Quantum field theory applications, Fall 2018, **Excercise 8.**

1. Generalize the fixed point analysis in the $\lambda\phi^4$-theory to the case where

$$\phi^2 = \sum_{i=1}^{N} \phi_i^2 \,.$$

That is, the theory has an $O(N)$-symmetry. Compute the beta function, the position of the WF-fixed point, the values of the anomalous mass dimension γ_m to first order and the anomalous dimension γ to second order in ϵ at the FW-fixed point. Compute also the critical exponent for the correlations ν, all these as a function of N. For guidance, and to see what you should get, see P&S Chapter 13, Eqs. (13.47-13.54). Note that PS has used a different normalization for the coupling λ in the $O(N)$-symmetric case ($\lambda/4! \to \lambda/4$).

2. Consider the scalar QED introduced in the problem 4/3:

$$\mathcal{L}_{\rm SED} = |D_\mu \Phi|^2 - m^2|\Phi|^2 - \frac{\lambda}{6}|\Phi|^4 - \frac{1}{4}F_{\mu\nu}^2,$$

where $D_\mu = \partial_\mu + ieA_\mu$ and $\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$. Show that the Callan-Symanzik beta functions for this theory are

$$\beta_e = \frac{e^3}{48\pi^2} \qquad \text{and} \qquad \beta_\lambda = \frac{1}{24\pi^2}(5\lambda^2 - 18e^2\lambda + 54e^4)\,.$$

Sketch the renormalization group flows in the (λ, e^2) plane. Show that every renormalization group trajectory passes through a region where λ is very small, of order $\lambda \sim (e^2)^2$.

3. Compute the renormalized one-loop effective potential for the $\lambda\phi^4$-theory in the case of spontaneous symmetry breaking:

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 - \tfrac{1}{4}\lambda\phi^4 + \tfrac{1}{2}\delta_2(\partial_\mu\phi)^2 - \tfrac{1}{2}\delta_m\phi^2 - \tfrac{1}{4}\delta_\lambda\phi^4\,,$$

where $\mu^2 < 0$. Show first that the minimum of the tree-level potential is asymmetric and the vacuum expectation of the field given by $v^2 = -\mu^2/\lambda$. Show that the tree level mass at the broken minimum is $m^2 = 2\lambda v^2$. Compute the one-loop expression for the derivative of the potential from the tadpole in the shifted theory. Note that this is exactly the same computation we did in the lectures for the case $\mu^2 > 0$. The only difference between the the symmetric and SSB-case is in the renormalization conditions for fixing the counter terms. Here impose the following conditions for the loop corrected effective potential:

$$\frac{\mathrm{d}V_{\rm eff}}{\mathrm{d}\phi}\,|_{\phi=v} \equiv 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2V_{\rm eff}}{\mathrm{d}\phi^2}\,|_{\phi=v} \equiv m^2\,.$$

where v is still the tree level vev and m^2 the tree level mass. Find the explicit expression for $V_{\rm eff}$ by integration.