Practicalities

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Schedule : 2.9 – 11.12 2024

Mondays & Wednesdays 10–12 FYS 5

Excercises: Mondays 12.15 – 14.00 YFL 226

Exams : Final 17.12 2024

Grading :

$$\text{Final points} = \text{ceil}\left\{30 \times \left(\frac{\text{excercise pts}}{\text{max exc pts}}\right) + 30 \times \left(\frac{\text{exam pts}}{\text{max exm pts}}\right)\right\} \leq 60$$

Final grade : Grad

1	30 – 35
2	36 – 41
3	42 – 47
4	48 – 53
5	54 – 60

1. Systematics of the Renormalization

The purpose of this chapter is to present a coherent scheme for the renormalization procedure, after we have been exposed to most of the practical and conceptual problems in previous chapters. We first do this for ϕ^4-theory and then proceed to renormalization of QED.

1.1. Renormalized p.t. for $\lambda\phi^4$-theory.

Start from the Lagrangian (written in terms of bare, unmeasurable quantities)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_0)^2 - \frac{m_0^2}{2}\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4 \qquad (1.1)$$

- Rescale the field

$$\phi_0 = Z_\phi^{1/2}\phi_r = (1-\delta_\phi)^{1/2}\phi_r \qquad (1.2)$$

$$\Rightarrow \mathcal{L} = \frac{Z_\phi}{2}(\partial_\mu\phi_r)^2 - \frac{m_0^2 Z_\phi}{2}\phi_r^2 - \frac{\lambda_0 Z_\phi^2}{4!}\phi_r^4 \qquad (1.3)$$

- Redefine the mass and the coupling

$$m^2 \equiv \underbrace{m_0^2 Z_\phi}_{(m^2+\delta m^2)Z_\phi} - \delta_m \qquad \Rightarrow \delta_m = m^2\delta_\phi + \delta m^2 Z_\phi \quad \Leftrightarrow \quad \delta m^2 = \frac{\delta_m - m^2\delta_\phi}{1+\delta_\phi}$$

$$\lambda \equiv \underbrace{\lambda_0}_{\lambda+\delta\lambda} Z_\phi^2 - \delta_\lambda \qquad \Rightarrow \delta_\lambda = (Z_\phi^2-1)\lambda + \delta\lambda Z_\phi^2 \quad \Rightarrow \delta\lambda = \frac{\delta_\lambda - (Z_\phi^2-1)\lambda}{Z_\phi^2} \qquad (1.4)$$

- Split the Lagrangian to renormalized theory + counter-terms:

$$\boxed{\begin{aligned}\mathcal{L} = {} & \frac{1}{2}(\partial_\mu\phi_r)^2 - \frac{1}{2}m^2\phi_r^2 - \frac{\lambda}{4!}\phi_r^4 \\ & + \frac{1}{2}\delta_Z(\partial_\mu\phi_r)^2 - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4\end{aligned}} \qquad (1.5)$$

- Treat the counter-term Lagrangian as interaction. The values of parameters δ_Z, δ_m & δ_λ are not known, however.

- Choose a renormalization prescription to define the (divergent) numbers δ_Z, δ_m and δ_λ order by order in the perturbation theory. For example

$$\left\{\begin{aligned} & \Delta_R^{-1}(p^2=m^2) = 0 \\ & \frac{d}{dp^2}\Delta_R^{-1}(p^2=m^2) = 1 \\ & \Gamma^{(4)}(s=4m^2;0,0) \equiv \lambda \end{aligned}\right. \qquad (1.6)$$

will fix δ_m & δ_Z (on-shell).

– will fix δ_λ

In order to compute n-point-functions we need the Feynman rules. Let us write the Generating function

$$Z[J] = \int \mathcal{D}\phi_r \, e^{i S_I^1[\phi_r; \lambda, \delta_Z, \delta_m, \delta_\lambda] + i\int d^4x \left(\frac{1}{2}(\partial_\mu \phi_r)^2 - \frac{m^2}{2}\phi_r^2 + J\phi \right)} \tag{1.7}$$

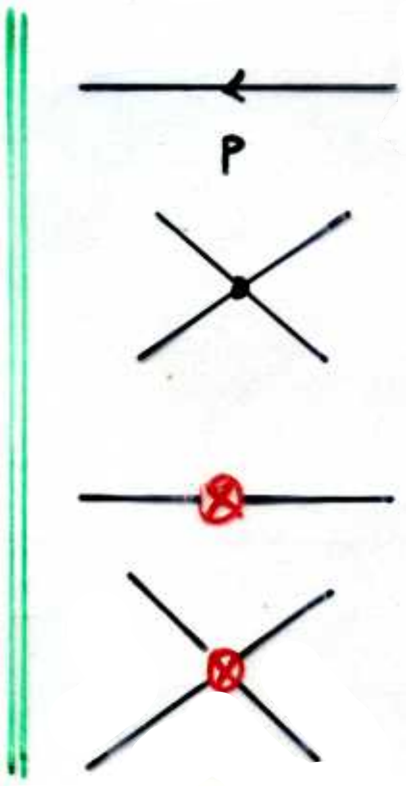
(Note that $\mathcal{D}\phi_0$ = Infinite constant $\times\ \mathcal{D}\phi_r$). When we treat terms with λ, δ_i as interactions, we easily get

$$Z[J] = e^{i S_I^1\left(-\frac{i\delta}{\delta J}; \lambda, \delta_Z, \delta_m, \delta_\lambda\right)} \int \mathcal{D}\phi_r \, e^{i\int d^4x \left(\mathcal{L}_{free} + J\phi\right)}$$

$$= e^{i S_I^1\left(-\frac{i\delta}{\delta J}; \lambda, \delta_Z, \delta_m, \delta_\lambda\right)} \, e^{\frac{i}{2}\int d^4x \, J(x)\, \Delta_R(x-y)\, J(y)} \tag{1.8}$$

the renormalized propagator!

Obviously the Feynman rules are:



p: $\dfrac{i}{p^2 - m^2 + i\epsilon}$

$-i\lambda$

$+i(p^2\delta_Z - \delta_m)$

$-i\delta_\lambda$

(1.9)

$i\int d^4x \, -\phi(\delta_\phi \Box + \delta_m)\phi$

$\to +i(p^2\delta_\phi - \delta_m)$

One-loop renormalization. It is now easy to collect results at the one loop level.

$(\Gamma^{(2)} =)$

$-i\Pi(p^2) =$  $\approx$ 

$+$ 

$$= -i\Pi_{1\text{-loop}}(p^2, m^2) + i(p^2\delta_Z - \delta_m)$$

$$\Rightarrow \quad \Pi(p^2) \approx \Pi(m^2) + \Pi'(m^2)(p^2-m^2) + \tilde{\Pi}(p^2) - (p^2\delta_Z - \delta_m)$$

↑ renormalized mass! ↳ $\tilde{\Pi}(m^2) = \tilde{\Pi}'(m^2) = 0$

(1.10)

The renormalization condition (1.6) now gives(*)

$(\Delta^{-1} = p^2 - m^2 - \Pi)$

$$\left.\begin{aligned} 0 &\equiv \Delta_R^{-1}(p^2 = m^2) = -\Pi(m^2) + m^2\delta_2 - \delta_m \\ 1 &\equiv \frac{d}{dp^2}\Delta_R^{-1}(p^2 = m^2) = 1 - \Pi'(m^2) + \delta_2 \end{aligned}\right\} \Rightarrow \begin{aligned} \delta_2 &= \Pi'(m^2) \\ \delta_m &= -\Pi(m^2) + m^2\Pi'(m^2) \end{aligned}$$

(1.11)

These conditions actually hold true, when interpreted recursively, to all orders. To first order $\Pi(m^2)$ is given by (6.67) in D-reg, and $\Pi'(m^2) = 0$. Note that definition for $Z_\phi = 1 - \delta_2 = 1 + \Pi'(m^2)$ agrees with (6.48) in QFT-1 lecture notes.

(*) Note that after imposing (1.6)

$$\Pi_{full} = \Pi_{loop} + p^2\delta_2 + \delta_m = \tilde{\Pi}(p^2)$$

$$\Rightarrow \Delta_R^{-1} = p^2 - m^2 - \tilde{\Pi}(p^2)$$

Similarly:

$-i\Gamma^{(4)} =$  $+$ ( $+ \cdots$) $+$ 

4

$iT = i\mathcal{M}$.

$$= -i\lambda + \left(\Gamma(s) + \Gamma(t) + \Gamma(u) \right) - i\delta_\lambda \qquad (1.12)$$

So the renormalization condition $\lambda \equiv -i\Gamma^{(4)}(s_0, t_0, u_0)$ sets:

$$\underline{\delta_\lambda \equiv i\left(\Gamma(s_0) + \Gamma(t_0) + \Gamma(u_0)\right)} \qquad (1.13)$$

$$= \begin{cases} 3i\,\Gamma(0) & \text{if } s_0 = t_0 = u_0 = 0 \\ i\left(\Gamma(4m^2) + 2\Gamma(0)\right) & \text{if } s_0 = 4m^2,\ t_0 = u_0 = 0 \end{cases}$$

The physical amplitude is then immediately finite:

Because $\Gamma \propto \log\Lambda$

$\frac{d\Gamma}{dp^2} = \text{finite}$

$\Rightarrow \Gamma(s) - \Gamma(s_0) = \text{finite}$

$$i\mathcal{M} = -i\lambda + \left(\Gamma(s) - \Gamma(s_0) + \text{perm} \right) \quad \text{finite!} \qquad (1.14)$$

Compare this with expressions (6.104) and (6.107) in QFT 1.

Quite clearly the PT. defined by (1.8) generates the diagrammatic expansion depicted in p. 229 of part I of these lectures, ie

2-loops:

$-i\Pi_{2\text{-loop}} =$ 

etc...

(The recursive nature of defining δ_i:s should be clear from this.)

1.2 Renormalization of QED

We now proceed to a systematic renormalization of QED in the context of renormalized perturbation expansion (the BPHZ-scheme).

- Start from bare Lagrangian

$$\mathcal{L}_{QED} = -\frac{1}{4}(F_{\mu\nu})_0^2 - \bar{\Psi}_0(i\not\partial - m_0)\Psi_0 - e_0\bar{\Psi}_0\gamma^\mu\Psi_0 A_\mu^0 \quad (1.15)$$

- Rescale fields

$$\Psi_0 \equiv Z_2^{1/2}\Psi_r \quad ; \quad A_\mu^0 \equiv Z_3^{1/2}A_\mu^r \quad (1.16)$$

$$\Rightarrow \mathcal{L}_{QED} = -\frac{1}{4}Z_3(F_{\mu\nu})_r^2 - Z_2\bar{\Psi}_r(i\not\partial - m_0)\Psi_r - e_0 Z_2 Z_3^{1/2}\bar{\Psi}_r \not{A}_r \Psi_r \quad (1.15b)$$

- Redef. coupling & mass $\quad \Leftrightarrow Z_\psi(m+\delta m) = \delta_m + m \Leftrightarrow \delta m = \dfrac{\delta_m - \delta_\psi m}{1+\delta_\psi}$

$$\left|\begin{aligned} Z_2 m_0 &\equiv \delta_m + m \\ e_0 Z_2 Z_3^{1/2} &\equiv e Z_1 \end{aligned}\right. \quad (1.17)$$

$= e + \delta e$

After these definitions we have (from now drop indices "r")

$$\boxed{\begin{aligned} \mathcal{L}_{QED} = & -\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2\xi}(\partial A^\mu)^2 + \bar\psi(i\partial\!\!\!/ - m)\psi - e\bar\psi A\!\!\!/\psi \\ & - \delta_3 \frac{1}{4}(F_{\mu\nu})_r^2 \\ & + \bar\psi(i\delta_2\partial\!\!\!/ - \delta_m)\psi - e\delta_1\bar\psi A\!\!\!/\psi \end{aligned}}$$

→ fix gauge of the ren. field.

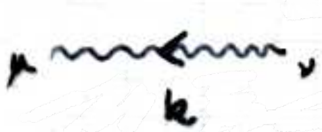
counter-term lagrangean (interaction)

(1.18)

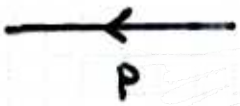
Implicit in this construction is the assumption that all divergences in the theory can be absorbed into four infinite numbers (counter-terms):

$$\begin{aligned} \delta_3 &\equiv Z_3 - 1 \\ \delta_2 &\equiv Z_2 - 1 \\ \delta_m &\equiv Z_2 m_0 - m \\ \delta_1 &\equiv Z_1 - 1 = \frac{e_0}{e} Z_2 Z_3^{1/2} - 1 \end{aligned} \qquad (1.19)$$

Let us for the moment assume that this is true. It then follows that the full set of Feynman rules for the renormalized theory is:



: $-\frac{i}{k^2+i\epsilon}\left(g_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$



: $\frac{i}{\not p - m + i\epsilon}$

: $ie\gamma^\mu$

(1,20 a)

In addition to these primitive Feynman rules, we have three additional counter-term interactions

: $-i\left(g^{\mu\nu}k^2 - k^\mu k^\nu\right)\delta_3$ *

$\langle\Omega|\hat a^{\alpha}_{k}\ \underbrace{\frac{-i}{2}\int_{d\omega}\hat A^{\mu}(-g_{\mu\nu}\partial^2+\partial_\mu\partial_\nu)\hat A^{\nu}}\ \hat a^{\beta\dagger}_{k}|\Omega\rangle$

$\int_p\int_{p'}\hat a^{+}_{p}\,(g_{\mu\nu}p'^2 - p'_\mu p'_\nu)\,\hat a^{\nu}_{p'}$

$= -\frac{i}{2}$

$+i(\not p\,\delta_2 - \delta_m)$

$-ie\delta_1\gamma^\mu$

(1.20 b)

These rules are of course unknown apart from their Lorentz structure. The coefficients $\delta_{1,2,3,m}$ must be defined from

$-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$

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* This structure follows from partially integrating $-\frac{1}{4}F^2\cdot\delta_3$ to give $= -\frac{1}{2}A_\mu\left[-g^{\mu\nu}\partial^2 + \partial^\mu\partial^\nu\right]A_\nu\cdot\delta_3$ + going to momentum space.

suitable renormalization conditions order by order in p.t.

From our experience with the $\lambda\phi^4$- theory, we would expect that the ct-lagrangian is uniquely set by the need to renormalize the 2- and 3-point functions in the theory

$$= i\Pi^{\mu\nu}(k) = i(g^{\mu\nu}k^2 - k^\mu k^\nu)\,\Pi(k^2)$$

↑ Necessary form, to be proven shortly !

$$= -i\Sigma(\not p)$$

$$\Big(\ \ \Big)_{\text{amputed}} = -ie\,\Gamma^\mu(p,p') \qquad (1.21)$$

From these we would deduce a choice :

$$\Sigma_r(\not p = m) = 0$$ ← on shell mass renormalization

$$\frac{d}{d\not p}\Sigma_r(\not p = m) = 0$$

$$\Pi_r(q^2=0) = 0$$

keep the residue of the propagator at 1.

$$\Gamma^\mu_r(p-p'=0) = \gamma^\mu \qquad (1.22)$$

← e = physical renormalized charge measured at $q^2=0$

The rules (1.19, 1.22) should suffice to make all computations of physical processes finite and unique in the theory expressed by Lagrangian (1.18).

Well, does it?

Possible sources of problems.

- Add another fermion ψ' to the theory. (say $\psi \hat{=} e$, $\psi' \hat{=} \mu$)

$\Rightarrow$ we have $\underline{\underline{2}}$ ways to express $e_0 - e$ - relation (1.17)

$$\frac{e_0 Z_3^{1/2}}{e} = \frac{Z_1}{Z_2} = \frac{Z_1'}{Z_2'} \qquad (1.23)$$

depends only on A_μ — depends on m — depends on m' ??

$\Rightarrow$ For the theory to make sense we would need

$Z_1 / Z_2 = \text{const}$! (will be: $Z_1 = Z_2$ (Ward-identity!))

- There are other superficially divergent 1PI-diagrams.

(again, cure will be Ward-identity)

Superficial degree of div. of QED n-point functions

Rehearsal: $\lambda\phi^4$-theory

$$D = dL - 2P_\phi = d(P_\phi - V + 1) - 2P_\phi$$
$$4V = 2P_\phi + n_\phi$$
$$L = P_\phi - V + 1$$
$$= (d-2)P_\phi - dV + d$$
$$= \frac{d-2}{2}(4V - n_\phi) - dV + d$$
$$= (d-4)V + d - \frac{d-2}{2}n_\phi$$
$$\xrightarrow{d=4} d - n_\phi$$

We have

- $$\underbrace{D}_{\text{degree of div}} = \underbrace{d}_{\text{dim. of space-time}}\underbrace{L}_{\text{\# of loops}} - \underbrace{P_e}_{\text{\# of e-propagators}} - 2\underbrace{P_\gamma}_{\text{\# of }\gamma\text{-propagators}}$$

Additionally

- $$L = P_e + P_\gamma - \overbrace{V}^{\text{\# of vertices}} + 1$$
- $$V = 2P_\gamma + \underbrace{N_\gamma}_{\text{\# of external }\gamma\text{-lines}} = \frac{1}{2}(2P_e + \underbrace{N_e}_{\text{\# of external e-lines.}}) \Rightarrow \begin{array}{l} P_\gamma = \frac{1}{2}(V - N_\gamma) \\ P_e = V - \frac{N_e}{2} \end{array}$$

$$\Rightarrow D = (d-1)P_e + (d-2)P_\gamma - dV + d$$
$$= \left[(\cancel{d}-1) + \tfrac{1}{2}(d-2) - \cancel{d}\right]V - \tfrac{1}{2}(d-1)N_e - \tfrac{1}{2}(d-2)N_\gamma + d$$
$$= d + \underbrace{\left[\frac{d}{2} - 2\right]}_{\begin{cases} =0 & d=4 \\ <0 & d<4 \\ >0 & d>4 \end{cases}}V - \tfrac{1}{2}(d-2)N_\gamma - \tfrac{1}{2}(d-1)N_e \qquad (1.24)$$

We now see that QED is

- super-renormalizable if $d<4$ (only finite number of divergencies to all loops)
- renormalizable when $d=4$ (# of div. indep. of V) only finite # of div. greens functions
- non-renormalizable if $d>4$ → all greens functions eventually divergent.

In d=4 in particular we get:

$$\boxed{D = 4 - N_\gamma - \tfrac{3}{2} N_e} \qquad (1.25)$$

However, according to (1.25) the # of divergent 1PI-graphs is more than 3!! :

~~(a)~~ vacuum  $D=4$

Furry.
~~(d)~~  $D=3$

(b)  $D=2$

~~(c)~~  $D=1$

~~(e)~~  $D=0$
Ward.

(f)  $D=1$

(g)  $D=0$

□ the ones we want.

All the rest are finite by dimensional counting.

We can easily get rid of diagrams (a), (d) and (e). The first of these represents only an unobservable (*) shift in vacuum energy, which never shows up in the S-matrix. Second, by Furry's theorem any diagram with odd number of external photon legs vanish.

Proof: An external photon must couple to a fermion line, so that the amputed amplitude is

$$\Big(\quad \Big)_{\text{amputed}} \propto (-ie)^{N_\gamma} \int \prod_{i=1}^{N_\gamma} d^4x_i \, e^{-i\sum q_i \cdot x_i} \langle \Omega | T\big(j_{\mu_1}(x_1) \cdots j_{\mu_{N_\gamma}}(x_N)\big) | \Omega \rangle$$

Now under charge conjugation

- $C|\Omega\rangle = |\Omega\rangle$ (QED C-symmetric)
- $C j_\mu(x) C^\dagger = -j_\mu(x)$ (P&S. page 70) (1.26)

$$\Rightarrow \langle \Omega | j_1 \cdots j_{N_\gamma} | \Omega \rangle = \langle \Omega | C^\dagger C j_1 C^\dagger C j_2 \cdots C^\dagger C j_{N_\gamma} C^\dagger C | \Omega \rangle$$

$$= (-1)^{N_\gamma} \langle \Omega | j_1 \cdots j_{N_\gamma} | \Omega \rangle$$

$$\Rightarrow \Gamma^{N_\gamma}_{N_e=0} = (-1)^{N_\gamma} \Gamma^{N_\gamma}_{N_e=0} \Rightarrow \Gamma^{N_\gamma = \text{odd}}_{N_e=0} = 0 \qquad (1.27)$$

∎

Thus $N_\gamma=1$ & $N_\gamma=3$-point-functions (d) and (e) vanish.

(*) As long as we do not fancy of quantum gravity.

We still have three issues left unexplained about our construction.

1) The remaining diagram (c).
2) The particular form of $\Pi_{\mu\nu}$ in (1.21)
3) Need for $Z_1/Z_2 = \text{const}$ in (1.23)

All these are taken care of by the Ward identity, which arises as a consequence of the gauge-invariance of the theory
$\Updownarrow$ current conservation

The Ward identity

There are many ways to obtain Ward identities, which also come in various forms. We saw in the QFT I one particular form of it

$$k^\mu \mathcal{M}_\mu = 0 \qquad (1.23)$$

q^μ  $\sim \bar{u}(p)\, \not{q}\, u(p')$
$= \bar{u}(p)(\not{p} - \not{p}')u(p') = 0$

while $Z_1 = Z_2$ is another. The most general and direct proof uses path integrals.

Indeed the Generating functional

$$Z = \int \underbrace{\mathcal{D}\bar{\psi}_0 \mathcal{D}\psi_0}_{\equiv \mathcal{D}\mu} \mathcal{D}A_0\, e^{iS[\psi_0, \bar{\psi}_0, A_0] + i\int d^4x \overbrace{[A^0_\mu J^\mu_0 + \bar{\eta}_0 \psi_0 + \bar{\psi}_0 \eta_0]}^{\equiv \mathcal{L}_J}} \qquad (1.28)$$

(work in terms of bare fields here) is just a definite integral, and hence must be invariant in the shift

$$\begin{cases} \psi_0 \to \psi_0' = e^{i\alpha(x)} \psi_0(x) & \approx (1+i\alpha)\psi_0 \\ \bar{\psi}_0 \to \bar{\psi}_0' = e^{-i\alpha(x)} \bar{\psi}_0(x), & \approx (1-i\alpha)\bar{\psi}_0 \end{cases} \quad \text{if } |\alpha| \ll 1. \qquad (1.30)$$

ie:

$$\int d\mu \; e^{iS[\psi_0,\bar{\psi}_0,A_0] + i\int d^4x \, \mathcal{L}_J(\psi_0,\bar{\psi}_0,A_\mu)}$$

$$= \int d\mu' \; e^{iS[\psi_0',\bar{\psi}_0',A_0] + i\int d^4x \, \mathcal{L}_J(\psi_0',\bar{\psi}_0',A_\mu)} \qquad (1.31)$$

Then, using the obvious invariance of the measure $\boxed{d\mu = d\mu'}$, and employing an infinitesimal shift, we get

$i\bar{\psi}\, i\partial\!\!\!/ \psi \to -i(\partial_\mu\alpha)\bar{\psi}\gamma^\mu\psi$

- $$iS[\psi_0',\bar{\psi}_0',A_0] = iS[\psi_0,\bar{\psi}_0,A_0] \underbrace{-i\int d^4x \, (\partial_\mu\alpha)\bar{\psi}_0\gamma^\mu\psi_0}_{= +i\int d^4x \, \alpha(x)\, \partial_\mu[\bar{\psi}\gamma^\mu\psi]}$$

- $$i\mathcal{L}_J[\psi_0',\bar{\psi}_0',A_0] = i\mathcal{L}_J[\psi_0,\bar{\psi}_0,A_0] + \alpha(x)\left[\bar{\psi}_0(x)\eta_0(x) - \bar{\eta}_0(x)\psi_0(x)\right]$$

Putting these back to (1.18) together with $d\mu' = d\mu$ we get

$$\int d^4x \, \alpha(x) \left\langle i\partial_\mu[\bar{\psi}_0(x)\gamma^\mu\psi_0(x)] - \bar{\eta}_0(x)\psi_0(x) + \bar{\psi}_0(x)\eta_0(x) \right\rangle_J = 0$$

Since $\alpha(x)$ is arbitrary we get the general Ward identity:

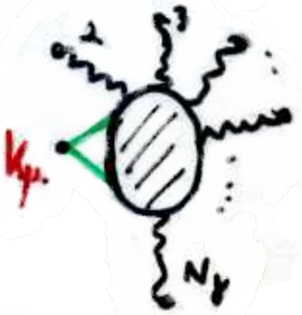
$$\boxed{\partial_\mu \langle \bar{\psi}_0(x)\gamma^\mu\psi_0(x)\rangle_J = i\langle \bar{\psi}_0(x)\eta_0(x)\rangle_J - i\langle \bar{\eta}_0(x)\psi_0(x)\rangle_J} \qquad (1.32)$$

It might be more appropriate to call this the "generating functional for all Ward identities". Anyway, from (1.32) we get all that we need.

↓ physically it is just the divergence of the current in presence of sources.

1) Differentiate (1.18) $N_\gamma - 1$ times wrt J_μ & set $J = \eta = \bar{\eta} = 0$

$$\Rightarrow \langle \Omega | T\left(\partial_\mu j^\mu \prod_{i=1}^{N_\gamma} A_{\mu_i}(x_i) \right) | \Omega \rangle = 0 \qquad (1.33)$$



$$= k_\mu M^\mu_{N_\gamma} = 0$$

(cf. 1.15)

(with no fermionic external lines. Generalizes further...)

That is, any amplitude, where at least one external photon line is contracted with 4-momentum k_μ vanishes.

2. Gauge invariant form of 2n-point functions

⇒ When contracted with ϵ^μ such amplitudes must be invariant under scaling $\epsilon^\mu \to \epsilon^\mu + \lambda k^\mu$, and hence can depend on ϵ^μ only through combination

$$f^{\mu\nu} = \epsilon^\mu k^\nu - \epsilon^\nu k^\mu \qquad (1.34)$$

Then for example

$$\epsilon^\mu \epsilon^\nu \Pi_{\mu\nu} \underset{\text{"must be"}}{\equiv} \overbrace{(\epsilon^\mu k^\nu - \epsilon^\nu k^\mu)(\epsilon_\mu k_\nu - \epsilon_\nu k_\mu)}^{\text{only possible scalar}} \cdot \frac{1}{2} \Pi(k^2)$$

$\epsilon^2 k^2$ $\qquad -\epsilon_\mu \epsilon_\nu k^\mu k^\nu = -(\epsilon \cdot k)^2$

$$= 2\left(\epsilon^2 k^2 - (\epsilon\cdot k)^2\right) \tfrac{1}{2}\Pi(q^2)$$

$$= \epsilon_\mu \epsilon_\nu \underbrace{\left(g^{\mu\nu} k^2 - k^\mu k^\nu \right) \Pi(q^2)}_{=\Pi^{\mu\nu}\ !}$$

- This proves the issue 2) on p.14 & justifies the form for the 2-point function in (1.21). Note that the degree of divergence for the $\Pi(q^2)$-function is only $D=0$, so that it is only log-divergent. (Hence the need for only one ct. δ_3.)

- Similarly we see that each of the contracted legs in diagram (c) on p.17 must contain a factor $f^{\mu\nu} \propto k^\mu$ as a result of which

$$\propto \underbrace{\prod_{i=1}^{4} k_i}_{\propto q^4} \cdot \Pi(s,t,u) \ldots$$

naive D.of div $=0$

Degree of divergence is $\underline{\underline{D=-4}}$

FINITE !

That is, as a result of Ward identity, the diagram (c) is in fact finite.

- We still have the issue about necessity to have a fixed relation between Z_1 & Z_2. To this end differentiate (1.32) w.r.t $\frac{\delta^2}{\delta\bar{\eta}_0(x_1)\,\delta\eta_0(x_2)}$; and then set $\eta = \bar{\eta} = J_\mu = 0$.

$S_F(x_1 - x_2) = \langle \Omega | T(\psi(x_1)\bar{\psi}(x_2)) | \Omega \rangle$

We now easily find:

$$\boxed{\langle\Omega|T(\partial_\mu j_0^\mu(x)\,\psi_0(x_1)\,\bar\psi_0(x_2))|\Omega\rangle = +\overbrace{\langle\Omega|T(\bar\psi_0(\overset{\to x_2}{x})\,\psi_0(x_1))|\Omega\rangle}^{=\,-S_F(x_1,x_2)}\,\delta^4(x-x_2) + \langle\Omega|T(\bar\psi_0(x_2)\,\psi(\underset{\to x_1 \text{ due to } \delta\text{-fct.}}{x}))|\Omega\rangle\,\delta^4(x-x_1)}$$

Connected 3-point amplitude.

(1,35)

Fourier-transform this w.r.t. all arguments:

$S_F(x-y) = \langle\Omega|T(\psi(x)\bar\psi(y))|\Omega\rangle$

$$\int d^4x\, d^4x_1\, d^4x_2\, \exp\left[-ik\cdot x + iq\cdot x_2 - ip\cdot x_1\right].$$

& use the translational invariance we get ($\tilde{dx_i} \equiv d^4x_i$)

~ refs. to vertex without coupling

- **LHS** $= \displaystyle\int d^4x\, d^4x_1\, d^4x_2 \left[\partial^x_\mu \tilde W^\mu_{(3)}(x, x_1, x_2)\right] e^{-ikx+iqx_2-ipx_1}$

$$= ik_\mu \int d^4x\, d^4x_1\, d^4x_2\; \tilde W^\mu_{(3)}(0, x_1-x, x_2-x)\, e^{-i(k+p-q)x + iq(\underbrace{x_2-x}_{\equiv x_2'}) - ip(\underbrace{x_1-x}_{\equiv x_1'})}$$

$$= (2\pi)^4\delta^4(k+p-q)\; ik_\mu\, \tilde W_{(3)}(p, k+p)$$

$$= (2\pi)^4\delta^4(k+p-q)\; S(p)\; ik_\mu \tilde\Gamma_3^\mu(p,k+p)\; S(k+p)$$

without coupling.

- **RHS** $= \displaystyle -\int d^4x\, d^4x_1\, d^4x_2\; S_F(x_1-x_2)\left(\delta^4(x-x_2) - \delta^4(x-x_1)\right) e^{-ikx+iqx_2-ipx_1}$

$$= -\int d^4x_1\, d^4x_2\; S_F(x_1-x_2)\left(e^{i(q-k)x_2 - ipx_1} - e^{iqx_2 - i(k+p)x_1}\right)$$

$\begin{cases} x_2 \equiv z + x \\ x_1 \equiv x \end{cases}$ $\qquad$ $\begin{cases} x_1 = z + x \\ x_2 = x \end{cases}$

$$= -\int d^4x\; e^{-i(p+k-q)x} \int \tilde{dz}\left(\underbrace{S(-z)}_{z\to -z}\, e^{i(q-k)z} - S(z)\, e^{-i(k+p)z}\right)$$

$$= -(2\pi)^4 \delta^4(p+k-q) \left(S(q-k) - S(k+p) \right)$$

$$= -(2\pi)^4 \delta^4(p+k-q) \left(S(p) - S(p+k) \right)$$

Putting LHS & RHS back to (1.35) we get

$$S(p)\left[-ik^\mu \tilde{\Gamma}^{(3)}_\mu(p,p+k)\right] S(p+k) = S(p) - S(p+k)$$

All quantities here are bare.

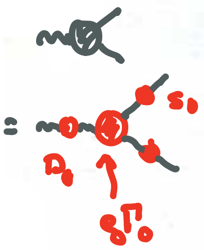
$$\Leftrightarrow \quad \boxed{-ik^\mu \tilde{\Gamma}_\mu(p,p+k) = S^{-1}(p+k) - S^{-1}(p)} \qquad (1.36)$$

We will later find this in a different way (current algebra)

Now remember that we have so far worked in terms of non-renormalized fields. The general connection between connected n-point functions now gives:

defined without coupling.

$$W_0^3 = Z_3^{1/2} Z_2 W_r^3 = Z_3^{1/2} Z_2 \left(S_r^2 D_r e \tilde{\Gamma}_r^{(3)} \right)$$



$$= S_0^2 D_0 e \tilde{\Gamma}^{(3)}_{0\mu} = Z_3 Z_2^2 (S_r^2 D_r) g \tilde{\Gamma}^{(3)}_0$$

$$\Rightarrow \tilde{\Gamma}^{(3)}_{0\mu} = \frac{e}{g Z_2 Z_3^{1/2}} \tilde{\Gamma}^{(3)}_{r\mu} = \frac{1}{Z_1} \tilde{\Gamma}^{(3)}_{r\mu} \xrightarrow[\text{point.}]{\text{ren.}} \frac{1}{Z_1} \gamma^\mu$$

Our renormalization point was just $p \to 0$, so that (1.36) becomes there:

$$-ik \frac{1}{Z_1} = -ik \frac{1}{Z_2} \quad \Leftrightarrow \quad \boxed{Z_1 = Z_2} \qquad (1.37)$$

Let us finally mention that it is a simple matter to generalize the identity (1.35) to one involving an arbitrary number of fermion lines. Just differentiating the identity (1.32) $2n+m$ – times

$$\prod_{i=1}^{n} \frac{\delta^2}{\delta\bar\eta(x_{i_1})\,\delta\eta(x_{i_2})} \cdot (-1)^m \prod_{j=1}^{m} \frac{\delta}{\delta J_{\mu_j}(x_j)} \;\cdots$$

one finds $(M^\mu = e\tilde\Gamma^\mu)$

$$k_\mu M^\mu(k, p_1 \dots p_n; q_1, \dots, q_n)$$

$$= e\sum_{i=1}^{n} \Big(M_0(p_1 \dots p_n; q_1 \dots (q_i - k) \dots q_n) - M_0(p_1 \dots, p_i + k, \dots p_n; q_1 \dots q_n) \Big)$$

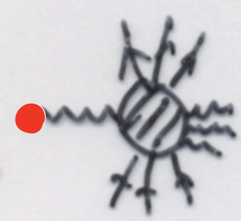
on-shell

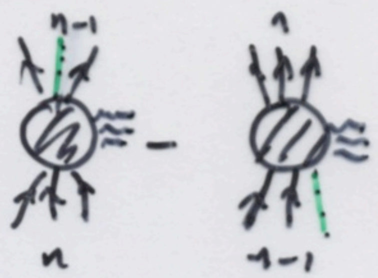
$$= 0 \qquad (1.38)$$

do not contribute to S-matrix of the State on the r.h.s. because of missing poles. LSZ-reduction kills on-shell.

i.e. any QED matrix element with one each photon line contracted with k^μ vanishes in S-matrix. (current conservation)

Graphically:

$k_\mu \sum_{\text{inserted}}$ ()

$= e \sum_{i=1}^{n}$ ()

(vs. P&S chapt. 7.4)

1.3 1-luuppi-renormalisaatio QED:ssä

Now that we have proven that QED can be renormalized, let us carry the procedure to 1st order. The primitively divergent graphs are

(a)

electron self-energy

(b)

photon self-energy

(c)

vertex correction.

Because of Ward-identity (1.37) we have only three divergent numbers in QED. This means that we only need 3 out of 4 conditions in (2?); (renormalizing (a) renders also (c) finite).

↳ or rather: renormalizing (a) also renormalizes = defines (c).

As we proceed, we shall encounter two more complications.

1) n-point functions are in general gauge-dependent

2) —"— —"— —"— infrared divergent

Neither of these shows up in any physical predictions, however.

a) Electron propagator renormalization

We wish to compute the full physical propagator regularized at the physical mass shell.

$\Gamma^{(2)} \equiv -i\Sigma_2^{\text{full}}$ = full, renormalized self-energy

$S(p) =$ 

$$= \frac{i}{\not p - m} + \frac{i}{\not p - m}(-i\Sigma_2)\frac{i}{\not p - m} + \cdots = \frac{i}{\not p - m}\left(1 + i\Sigma_2 \frac{i}{\not p - m}\right)^{-1}$$

$$\overset{*}{=} \frac{i}{\not p - m - \Sigma_2}\,.$$

We require that $S(p) \approx \frac{i}{\not p - m}$ on shell, which leads to renormalization conditions (1.22):

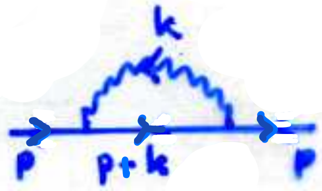
$$\Sigma_2^{\text{full}}(\not p = m) = 0 \quad ; \quad \frac{d}{d\not p}\Sigma_2^{\text{full}}(\not p = m) = 0 \qquad (1.39)$$

At 1-loop:

$-i\Sigma_2^{\text{full}} =$ 

$$= \underline{-i\Sigma_2^{\text{1-loop}} + i(\not p\,\delta_2 - \delta_m)} \qquad (1.40)$$

Here the <u>1-loop self energy</u> is



$$-i\Sigma_2^{\text{1-loop}} = (-ie)^2 \int \frac{d^4k}{(2\pi)^4}\,\gamma^\mu \frac{i}{\not p + \not k - m}\gamma^\nu \frac{-i}{k^2 - \mu_p^2 + i\epsilon}\left(g_{\mu\nu} + \eta\,\underbrace{\frac{k_\mu k_\nu}{k^2}}_{\text{gauge-dependence!}}\right) \qquad (1.41)$$

photon mass; <u>infrared regulator</u>

---

* Necessarily $\Sigma_2 = a\not p + bm \Rightarrow [\Sigma, \not p - m] = 0$

We immediately realize that $-i\Sigma_2$ is gauge-dependent.
Gauge-dependence does not show up on shell, however:

$$-i\Sigma_2^\eta = -\eta e^2 \int \frac{d^4k}{(2\pi)^4}\, \not k \,\frac{1}{\not p + \not k - m}\, \not k \,\frac{1}{(k^2-\mu_p^2+i\epsilon)k^2}$$

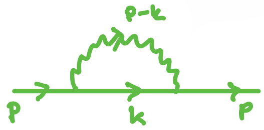
$$\xrightarrow{\not p - m = 0} -\eta e^2 \int \frac{d^4k}{(2\pi)^4}\frac{\not k}{k^2(k^2-\mu_p^2)} = 0 \quad \text{(odd integral)}$$

$$\Rightarrow \quad \underbrace{-i\Sigma_2^\eta = \eta\, a\, (\not p - m)}_{} \qquad (1,42)$$

a: Some number (divergent)

Σ^η thus only affects the wave function renormalization Z_2 (It can be used to set $\delta Z_2 = 0$ at one loop (Landau gauge $\eta = 1$).) Homework

In Feynman Gauge $\eta = 0$, and we get:



$$\frac{1}{ab} = \int_0^1 dx \frac{1}{[ax+(1-x)b]^2}$$

$$4 \to d = 4-\epsilon$$

$$-i\Sigma_2^F = -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu(\not k + m)\gamma_\mu}{((p-k)^2-\mu^2+i\epsilon)(k^2-m^2+i\epsilon)}$$

$$\equiv -e^2\mu^{4-d}\int_0^1 dx \int d^dk \frac{(2-d)\not k \;+ dm}{[(1-x)(k^2-m^2)+x(k^2+p^2+2k\cdot p-\mu_p^2)+i\epsilon]^2}$$

$\not k \to \not \ell + x\not p$

$$k^2 + x(p^2-2k\cdot p-\mu^2)+i\epsilon + (1-x)m^2$$

$$= (k-xp)^2 - \underbrace{[x(x-1)p^2+(1-x)m^2+x\mu_p^2-i\epsilon]}_{=\Delta_2}$$

$$= \ell^2 - \Delta_2 = -(\ell_E^2+\Delta_2)$$

Passarino - Veltman:

$$-i\Sigma_2^F = +e^2\big((4-d)\not p\, B_1(p^2;\mu_p,m_e^2) + 4m_e B_0(p^2;\mu_p^2,m_e^2)\big)$$

$$= -i\, e^2\mu^\epsilon \int_0^1 dx \int d^d\ell_E \frac{-(2-\epsilon)\,x\not p + (4-\epsilon)m}{(\ell_E^2+\Delta_2)^2}$$

$\ell \equiv k - xp$
$\Leftrightarrow k = \ell + xp$

went to E-space with ℓ (not p! That is not necessary)

Using: $$\int \frac{d^d \ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2+\Delta^2)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \Delta^{\frac{d}{2}-n} \qquad (1.43b)$$

We get:

$$\Sigma_2^{F,\text{loop}} = \frac{e^2}{16\pi^2} (4\pi\mu^2)^{\epsilon/2}\,\Gamma\!\left(\frac{\epsilon}{2}\right) \int_0^1 dx \, \frac{(4-\epsilon)m - (2-\epsilon)x\not p}{\left(x(x-1)p^2 + (1-x)m^2 + x\mu_p^2\right)^{\epsilon/2}} \qquad (1.43)$$

Minkowski

(Clearly Σ_0^{η} would contribute only here!)

$$\equiv \Sigma_2^{F,\text{loop}}(\not p = m) + (\not p - m) \left.\frac{d\Sigma_2^{F,\text{loop}}}{d\not p}\right|_{\not p = m} + \tilde{\Sigma}_2^{F,\text{loop}}(\not p) \qquad (1.44)$$

- The full self-energy correction of course contains also the counter-term parts (cf. (1.40))

$$\Sigma_2^{\text{full}} = \Sigma_2^{F,\text{loop}} - \not p\, \delta_2 + \delta_m \qquad (1.44b)$$

Our renormalization conditions then set:

$$0 = \left.\frac{d\Sigma_2^{\text{full}}}{d\not p}\right|_{\not p=m} = \underbrace{-\delta_2}_{1-Z_2} + \left.\frac{d\Sigma_2^{F,\text{loop}}}{d\not p}\right|_{\not p=m} \Leftrightarrow \boxed{Z_2 = 1 + \left.\frac{d\Sigma_2^{F,\text{loop}}}{d\not p}\right|_{\not p=m}} \qquad (1.45)$$

$$0 = \Sigma_2^{\text{full}}(\not p = m) = -m\delta_2 + \delta_m + \Sigma_2^{F,\text{loop}}(\not p = m) \Rightarrow \delta_m = m\frac{d\Sigma_2^{\text{loop}}}{d\not p} - \Sigma_2^{\text{loop}}(m) \qquad (1.46)$$

Let us look explicitly at Z_2:

$$\frac{d\Sigma_2}{d\not p}\Big|_{\not p=m} = \frac{\alpha}{4\pi}(4\pi\mu^2)^{\epsilon/2}\underbrace{\Gamma\left(\frac{\epsilon}{2}\right)}_{=\frac{2}{\epsilon}+\gamma_E+\mathcal{O}(\epsilon)}\int_0^1 dx\,\frac{1}{\bar{\Delta}_2^{\epsilon/2}}\left(-(2-\epsilon)x - \epsilon x(x-1)m\cdot(1-2x)m\frac{1}{\bar\Delta}\right)$$

$$= -\frac{\alpha}{2\pi}(4\pi\mu^2)^{\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dx\,\frac{x}{\bar{\Delta}_2^{\epsilon/2}}\left(1-\frac{\epsilon}{2}\left(1+\frac{2(1-x)(2-x)}{\bar\Delta_2}\right)\right)$$

$$= \delta Z_2 = Z_2 - 1 = \delta_2 \qquad (1.47)$$

$$\bar\Delta_2 \equiv \Delta_2(\not p = m) = (1-x)^2m^2 + x\mu_\gamma^2$$

The UV-divergent part is:

$$Z_2^\epsilon = -\frac{\alpha}{2\pi}\cdot\frac{2}{\epsilon}\int_0^1 dx\,x = -\frac{\alpha}{2\pi}\frac{1}{\epsilon} \qquad \left(\Gamma(\epsilon)\simeq\frac{1}{\epsilon}-\gamma_E\right) \qquad (1.48)$$

and alltogether

$$\boxed{\delta Z_2 = -\frac{\alpha}{2\pi}\left[\underbrace{\frac{1}{\epsilon}-\frac{1}{2}(\gamma_E-\ln 4\pi)}_{\equiv\frac{1}{\epsilon_2}} - \int_0^1 dx\,x\left(\log\frac{\bar\Delta_2}{\mu^2}+1+\frac{2(1-x)(2-x)}{\bar\Delta_2}\right)\right]}$$

To be compared with δ_1 to check WT-identity (1.47b)

- Propagator is now finite

$$S_F^{ren}(p) \equiv \frac{i}{\not p - m + \Sigma_2^{full}(p)} \xrightarrow{p^2\to m^2} \frac{i}{\not p - m}$$

note: no renormalization factors here. Just Σ^{full}!

- Z_2 contains UV-pole but also an infrared divergence

$$\sim \int_0^1 dx\,x\,\frac{2(1-x)(2-x)}{\bar\Delta} \overset{y=1-x}{\sim} \int_0^1 dy\,\frac{4y}{m^2y^2+\mu^2} \sim \log\frac{\mu^2}{m^2}.$$

Moreover (add to 1.47)

$$\Sigma_1(\not p = m) = \frac{\alpha}{4\pi}(4\pi\mu^2)^{\epsilon/2}\,\Gamma(\tfrac{\epsilon}{2})\int_0^1 dx\,\frac{[4-\epsilon-(2-\epsilon)x]m}{\Delta^{\epsilon/2}}$$

So that from (1.44b), (1.45) and (1.46):

$$\Sigma_2^{full} = \Sigma_2(\not p) - \not p\,\frac{d\Sigma}{d\not p}\Big|_{\not p = m} + m\,\frac{d\Sigma}{d\not p}\Big|_{\not p = m} - \Sigma_2(m)$$

$$= \Sigma_2(\not p) - \Sigma_2(m) - (\not p - m)\,\frac{d\Sigma_2}{d\not p}\Big|_{\not p = m}$$

IR-divergent $\mu_r^2 \to 0$ does not affect pole!

(IR-finite)

$$= \ldots = \frac{\alpha}{2\pi}\int_0^1 dx\left[(2m - x\not p)\log\frac{\Delta_2}{\bar\Delta_2} - (\not p - m)\,\frac{2m\,x(1-x)(2-x)}{\bar\Delta_2}\right]$$

(1.48 b)

with $\Delta_2 = -x(1-x)p^2 + (1-x)m^2 + x\mu_r^2$

$\bar\Delta_2 = (1-x)^2 m^2 + x\mu_r^2$; finite ; $\to 0$ when $\not p \to m$.

- The term $\log\Delta^2$ contains a cut whenever

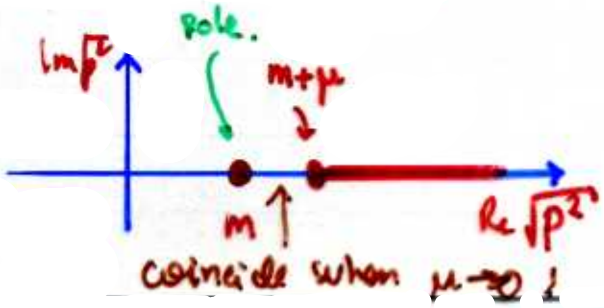
$$x(1-x)p^2 > (1-x)m^2 + x\mu_r^2$$

That is, when $x_- \le x \le x_+$ with

$$x_\pm = \frac{1}{2} + \frac{m^2-\mu^2}{2p^2} \pm \frac{1}{2p^2}\sqrt{(p^2-(m+\mu)^2)(p^2-(m-\mu)^2)}$$

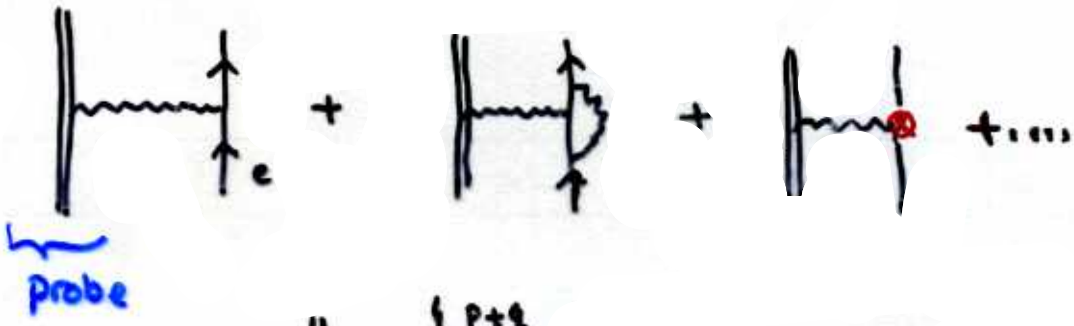
The smallest p^2 for which there is a cut is thus

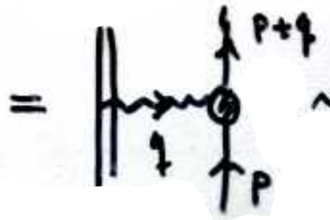
$$p_{min} = (m+\mu)^2 \quad (1.48c)$$



Electron vertex function

Here, instead of trying to figure out the arbitrary phase space structure of the vertex function, we concentrate on the parts of Γ_μ which are relevant for measurements. That is consider a process





$$\sim -ie\,\bar{u}(p+q)\,\Gamma^\mu(p,p+q)\,u(p)$$

If we now assume that electrons are on shell*, then the Ward-identity (1.36) implies $q_\mu \Gamma^\mu = 0$. Given this we may write. (see P&S chap. 6)

$$\Gamma^\mu(p,p+q) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2) \qquad (1.49)$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$. Our renormalization condition (1.22) requires

$$F_1(0) = 1. \qquad (1.50)$$

---

* On shell $S_0^{-1}(p) = \not{p} - m_0 - \Sigma_0 = Z_2^{-1}(\not{p} - m) + \tilde{\Sigma} = Z_2^{-1}\cdot 0 + 0 = 0$.

There is no renormalization condition for $F_2(q^2)$. It must be finite by itself. In fact this is a physical quantity, which is related to electron magnetic moment. One can easily show from the form (1.49) (see PS p 187-189) that

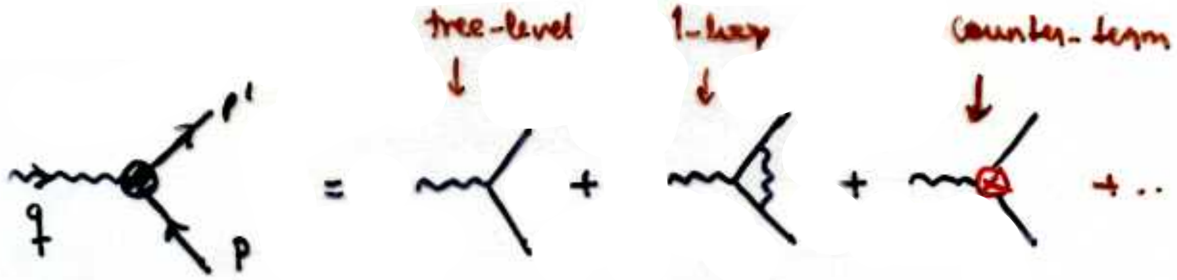
$$\vec{\mu}_e = g\left(\frac{e}{2m_e}\right)\vec{S} \qquad (1.51)$$

where

$$\underset{\uparrow\ \text{observable}}{g} = 2\left[F_1(0) + F_2(0)\right] = 2 + \underset{\uparrow\ \text{(finite) after renormalization } (\Rightarrow \text{ definition of charge})}{2F_2(0)} \qquad (1.52)$$

It is easiest to keep track of the on-shell requirement by keeping Γ^μ sandwiched by $\bar{u}\cdot\cdot u$'s.

One-loop calculation

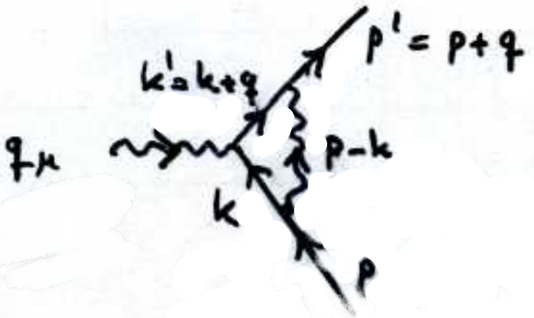


$$= \bar{u}(p')\left[-ie\gamma^\mu - ie\delta\Gamma^\mu_{1\text{-loop}} - ie\,\delta_1\gamma^\mu\right]u(p) \qquad (1.53)$$

$\delta_1 = Z_1 - 1 \overset{\text{Ward}}{=} Z_2 - 1$

because of Ward identity this is already perfectly finite operator. We compute it as a check.

not a free parameter anymore. !



$\sim I^\mu$

In Feynman gauge:

$$I^\mu \equiv \bar u(p')\,\delta\Gamma^\mu_{1\text{-loop}}\,u(p)$$

$$= (-ie)^2 \int \frac{d^4k}{(2\pi)^4}\,\frac{-ig^{\alpha\beta}}{\underbrace{(p-k)^2}_{-\mu_P^2}+i\epsilon}\,\bar u(p')\,\gamma_\alpha\,\overbrace{\frac{i(\not k'+m)}{k'^2-m^2+i\epsilon}\gamma^\mu\frac{i(\not k+m)}{k^2-m^2+i\epsilon}}^{\sim\,\gamma^\alpha(\not k'+m)\gamma^\mu(\not k+m)\gamma_\alpha}\gamma_\beta\,u(p) \qquad (1.54)$$

When using d-regularization, we must account for ε-terms arising from dirac algebra in all divergent terms. In practice this means only the quadratic term

$$\gamma^\alpha \not k' \gamma^\mu \not k \gamma_\alpha = -2\not k\gamma^\mu\not k' + \epsilon\,\not k'\gamma^\mu\not k$$

Then after some algebra: $(k' = k+q)$

here pick only divergent part.

finite

$$I^\mu_{1\text{-loop}} = 2i\,e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\,\frac{\bar u(p')\left[\not k\gamma^\mu\not k' - 2m(k+k')^\mu + m^2\gamma^\mu - \frac{\epsilon}{2}\not k'\gamma^\mu\not k\right]u(p)}{\underbrace{(k'^2-m^2+i\epsilon)}_{a}\,\underbrace{(k^2-m^2+i\epsilon)}_{b}\,\underbrace{((p-k)^2}_{c}\,\underbrace{}_{-\mu_P^2}+i\epsilon)} \qquad (1.55)$$

Now introduce Feynman parametrization in the usual way

$$\frac{1}{abc} = \int_0^1 dx\,dy\,dz\;\delta(x+y+z-1)\,\frac{2}{D^3}$$

$$D = x(k^2-m^2) + y(\underbrace{k'^2}_{(k+q)^2}-m^2) + z\left[(k-p)^2-\mu_P^2\right] + \overbrace{(x+y+z)}^{=1}i\epsilon$$

$$= k^2 + 2k\cdot(qy-zp) + yq^2 + zp^2 - \overbrace{(x+y)}^{1-z}m^2 + i\epsilon - z\mu_P^2 \qquad (1.56)$$

Completing the square:

$$l \equiv k + yq - zp \qquad (k = l - yq + zp) \qquad (1.57)$$

one gets

$$D = l^2 - \Delta_3 + i\epsilon \qquad (1.58)$$

with

$$\Delta_3 = -xyq^2 + (1-z)^2 m^2 + z\mu_p^2 \qquad (1.60)$$

With this shift the [ ]-term in the numerator becomes

$$[\;] \simeq \underbrace{\not l \gamma^\mu \not l} + \Big[(-y\not q + z\not p)\gamma^\mu((1-y)\not q + z\not p) + m^2\gamma^\mu - 2m((1-2y)q^\mu + 2zp^\mu \Big] - \cancel{\frac{\epsilon}{2}} \underbrace{\not l \gamma^\mu \not l}_{\simeq -\frac{1}{4} l^2 \gamma^\mu}$$

$$= -l^2\gamma^\mu + 2 l^\mu \not l$$

$$= \left(\frac{2}{d} - 1\right) l^2 \gamma^\mu = \left(-\frac{1}{2} + \frac{\epsilon}{8}\right) l^2\gamma^\mu$$

$$= \underbrace{-\frac{1}{2} l^2 \gamma^\mu}_{\text{divergent.}} + \underbrace{[\;\ldots\ldots\;]}_{\text{finite}} + \underbrace{\frac{3}{8}\epsilon\, l^2 \gamma^\mu}_{\text{finite, pick only the } \frac{1}{\epsilon}\text{-pole here}}$$

This form already tells us that the only divergence occurs in the F_1-term $\sim \gamma^\mu$ in (1.49). After quite a bit of algebra can be recast in the form (using $\not p u(p) = m u(p)$ & $\bar u(p') \not q u(p) = 0$ etc.)

$$\bar u(p') \left[A\gamma^\mu + \frac{B}{2m}(p'^\mu + p^\mu) + \cancel{C q^\mu} \right] u(p) \qquad (1.61)$$

this must integrate to zero

where

$$A = \left(\tfrac{1}{2} - \tfrac{3}{8}\epsilon\right) \ell_E^2 + (1-x)(1-y)\, q^2 + (1-2z-z^2)\, m^2$$

?!

$$B = 2m^2 z(z-1)$$

$$C = m(z-2)(x-y) \qquad (1{,}62)$$

That C-term does integrate to zero follows from symmetry of Δ and antisymmetry of C under $x \leftrightarrow y$.

Then, using Gordon-identity

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}}{2m} q_\nu\right] u(p) \qquad (1.63)$$

one can rewrite (1.61) as

$$\bar{u}(p')\left[(A+B)\gamma^\mu - B\,\frac{i\sigma^{\mu\nu} q_\nu}{2m}\right] u(p) \qquad (1.64)$$

This shows explicitly the division into Form-factors. Collecting results

$$\boxed{\begin{aligned} F_1(q^2) &= 1 + 4e^2\mu^\epsilon \int dx\,dy\,dz\, \delta(x+y+z-1) \int \frac{d^d \ell_E}{(2\pi)^d} \frac{A+B}{(\ell_E^2+\Delta_3)^3} + \delta_1 \\ F_2(q^2) &= -4e^2 \int dx\,dy\,dz\, \delta(x+y+z-1) \int \frac{d^4 \ell_E}{(2\pi)^4} \frac{B}{(\ell_E^2+\Delta_3)^3} \end{aligned}} \qquad (1{,}65)$$

(where we also remembered $\int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta)^3} = (-1)^3 i \int \frac{d^d \ell_E}{(2\pi)^d} \frac{1}{(\ell_E^2+\Delta)^3}$)

Our fourth renormalization condition (1.22) now sets

$$\delta_1 = -\delta F_{loop}(0), \qquad (1.66)$$

This is of course not free, but set by the Ward identity, which now requires

$$\delta_1 = \delta_2 \iff \frac{\partial \Sigma_2}{\partial \not p}\Big|_{\not p = m} = -\delta F_{loop}(0) \qquad (1.67)$$

Now write

$c = (1-x)(1-y)$ $\qquad g = 1-4z+z^2$

$$\delta F_1(q^2) = 4e^2\mu^\epsilon \int dx\,dy\,dz\,\delta(x+y+z-1)\int \frac{d^d l_E}{(2\pi)^d}\,\frac{\frac{1}{2}(1-\frac{3}{4}\epsilon)l_E^2 + c(y,z)q^2 + g(z)m^2}{(l^2+\Delta_3)^3}$$

$$= \frac{2}{16\pi^2}(4\pi\mu^2)^{\epsilon/2}\Gamma(\tfrac{\epsilon}{2})\int_0^1 dz\int_0^{1-z} dy\left(\frac{(1-\frac{3}{4}\epsilon)(1-\frac{\epsilon}{4})}{\Delta_3^{\epsilon/2}} + \frac{\epsilon}{2}\,\frac{cq^2+gm^2}{\Delta_3^{\frac{\epsilon}{2}+1}}\right)$$

$$\underbrace{=}_{=\frac{\alpha}{2\pi}}\ \frac{2}{16\pi^2}(4\pi\mu^2)^{\epsilon/2}\Gamma(\tfrac{\epsilon}{2})\int_0^1 dz\int_0^{1-z}dy\,\frac{1}{\Delta_3^{\epsilon/2}}\left(1-\frac{\epsilon}{2}\left(2-\frac{cq^2+gm^2}{\Delta_3}\right)\right) \qquad (1.68)$$

And in particular then (note that $\Delta_3(q^2=0) = \Delta_2$)

$$\delta F_1(0) = \frac{\alpha}{2\pi}(4\pi\mu^2)^{\epsilon/2}\Gamma(\tfrac{\epsilon}{2})\int_0^1 dz\,(1-z)\,\frac{1}{\Delta_2^{\epsilon/2}}\left(1-\frac{\epsilon}{2}\left(2-\frac{1-4z+z^2}{\Delta_2/m^2}\right)\right)$$

$$= \frac{\alpha}{2\pi}\left(\frac{1}{\epsilon}-\frac{1}{2}(\gamma_E+\ln 4\pi) - \int_0^1 dz\,(1-z)\left(\log\frac{\Delta_2}{\mu^2}+2-\frac{1-4z+z^2}{\Delta_2/m^2}\right)\right)$$

$$= -\frac{\alpha}{2\pi}\left(\frac{1}{\epsilon_2}+\int_0^1 dz\,(1-z)\left(\log\frac{\Delta_2}{\mu^2}+2-\frac{(1-4z+z^2)}{\Delta_2/m^2}\right)\right) \qquad (1.69)$$

After one partial integration with log.-term, one can show that (1.67) indeed holds.

Then the UV-finite part is then:

$$\boxed{\hat{F}_1(q^2) = 1 - \frac{\alpha^2}{2\pi}\int_0^1 dz \int_0^{1-z} dy \left(\log\left(\frac{\Delta_3}{\bar{\Delta}_2}\right) - \frac{cq^2 + g\,m^2}{\Delta_3} + \frac{g\,m^2}{\bar{\Delta}_2} \right)}$$

with $c = (1-y)(z+y)$, $g \equiv 1-4z+z^2$ (1.70)

$\bar{\Delta}_2 \equiv (1-z)^2 m^2 + z\mu_r^2$; $\Delta_3 = -q^2 y(1-y-z) + \bar{\Delta}_2$

The function $\hat{F}_2(q^2)$ is finite as such. It is also infrared-finite:

$$\hat{F}_2(q^2) = -4e^2 \int_0 dz\,(1-z)\,2m^2 z(z-1) \int \frac{d^4 l_E}{(2\pi)^4} \frac{1}{(l_E^2-\Delta)^3}$$

$$= +\frac{4e^2}{16\pi^2} \cdot \frac{\Gamma(1)}{\Gamma(2)\,\Gamma(3)} \int_0^1 dz\, \frac{2\cancel{m^2} z\,\cancel{(1-z)^2}}{\cancel{m^2}\cancel{(1-z)^2}}$$

$$= \frac{\alpha}{\pi}\int_0^1 dz\, z = \frac{\alpha}{2\pi}. \qquad (1.71)$$

The lowest order anomalous magnetic moment is then:

$$a_e \equiv \frac{g_e - 2}{2} = F_2(0) = \frac{\alpha}{2\pi} \approx 0.0011614$$

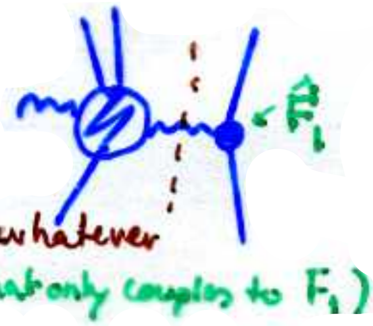
(experimentally: 0.0011597...)

Infrared divergences

We were able to define unambiguously both the free electron propagator on shell, and the physical charge of this electron. However, the the finite parts of the renormalized n-point functions remained IR-divergent.

- In the electron self-energy IR-divergence does not affect the pole, but merging of pole and cut when $\mu_P^2 \to 0$ signals problems in identifying free electron as a physical observable state.

- The 3-point function is seriously IR-divergent. If we want to improve some scattering process involving electron, we have



$$d\sigma = d\sigma_0 \cdot \left[\hat{F}_1(q^2)\right]^2$$

$$= d\sigma_0 \cdot \underbrace{\left(1 + 2\delta\hat{F}(q^2)\right)}_{O(\alpha)} \qquad (1.71)$$

where $\delta\hat{F}(q^2)$ is the integral given by (1.70).
It turns out that this correction is infinite when $\mu^2 \to 0$.

Indeed by change of variables

$$\begin{cases} y \equiv w\xi \\ z \equiv (1-w) \end{cases} \qquad \int_0^1 dz \int_0^{1-z} dy \;\to\; \frac{1}{2}\int_0^1\int_0^1 d\xi\, dw^2 \qquad (1.72)$$

we can rewrite $\delta\hat{F}(q^2)$ as

$$\Delta_3 \to w^2\left(m^2 - q^2\xi(1-\xi)\right) + (1-w)\mu^2$$

$$g \to w^2 - 2(1-w)$$

$$c \to 1 - w + w^2\xi(1-\xi)$$

$$\delta\hat{F}_1(q^2) = -\frac{\alpha}{4\pi}\int d\xi\, dw^2 \left[\log\left(1-\frac{q^2}{m^2}\xi(1-\xi)\right) + \frac{(-2+2w+w^2)m^2}{w^2m^2+\mu^2} - \frac{(-2+2w+w^2)m^2 + (1-w-w^2\xi(1-\xi))q^2}{w^2(m^2-q^2\xi(1-\xi))+\mu^2}\right] \quad (1.73)$$

IR singular when $\mu^2 \to 0$

where we have set $(1-w)\mu^2 = \mu^2$ since when $\mu^2 \to 0$ it only contributes at singularity which needs $w \approx 0$. We also dropped μ^2-terms in the log, which is IR-finite. The integral over w^2 can be easily done using

- $$\int_0^1 dw^2 \frac{c}{aw^2+\mu^2} = \frac{c}{a}\log\frac{a+\mu^2}{\mu^2} \approx \frac{c}{a}\log\frac{a}{\mu^2} \quad (1.74)$$
- $$\int_0^1 dw^2 \frac{w^2}{aw^2+\mu^2} = \frac{1}{a}$$
- $$\int_0^1 dw^2 \frac{w}{aw^2+\mu^2} = \frac{2}{a}$$

One finds

$$\boxed{\delta\hat{F}_1(q^2,\mu^2) = -\frac{\alpha}{2\pi}\int_0^1 d\xi \left[\log\left(1-\frac{q^2}{m^2}\xi(1-\xi)\right) - \frac{q^2}{m^2}\,\frac{3\xi(1-\xi)-1}{1-\xi(1-\xi)\frac{q^2}{m^2}} + \frac{m^2-q^2/2}{m^2-\xi(1-\xi)q^2}\left(\log\left(\frac{m^2-q^2\xi(1-\xi)}{\mu^2}\right) - \log\left(\frac{m^2}{\mu^2}\right)\right)\right]} \quad (1.75)$$

- $\log\left(1-\frac{q^2}{m^2}\xi(1-\xi)\right)$ contains a cut for $q^2 > 4m^2$ corresponding to virtual photon decay to two electrons. ok

It is now a simple matter to extract μ^2-dependence from the finite part. Introducing a finite scale Q^2:

choose optimally; m^2, $-q^2$, ...

$$\underbrace{\delta\hat{F}_1(q^2) = \delta\hat{F}_1(q^2, Q^2) - \frac{\alpha}{2\pi} f_{IR}(q^2) \log\left(\frac{Q^2}{\mu^2}\right)}_{\text{finite}} \tag{1.76}$$

IR- singularity extracted.

where

$$f_{IR}(q^2) = \int_0^1 d\xi \, \frac{m^2 - q^2/2}{m^2 - \xi(1-\xi)q^2} - 1 \tag{1.77}$$

For $-q^2 \to \infty$ one can see that $f_{IR}(q^2) \longrightarrow \log\left(-\frac{q^2}{m^2}\right)$, so that (taking $Q^2 = -q^2$)

$$\delta\hat{F}_1(q^2) \approx -\frac{\alpha}{2\pi} \log\left(-\frac{q^2}{m^2}\right) \log\left(-\frac{q^2}{\mu^2}\right) \tag{1.78}$$

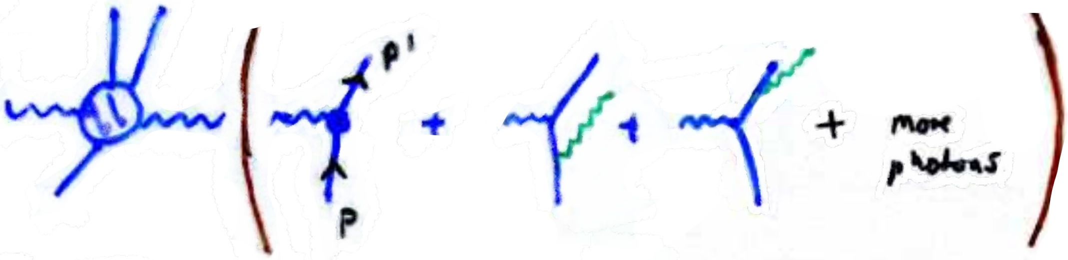
Sudakov - double logarithm.

Clearly, for sufficiently large $-q^2/\mu^2$ we can make (1.71) negative! = disaster.

Remedy to IR-disaster. Soft brehmsstrahlung:

Due to IR-singularity in electrons self-energy a free electron is an idealization; it can never be observed. No matter how gently nudged, any external probe makes it irradiate infinite number of soft photons. So the observable state always contains an electron + arbitrary number of soft photons.

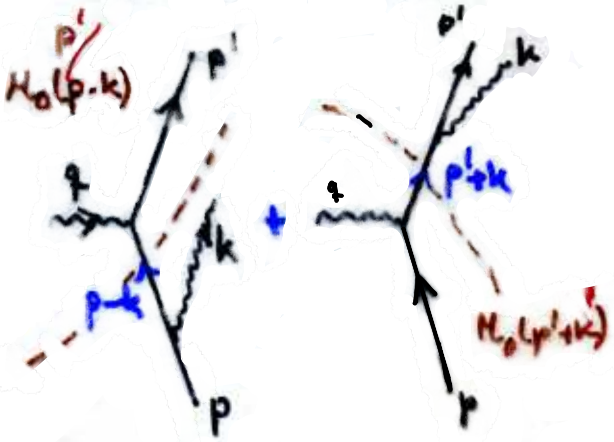
Indeed, for our physical process (1.71) the observable cross section is really a sum:



i.e. to order α^2:

$$\boxed{d\sigma_{\text{measured}}(p\to p') = d\sigma_0(p\to p')\left(1-2\delta\hat{F}_1(q^2)\right) + d\sigma(p\to p'+\gamma)} \qquad (1.79)$$

The amplitude of the soft photon emission process is:



~amplitude without photon emission for $d\sigma_0$

$$i\mathcal{M} = -ie\,\bar{u}(p')\left[i\tilde{\mathcal{M}}_0(p',p-k)\frac{i}{\not{p}-\not{k}-m}\not{\epsilon}^*_\mu(k) + \not{\epsilon}^*_\mu(k)\frac{i}{\not{p}'+\not{k}-m}i\tilde{\mathcal{M}}_0(p'+k,p)\right]u(p) \qquad (1.80)$$

Now take the limit $|\vec{k}| \ll |\vec{p}-\vec{p}'| = |\vec{q}|$.

$$\Rightarrow \tilde{\mathcal{M}}_0(p',p-k) \approx \tilde{\mathcal{M}}_0(p'+k,p) \approx \tilde{\mathcal{M}}_0(p',p)$$

$$\Rightarrow i\mathcal{M} = e\,\bar{u}(p')\left[i\tilde{\mathcal{M}}_0\frac{(\not{p}-\not{k}+m)\not{\epsilon}^*}{(p-k)^2-m^2} + \frac{\not{\epsilon}^*(\not{p}'+\not{k}+m)}{(p'+k)^2-m^2}i\tilde{\mathcal{M}}_0\right]u(p) \qquad (1.81)$$

- $\not{k} \ll \not{p}, m$ in numerator
- $(\not{p}+m)\not{\epsilon}^* u(p) = 2\epsilon\cdot p\, u(p)$; $\bar{u}(p')\not{\epsilon}^*(\not{p}'+m) = 2\epsilon\cdot p'\,\bar{u}(p')$

Also noting that $(k-p)^2-m^2=-2k\cdot p$, $(k+p')^2-m^2=+2k\cdot p'$, we find

$$\Rightarrow \quad iM = \underbrace{\bar{u}(p')\, i\tilde{M}_0\, u(p')}_{iM_0} \cdot e\left(\frac{\epsilon\cdot p'}{k\cdot p'} - \frac{\epsilon\cdot p}{k\cdot p}\right) \qquad (1.82)$$

Let us now suppose that the measuring apparatus has the resolution E_γ to observe a free-travelling photon. Then the observed cross-section (1.79) becomes:

$$d\sigma_{\text{Measured}} = d\sigma_0(p\to p')\left(1+2\delta F_1(q^2,\mu^2) + \delta I_{sp}(\mu^2; E_\gamma) + \mathcal{O}(\alpha^2)\right) \qquad (1.83)$$

where

$$\delta I_{sp} = e^2 \int_\mu^{E_\gamma} \frac{d^3k}{(2\pi)^3}\frac{1}{2E_\gamma}\sum_\lambda \left|\frac{\epsilon^\lambda\cdot p'}{k\cdot p'} - \frac{\epsilon^\lambda\cdot p}{k\cdot p}\right|^2 \quad ; \quad \sum_\lambda \epsilon^\lambda_\mu \epsilon^\lambda_\nu = -g_{\mu\nu}$$

$$= \frac{e^2}{16\pi^3}\int_\mu^{E_\gamma} d\Omega_k \frac{k^2 dk}{E_\gamma}\cdot\left(\frac{2p\cdot p'}{k\cdot p'\, k\cdot p} - \frac{m^2}{(k\cdot p')^2} - \frac{m^2}{(k\cdot p)^2}\right)$$

$k = E_\gamma$, $\hat{k}_\mu = \frac{1}{E_\gamma}k_\mu$

$$= \frac{\alpha}{\pi}\underbrace{\int_\mu^{E_\gamma} dE_\gamma \frac{1}{E_\gamma}}_{=\log\frac{E_0}{\mu}} \cdot \underbrace{\int \frac{d\Omega_k}{4\pi}\left(\frac{2p\cdot p'}{\hat{k}\cdot p'\,\hat{k}\cdot p} - \frac{m^2}{(\hat{k}\cdot p')^2} - \frac{m^2}{(\hat{k}\cdot p)^2}\right)}_{\equiv I_v(\vec{v},\vec{v}')}$$

$$= \frac{\alpha}{2\pi}\log\frac{E_\gamma^2}{\mu^2}\cdot I_v(\vec{v},\vec{v}') \qquad (1.84)$$

↑ promising! This cross section is also IR-divergent!! (as expected)

$I(\vec{v},\vec{v}')$ is straightforward to compute:

$$\bullet \int \frac{d\Omega_k}{4\pi} \frac{m^2}{(\hat{k}\cdot p)^2} = \frac{m^2}{2}\int_{-1}^{1} d\cos\theta \, \frac{1}{(p_0 - p\cos\theta)^2} = \frac{m^2}{p^2} = 1$$

↑ 4-momentum: $p_0^2 - p^2$

$$\bullet \int \frac{d\Omega_k}{4\pi} \frac{m^2}{(\hat{k}\cdot p')^2} = 1$$

$$\bullet \int \frac{d\Omega_k}{4\pi} \frac{2p\cdot p'}{k\cdot p' \, k\cdot p} = \int_0^1 d\xi \int \frac{d\Omega_k}{4\pi} \frac{-q^2+2m^2}{[\xi \hat{k}\cdot p' + (1-\xi)\hat{k}\cdot p]^2}$$

$$= \int_0^1 d\xi \int \frac{d\Omega_k}{4\pi} \frac{-q^2+2m^2}{(\hat{k}\cdot(\xi p' + (1-\xi)p))^2}$$

$$= \int_0^1 d\xi \, \frac{-q^2+2m^2}{m^2 - \xi(1-\xi)q^2}$$

Thus

$$I(\vec{v},\vec{v}') = 2\left(\int_0^1 d\xi \, \frac{m^2 - q^2/2}{m^2 - \xi(1-\xi)q^2} - 1 \right) = 2 f_{IR}(q^2) \; ! \qquad (1.88)$$

$$\Rightarrow \delta I_{sp} = \frac{\alpha}{\pi} \log \frac{E_\gamma^2}{\mu^2} \cdot f_{IR}(q^2) \qquad (1.89)$$

Combining this with (1.74) & (1.83) eventually gives to order α^2:

$$d\sigma_{\text{Measured}} = d\sigma_0 \left(1 + 2\delta \hat{F}_1(q^2, Q^2) - \frac{\alpha}{\pi} \log\left(\frac{Q^2}{E_\gamma^2}\right) f_{IR}(q^2) \right)$$

$$= d\sigma_0 \left(1 + 2\delta \hat{F}_1(q^2, E_\gamma^2) \right) \qquad (1.90).$$

cf. eqn (1.75)

↑ perfectly finite, μ^2-dep. traded to E_γ^2-dep!

The Sudakov form factor

While useful, (1.90) is still badly behaved for extremely small E_γ. That is, making E_γ small one can still make $d\sigma_{measured}$ negative.

Indeed, for $-q^2 \gg m^2$ one can show that $\delta \hat{F}_1(q^2, -q^2) \to 0$ and

$$d\sigma_{measured}(q^2 \to \infty) \simeq d\sigma_0 \left(1 - \underbrace{\frac{\alpha}{\pi} \log\left(\frac{-q^2}{E_\gamma^2}\right) \log\left(-\frac{q^2}{m^2}\right)}_{\text{possibly negative.}}\right) \qquad (1.91)$$

This can be further accounted by including not only one, but arbitrarily many photons in the final state, while at the same time resumming all IR-divergent parts of quantum vertex corrections. The result is (see. P&S. 6.5) the Sudakov-form factor:

$$\left(\frac{d\sigma}{d\Omega}\right)_{measured} = \left(\frac{d\sigma}{d\Omega}\right)_0 e^{-\frac{\alpha}{\pi} f_{IR}(q^2) \log\left(-\frac{q^2}{E_\gamma^2}\right)} \qquad (1.92)$$

- finite,
- independent of μ^2
- making $E_\gamma \to 0$ puts $\left(\frac{d\sigma}{d\Omega}\right)_{measured} = 0$.

This merely proves our observation that in QED free electrons are an idealization, which cannot be observed in practice. !

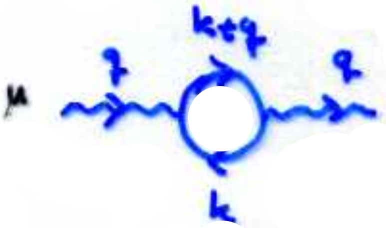
Charge renormalization

As a result of Ward identity $Z_1 = Z_2$ the combined effect of vertex and electron w/f corrections fail to contribute to the renormalization of charge e. Indeed this had to be so for the theory to make sense (see p.10) and from (1.23) and (1.17) we see that

$$\boxed{e = Z_3^{1/2} e_0} \qquad (1.93)$$

i.e. charge renormalization (running of e) only depends on the EM-field itself, i.e. on photon. At one loop (1.93) follows from $\Pi^{loop}_{\mu\nu}$ given by

because there is $-ig_{\mu\nu}$ in def. of prop



$$\equiv + i\Pi^{loop}_{\mu\nu}(q^2) \; -$$

$$= (-ie)^2(-1)\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left(\gamma^\mu \frac{i}{\not{k}-m}\gamma^\nu \frac{i}{\not{k}+\not{q}-m}\right)$$

By Ward-identity we expect $\Pi_{\mu\nu} \propto (q^2 g_{\mu\nu} - q_\mu q_\nu)$ so that the diagram should be log-divergent. Momentum cut-off breaks the gauge-invariance at the quantum level, and this is manifested by a leading $\Lambda^2 g_{\mu\nu}$-dependence of $\Pi_{\mu\nu}$ in naive Λ-cutoff. Dimensional- or Pauli-Villars regularization respect gauge-symmetry however, and we shall use the former.

$$\Rightarrow i\Pi^{loop}_{\mu\nu} = -e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{Tr}\left(\gamma^\mu(\not{k}+m)\gamma^\nu(\not{k}+\not{q}+m)\right)}{(k^2-m^2+i\epsilon)((k+q)^2-m^2+i\epsilon)} \qquad (1.94)$$

$$= -e^2\mu^\epsilon \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{4\left(k^\mu(k^\nu+q^\nu) + k^\nu(k^\mu+q^\mu) + (m^2 - k\cdot(k+q))g^{\mu\nu}\right)}{\left[\underbrace{(1-x)(k^2-m^2) + x((k+q)^2-m^2) + i\epsilon}\right]^2}$$

$$= k^2 + 2xk\cdot q + x^2q^2 - m^2 + i\epsilon$$

$$= \underbrace{(k+xq)^2}_{\equiv l} + \underbrace{x(1-x)q^2 - m^2}_{\equiv -\Delta} + i\epsilon$$

$$= -i4e^2\mu^\epsilon \int_0^1 dx \int \frac{d^dl_E}{(2\pi)^d} \frac{-2l_E^\mu l_E^\nu + l_E^2 g^{\mu\nu} - 2x(1-x)q^\mu q^\nu + (m^2 + x(1-x)q^2)g^{\mu\nu}}{[l_E^2+\Delta]^2}$$

- $$\int \frac{d^dl_E}{(2\pi)^d} \frac{-2l_E^\mu l_E^\nu + l_E^2 g^{\mu\nu}}{[l_E^2+\Delta]^2} = \left(1-\frac{2}{d}\right)\int\frac{d^dl_E}{(2\pi)^d}\frac{l_E^2 g^{\mu\nu}}{[l_E^2+\Delta]^2}$$

$$= \left(1-\frac{2}{d}\right)\cdot\frac{1}{(4\pi)^{d/2}}\frac{d}{2}\Gamma\left(1-\frac{d}{2}\right)\frac{g^{\mu\nu}}{\Delta^{1-\frac{d}{2}}} = -\left(1-\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)\frac{1}{(4\pi)^{d/2}}\frac{g^{\mu\nu}}{\Delta^{1-\frac{d}{2}}}$$

$$= -\frac{1}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\frac{\Delta g^{\mu\nu}}{\Delta^{2-\frac{d}{2}}}$$

$$\Rightarrow i\Pi^{\text{loop}}_{\mu\nu} = -i\frac{4e^2}{16\pi^2}(4\pi\mu^2)^{\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dx\frac{d}{\Delta^{\epsilon/2}}\left(\frac{1}{2}\overbrace{(m^2+x(1-x)q^2-\Delta)}^{2x(1-x)q^2}g^{\mu\nu} - x(1-x)q^\mu q^\nu\right)$$

$$= +i(q^2g^{\mu\nu} - q^\mu q^\nu)\cdot\underline{\left(-\frac{2\alpha}{\pi}\right)\cdot(4\pi\mu^2)^{\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dx\frac{x(1-x)}{\Delta^{\epsilon/2}}}$$

$$\equiv \underline{i(q^2g^{\mu\nu}-q^\mu q^\nu)}\cdot\underline{\Pi_2^{\text{loop}}(q^2)} \qquad (1.95)$$

↑
WT-identity works.

For the full 2-point function we still need to include the counter-term with δ_3:

 $\Rightarrow \quad i\Pi^{\mu\nu}_{ct} = -i(g^{\mu\nu}q^2 - q^\mu q^\nu)\,\delta_3$

$$\Rightarrow i\Pi^{\mu\nu}_{full} = \underbrace{i(g^{\mu\nu}q^2 - q^\mu q^\nu)}_{\equiv \Delta^{\mu\nu}\cdot q^2}\overbrace{\left(\Pi_2^{loop}(q^2) - \delta_3\right)}^{\equiv \hat{\Pi}(q^2)} \qquad (1.96)$$

With this form the photon propagator can be resummed

$$D_{\mu\nu} = -\frac{i}{q^2}\left(g_{\mu\nu} + \eta\frac{q_\mu q_\nu}{q^2}\right) - \frac{i}{q^2}g_{\mu\alpha}\Big[\Delta^{\alpha\rho}\hat{\Pi}(-ig_{\rho\nu}) + \overbrace{\Delta^{\alpha\rho}\hat{\Pi}(-ig_{\beta\rho})\Delta^{\rho\sigma}\Pi(-ig_{\sigma\nu})}^{\hat{\Pi}^2\Delta^\alpha_{\ \rho}\Delta^\rho_{\ \nu} = \hat{\Pi}^2\Delta^\alpha_{\ \nu}} + \dots\Big]$$

$$= -\frac{i}{q^2}\Delta_{\mu\nu}\left(1 + \hat{\Pi} + \hat{\Pi}^2 + \dots\right) - \frac{i}{q^2}\eta\frac{q_\mu q_\nu}{q^2}$$

$$= -\frac{i}{q^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\left[\frac{1}{1-\hat{\Pi}}\right] - i\eta\frac{q_\mu q_\nu}{q^4} \qquad (1.97)$$

Irrelevant in QED S-matrix

Our renormalization condition (1.22) : $\Pi_2(q^2) = 0$ thus corresponds to setting residue of physical propagator pole to 1. That is

$$\delta_3 \equiv -\hat{\Pi}_2^{loop}(0)$$

$$= \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\left\{\frac{2}{\epsilon} - \gamma + \log 4\pi - \log\left(\frac{m^2}{\mu^2}\right)\right\} \qquad (1.98)$$

and

$$i\Pi^{full}_{\mu\nu} = \underbrace{i(q^2 g^{\mu\nu} - q^\mu q^\nu)}_{\text{WT-id.}}\left(+\frac{2\alpha}{\pi}\right)\underbrace{\int_0^1 dx\, x(1-x)\log\left(1 - x(1-x)\frac{q^2}{m^2} - i\epsilon\right)}_{\hat{\Pi}_2} \qquad (1.99)$$

$\hat{\Pi}_2$ • $\to 0 \quad q^2 \to 0$,
• has a cut for $-q^2 > 4m^2$

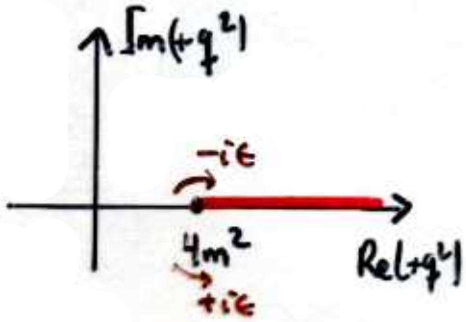
Bare charge:

Since $Z_3 = 1 + \delta_3$ & $e^2 = Z_3 e_0^2 = (1+\delta_3) e_0^2$

$$\Rightarrow \quad \frac{e^2 - e_0^2}{e_0^2} = \delta_3 \approx \frac{2\alpha}{3\pi\epsilon} \qquad (1.100)$$

i.e. bare charge is infinitely larger than e^2 (as expected).

- The cut in the finite part signals decay of the virtual photon with $-q^2 > 4m^2$ to an electron-positron pair:



$$\mathrm{Im}\,\hat{\Pi}_2(+q^2+i\epsilon) = \pi\left(-\frac{2\alpha}{\pi}\right)\int_{\frac{1}{2}-\beta}^{\frac{1}{2}+\beta} dx\, x(1-x) \; ; \quad x(1-x)\frac{q^2}{m^2} \equiv 1$$

$$\Leftrightarrow x_\pm = \frac{1}{2}\left(1 \pm \sqrt{1-\frac{4m^2}{q^2}}\right) \equiv \frac{1}{2}(1\pm\beta)$$

$y = x - \frac{1}{2}$

$$= 2\alpha \int_{-\beta/2}^{\beta/2} dy \left(\frac{1}{4} - y^2\right)$$

$$= 4\alpha \cdot \left(\frac{\beta}{8} - \frac{\beta^3}{8\cdot 3}\right) = \frac{\alpha}{2}\beta\left(1 - \frac{\beta^2}{3}\right)$$

$$= \frac{\alpha}{3}\left(1 + \frac{2m^2}{q^2}\right)\sqrt{1 - \frac{4m^2}{q^2}} \qquad (1.101)$$

Running coupling

$$\sim -i\frac{g_{\mu\nu}}{q^2}\frac{e^2}{1-\Pi(q^2)_{\text{full}}}$$

Since each virtual photon is connected to two charges, we can absorb the finite part of the q^2-dependent propagator to charge by defining

$$\alpha_{\text{eff}}(q^2) \equiv \frac{\alpha(0)}{1 - \hat{\Pi}_2(q^2)} \qquad (1.102)$$

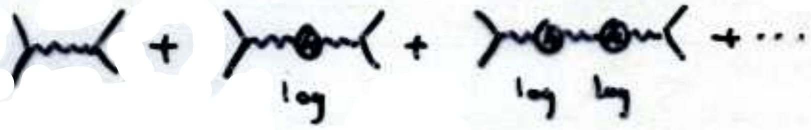
• For very large $-q^2 \gg m^2$ (small distance limit) (hard hit)

$$\hat{\Pi}_2^{\text{full}}(q^2) = \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\left(\log\left(\frac{-q^2}{m^2}\right) + \log x(1-x)\right) + \mathcal{O}\left(\frac{m^2}{q^2}\right)$$

$$= \frac{\alpha}{3\pi}\left[\log\left(\frac{-q^2}{m^2}\right) - \frac{5}{3} + \mathcal{O}\left(\frac{m^2}{q^2}\right)\right]$$

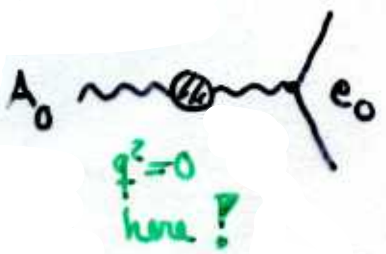
$$\Rightarrow \boxed{\alpha_{\text{eff}}(q^2) = \frac{\alpha}{1 - \frac{\alpha}{3\pi}\log\left(\frac{-q^2}{A m^2}\right)}} \quad ; \; A = e^{5/3} \qquad (1.103)$$

Note that physically this asymptotic form of the running coupling involved resumming the leading logarithmic terms of the expansion



In case of an external photon one $\sqrt{Z_3}$ renormalizes A_μ-field and the other goes to charge:

(Here I work first in bare expansion)



$$A_0^\mu \, \frac{1}{1-\hat{\Pi}(0)} \, e_0$$

$$= \underbrace{\sqrt{Z_3}\, A_0^\mu}_{\text{ren. photon field}} \; \underbrace{(\sqrt{Z_3}\, e_0)}_{\text{ren. charge.}} = A_\mu^\mu e$$

(no q^2-dependence of course)

• Charge screening

In QFT-I we have seen how the $2\to 2$ coulomb scattering gives rise to scattering from $1/r$-potential in the nonrelativistic limit (Eq. 3.157):

$$S_{f\neq i} \approx -i\int d^4x\, \bar{\psi}_k(x)\, A_0\gamma^0\, \psi_{k'}(x) . \qquad (1.104)$$

The obvious modification due to charge renormalization is the modification of the potential eqn. (3.155) to

$$A_0(\vec{x}) = 4\pi\alpha\, Q_a Q_b \int \frac{d^3q}{(2\pi)^3}\, \frac{e^{-i\vec{q}\cdot\vec{x}}}{|\vec{q}|^2\left(1-\hat{\Pi}(q^2)\right)} \qquad (1.105)$$

$q^2 \approx -|\vec{q}|^2$

Obviously when $Q_a = -Q_b = 1$ & $\Pi \to 0 \Rightarrow A_0(\vec{x}) = -e^2/|\vec{x}|$.

In the limit $-q^2 \ll m^2$

$$\hat{\Pi}_2(q^2) \approx \frac{2\alpha}{\pi}\int_0^1 dx\, x^2(1-x^2)\,\frac{q^2}{m^2} \approx -\frac{\alpha}{15\pi}\,\frac{|\vec{q}|^2}{m^2}$$

$Q_a = Q_b = -1$

$$\Rightarrow A_0(x) = -\frac{\alpha}{|\vec{x}|} + 4\pi\alpha\int_0^\infty \frac{d^3q}{(2\pi)^3}\cdot e^{-i\vec{q}\cdot\vec{x}}\,\frac{1}{\cancel{|\vec{q}|^2}}\left(-\frac{\alpha}{15\pi}\right)\frac{\cancel{|\vec{q}|^2}}{m^2}$$

$$= -\frac{\alpha}{|\vec{x}|} - \frac{4\alpha^2}{15m^2}\,\delta^3(\vec{x}) \qquad (1.106)$$

a contact term:

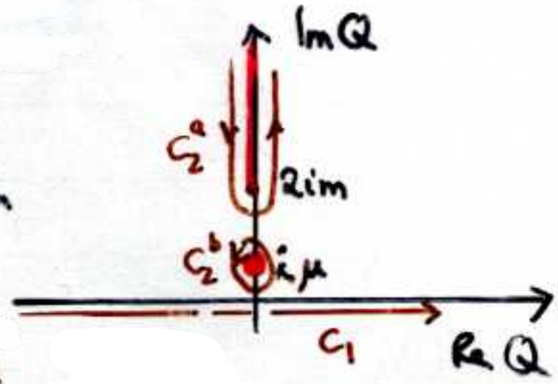
For hydrogen levels this causes a shift (Lamb) $\neq 0$ only for $l=0$-states.

$$\Delta E_{nlm} = +\int d^3x\, |\psi_{nlm}(x)|^2\,\delta A_0 = -\frac{4\alpha^2}{15m^2}\,|\psi_{nlm}(0)|^2 \qquad (1.107)$$

For more general correction one regulates the $|\vec{q}|^2=0$ pole by a photon mass (this term came from $-q^2 \to -q^2+\mu^2 \to |\vec{q}|^2+\mu^2$)

$$\Rightarrow A_0(x) \approx -4\pi\alpha \frac{1}{(2\pi)^2}\int_0^\infty dQ \frac{Q^2}{Q^2+\mu^2}\left(1+\hat{\Pi}_2(-Q^2)\right)\underbrace{\int_{-1}^{1} dz\, e^{-iQrz}}_{=\frac{1}{-iQr}\left(e^{-iQr}-e^{iQr}\right)}$$

$$= \frac{4\pi\alpha i}{(2\pi)^2 r}\int_{-\infty}^{\infty} dQ \frac{Q e^{+iQr}}{Q^2+\mu^2}\left(1+\hat{\Pi}_2(-Q^2+i\epsilon)\right)$$



Pushing contour up the pole (after $\mu\to 0$) gives the Coulomb-potential. In addition the imaginary part of $\hat{\Pi}$ contributes on the cut-path C_2^a

$Q \equiv iK : -Q^2 = K^2$

$$\delta A_0^{cut}(x) = \frac{4\pi\alpha}{(2\pi)^2 r}\, i\cdot i\cdot 2\int_{2m}^{\infty} dK \frac{e^{-Kr}}{K}\, \mathrm{Im}\left(\hat{\Pi}_2(K^2-i\epsilon)\right)$$

$$= -\frac{2\alpha^2}{3\pi r}\int_{2m}^{\infty} dK \frac{e^{-Kr}}{K}\sqrt{1-\frac{4m^2}{K^2}}\left(1+\frac{2m^2}{K^2}\right)$$

for $r \gg 1/m$ the lower limit dominates & defining $t \equiv K-2m$ one gets

$$\delta A_0^{cut}(x) \approx -\frac{\alpha^2}{\pi r}\int_0^\infty dt\, \frac{e^{-(t+2m)r}}{2m}\sqrt{\frac{t}{m}} \approx -\frac{\alpha^2}{4\sqrt{\pi}\, r}\cdot\frac{1}{(mr)^{3/2}}e^{-2mr}$$

$$\Rightarrow A_0 = -\frac{\alpha}{r}\left(1+\frac{\alpha}{4\sqrt{\pi}}\frac{e^{-2mr}}{(mr)^{3/2}}+\dots\right) \qquad (1.108)$$

↑ potential gets stronger at smaller r
⇒ bare charge is screened.

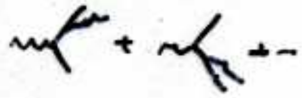
Uehling potential.

- scale of screening $\sim 1/m \sim \lambda_{\text{compton}}(e)$

QED-renormalization summary.

- Systematic renormalization counterterm interactions
  renormalization conditions
  Ward-identity $Z_1 = Z_2$
  $\Rightarrow$ UV-finite pert. theory.

(Gauge dependence yes, but only off-shell.)

- IR-divergences in  & in .
  - free electron an unobservable identification
  - soft photon emission & cancellation of IR-div.
    - Sudakov form factors. 

- Anomalous magnetic moment $= F_2(0)$ finite!

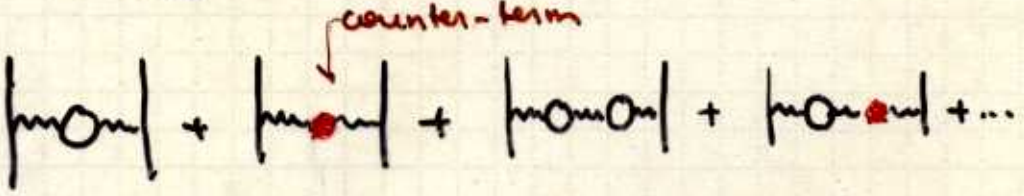
- Charge renormalization
  - Bare charge - unobservable ∞
  - Running coupling $\alpha_{eff}(q^2)$
  - Charge screening

later from RG-equation.

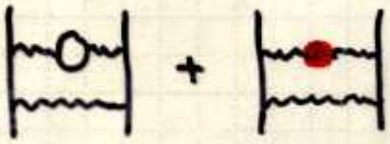
Higher loop effects?

Renormalization beyond leading order

Most of the time getting to higher orders is simple.
We already have resummed f.ex.

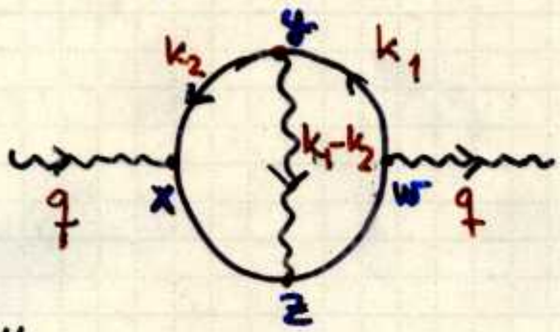


Also diagrams like with sub-divergence

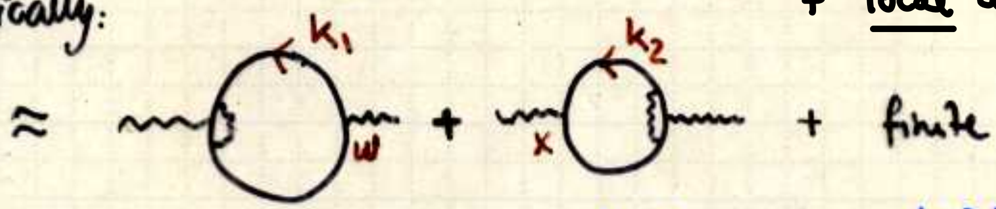


add up to a finite result. Situation is more complicated when there are overlapping or nested singularities.

Consider a diagram



Schematically: + local divergent part

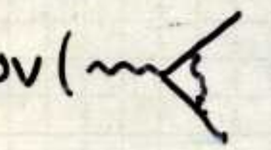


k_2 large — k_1 large — k_1 & k_2 small

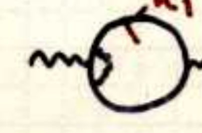
$\not\approx$ ⇒ xyz close to each other, w possibly far

⇒ yzw close to each other. x possibly far

If k_2 is large the first term behaves like the UV-part of the vertex function

$$UV\Big($$  $$\Big) \sim -ie\gamma^\mu \cdot \alpha \log \Lambda^2$$

embedded into a vacuum polarization diagram. Hence

 k_1

$$\sim \alpha(g^{\mu\nu}q^2 - q^\mu q^\nu)\Big(\underset{\uparrow}{\log \Lambda^2} + \underset{\uparrow}{\log q^2}\Big)\log \Lambda^2$$

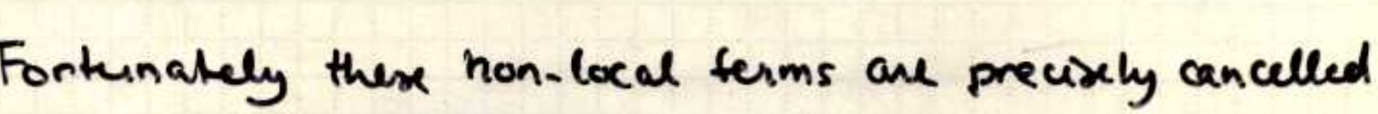
also k_1 large $\qquad$ k_1 small

- $(\log \Lambda^2)^2$ multiply polynominals of q^2 $\Rightarrow$ local terms
  (all $xyzw$ arbitrarily close to each other)
  OK

- $\log \Lambda^2 \log q^2$ -terms are non-local $\Rightarrow$ PROBLEM.
  (w far from xyz.)
  $F(\log q^2)$ is non-local

Fortunately these non-local terms are precisely cancelled by the vertex function counter terms, such that

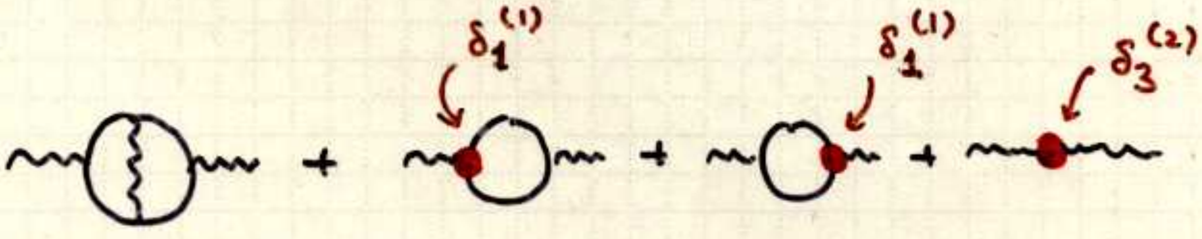
$\alpha \log \Lambda^2$ $\qquad\qquad$ $\sim \alpha \log \Lambda^2$



local, containing $(\log \Lambda^2)^2 (g^{\mu\nu}q^2 - q^\mu q^\nu)$ divergence only

local $\alpha^2[\log \Lambda^2]^2$ counter-term.

The previous breakup into subdiagrams was of course just illustrative, but it helped to see that the total sum of



is indeed local and finite after renormalization condition $\Pi_2(0) = 0$ has been set (recursively).

Go through the explicit 2-loop example in the $\phi\phi \to \phi\phi$ scattering given in Peskin & Scröder, chapter 10.5 !