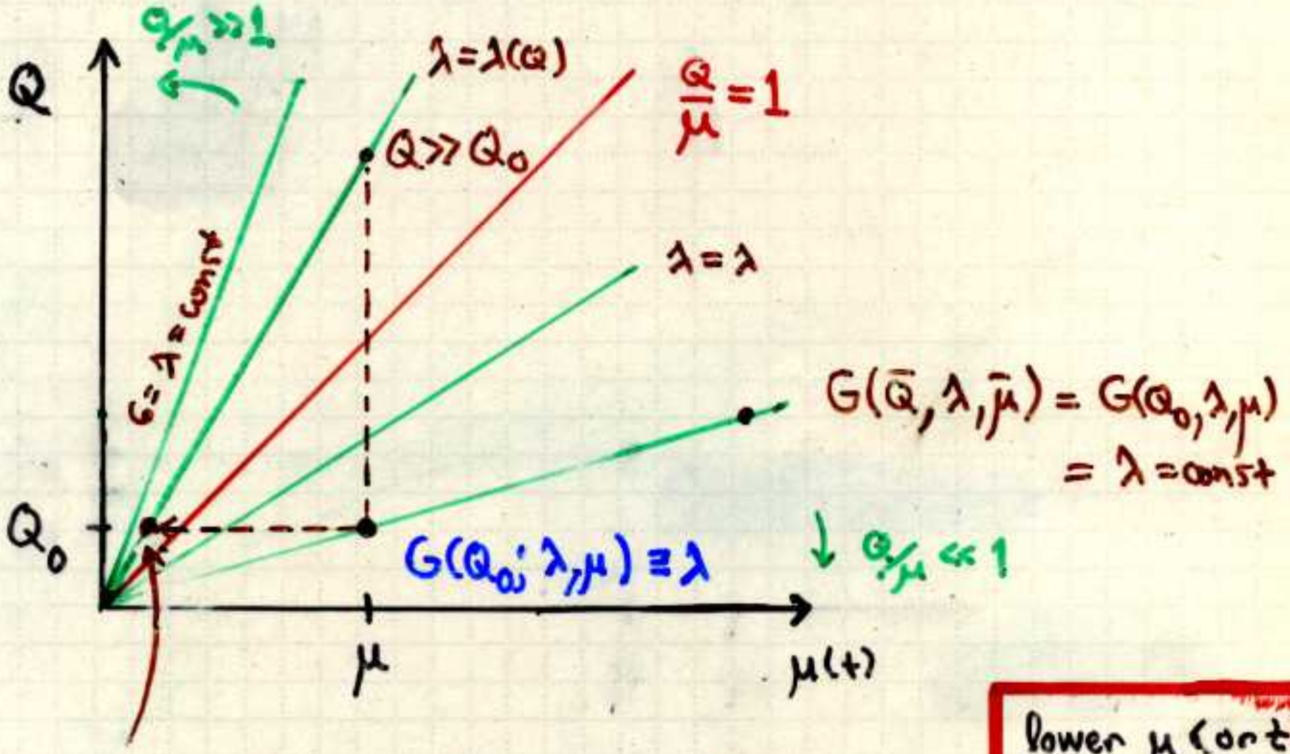
$$G(p_\lambda^0;\lambda_r,\mu) \equiv \lambda_r \qquad (2.78)$$

We can then use the scaling property (2.76) to compute G at some other scale:

$$G(e^t p_i^\circ;\lambda,\mu) = G(p_i^\circ;\lambda,\mu e^{-t})$$

$$= G(p_i^\circ;\bar\lambda(t),\mu) = \bar\lambda(t) \qquad (2.79)$$

That is, the coupling $\bar\lambda(t)$ defined by eqns. (2.77) and (2.78) is the running coupling of the theory at scale $Q = Q_0 e^t$ when $\lambda(0)$ was defined on scale $Q = Q_0$.



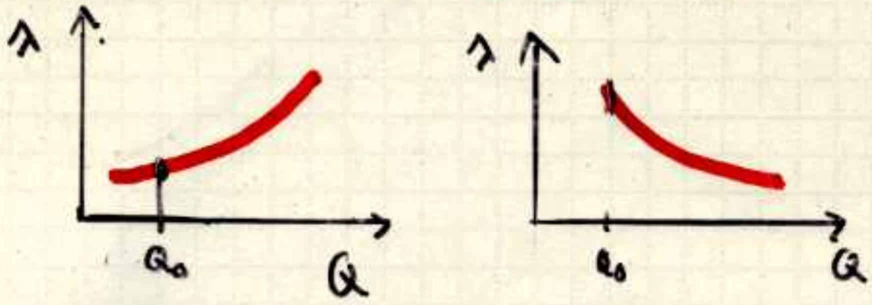
$\boxed{\text{lower } \mu \text{ (or } t) \Leftrightarrow \text{increase } Q}$

$$= G(Q_0;\lambda,\mu e^{-t}) = G(Q_0 e^t,\lambda,\mu)$$
$$= G(Q_0;\bar\lambda,\mu) = \lambda(Q_0 e^t) = \lambda(Q).$$

Fixed point structure

The main use of RG is to shed light on the large, or small momentum behaviour of the QFT

From (2.77) we see that $\bar\lambda$ increases decreases as t is increased decreased (Q is increased decreased) if $\beta(\lambda) > 0$. Ie. in this case the coupling becomes stronger at larger Q. In contrast, λ decreases as a function of Q if $\beta(\lambda) < 0$.



$\beta(\lambda) > 0$ $\qquad\qquad$ $\beta(\lambda) < 0$

We alreaby saw that $\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} > 0$ for $\lambda\phi^4$- theory, so that the behaviour is like that shown in the left panel above.

Let us redo the β-function calculation in the $\overline{MS}$-scheme:

$$\lambda = \lambda_r(\lambda_0\mu^{-\epsilon}, \epsilon) = Z_1^{-1}\big(\lambda(\lambda_0\mu^{-\epsilon}), \epsilon\big) \cdot \lambda_0 \mu^{-\epsilon}$$

see p. 82

where

$$Z_1 \simeq 1 + \frac{3\lambda_{\overline{MS}}}{16\pi^2}\frac{1}{\epsilon_{\overline{MS}}}$$

Then

$$\beta(\lambda) \equiv \mu \frac{\partial \lambda_r}{\partial \mu}\bigg|_{\lambda_0,\epsilon} = -\epsilon\lambda - \lambda\mu\frac{\partial}{\partial\mu}\log Z_1\big(\lambda(\lambda_0\mu^{-\epsilon}), \epsilon\big)\bigg|_{\lambda_0,\epsilon}$$

$$\simeq -\epsilon\lambda\left(1 - \lambda\frac{\partial}{\partial\lambda}\log Z_1\right)$$

$$\simeq -\epsilon\lambda + \epsilon\lambda^2\frac{\partial}{\partial\lambda}\left(+\frac{3\lambda}{16\pi^2}\frac{1}{\epsilon_{\overline{MS}}}\right)$$

$$= +\frac{3\lambda^2}{16\pi^2} + O(\epsilon, \lambda^3) \qquad (290)$$

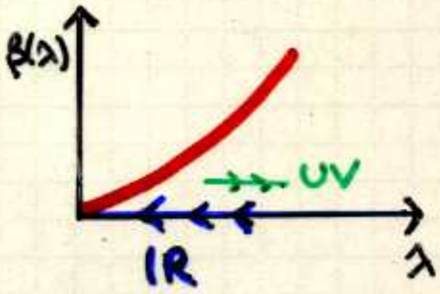
That is, $\overline{MS}$-scheme provides exactly the same scaling as M-subtraction scheme !

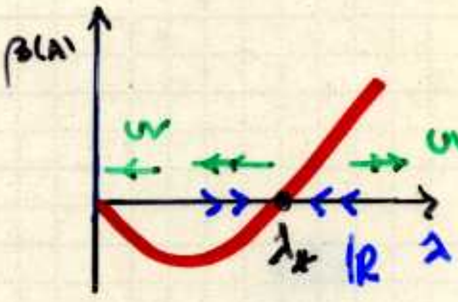
$$-\left(\frac{1}{\lambda(t)} - \frac{1}{\lambda(0)}\right) = \frac{3t}{16\pi^2} \Rightarrow \lambda(t) = \frac{\lambda(0)}{1-\frac{3\lambda(0)}{16\pi^2}t}$$

If we now let $\lambda(0) = \lambda(Q_0) = G(Q_0, \lambda(Q_0)\,\mu)$ and furthermore $Q \equiv Q_0 e^t \Leftrightarrow t = \log\frac{Q}{Q_0}$ we get

$$\lambda(Q) = \frac{\lambda(Q_0)}{1 - \frac{3\lambda(Q_0)}{16\pi^2}\log\left(\frac{Q}{Q_0}\right)} \qquad (2.81)$$

That is $\lambda = 0$ is an IR-fixed point in $\lambda\phi^4$-theory, just as we saw in the Wilson analysis:





$\lambda_* = \frac{16\pi^2}{3}\epsilon$

However, if we retain the ϵ-term (stay in "$4-\epsilon$-dimensions") we have:

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} - \epsilon\lambda + \mathcal{O}(\epsilon\lambda^2) \qquad (2.82)$$

which has an IR-fixed point at $\lambda_* = \frac{16\pi^2}{3}\epsilon \neq 0$, again in complete agreement with Wilsons method.

At the same time, $\lambda = 0$ becomes an UV-fixed point when $\epsilon \neq 0$. Thus $\underbrace{\lambda\phi^4}_{\text{a regulated}}$- theory with $\epsilon < 0$ is our first example of an asymptotically free field theory. (When the collision energy increases particles become transparent to each other.) This is an important observation:

> If $\beta'(g)|_{g=0} < 0$, the theory is asymptotically free (interactions are turn off at high Q).

We will see later that this is precicely the situation in the case of the physical theory of strong interactions between the quarks. (QCD).

More generically, in the neigbourhood of a fixed point $\beta(\lambda_*)=0$, one may expand

$$\beta(\lambda) \simeq B(\lambda-\lambda^*) \quad ; \quad B \equiv \beta'(\lambda_*)$$

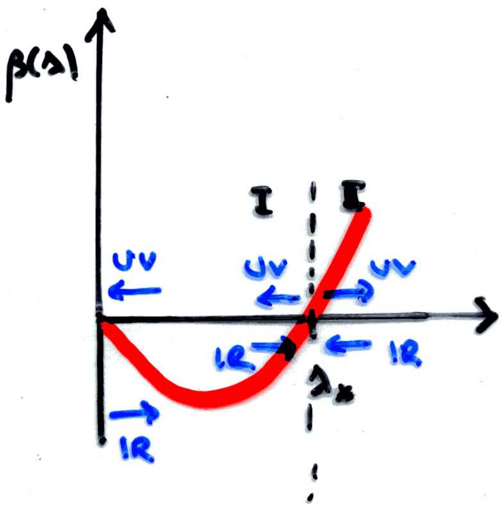
$\Delta\lambda \equiv \bar{\lambda}(0) - \lambda_*$

whereby

$$\frac{d\bar{\lambda}}{dt} = B(\bar{\lambda}-\lambda_*) \Rightarrow \frac{d\bar{\lambda}}{\bar{\lambda}-\lambda_*} \simeq \beta' dt \Rightarrow \bar{\lambda} = \lambda_* + \Delta\lambda\, e^{\beta' t}$$

$t \equiv \log \frac{Q}{Q_0}$

$$\Rightarrow \quad \bar{\lambda} = \lambda_* + \Delta\lambda \left(\frac{Q}{Q_0}\right)^B$$



1) $\lambda_* \equiv 0 \quad \Rightarrow \Delta\lambda > 0; \quad \beta' = B < 0$

$$\Rightarrow \bar{\lambda} \to \begin{cases} \lambda_* = 0 & Q \to \infty \\ \infty & Q \to 0 \end{cases}$$

2) $\lambda_* = \lambda_* \neq 0$ $\quad$ $\Delta\lambda < 0$ Reg I $\quad$ and $\beta' = B > 0$.
$\Delta\lambda > 0$ Reg II

$$\Rightarrow \quad \bar{\lambda} \to \begin{cases} \lambda_* & ; \; Q \to 0 \qquad \text{Region II} \\ \infty & ; \; Q \to \infty \end{cases}$$

$$\bar{\lambda} \to \begin{cases} \lambda_* & : \; Q \to 0 \qquad \text{Region I} \\ \lambda_* & : \; \text{decreases (until catched by } \lambda = 0 \text{ fp)} \end{cases}$$

Digression: Let me clarify one (potentially) confusing point. Throughout the $p^2 = -M^2$ subtraction section I was using renormalized perturbation theory, whereas in discussing $\overline{MS}$-scheme I have been writing relations (such as (3.84)) apparently in terms of the bare coupling expansion. Indeed, in the bare coupling expansion μ-dependence is explicit:

$$Z_i^{bare} = Z_i^{bare}(\underbrace{\lambda_0 \mu^{-\epsilon}}_{\text{dimensionless}}, m_0, \epsilon) \qquad (D.1)$$

In the renormalized pt. we have (or rather) μ-dependence implicit.

$$Z_i^{rn} = Z_i^{rn}(\lambda, \epsilon) \qquad (D.2)$$

$\lambda = \lambda(\lambda_0 \mu^{\epsilon})$

Shifting between these definitions does not change the form of the Callan-Symanzik equations, but it does change the way β, γ and γ_m-functions are related to Z_i's.

Let us see this for example for β-function. It can be derived just from

$$\lambda \equiv Z_1^{-1}(\dots)\underbrace{\lambda_0\mu^{-\epsilon}}_{\equiv \tilde{\lambda}_0} \qquad (D.3)$$

without accurate specification of μ-dependence of Z_1. If we assume (D.1), i.e. bare expansion

$$Z_1 \equiv Z_1^{bare}(\tilde{\lambda}_0) \Rightarrow \beta(\lambda) = -\epsilon\lambda - \lambda\mu\frac{\partial}{\partial\mu}\log Z_1(\tilde{\lambda}_0)_{bare}$$

$$= -\epsilon\lambda\left(1 - \tilde{\lambda}_0\frac{\partial}{\partial\tilde{\lambda}_0}\log Z_1(\tilde{\lambda}_0)_{bare}\right) \qquad (D.4)$$

If on the other hand

μ-dependence now implicit

$$Z_1 \equiv Z_1^{ren}[\lambda] \Rightarrow \mu\frac{\partial\tilde{\lambda}_0}{\partial\mu} = -\epsilon\tilde{\lambda}_0 = \mu\frac{\partial}{\partial\mu}\left[Z_{ren}(\lambda)\cdot\lambda\right]$$

$$= \beta(\lambda)\cdot\left[Z_{ren} + \lambda\frac{\partial}{\partial\lambda}Z_{ren}\right]$$

$$\Rightarrow \beta(\lambda) = -\epsilon\tilde{\lambda}_0\left(Z_{ren} + \lambda\frac{\partial}{\partial\lambda}Z_{ren}\right)^{-1}$$

$$= \frac{-\epsilon\lambda}{1 + \lambda\frac{\partial}{\partial\lambda}\log Z_{ren}} \qquad (D.5)$$

Of course both definitions lead to identical β-functions.*

* The connection between functions is of course

$$\tilde{\lambda}_0 = Z_{ren}[\lambda]\,\lambda = \frac{\tilde{\lambda}_0}{Z_{bare}[\tilde{\lambda}_0]}\,Z_{ren}\left[\frac{\tilde{\lambda}_0}{Z_{bare}[\tilde{\lambda}_0]}\right]$$ ⇒ iterative def of parameters.

The role of the anomalous dimension $\gamma(\lambda)$

Remember the formula for the massless 2-point function:

$$G^{(2)}(p) = \frac{i}{p^2}\tilde{g}_2(\bar{\lambda})\, e^{2\int_1^{\log p/M} d\log x\, \gamma(\bar{\lambda})}$$

If $\bar{\lambda}$ tends to a fixed point already at $p \approx M$, γ remains constant thereafter and one simply gets

$$G^{(2)}(p) \sim \frac{i}{p^2}\tilde{g}(\lambda_*)\, e^{2\gamma(\lambda_*)\log\frac{p}{M}} \qquad \text{j rem.: } \gamma \equiv -\frac{\mu}{2Z_\phi}\frac{dZ_\phi}{d\mu}$$

$$= C\left(\frac{1}{p^2}\right)^{1-\gamma(\lambda_*)}. \qquad (D6)$$

Thus, if the fixed point is trivial $\lambda_* = 0$ (perturbatively $\gamma(\lambda)$ = expansion in λ), then $\gamma(\lambda_*) \to 0$ and one has the usual scaling of the propagator. However, if $\lambda_* \neq 0$, then generically $\gamma(\lambda_*) \neq 0$ and we will have a scale-invariant QFT, in which interactions affect the law of rescaling. Hence γ is called anomalous dimension.

Application to QED

Now consider the C-Z-equation for the QED- 3-point-function. The invariant charge defined in analogy with (2.73) now obeys the equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(e)\frac{\partial}{\partial e} + \gamma_{m_e} m_e\frac{\partial}{\partial m_e}\right) G_3(p_i, e, m_e, \mu) = 0 \qquad (2.83)$$

where

$$\beta(e) = \mu\frac{\partial e}{\partial\mu} \quad \text{with} \quad e = \frac{Z_2}{Z_1}\sqrt{Z_3}\,\tilde{e}_0 \overset{\text{ward}}{=} \sqrt{Z_3}\,\tilde{e}_0$$

$$\tilde{e}_0 = e_0\mu^{-\epsilon/2}$$

$$\gamma_m(e) = \frac{\mu}{m_e}\frac{\partial m_e}{\partial\mu} = -\mu\frac{\partial}{\partial\mu}\log Z_m .$$

$m_0 = Z_m m$ running of mass. Later!

$[\psi] = M^{\frac{d-1}{2}}$
$[A] = M^{\frac{d-2}{2}}$
$\Rightarrow [\bar\psi A\psi] = M^{3\frac{d}{2}-2}$
$\Rightarrow [e_0] = M^{2-\frac{d}{2}} = M^{\epsilon/2}$!

Solution is again

$$\bar{e} = G(e^t p_i; e, m, \mu) = G(p_i^0, \bar{e}, \underbrace{e^{-t}\bar{m}(t)}_{\to 0 \text{ (by assumption)}}, \mu)$$

where

$$\frac{\partial\bar{e}}{\partial t} = \beta(\bar{e}) . \qquad (2.84)$$

Now $\beta(e) = \mu\frac{\partial e}{\partial\mu} = -\frac{\epsilon}{2}e + \tilde{e}_0\mu\frac{\partial e}{\partial\mu}\frac{\partial\sqrt{Z_3}}{\partial e}$, with $\tilde{e}_0 = \frac{e}{\sqrt{Z_3}}$

$$\Rightarrow \beta(e) = \frac{-\epsilon e/2}{1+\frac{1}{2}e\frac{\partial}{\partial e}\log Z_3}$$

From (1.98) we see that $\delta_3^{\overline{MS}} = Z_3^{\overline{MS}} - 1 = -\frac{\alpha}{3\pi}\cdot\frac{2}{\epsilon_{\overline{MS}}} = -\frac{2\alpha}{3\pi\epsilon} + \text{finite}$

$$\Rightarrow \beta(e)_{\overline{MS}} \approx -\frac{\epsilon}{2}e\left(1 - \frac{2\alpha}{3\pi}\frac{1}{\epsilon_{\overline{MS}}} + \cdots\right) \approx +\frac{e^3}{12\pi^2} \qquad (2.85)$$

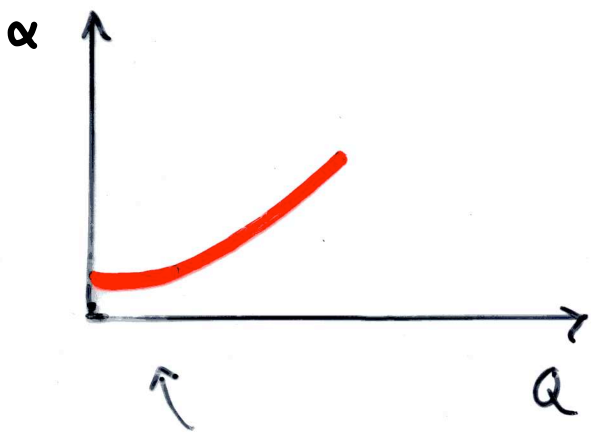
Then from (2,95)

$$\frac{\partial e}{\partial t} = \frac{e^3}{12\pi^2} \Rightarrow \left(\frac{1}{e^2} - \frac{1}{\bar{e}^2}\right) = \frac{t}{24\pi^2}$$

$$\Leftrightarrow \quad \bar{e}(t)^2 = \frac{e^2}{1 - \frac{e^2}{24\pi^2}t} \quad ; \text{ set } \begin{array}{l} e = e(Q_0) \\ Q = e^t Q_0 \end{array}$$

$$\Rightarrow \quad \alpha(Q) = \frac{\alpha(Q_0)}{1 - \frac{\alpha^2}{3\pi}\log\left(\frac{Q}{Q_0}\right)^2} \qquad (2.86)$$

This is just the scaling law we obtained in eqn. (1.103) resumming the leading logs of photon polarization tensor to a running electron charge. (taking $Q_0 = Am^2$; $Q^2 \to -Q^2$ match is exact).



not valid at very small Q_0.

Generalization to massive theories ; critical exponents.

Generalizing $p^2=-M^2$-subtraction scheme RGE to the case with $m \neq 0$ requires additional work (renormalization of composite operators). See P&S. 12.4 - 12.5. The $\overline{MS}$-scheme RGE is more easily extended to this case, simply because $\overline{MS}$-counter terms are independent of mass.

By an argument similar to the one that led to (2.71) and (2.72) we get:

$$\boxed{\left[\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+n\gamma(\lambda)+\gamma_m m\frac{\partial}{\partial m}\right]\Gamma_R^{(n)}(\lambda,m,\mu)=0} \qquad (2.87)$$

with

$$\beta=\mu\frac{\partial\lambda}{\partial\mu}\bigg|_{\lambda_0,m_0,\epsilon}=\mu\frac{\partial}{\partial\mu}\left[Z_1^{-1}(\lambda_0\mu^{-\epsilon},\epsilon)\cdot\lambda_0\mu^{-\epsilon}\right] \qquad (2.88)$$

$$=-\epsilon\lambda-\lambda\mu\frac{\partial}{\partial\mu}\log Z_1\bigg|_{\lambda_0,\epsilon,\mu}$$

RGE-running of λ.

$$\gamma(\lambda)=-\frac{1}{2}\mu\frac{\partial}{\partial\mu}\log Z_\phi\bigg|_{\lambda_0,m_0,\epsilon} \qquad (2.89)$$

$$\gamma_m(\lambda)=\frac{\mu}{m}\frac{\partial m}{\partial\mu}\bigg|_{m_0,\lambda_0}=-\mu\frac{\partial}{\partial\mu}\log Z_m\bigg|_{m_0,\lambda_0,\epsilon} \qquad (2.90)$$

← Defines RGE-running of mass

where

$$m_0\equiv Z_m m \qquad (2.90a)$$

These eqns must be solved simultaneously.

Solution proceeds as before, and one finds the invariant charge ($p_i^2 \to p_i^2 - m^2$ in (2.73))

$$G(p_i; \lambda, m, \mu) = G(p_i; \bar{\lambda}(t), \bar{m}(t), \bar{\mu}(t) = e^t \mu) \qquad (2.91)$$

where $\bar{\lambda}$ obeys the same equation as before and

$$\frac{\partial \bar{m}}{\partial t} = \bar{m}(t) \cdot \gamma_m(\bar{\lambda}(t))$$

$$\Rightarrow \bar{m}(t) = m\, e^{\int_0^t dt' \, \gamma_m(\bar{\lambda}(t'))}$$

running of mass

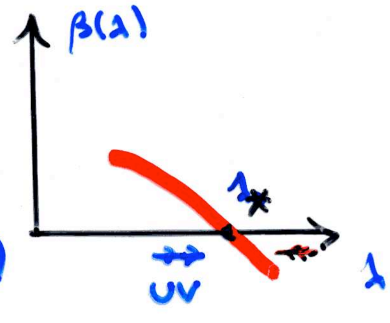
$$= m \cdot \exp\left[\int_{\lambda}^{\bar{\lambda}} dx \, \frac{\gamma_m(x)}{\beta(x)} \right] \qquad (2.92)$$

The analog of the scaling relation (2.79) now reads

rel. imp. of m w.r.t. p^0 reduces for large momenta even if $\bar{m} \gg m$.

$$G(e^t p_i^0; \lambda, m, \mu) = G(p_i^0; \bar{\lambda}, e^{-t}\bar{m}, \mu) \qquad (2.93)$$

so the dynamics at momenta $e^t p_i^0$ will be controlled by effective charge $\bar{\lambda}$ and mass parameter $e^{-t}\bar{m}$. Suppose now that theory has an UV-fixed point $\lambda = \lambda_*$. Then from (2.92)



$$\Rightarrow e^{-t}\bar{m}(t) \to e^{-t(1-\gamma_m(\lambda_*))} m$$

$$\xrightarrow[\text{limit}]{\text{UV}} 0 \quad \text{if } \gamma_m(\lambda_*) < 1 \qquad (2.94)$$

(This assumption is not necessary in asympt. free theories with $\gamma_m(0) = 0$.)

condition for effectively massless UV-limit of the theory $\hat{=}$ cond. for deducing UV without knowing ∞-mass limit.

Suppose now that $d<4$. In the neighbourhood of the Fischer-Wilson-fixed point $\lambda_* \simeq \frac{16\pi^2}{3}\epsilon$.

On the other hand $m_0^2 = m^2 + \Delta m^2$ & $m_0^2 Z_\phi \equiv m^2 + \delta_m = m^2\left(1+\frac{\delta_m}{m^2}\right)$

$$m_0^2 = m^2 Z_\phi^{-1}\left(1+\frac{\delta_m}{m^2}\right) \equiv Z_m^2 m^2 \quad \Rightarrow \quad Z_m = \left(1+\frac{\delta_m}{m^2}\right)^{1/2} Z_\phi^{-1/2} \qquad (2.95)$$

In $\lambda\phi^4$-theory at 1-loop (Ω)

or, $\mu\frac{\partial\lambda}{\partial\mu} = -\epsilon\lambda$ to lowest order!

λ to lowest order

$$\begin{cases} \Pi'(m^2) = 0 \\ \Pi(m^2) = -\frac{\underbrace{\lambda_0\mu^{-\epsilon}}}{16\pi^2} m^2\left(\frac{1}{\epsilon}+\cdots\right) \end{cases} \qquad (2.96)$$

$$\Rightarrow Z_\phi = 1 \quad \text{and} \quad \delta_m^{\overline{MS}} = +\frac{\lambda_0\mu^{-\epsilon}}{16\pi^2}\frac{m^2}{\epsilon_{\overline{MS}}}. \qquad (2.97)$$

Then

$$\gamma_m = -\mu\frac{\partial}{\partial\mu}\log Z_m = -\frac{1}{2}\mu\frac{\partial}{\partial\mu}\log\left(1+\frac{\delta_m}{m^2}\right)$$

$$\approx -\frac{1}{2m^2}\mu\frac{\partial\delta_m}{\partial\mu} = +\frac{\lambda}{32\pi^2} \qquad (2.98)$$

Then, close to FW-fixed point, the mass-operator scaling in (2.94) is

$$\bar{m}^2(t) = m(t_0)^2\left[e^{+(t-t_0)}\right]^{+2\gamma_m} \quad ; \quad \frac{Q}{Q_0} \equiv e^{t-t_0}$$

$$\Leftrightarrow \quad \bar{m}^2(Q) = m(Q_0)^2\left(\frac{Q}{Q_0}\right)^{+2\gamma_m} \quad ; \quad 2\gamma_m = \frac{1}{3}\epsilon \qquad (2.99)$$

at WF-fp.

In (2.93) the relevant mass scaling was $e^{-t}\bar{m}$ however. This is the scaling law of a dimensionless mass operator. Indeed, if we rewrite the mass term in the Lagrangian as

$$m^2\phi^2 \equiv \rho_m \mu^2 \phi^2 \qquad ; \quad m^2 \equiv \rho_m \mu^2 \qquad (2.100)$$

we would write instead of (2.91) we write

$$G(\lambda, \rho_m, \mu) \rightarrow G(\bar{\lambda}, \bar{\rho}_m, \bar{\mu}) \qquad (2.101)$$

Since ρ_m is now dimensionless one should find

$$G(e^t p^0, \lambda, \rho_m, \mu) = G(p_0, \bar{\lambda}, \bar{\rho}_m, \mu) = \bar{\lambda} \qquad (2.102)$$

So $\bar{\rho}_m$ is a coupling more like λ. Writing CS-equation with ρ_m, one finds

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + \underbrace{\left(\frac{-\mu}{\rho_m}\frac{\partial\rho_m}{\partial\mu}\right)}_{=\tilde{\beta}_m}\rho_m\frac{\partial}{\partial\rho_m}\right] G(\lambda, \mu, \rho_m) = 0 \qquad (2.103)$$

Thus we get

$$\frac{\mu}{\bar{\rho}_m}\frac{\partial\bar{\rho}_m}{\partial\mu} = \tilde{\beta}_m \quad \overset{\text{if } \tilde{\beta}_m = \text{const}}{\Rightarrow} \quad \bar{\rho}_m = \bar{\rho}_m(0)\, e^{\tilde{\beta}_m t} \qquad (2.104)$$

From (2.100) we see that

$$\tilde{\beta}_m = -\frac{\mu}{\rho_m}\frac{\partial\rho_m}{\partial\mu} = -\frac{\mu^3}{m^2}\frac{\partial}{\partial\mu}\left(\frac{m^2}{\mu^2}\right) = -2 + 2\frac{\mu}{m}\frac{\partial m}{\partial\mu}$$

$$= -2 + 2\gamma_m \qquad (2.105)$$

putting this back, one finds

$$\underline{\bar{\rho}_m} = \rho_m e^{-2(1+\gamma_m)t} = \underline{\rho_m \left(\frac{Q}{Q_0}\right)^{-2+2\gamma_m}} \qquad (2.106)$$

where at last $t = \log Q/Q_0$ was used. So, if we are close to a trivial fixed point $\lambda = 0$, we get $\gamma_m(\lambda) \to 0$, and (2.106) reduces to a fancy way of saying that m is irrelevant for very large momenta. However, near nontrivial fixed point the scaling law is changed: $\gamma_m(\lambda_*) \neq 0$.

Correlation length

One expects that correlation length for a fluctuation in scale Q is

$$\xi \sim \frac{1}{Q} \qquad (2.107)$$

if the mass operator remains small. (If mass is large, then it will cut correlations off exponentially.) The smallest scale for which (2.107) then holds is Q corresponding to

$$\bar{\rho}_m(Q) = \rho_m \left(\frac{Q}{Q_0}\right)^{-2+2\gamma_m} \equiv 1 \qquad (2.108)$$

Hence the maximum correlation length is

$$\xi \sim \frac{1}{Q} \sim (\rho_m)^{-\nu} \qquad (2.109)$$

with

$$\boxed{\nu \equiv \frac{1}{2-2\gamma_m}} \qquad (2.110)$$

We are not interested in the factor in front of (2.109), but only on the scaling of ξ as one adjusts the parameter ρ_m.

In QFT ρ_m can be adjusted by changing μ. In what follows, we shall see that for a statistical system near critical point the mass is proportional to the distance from the critical point, so that $\rho_m \sim (T_c - T)/T_c$. Then

$$\xi \sim (T_c - T)^{-\nu} \sim (T_c - T)^{-\frac{1}{2}\left(\frac{1}{1-\gamma_m}\right)} \qquad (2.111)$$

We shall argue that certain statistical systems near critical point in described (at IR-limit) by a 3-dimensional Euclidean QFT. Thus we should evaluate γ_m at 3-d Wilson-Fisher fixed point. From (2.99) then : $\gamma_m = \frac{1}{6}\epsilon = \frac{1}{6}$. So that

$$\nu = +\frac{1}{2}\left(\frac{1}{1-\frac{1}{6}}\right) = \frac{1}{2}\frac{6}{5} = \frac{3}{5} = 0.6 \qquad (2.112)$$