Home exam   13.12.2018   (Answers 14.1.2019)

I Gross-Neveu model: $\mathcal{L} = \sum_j \bar\psi_j i\partial\!\!\!/ \psi_j + \frac{1}{2} g^2 \left(\sum \bar\psi_j \psi_j\right)^2$ in 2D.

Part one: Condensate; Chirality breaking.

a) $\psi \to \gamma^5\psi \Rightarrow \bar\psi = \psi^\dagger \gamma^0 \to \psi^\dagger \gamma^5 \gamma^0 = -\bar\psi\gamma_5 \Rightarrow (\bar\psi\psi)^2 \to (-\bar\psi\psi)^2 = (\bar\psi\psi)^2$

$\bar\psi \partial\!\!\!/ \psi \to -\bar\psi \gamma^5 \partial\!\!\!/ \gamma^5 \psi = \bar\psi \partial\!\!\!/ \psi$ ✓

Mass term $m\bar\psi\psi \to -m\bar\psi\psi$. So, if not present initially, will not be created. (Protected by discrete symmetry.)

b) $[S] = [d^dx][\bar\psi\partial\!\!\!/\psi] = L^{d-1}[\psi]^2 \equiv L^0 \Rightarrow [\psi] = L^{\frac{1-d}{2}} \Rightarrow$

$\Rightarrow [d^dx][g^2(\bar\psi\psi)^2] = L^{d+2-2d}[g^2] \Rightarrow [g^2] = L^{d-2} \xrightarrow{d\to 2} L^{-\epsilon} \to L^0$

Thus g is dimensionless in 2d, so theory is renormalizable by power counting.

$$D = 2L - P_\psi \;;\quad \left.\begin{array}{l} L = P_\psi - V + 1 \\ 4V = 2P_\psi + N_\psi \end{array}\right\} \Rightarrow D = 2P_\psi - 2\left(\frac{1}{4}(2P_\psi + N_\psi) + 1\right) - P_\psi$$

$$= 2 - \frac{1}{2}N_\psi$$

$\Rightarrow$ $N = 0, 2$ & 4 are only primitively divergent N-pt. functions

$D=2$ $D=1$ $D=0$

$N=0$ $N=2$ $N=4$

$\sim \Lambda^2$ $\sim \Lambda$ $\sim \log \Lambda$

c) $$\int \mathcal{D}\sigma \exp\left(-i\sigma\bar\psi\psi - \frac{i}{2g^2}\sigma^2\right) = \int \mathcal{D}\sigma \exp\left(-\frac{i}{2g^2}\left(\sigma + g^2\bar\psi\psi\right)^2 + \frac{i}{2}g^2(\bar\psi\psi)^2\right)$$

$$= \# \exp\left(+\frac{i}{2}g^2(\bar\psi\psi)^2\right) \Rightarrow \boxtimes$$

$\uparrow$ indep. of σ and $\bar\psi\psi$.

Note $\bar\psi\psi$ is a scalar.

d) $\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi \exp \sum_j^N i\int d^2x \left(\bar{\Psi}_j i\partial\!\!\!/ \Psi_j - \sigma\bar{\Psi}_j\Psi_j\right) = \left[\det i(i\partial\!\!\!/ - \sigma)\right]^N$; $\gamma^0 = \sigma^2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$, $\gamma^1 = i\sigma^1 = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$

$$= \left[-\det\begin{pmatrix}-\sigma & \partial_t-\partial_x\\ -\partial_t+\partial_x & -\sigma\end{pmatrix}\right]^N = \det\left(-\partial_t^2+\partial_x^2-\sigma^2\right)^N = \det\left(-\partial^2-\sigma^2\right)^N$$

$$= \exp N\,\mathrm{Tr}\log(-\partial^2-\sigma^2) = \exp N\int\frac{d^2k}{(2\pi)^2}\log(k^2-\sigma^2) = \exp\left(iN\int\frac{d^2k_E}{(2\pi)^2}\log(k_E^2+\sigma^2)\right)$$

Now:

$d = 2-\epsilon$

$$\mu^\epsilon\int\frac{d^dk_E}{(2\pi)^d}\log(k_E^2+\sigma^2) = \mu^\epsilon\int^{\sigma^2}d\alpha\int\frac{d^dk_E}{(2\pi)^d}\frac{1}{k_E^2+\alpha} = \int^{\sigma^2}d\alpha\,\frac{(4\pi\mu^2)^{\epsilon/2}}{4\pi}\overset{\sim\epsilon/2}{\Gamma\left(1-\frac{d}{2}\right)}\alpha^{\frac{d}{2}-1}$$

$$= \frac{1}{4\pi}(4\pi\mu^2)^{\frac{\epsilon}{2}}\frac{2}{d}\Gamma\left(\frac{\epsilon}{2}\right)\sigma^{2-\epsilon}$$

$\int d\alpha\,\alpha^{\frac{d}{2}-1} = \frac{2}{d}\alpha^{\frac{d}{2}}$

$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon}-\gamma_E$

$$= \frac{\sigma^2}{4\pi}\left(\frac{2}{\epsilon}-\gamma_E\right)\left(4\pi\frac{\mu^2}{\sigma^2}\right)^{\epsilon/2}\frac{1}{1-\frac{\epsilon}{2}} \approx \frac{\sigma^2}{4\pi}\left(\frac{2}{\epsilon}-\gamma_E+1+\log 4\pi-\log\frac{\sigma^2}{\mu^2}\right)$$

$$= \frac{\sigma^2}{4\pi}\left(\frac{2}{\epsilon_{\overline{MS}}}-\log\frac{\sigma^2}{\mu^2}+1\right) \xrightarrow{\overline{MS}} -\frac{\sigma^2}{4\pi}\left(\log\frac{\sigma^2}{\mu^2}-1\right)$$

each field gives the same.

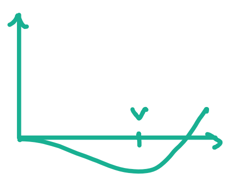
Thus:

$$Z_{GN} = \int\mathcal{D}\bar{\Psi}\mathcal{D}\Psi\, e^{iS[\Psi]} = \int\mathcal{D}\sigma\mathcal{D}\bar{\Psi}\mathcal{D}\Psi\, e^{i\int d^2x\sum_j\bar{\Psi}_j i\partial\!\!\!/\Psi_j-\sigma\bar{\Psi}_j\Psi_j-\frac{1}{2g^2}\sigma^2+\text{c.t.}}$$

$= \int\mathcal{D}\sigma\, e^{-i\int d^2x V_{eff}(\sigma)}$, where $V_{eff}(\sigma) = \frac{1}{2g^2}\sigma^2\left(1+\frac{Ng^2}{2\pi}\left(\log\frac{\sigma^2}{\mu^2}-1\right)\right)$

e) $\frac{\partial V_{eff}}{\partial\sigma} = \frac{\sigma}{g^2}\left(1+\frac{Ng^2}{2\pi}\left(\log\frac{\sigma^2}{\mu^2}-\cancel{1}+\cancel{1}\right)\right) = \frac{\sigma}{g^2}\left(1+\frac{Ng^2}{2\pi}\log\frac{\sigma^2}{\mu^2}\right)$

$\Rightarrow \sigma = 0 \vee \sigma = \mu^2 e^{-\frac{\pi}{Ng^2}} \equiv v$



Furthermore:

$$\left.\frac{\partial^2 V_{eff}}{\partial\sigma^2}\right|_{\sigma=v} = \frac{N}{\pi} > 0 \Rightarrow v \text{ is minimum} : \bar{\Psi}\Psi \neq 0.$$

While chirality was conserved perturbatively to all orders by chiral symmetry, fermions get mass term $\sim \sigma \bar{\Psi}_i \Psi_i$ at nonperturbative level. $\Rightarrow$ chirality broken.

σ corresponds to $\langle \bar{\Psi}\Psi \rangle$ $\quad \frac{1}{2g^2}(\bar{\Psi}\Psi)^2 \equiv \frac{1}{2g^2}\big(\bar{\Psi}\Psi - \underbrace{\langle\bar{\Psi}\Psi\rangle}_{v}\big)^2 - \frac{1}{2g^2}(\langle\bar{\Psi}\Psi\rangle)^2.$

Pause for some generic results: | (this is not needed in solution)

$$\gamma^0 \equiv i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; ; \; \gamma^1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; \Rightarrow \; \gamma^5 \equiv \gamma^0\gamma^1 = i\sigma^1\sigma^2 = \sigma^3$$

Obviously: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

This is slightly different set than we usually use for chirality (Normal would be $\gamma^0 = \sigma^1$, $\gamma^1 = i\sigma^2$ and $\gamma^5 = -\sigma^3$)

In chiral basis $\gamma^5 \Psi_\pm = \sigma_3 \Psi_\pm \equiv \pm \Psi_\pm \; ; \Rightarrow \Psi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} a_+$ and $\Psi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} a_-$

$$\gamma^0 \not{\partial} \Psi_\pm = (\partial_t + \gamma^0\gamma^1 \partial_z)\Psi_\pm = (\partial_t + \gamma^5 \partial_z)\Psi_\pm = (\partial_t \pm \partial_z)\Psi_\pm$$

Now $a_\pm \propto e^{-i\omega t + ik_\pm z}$, or in the massless limit: $a_\pm \propto e^{-i|k_\pm| t + ik_\pm x}$,

whence $(\partial_t \pm \partial_z)\Psi_\pm = 0 \Rightarrow k_\pm = \pm |k| \; ; \; \begin{cases} + & \text{right mover} \\ - & \text{left mover} \end{cases}$

A mass term $m\bar{\Psi}\Psi$ would mix chirality:

$$\begin{pmatrix} i\partial_t + i\partial_z & -m \\ -m & i\partial_t - i\partial_z \end{pmatrix} \chi = \begin{pmatrix} p_0 - p & -m \\ -m & \omega + p \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \Rightarrow \begin{pmatrix} p_0 - h|p| & -m \\ -m & \omega + h|p| \end{pmatrix}\begin{pmatrix} a_1^h \\ a_2^h \end{pmatrix} = 0$$

We know these solutions: $\Psi_h^\pm = \begin{pmatrix} \sqrt{\omega + h|p|} \\ \overset{+}{\scriptstyle(-)}\sqrt{\omega - h|p|} \end{pmatrix} e^{\overset{+}{\scriptstyle(-)}(i\omega t - ih|p|z)}$; $u(h,p)$ $v(h,p)$

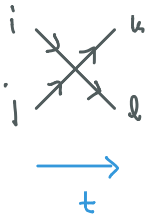
$h = 1 \equiv$ right mover

$h = -1 \equiv$ left mover

Feynman rules

$$\psi_i(x) = \sum_h \int \frac{dp}{4\pi\omega_p}\left(a^h_{pi} u_{ph} e^{-ip\cdot x} + b^{h\dagger}_{pi} v_{ph} e^{ip\cdot x}\right)$$

$$\bar\psi_i(x) = \sum_h \int \frac{dp}{4\pi\omega_p}\left(a^{h\dagger}_{pi} \bar u_{ph} e^{ip\cdot x} + b^h_{pi} \bar v_{ph} e^{-ip\cdot x}\right)$$

- 

$$\sim \frac{ig^2}{2}\sum_{n,m}\langle ij|(\bar\psi_n\psi_n)(\bar\psi_m\psi_m)|kl\rangle$$

$$= ig^2 \sum_{n,m}\langle\Omega|\, a_i a_j\, \bar\psi_n \psi_n\, \bar\psi_m \psi_m\, a_k^\dagger a_l^\dagger\,|\Omega\rangle$$

$$= -ig^2\sum_{mn}\delta_{im}\delta_{jn}\left(\delta_{nl}\delta_{mk} - \delta_{nk}\delta_{ml}\right) = ig^2\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right)$$

two distinct fermion flows
note - sign!

- $$\langle \psi_i \bar\psi_j\rangle = \frac{i}{\not k}\delta_{ij}$$

Part two: β-function

$$\left([g^2] = \mu^\epsilon \;;\; g^2 \to \overset{d\text{-dim}}{g^2}\mu^\epsilon \right)$$

Renormalization $\quad \psi_0 = Z_\psi^{1/2}\psi_R \qquad g_0^2\mu^{-\epsilon} \equiv g^2 + \delta g^2 = \bar Z_{g^2} g^2 \iff g^2 = \bar Z_{g^2}^{-1} g_0^2 \mu^{-\epsilon}$

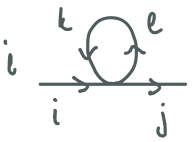
$$g_0^2\mu^{-\epsilon}(\bar\psi_0\psi_0)^2 = (g^2+\delta g^2) Z_\psi^2 (\bar\psi\psi)^2 = g^2 \bar Z_{g^2} Z_\psi^2 (\bar\psi\psi)^2 = g^2 Z_{g^2}(\bar\psi\psi)^2$$

$$\Rightarrow Z_{g^2} = \bar Z_{g^2} Z_\psi^2$$

$$\iff \bar Z_{g^2} \equiv Z_{g^2}/Z_\psi^2 \approx 1 + \delta_{g^2} - 2\delta_\psi$$

$$\beta(g^2) \equiv \mu\frac{\partial g^2}{\partial\mu} = \mu\frac{\partial}{\partial\mu}\left(g_0^2\mu^{-\epsilon}\bar Z_{g^2}^{-1}\right) = -\epsilon g^2 - g^2\frac{1}{\bar Z_{g^2}}\mu\frac{\partial}{\partial\mu}\bar Z_{g^2} = -\epsilon g^2 - g^2\beta(g)\frac{1}{\bar Z_{g^2}}\frac{\partial \bar Z_{g^2}}{\partial g^2}$$

$$\iff \beta(g^2) = -\frac{\epsilon g^2}{1 + g^2\frac{\partial}{\partial g^2}\log \bar Z_{g^2}} \approx -\epsilon g^2\left(1 - g^2\frac{\partial}{\partial g^2}\log\bar Z_{g^2}\right) \approx -\epsilon g^2\left(1 - g^2\frac{\partial}{\partial g^2}\left(\delta_{g^2} - 2\delta_\psi\right)\right)$$

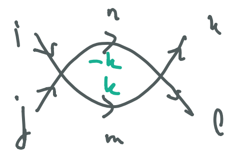
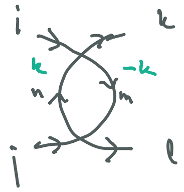
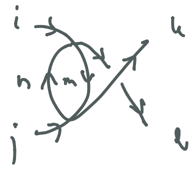
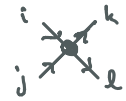
(I) 

$$= -ig^2 \sum_{k,l} \mu^\epsilon \int \frac{d^dk}{(2\pi)^d} \mathrm{Tr}\left(\frac{i}{\not k}\right)\delta_{kl}\left(\delta_{kl}\delta_{ij} - \delta_{il}\delta_{kj}\right)$$

$$= +g^2(N-1)\,\delta_{ij}\,\mu^\epsilon \underbrace{\int \frac{d^dk}{(2\pi)^d}\mathrm{Tr}\left(\frac{1}{\not k}\right)}_{=0} = 0$$

Also, would have been a tadpole with no γ-part anyway

$$\Rightarrow \delta_\psi = 0$$

(II)  +  +  + 

(a) (b) (c) $-i\delta_{g^2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$

a) $$-\frac{1}{2}\cdot 2\,(ig^2)^2 \sum_{n,m} \mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\,\frac{-i^2}{k^2}\left(\delta_{in}\delta_{jm}-\delta_{im}\delta_{jn}\right)\left(\delta_{nk}\delta_{ml}-\delta_{nl}\delta_{mk}\right)$$

$$= -2g^4\left(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}\right)\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2}$$

b) $$-\frac{1}{2}\cdot 2\,(ig^2)^2 \sum_{n,m} \mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\,\frac{-i^2}{k^2}\left(\delta_{ik}\delta_{mn}-\delta_{im}\delta_{kn}\right)\left(\delta_{jl}\delta_{nm}-\delta_{ml}\delta_{jn}\right)$$

$$= -g^4\left((N-2)\,\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2}$$

c) $$-\frac{1}{2}\cdot 2\,(ig^2)^2 \sum_{n,m} \mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\,\frac{-i^2}{k^2}\left(\delta_{il}\delta_{mn}-\delta_{im}\delta_{ln}\right)\left(\delta_{jk}\delta_{nm}-\delta_{mk}\delta_{nj}\right)$$

$$= -g^4\left((N-2)\,\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl}\right)\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2}$$

Combined:

$$a)+b)+c) = -g^4\left(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}\right)\Big(\overbrace{2+(N-2)-1}^{N-1}\Big)\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2}$$

$$\Rightarrow \text{1-loop + ct} = -\left(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}\right)\left(g^4(N-1)\,\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2} - i\delta_{g^2}\right) \equiv \text{finite}$$

Now, we want to extract the UV-divergence, so we regulate IR by some Δ:

- $$\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2} = \lim_{\Delta\to0}\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\Delta} = \lim_{\Delta\to0} -\frac{i}{(4\pi)^{d/2}}\mu^\epsilon\frac{\Gamma(1-\frac{d}{2})}{\Gamma(1)}\left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}}$$

$$\overset{d=2-\epsilon}{=} \lim_{\Delta\to0} -\frac{i}{4\pi}\left(4\pi\frac{\mu^2}{\Delta}\right)^{\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right) = -\frac{i}{2\pi\epsilon_{\overline{MS}}} + \text{UV-finite.}$$

We then get $\boxed{\delta_{g^2}^{\overline{MS}} = -g^4(N-1)\frac{1}{2\pi\epsilon}}$.

Then $$\beta(g^2) = -\epsilon g^2\left(1-g^2\frac{\partial}{\partial g^2}(\delta_{g^2}-2\delta_\psi)\right) = \epsilon g^2\frac{\partial}{\partial g^2}\delta_{g^2} = -\frac{N-1}{\pi}g^4$$

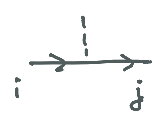
Now $$\beta(g^2) = \mu\frac{\partial g^2}{\partial\mu} = 2g\mu\frac{\partial g}{\partial\mu} = 2g\beta(g) \Rightarrow \boxed{\beta(g) = -\frac{N-1}{2\pi}g^3}$$

β-function is negative $\Rightarrow$ theory is asymptotically free.

Some additional comments:

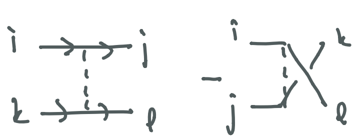
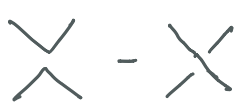
- An alternative way to perform loop calculations is to use Lagrangian with σ

$$\mathcal{L} = \sum_i \bar\psi_i i\partial\!\!\!/\psi - \sigma\bar\psi_i\psi_i - \frac{1}{2g^2}\sigma^2$$

$\Rightarrow$  $\sim -i\delta_{ij}$ $\sim \frac{i}{\not k-\sigma}$ $\sim \frac{i}{-1/g^2} = -ig^2$ obs.

Lowest order:

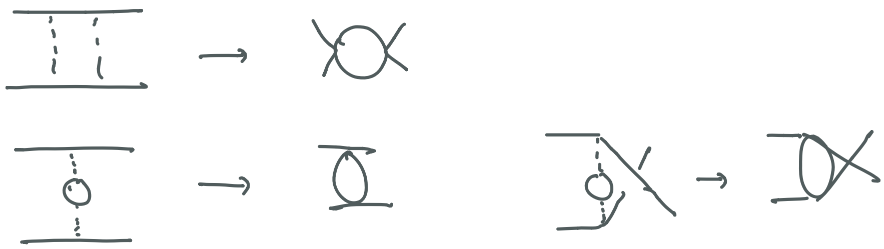
σ is not propagating ⇒ collapses

 $=$  $\Rightarrow$  $\sim ig^2(\delta_{ij}\delta_{kl}-\delta_{ik}\delta_{jl})$

amputated vertex

$=$ tree-level F-rule

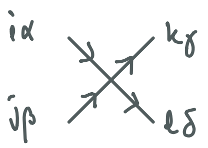
Similarly



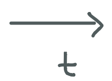
etc.

- There was no point in keeping the chiral index α in the computation. If one insists on including γ^0 in the vertex, then the F-rules are

$$\frac{ig^2}{2}(\bar\psi\psi)^2 = \frac{ig^2}{2}\left(-i\psi^\dagger i\sigma^2\psi\right)^2 = -\frac{ig^2}{2}\left(\psi^{+\alpha}\epsilon_{\alpha\beta}\psi^\beta\right)^2$$



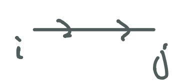
$$\sim -\frac{ig^2}{2}\sum_{n,m}\langle i\alpha\, j\beta|\left(\psi_n^{+a}\epsilon_{ab}\psi_n^b\right)\left(\psi_m^{+c}\epsilon_{cd}\psi_m^d\right)|k\gamma\,\ell\delta\rangle$$



$$= -ig^2\sum_{n,m}\langle\Omega|\, a_{i\alpha}a_{j\beta}\,\psi_n^{+a}\epsilon_{ab}\psi_n^b\,\psi_m^{+c}\epsilon_{cd}\psi_m^d\, a^+_{k\gamma}a^+_{\ell\delta}|\Omega\rangle$$

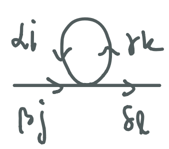
$$= ig^2\delta_{in}\delta_{jn}\left(\delta_{km}\delta_{\ell n}\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} - \delta_{kn}\delta_{\ell n}\epsilon_{\alpha\delta}\epsilon_{\beta\gamma}\right)$$

$$= ig^2\left(\delta_{ik}\delta_{j\ell}\epsilon_{\gamma\alpha}\epsilon_{\delta\beta} - \delta_{i\ell}\delta_{jk}\epsilon_{\delta\alpha}\epsilon_{\delta\beta}\right)$$ there is still a $-$ sign.

And in this scheme  $= \langle\psi^+\psi\rangle = \langle\bar\psi\psi\rangle\gamma^0 = \frac{i}{\not k}\sigma^2 = \left(\frac{1}{\not k}\right)_{\alpha\alpha}\epsilon_{\alpha\beta}$

Then eg:

$$\epsilon_{\alpha\phi}\left(\tfrac{1}{\not k}\right)_{\phi\gamma}$$

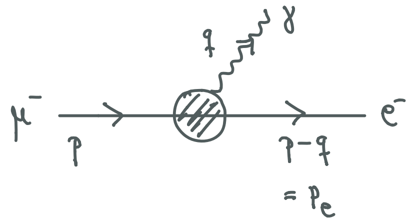


$$-ig^2\mu^\epsilon\sum_{\substack{ik\\ \alpha\gamma}}\int\frac{d^dk}{(2\pi)^d}\left(\frac{1}{\not k}\right)_{\alpha\phi}\epsilon_{\phi\gamma}\,\delta_{ik}\left(\delta_{ik}\delta_{j\ell}\epsilon_{\gamma\alpha}\epsilon_{\delta\beta} - \delta_{i\ell}\delta_{jk}\epsilon_{\delta\alpha}\epsilon_{\delta\beta}\right)$$

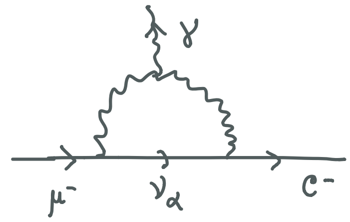
$$= ig^2\mu^\epsilon\delta_{j\ell}\int\frac{d^dk}{(2\pi)^d}\left(N\,\mathrm{Tr}\left(\frac{1}{\not k}\epsilon\epsilon\right)\epsilon_{\delta\beta} - \left(\epsilon\frac{1}{\not k}\overbrace{\epsilon\epsilon}^{-1}\right)_{\delta\beta}\right)$$

multiply by $\epsilon_{\beta\delta}$ & take trace $\Rightarrow$ old result. etc.

II $\mu^- \to e^- \gamma$ - decay. (This is possible, althoug *very* rare)



SM $\hat{=}$



Part 1: Photon couples to a vector index in current. The problem contains two independent vectors $q = p_\mu - p_e$ and $P = p_\mu + p_e$. We can combine these with $1, \gamma^5, \gamma^\lambda, \gamma^\lambda\gamma^5$ and $\sigma_{\mu\nu}$ to get vectors:

$$(A + B\gamma^5)\, q^\nu \sigma_{\lambda\nu} \quad ; \quad \gamma_\lambda (C + D\gamma_5) \qquad q_\lambda (E + F\gamma^5)$$

$$(\bar A + \bar B \gamma^5)\, P^\nu \sigma_{\lambda\nu} \quad ; \qquad\qquad P_\lambda (\bar E + \bar F\gamma^5)$$

However (generalizing Gordon identity) $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

- $\bar u_e(p')\Big(\overbrace{p^\mu + p'^\mu}^{P^\mu} - i\sigma^{\mu\nu}(\overbrace{p_\nu - p'_\nu}^{q^\mu})\Big)\begin{pmatrix}1\\ \gamma_5\end{pmatrix} u_\mu(p)$

$= \bar u_e(p')\left(\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}(p_\nu + p'_\nu) + \frac{1}{2}[\gamma^\mu,\gamma^\nu](p_\nu - p'_\nu)\right)\begin{pmatrix}1\\ \gamma_5\end{pmatrix} u_\mu(p)$

$= \bar u_e(p')\left(\gamma^\nu \not{p} + \not{p}'\gamma^\nu\right)\begin{pmatrix}1\\ \gamma_5\end{pmatrix} u_\mu(p) = \bar u_e(p')\left(m_\mu\gamma^\mu + m_e\gamma^\nu\right)\begin{pmatrix}1\\ \gamma_5\end{pmatrix} u_\mu(p)$

$\Rightarrow$ $(\bar A + \bar B\gamma^5) P^\nu \sigma_{\lambda\nu}$ can be absorbed to A, B, C & D.

- $\bar u_e(p')\, i\sigma^{\mu\nu} P_\mu \begin{pmatrix}1\\ \gamma_5\end{pmatrix} u(p) = \bar u_e(p')\, \frac{1}{2}[\gamma^\mu,\gamma^\nu](p_\mu + p'_\mu)\begin{pmatrix}1\\ \gamma_5\end{pmatrix} u_\mu(p)$

$= \frac{1}{2}\bar u_e(p')\left((\not{p} + \not{p}')\gamma^\nu - \gamma^\nu(\not{p} + \not{p}')\right)\begin{pmatrix}1\\ \gamma_5\end{pmatrix} u_\mu(p)$

$= \frac{1}{2}\bar u_e(p')\left(2m_e\gamma^\nu + 2(p-p')^\nu - 2\gamma^\nu\not{p}'\right)\begin{pmatrix}1\\ \gamma_5\end{pmatrix} u_\mu(p)$

$\hookrightarrow \pm 2\gamma^\nu m_\mu$

$\Rightarrow$ also $P^\mu(\bar A + \bar B\gamma^5)$ can be absorbed to A, B, C & D.

So we can write the most general form as

$$T_\lambda = \bar{u}_e(p')\left[i q^\nu \sigma_{\lambda\nu}(A+B\gamma^5) + \gamma_\lambda (C+D\gamma_5) + q_\lambda (E+F\gamma^5)\right] u_\mu(p)$$

Using gauge invariance (both μ & e are gauge-charged & EM-gauge field is flavour blind):

$$q^\lambda T_\lambda = \bar{u}_e(p')\left[\underbrace{i q^\lambda q^\nu \sigma_{\lambda\nu}}_{=0 \text{ identically}}(A+B\gamma_5) + \underbrace{\not{q}}_{=\not{p}-\not{p}'}(C+D\gamma_5) + q^2(E+F\gamma^5)\right] u_\mu(p) \equiv 0$$

$$\Rightarrow \quad m_e(C+D\gamma^5) + m_\mu(C-D\gamma_5) + q^2(E+F\gamma^5) = 0$$

$$\xRightarrow{q^2=0} \quad C = D = 0 .$$

Furthermore, given that $q\cdot\epsilon = 0$, E and F do not contribute to physical amplitude, and we get

$$\epsilon^\lambda T_\lambda = \bar{u}_e(p')\left[i \overbrace{\epsilon^\lambda q^\nu \sigma_{\mu\nu}}^{\frac{1}{2}(\not{q}\not{\epsilon}-\not{\epsilon}\not{q}) \simeq \not{q}\not{\epsilon} = \not{p}\not{\epsilon} = 2p\cdot\epsilon - \not{\epsilon}\not{p} = 2p\cdot\epsilon - m_\mu\not{\epsilon}}(A+B\gamma_5)\right] u_\mu(p) \qquad \text{(magnetic transition)}$$

- We shall set $m_e = 0$, so e^- is either L- or R-chiral. Since it must couple to ν via L-chiral vertex, only L-chiral part $\bar{u}_L = \bar{u}_e P_R$ contributes. This implies that $A = B$. Then

$$\epsilon^\lambda T_\lambda = A\, \bar{u}_e(p-q)(1+\gamma^5)(2p\cdot\epsilon - m_\mu\not{\epsilon})\, u_\mu(p) \qquad ;$$

$$\Rightarrow \sum |M|^2 = \cancel{\tfrac{1}{2}}|A|^2 \sum_\lambda \underbrace{\epsilon^\mu_\lambda \epsilon^{\nu *}_\lambda}_{\to -g^{\mu\nu}} \mathrm{Tr}\Big((\not{p}+m_\mu)(2p_\nu - m_\mu\gamma_\nu)\underbrace{(1-\gamma^5)(\not{p}-\not{q})(1+\gamma^5)}_{=\cancel{2}(\not{p}-\not{q})(1+\gamma_5)}(2p_\mu - m_\mu\gamma_\nu)\Big)$$

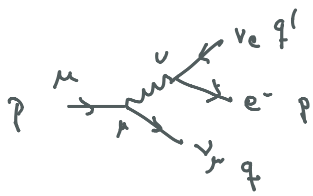
$$= -|A|^2\, \mathrm{Tr}\Big((\not p + m_\mu)\, \underbrace{(2p^\mu - m\gamma^\mu)(\not p - \not q)(2p_\mu - m_\mu\gamma_\mu)}_{(4-2)\, m^2(\not p - \not q) - 2m\{\not p,\, \not p - \not q\} \;=\; 2m^2(\not p - \not q) - 4m\, p\cdot(p-q)} \Big)$$

$$= |A|^2\, \mathrm{Tr}\big(-2m_\mu^2 \not p(\not p - \not q) + 4m^2\, p\cdot(p-q) \big)$$

$$= 8m_\mu^2 |A|^2\, p\cdot(p-q) = 8m_\mu^2|A|^2 \Big(m^2 + \frac{1}{2}\overbrace{(p-q)^2}^{m_e^2} - \frac{m^2}{2} \Big) = 4m_\mu^4 |A|^2$$

$$\Rightarrow \Gamma_{\mu\to e\gamma} = \frac{|M|^2}{16\pi m_\mu} = \frac{m_\mu^3}{4\pi}|A|^2 \qquad \square$$

Extra: Decay $\mu \to e\bar\nu_e\nu_\mu$: $\frac{ig}{2\sqrt2}\, \bar e \gamma^\mu P_L \nu + h.c$



$$= \Big(\frac{-ig}{2\sqrt2}\Big)^2\, \bar u_{\nu_\mu}(q)\gamma_\mu(1-\gamma_5)u_\mu(p)\, \frac{-ig^{\mu\nu}}{-M_W^2}\, \bar u_e(p')\gamma_\nu(1-\gamma_5)v_{\nu_e}(q')$$

$$\Rightarrow \overline{|M|^2} = \frac{1}{2}\Big(\frac{g^2}{8M_W^2}\Big)^2 \mathrm{Tr}\Big((\not p - m_\mu)\gamma_\mu \overbrace{(1-\gamma_5)\not q\gamma_\nu}^{=\not q\gamma_\nu(1-\gamma_5)}(1-\gamma_5)\Big)\cdot \mathrm{Tr}\Big(\not q'\gamma^\mu \overbrace{(1-\gamma_5)\not p'\gamma^\nu}^{\not p'\gamma^\nu(1-\gamma_5)}(1-\gamma_5)\Big)$$

$$= \frac{1}{2}\Big(\frac{G_F}{\sqrt2}\Big)^2 \cdot 16\, \big(p^\mu q^\nu + p^\nu q^\mu - g^{\mu\nu} p\cdot q + i\epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta \big)$$
$$\times \big(q'_\mu p'_\nu + q'_\nu p'_\mu - g_{\mu\nu}\, q'\cdot p' + i\epsilon_{\mu\nu\gamma\delta}\, q'^\gamma p'^\delta \big)$$

$$= 4G_F^2 \big(2p\cdot q'\, q\cdot p' + 2p\cdot p'\, q\cdot q' - 2p\cdot q'\, q\cdot p' + 2p\cdot p'\, q\cdot q' \big)$$

$$= 16 G_F^2\, p\cdot p'\, q\cdot q' = 16 G_F^2\, m_\mu p' q q' (1-\cos\theta_{qq'})$$

where in the last step I adopted rest frame of muon $p = (m_\mu, \vec 0)$. Using this frame,

$$\Gamma_{\mu\to e\nu\nu} = \frac{1}{2m_\mu}\int \frac{d^3p'}{(2\pi)^3 2p'}\frac{d^3q}{(2\pi)^3 2q}\frac{d^3q'}{(2\pi)^3 2q'}\underbrace{(2\pi)^4\delta^3(\vec{p}\,'+\vec{q}+\vec{q}\,')}_{\text{kill } p'\text{-integral}}\delta(m_\mu-p'-q-q')\,|M|^2$$

$$\Rightarrow p'^2 = q^2+q'^2+2qq'\cos\theta_{qq'}$$

$$\Rightarrow \delta(m_\mu-p'-q-q') = \delta\left(m_\mu-q-q'-\sqrt{q^2+q'^2+2qq'\cos\theta_{qq'}}\right) = \frac{m_\mu-q-q'}{qq'}\,\delta(\cos\theta-\cos\theta_*)$$

$$\text{where}\quad \cos\theta_* = \frac{(m_\mu-q-q')^2-q^2-q'^2}{2qq'} = \frac{m_\mu^2-2m_\mu(q+q')}{2qq'}+1$$

and

$$|q-q'| \leq m_\mu-q-q' \leq q+q' \quad \text{so } \delta\text{-function has support.}$$

from $q-q'=0$ in lower limit

$$\Leftrightarrow \quad |q-q'| \leq m_\mu-(q+q') \quad \text{and} \quad \frac{m_\mu}{2} \leq q+q' \leq m_\mu$$

Use $dq\,dq' = dQ\,dq_1$ where $Q = q+q'$ and $q = \frac{1}{2}(q-q')$. ; $|q| < 2(m-Q)$

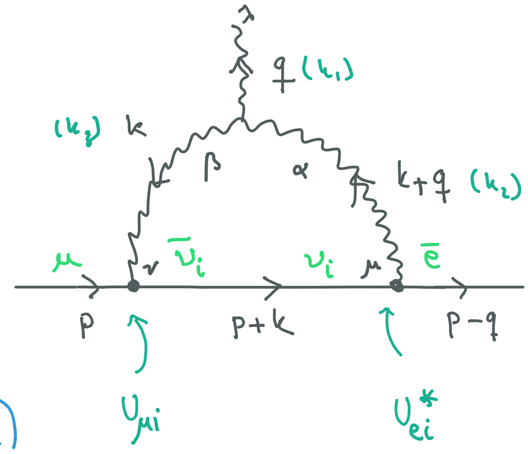
$$\Rightarrow \Gamma = \frac{1}{2\cancel{m_\mu}}\,16\,G_F^2\,\cancel{m_\mu}\,\frac{1}{2^3(2\pi)^5}\,8\pi^2\int dq\,dq'\,\frac{q^2q'^2}{p'qq'}\int d\cos\theta_{qq'}\,(1-\cos\theta_{qq'})\,p'qq'\,\frac{m-q-q'}{qq'}\,\delta(\cos\theta-\cos\theta_*)$$

$$= \frac{G_F^2}{4\pi^3}\int_{m_\mu/2}^{m_\mu} dQ\; 2\int_0^{2(m_\mu-Q)} dq\;\cancel{qq'}\,m_\mu\,\frac{2Q-m_\mu}{2\cancel{qq'}}\,(m-Q)$$

$$= \frac{G_F^2}{4\pi^3}m_\mu^5\int_{1/2}^{1} dx\,(2x-1)(1-x)\int_0^{2(1-x)} dy = \frac{G_F^2}{2\pi^3}m_\mu^5\underbrace{\int_{1/2}^{1} dx\,\overbrace{(2x-1)(1-x)^2}^{2x^3-5x^2+4x-1}}_{=1/96} = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

$\mu\to e\bar{\nu}_e\nu_\mu$ is dominating channel $\Rightarrow$ $Br_{\mu\to e\gamma} \approx \dfrac{\Gamma_{\mu\to e\gamma}}{\Gamma_{\mu\to e\bar{\nu}_e\nu_\mu}} \simeq 48\pi^2\dfrac{|A|^2}{(G_F m_\mu)^2}$.

Part 2. Unitary gauge evaluation of A.



$$\mathcal{L} = \frac{g}{2\sqrt{2}}\left(\bar{\nu}_e\gamma^\alpha(1-\gamma_5)e\,W^-_\alpha + \bar{e}\gamma^\alpha(1-\gamma_5)\nu_e W^+\right)$$

Flavour mixing: $\nu_a = \sum_i U^*_{ai}\,\nu_i$ $\quad (\nu_i \equiv \sum U_{ia}\nu_a)$

$\bar{\nu}_a = \sum_i U_{ai}\,\bar{\nu}_i$

and finally

$$-ieV_{\lambda\alpha\beta}(k_1,k_2,k_3) = -ie\left[(k_1-k_2)_\beta\, g_{\lambda\alpha} + (k_2-k_3)_\lambda\, g_{\alpha\beta} + (k_3-k_1)_\alpha\, g_{\beta\lambda}\right]$$

$\bar{\nu} = \nu^\dagger\gamma$

$$\Rightarrow T = \left(i\frac{g}{2\sqrt{2}}\right)^2(-ie)\sum_i U^*_{ei}U_{\mu i}\int\frac{d^dk}{(2\pi)^d}\frac{2\,i^3\,R}{[(k+p)^2-m_i^2]\,[(k+q)^2-M_W^2]\,[k^2-M_W^2]}$$

where

$$R = \frac{1}{\cancel{2}}\bar{u}_e(p-q)\Big(\overbrace{\gamma^\mu(1-\gamma^5)(\not{p}+\not{k})\gamma^\nu(1-\gamma^5)}^{=\cancel{2}\gamma^\nu(\not{p}+\not{k})\gamma^\mu(1-\gamma_5)}\;\overbrace{\epsilon^\lambda V_{\lambda\alpha\beta}(-q,k+q,-k_1)}^{\Gamma_{\alpha\beta}}\,\Delta_W^{\mu\alpha}(k+q)\,\Delta_W^{\nu\beta}(k)\Big)u_\mu(p)$$

$$= \Delta_W^{\mu\alpha}(k+q)\,\Delta_W^{\nu\beta}(k)\,\Gamma_{\alpha\beta}\,\underbrace{\bar{u}_e(p-q)\gamma^\nu(\not{p}+\not{k})\gamma^\mu(1-\gamma^5)u_\mu(p)}_{\equiv N_{\mu\nu}}$$

$$= \Delta_W^{\mu\alpha}(k+q)\,\Delta_W^{\nu\beta}(k)\,\Gamma_{\alpha\beta}\,N_{\mu\nu}. \qquad \text{where } \Delta_W^{\alpha\beta}(q) \equiv -g_{\alpha\beta} + \frac{q^\alpha q^\beta}{M_W^2}$$

- lowest-order term in m_i^2 vanishes, because $\sum_i U^*_{ei}U_{\mu i} = (U^\dagger U)_{e\mu} = 0$.
- Because m_i^2 is tiny, it makes sense to expand $\dfrac{1}{(k+p)^2-m_i^2} = \dfrac{1}{(k+p)^2} + \dfrac{m_i^2}{[(k+p)^2]^2} + \mathcal{O}(m_i^4)$

Hence to good approximation

$$T \simeq \frac{g^2 e}{4}\sum_i U^*_{ei}U_{\mu i}\,m_i^2\int\frac{d^4k}{(2\pi)^4}\frac{R}{[(k+p)^2]^2\,[(k+q)^2-M_W^2]\,[k^2-M_W^2]}$$

Because of the lowest term vanishes integral converges better. Still, in U-gauge the part that picks the $k^\mu k^\nu/M_W^2$ - term from both Δ_W's goes formally as

$$\sim \int d^4k \frac{k^4}{k^8} \sim \int \frac{dk}{k} \sim \log \Lambda \quad \Rightarrow \text{ danger of divergence}$$

However
$$\left(\Delta_W^{N\alpha}(k+q)\,\Delta_W^{\nu\beta}(k)\,\Gamma_{\alpha\beta}\right)_{\text{dangerous}}$$
$$\sim (k+q)^\alpha k^\beta \epsilon^\lambda\, V_{\lambda\alpha\beta}(-q, k+q, -k)$$
$$= (k+q)^\alpha k^\beta \epsilon^\lambda \Big[\underbrace{-(k+2q)}_{k_1-k_2}{}_\beta\, g_{\lambda\alpha} + \underbrace{(2k+q)}_{k_2-k_3}{}_\lambda\, g_{\alpha\beta} + \underbrace{(q-k)}_{k_3-k_1}{}_\alpha\, g_{\beta\lambda}\Big]$$
$$= -(k+q)\cdot\epsilon\; k\cdot(k+2q) + (2k+q)\cdot\epsilon\; k\cdot(k+q) + (k\cdot\epsilon)\,(q^2-k^2)$$
$$= (k\cdot\epsilon)\left(\cancel{-k^2} - \cancel{2k\cdot q} + \cancel{2k^2} + \cancel{2k\cdot q} + q^2 - \cancel{k^2}\right) + q\cdot\epsilon\left(-k^2 - 2q\cdot k + k^2 + q\cdot k\right)$$
$$= q^2\, k\cdot\epsilon - q\cdot k\; q\cdot\epsilon \;=\; 0 \quad \text{because } q^2 = 0 \;\&\; q\cdot\epsilon = 0$$

The remaining terms are finite & this is why U-gauge works here. Remaining contractions can be arranged as such

$$\Delta_W^{N\alpha}(k+q)\,\Delta_W^{\nu\beta}(k)\,\Gamma_{\alpha\beta}\,N_{\mu\nu} = \Gamma_{\mu\nu}\,N^{\mu\nu}$$
$$- \frac{1}{M_W^2}\left(k^\nu k^\beta\, \Gamma^\mu{}_\beta + (k+q)^\mu (k+q)^\alpha\, \Gamma_\alpha{}^\nu\right) N_{\mu\nu}$$
$$\equiv \underline{S_1 + S_2 + S_3}$$

Feynman parametrization:

$$\frac{1}{A_1^{m_1} A_2^{m_2} \cdots A_n^{m_n}} = \int_0^1 dx_1 \ldots dx_n\, \delta(\Sigma x - 1)\, \frac{\prod x_i^{m_i-1}}{(\Sigma x_i A_i)^{\Sigma m_i}}\, \frac{\Gamma(m_1+\ldots+m_n)}{\Gamma(m_1)\ldots\Gamma(m_n)} \qquad \text{(PS 6.42)}$$

$x_3 = 1 - x_1 - x_2$

$$\Rightarrow \frac{1}{[(k+p)^2]^2\,[(k+q)^2 - M_W^2]\,[k^2 - M_W^2]} = \frac{\Gamma(4)}{\Gamma(2)\Gamma(1)^2} \int_0^1 dx_1 dx_2 dx_3\, \frac{\delta(\Sigma x_i - 1)\, x_1}{\left(x_1 (k+p)^2 + x_2((k+q)^2 - M_W^2) + x_3(k^2 - M_W^2)\right)^4}$$

$$= 6 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_1}{(\underbrace{k^2 + 2x_1 k\cdot p + x_1 p^2 + 2x_2 k\cdot q - (1-x_1)M_W^2}_{(k+x_1p+x_2q)^2 + x_1(1-x_1)p^2 - (1-x_1)M_W^2 \;\equiv\; \ell^2 - a^2})^4}$$

$$= 6 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_1}{(\ell^2-a^2)^4} \quad ; \quad \ell = k + x_1 p + x_2 q \qquad a = (1-x_1)M_W^2 - x_1(1-x_1)m_\mu^2$$

$$k = \ell - x_1 p - x_2 q \qquad\qquad \to (1-x_1)M_W^2$$

$$\Rightarrow \int \frac{d^4k}{(2\pi)^4} \frac{R(k)}{[(k+p)^2]^2[(k+q)^2 - M_W^2][k^2 - M_W^2]} = 6 \int_0^1 dx_1 \int_0^{1-x_1} dx_2\, x_1 \int \frac{d^d\ell}{(2\pi)^d} \frac{R(\ell - x_1 p - x_2 q)}{(\ell^2 - a)^4}$$

Now the task is to compute $R = \sum S_i$. Observe that R will contain pieces

$$\sim \quad B_{\mu\nu}, \quad C_\mu \ell_\nu \xrightarrow{\text{odd}} 0, \quad D_{\mu\nu}\ell^2, \quad D\ell_\mu\ell_\nu \to \frac{D}{d}\ell^2 g_{\mu\nu}, \quad E_\mu \ell_\nu \ell^2 \xrightarrow{\text{odd}} 0,$$

also observe that

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2-a)^4} = \frac{i}{96\pi^2 a^2} \sim \frac{1}{M_W^4} \quad \text{and} \quad \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2-a)^4} = -\frac{i}{48\pi^2 a} \sim \frac{1}{M_W^2}$$

Because $S_{2,3}$ contain explicit $\frac{1}{M_W^2}$, one needs to account only for terms $\sim \ell^2 - \ell^4$ when computing them. (B, C-terms are suppressed by m_μ^2/M_W^2.) We shall take $a \equiv (1-x_1)M_W^2$.

Now
$$\Gamma_{\alpha\beta} = \epsilon^\lambda \left[-(k+2q)_\beta\, g_{\lambda\alpha} + (2k+q)_\lambda\, g_{\alpha\beta} + (q-k)_\alpha\, g_{\beta\lambda} \right]$$
$$= -\epsilon_\alpha (k+2q)_\beta + 2k\cdot\epsilon\, g_{\alpha\beta} + \epsilon_\beta (q-k)_\alpha$$

and
$$N^{\mu\nu} = \bar{u}_e(p-q)\, \gamma^\mu (\not{p} + \not{k}) \gamma^\nu (1-\gamma^5)\, u_\mu(p)$$

$$\gamma^\mu \not{\epsilon}\, \gamma_\mu = -2\not{\epsilon}$$

$$\Rightarrow S_1 = \Gamma_{\mu\nu} N^{\mu\nu} = \bar{u}_e(p-q)\left[-\not{\epsilon}(\not{p}+\not{k})(\not{k}+2\not{q}) - 4k\cdot\epsilon(\not{p}+\not{k}) + (\not{q}-\not{k})(\not{p}+\not{k})\not{\epsilon}\right](1-\gamma_5)\,u_\mu(p)$$

$$\text{use:}\quad \bar{u}_e(p-q)\,\not{q}\,\mathcal{O}\,u_\mu(p) = \bar{u}_e(q-p)\,(\underbrace{\not{p}}_{=m\,\to 0}-(\not{p}-\not{q}))\,\mathcal{O}\,u_\mu(p) \simeq \bar{u}_e(q-p)\,\not{p}\,\mathcal{O}\,u_\mu(p)$$

Additionally drop all terms that are scalar $\cdot\not{\epsilon}$ and also all terms that are linear in l (after $k \equiv l - x_1 p - x_2 q$) as well as $\sim q^2$ and $\sim \epsilon\cdot q$ - terms. Then

$$\not{\epsilon}\not{q}\not{p} = -\not{q}\not{\epsilon}\not{p} \simeq -\not{p}\not{\epsilon}\not{p} \simeq -2p\cdot\epsilon\not{p}$$

$\Rightarrow$ • $-\not{\epsilon}(\not{p}+\not{k})(\not{k}+2\not{q}) \mathrel{\hat=} \not{\epsilon}(\not{k}\not{p} + 2\not{q}\not{p} - k^2 + 2\not{q}\not{k}) \mathrel{\hat=} 2\epsilon\cdot p\,\not{p}\,(-2+2x_1+x_2)$

• $-4k\cdot\epsilon(\not{p}+\not{k}) \mathrel{\hat=} 4x_1 p\cdot\epsilon\,\not{p}(1-x_1-x_2) - 4\,l\cdot\epsilon\,\not{l} \mathrel{\hat=} 2x_1(1-x_1-x_2)\;2p\cdot\epsilon\,\not{p}$

• $(\not{q}-\not{k})(\not{p}+\not{k})\not{\epsilon} \mathrel{\hat=} (p^2 - k^2 + \underbrace{\not{p}\not{k} - \not{k}\not{p}}_{\not{p}\not{k}-\not{k}\not{p}})\not{\epsilon} \mathrel{\hat=} -2\not{q}\not{p}\not{\epsilon} \mathrel{\hat=} 0$

$$\Rightarrow S_1 = (-2+4x_1+x_2-2x_1(x_1+x_2))\,m_\mu\,\bar{u}_e(p-q)\;2\epsilon\cdot p\,(1+\gamma^5)\,u_\mu(p) \;+\; \not{\epsilon}\text{-terms}$$

$$S_2 = -\frac{1}{M_W^2}\,k^\nu N_{\mu\nu}\,k^\beta\,\Gamma^\mu{}_\beta$$

• $k^\nu N_{\mu\nu} = \bar{u}_e(p-q)\,\gamma_\mu(\not{p}+\not{k})\not{k}(1-\gamma^5)\,u_\mu(p)$

• $k^\beta\Gamma^\mu{}_\beta = -\epsilon^\mu(k^2+2k\cdot q) + 2k^\mu k\cdot\epsilon + k\cdot\epsilon(q-k)^\mu = -(k^2+2k\cdot q)\epsilon^\mu + k\cdot\epsilon\,(q+k)^\mu$

$$\Rightarrow S_2 = \frac{1}{M_W^2}\bar{u}_e(p-q)\left(\underbrace{\not{\epsilon}(\not{p}+\not{k})\not{k}}_{\mathcal{O}_1}(k^2+2k\cdot q) - \underbrace{(\not{q}+\not{k})(\not{p}+\not{k})\not{k}}_{\mathcal{O}_2}\,k\cdot\epsilon\right)(1-\gamma_5)\,u_\mu(p)$$

$$\mathcal{O}_1 = \not{\epsilon}(\not{p}\not{k}+k^2)(k^2+2k\cdot q) \mathrel{\hat=} -\not{\epsilon}\not{k}\not{p}(k^2+2k\cdot q)$$

$$\mathrel{\hat=} -\not{\epsilon}\not{l}\not{p}\left(-2(x_1p+x_2q)\cdot l + 2l\cdot q\right) + x_2\not{\epsilon}\not{q}\not{p}\,l^2$$

$$\mathrel{\hat=} l^2\,\underbrace{\not{\epsilon}\not{q}\not{p}}_{\to -2\epsilon\cdot p\not{p}}\left(x_2+\frac{2}{d}(x_2-1)\right) \mathrel{\hat=} -\frac{1}{2}(3x_2-1)\,l^2\,2\epsilon\cdot p\,\not{p}$$

$$\mathcal{O}_2 \triangleq -k\cdot\epsilon\,(\overset{\slashed{p}-\slashed{q}}{\cancel{\slashed{q}}}+\slashed{k})(\slashed{p}+\slashed{k})\slashed{k} \triangleq -k\cdot\epsilon\,(p+k)^2\,\overset{\slashed{\ell}-(x_1+x_2)\slashed{p}}{\slashed{k}}$$

$$\triangleq (x_1+x_2)\,\slashed{p}\;k\cdot\epsilon\left(k^2+2k\cdot p\right) - k\cdot\epsilon\,\slashed{\ell}\left(p^2+2p\cdot k+k^2\right)$$

$$\triangleq (x_1+x_2)\,\slashed{p}\left(\ell\cdot\epsilon - x_1 p\cdot\epsilon\right)\left(\ell^2-2(x_1p+x_2q)\cdot\ell+2\ell\cdot p\right) + x_1 p\cdot\epsilon\,\slashed{\ell}\left(2p\cdot\ell-2(x_1p+x_2q)\cdot\ell\right)$$

$$\triangleq (x_1+x_2)\slashed{p}\;p\cdot\epsilon\left(-\ell^2x_1+\frac{2}{d}\ell^2(1-x_1)\right) + \frac{2\ell^2}{d}x_1(1-x_1-x_2)\,p\cdot\epsilon\,\slashed{p}$$

$$\triangleq \frac{\ell^2}{2}\left[(1-3x_1)(x_1+x_2)+x_1(1-x_1-x_2)\right]p\cdot\epsilon\,\slashed{p} = \frac{\ell^2}{2}\left[(1-4x_1)(x_1+x_2)+x_1\right]p\cdot\epsilon\,\slashed{p}$$

$$= \frac{\ell^2}{2}\left(x_1(1-2x_1)-2x_1x_2+\tfrac{1}{2}x_2\right)2p\cdot\epsilon\,\slashed{p}$$

$$\Rightarrow\; S_2 = \frac{\ell^2}{M_W^2}\left(-\frac{1}{2}(3x_2-1)+\frac{1}{2}\left(x_1(1-2x_1)-2x_1x_2+\tfrac{1}{2}x_2\right)\right) m_\mu\,\bar{u}_e(p-q)\,2\,\epsilon\cdot p\,(1+\gamma^5)\,u_\mu(p)$$

$$+\ \slashed{\epsilon}\text{-terms}$$

$$S_3 = -\frac{1}{M_W^2}(k+q)^\mu(k+q)^\alpha\,\Gamma_\alpha{}^\nu\,N_{\mu\nu}$$

- $(k+q)^\mu N_{\mu\nu} = \bar{u}(p-q)\left((\slashed{k}+\slashed{q})(\slashed{p}+\slashed{k})\gamma_\nu(1-\gamma_5)\right)u_\mu(p)$
- $(k+q)^\alpha\Gamma_\alpha{}^\nu = (k+q)^\alpha\left(-\epsilon_\alpha(k+2q)^\nu + 2k\cdot\epsilon\, g_\alpha{}^\nu + \epsilon^\nu(q-k)_\alpha\right)$

$$= -k\cdot\epsilon\,(k+2q)^\nu + 2k\cdot\epsilon\,(k+q)^\nu + (q^2-k^2)\epsilon^\nu = k\cdot\epsilon\,k^\nu - k^2\epsilon^\nu$$

$$\Rightarrow\; S_3 = -\frac{1}{M_W^2}\bar{u}(p-q)\,(\slashed{k}+\slashed{q})(\slashed{p}+\slashed{k})\left(k\cdot\epsilon\,\slashed{k}-k^2\slashed{\epsilon}\right)(1-\gamma_5)\;u_\mu(p)$$

- $-k\cdot\epsilon\,(\slashed{k}+\slashed{q})(\slashed{p}+\slashed{k})\slashed{k} \;\triangleq\; \frac{1}{2}\left(x_1(1-2x_1)-2x_1x_2+\tfrac{1}{2}x_2\right)2p\cdot\epsilon\,\slashed{p}$
- $(\slashed{k}+\slashed{q})(\slashed{p}+\slashed{k})k^2\slashed{\epsilon} = (p+k)^2k^2\slashed{\epsilon} \triangleq 0$

$$\Rightarrow\; S_3 = \frac{\ell^2}{M_W^2}\frac{1}{2}\left(x_1(1-2x_1)-2x_1x_2+\tfrac{1}{2}x_2\right) m_\mu\,\bar{u}_e(p-q)\,2\,\epsilon\cdot p\,(1+\gamma^5)\,u_\mu(p)$$

$$\Rightarrow\; R = \Big[\overbrace{-2+4x_1+x_2-2x_1(x_1+x_2)}^{a(x_1,x_2)} + \frac{\ell^2}{M_W^2}\overbrace{\left(-\tfrac{3}{2}x_2+\tfrac{1}{2}+\ x_1(1-2x_1)-2x_1x_2+\tfrac{1}{2}x_2\right)}^{b(x_1,x_2)}$$

$$\times\, m_\mu\,\bar{u}_e(p-q)\,2\,\epsilon\cdot p\,(1+\gamma^5)\,u_\mu(p) + \slashed{\epsilon}\text{-terms}$$

A.

$$= \left(a(x_1,x_2) + \frac{\ell^2}{M_W^2}\, b(x_1,x_2)\right) m_\mu\, \bar{u}_e(p-q)\, 2\,\epsilon\cdot p\,(1+\gamma^5)\, u_\mu(p) + \not{q}\text{-terms}$$

- $a(x_1,x_2) = -2+4x_1+x_2-2x_1(x_1+x_2) = -2(1-x_1)^2 + x_2(1-2x_1)$
- $b(x_1,x_2) = -\frac{3}{2}x_2 + \frac{1}{2} + x_1(1-2x_1) - 2x_1x_2 + \frac{1}{2}x_2 = \frac{1}{2} + x_1(1-2x_1) - x_2(1+2x_1)$

Combining all results

$$T \simeq \frac{g^3 e}{4} \sum_i U_{ei}^* U_{\mu i}\, m_i^2\; 6 \int_0^1 dx_1 \int_0^{1-x_1} dx_2\, x_1 \int \frac{d^d\ell}{(2\pi)^d} \frac{R(\ell - x_1 p - x_2 q)}{(\ell^2-a)^4}$$

$$= \frac{3g^3 e}{2} \sum_i U_{ei}^* U_{\mu i}\, m_i^2\, \frac{i}{96\pi^2} \frac{1}{M_W^4} \int_0^1 dx_1 \int_0^{1-x_1} dx_2\, x_1 \left(\frac{a(x_1,x_2)}{(1-x_1)^2} - 2\,\frac{b(x_1,x_2)}{1-x_1} \right)$$

$$\times\, m_\mu\, \bar{u}_e(p-q)\cdot 2\epsilon\cdot p\,(1+\gamma_5)\, u_\mu(p) \; + \not{q}\text{-terms}$$

Hence

$$A = i\, \frac{g^2 e}{64\pi^2} \sum_i U_{ei}^* U_{\mu i}\, \frac{m_i^2 m_\mu}{M_W^4} \times \left(I_a + I_b\right)$$

with

$$I_a = \int_0^1 dx_1\, x_1 \int_0^{1-x_1} dx_2\, \frac{1}{(1-x_1)^2}\left(-2(1-x_1)^2 + x_2(1-2x_1)\right)$$

$$= \int_0^1 dx_1\, x_1 \left(-2(1-x_1) + \frac{1}{2}(1-2x_1)\right) = \int_0^1 dx \left(-\frac{3}{2}x + x^2\right) = -\frac{3}{4} + \frac{1}{3} = \frac{-5}{12}$$

$$I_b = -2\int_0^1 dx_1\, x_1 \int_0^{1-x_1} dx_2\, \frac{1}{(1-x_1)} \left(\frac{1}{2} - x_2(1+2x_1) + x_1(1-2x_1)\right)$$

$$-\tfrac{1}{2}(1+x_1-2x_1^2)$$

$$= -2\int_0^1 dx_1\, x_1 \left(\frac{1}{2} + x_1(1-2x_1) - \frac{1}{2}(1-x_1)(1+2x_1)\right)$$

$$= -2\int_0^1 dx \left(\frac{1}{2}x^2 - x^3\right) = -2\left(\frac{1}{4} - \frac{1}{3}\right) = \frac{1}{6}$$

Then $I_a + I_b = -\frac{5}{12} + \frac{1}{6} = -\frac{3}{12} = -\frac{1}{4} \Rightarrow$ $\underline{A = -\frac{i}{64\pi^2}\frac{g^2 e}{4}\sum_i U_{ei}^* U_{\mu i}\frac{m_i^2 m_\mu}{M_W^4}}$

Using notation $\sum_i U_{ei}^* U_{\mu i}\frac{m_i^2}{M_W^2} \equiv \delta_\nu$

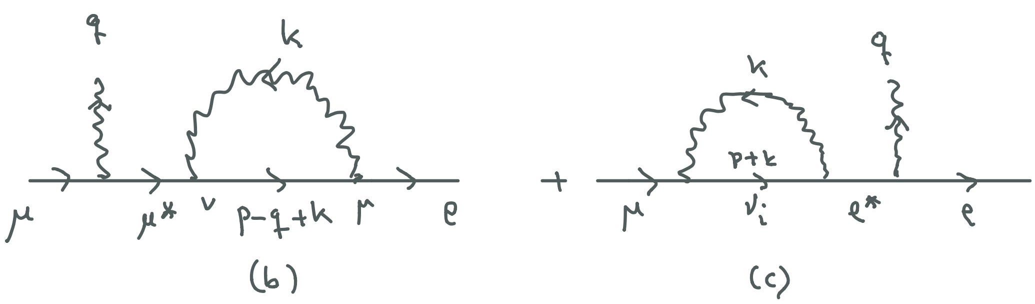
$$\Rightarrow A = -\frac{i}{64\pi^2}\frac{g^2 e}{4}\frac{m_\mu}{M_W^2}\delta_\nu = -\frac{i\sqrt{2}}{64\pi^2} e G_F m_\mu \delta_\nu \qquad \left(\frac{g^2}{8M_W^2} = \frac{G_F}{\sqrt{2}}\right)$$

$$\text{and } \Gamma_{\mu\to e\gamma} = \frac{m_\mu^3}{4\pi}|A|^2 = \frac{2}{(64\pi^2)^2}\frac{e^2}{4\pi} G_F^2 m_\mu^5 \delta_\nu^2 = \frac{\alpha}{32\cdot 64\pi^4} G_F^2 m_\mu^5 \delta_\nu^2$$

$$\Rightarrow \underline{Br_{\mu\to e\gamma}} \simeq \frac{\overset{\overset{3}{\cancel{96}}}{\cancel{192}}\pi^3}{\underset{32}{\cancel{64}}\cdot\cancel{2}\pi^4}\,\alpha\,\frac{G_F^2 m_\mu^5}{G_F^2 m_\mu^5}\delta_\nu^2 = \underline{\frac{3\alpha}{32\pi}\delta_\nu^2} \qquad \boxtimes$$

The End

A final note: There are two other terms in U-gauge, but these are $\not{q}$-terms, so they are not needed in our calculation.



$$\Rightarrow T_b + T_c = -ie\,\bar{u}_e(p-q)\left(\Sigma^{p-q}\frac{i}{\not{p}-\not{q}-m_\mu}\not{\epsilon} + \not{\epsilon}\frac{i}{\not{p}-m_e}\Sigma^{p}\right)u_\mu(p)$$

where $\Sigma^{q} = \#\,\not{q}\,P_L$ on general grounds

$$\cancel{Q}\text{-term}$$

$$\Rightarrow \underline{T_b + T_c} = -ie\# \bar{u}_e(p-q)\left((\cancel{p}-\cancel{q})P_L \frac{i}{\cancel{p}-\cancel{q}-m_\mu} \cancel{\epsilon} + \cancel{\epsilon} \frac{i}{\cancel{q}-m_e} \cancel{p} P_L \right) u_\mu(p) = \underline{\cancel{\epsilon}\text{-terms.}} \quad \blacksquare$$

The self energy is actually easily computed:

$$\Sigma_Q = \left(\frac{ig}{2\sqrt{2}}\right)^2 \sum_i U_{ei}^* U_{\mu i} \int \frac{d^dk}{(2\pi)^d} \underbrace{\gamma^\mu (1-\gamma_5) \frac{i}{\cancel{Q}+\cancel{k}-m_i} \gamma^\nu (1-\gamma^5)}_{} \frac{-i\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M_W^2}\right)}{k^2 - M_W^2}$$

$$= \gamma^\mu (1-\gamma^5) \frac{\cancel{Q}+\cancel{k}+m_i}{(Q+k)^2 - m_i^2} (1+\gamma_5)\gamma^\nu = 2\gamma^\mu \frac{\cancel{Q}+\cancel{k}}{(Q+k)^2-m_i^2} \gamma^\nu (1-\gamma^5)$$

again lowest order term drops and we get

$$\Sigma_Q = -\frac{ig^2}{4} \sum_i U_{ei}^* U_{\mu i} m_i^2 \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^\mu (\cancel{Q}+\cancel{k})\gamma^\nu}{[(Q+k)^2]^2 (k^2-M_W^2)} (1-\gamma_5) \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2}\right)$$

- $$\gamma^\mu (\cancel{Q}+\cancel{k})\gamma^\nu \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M_W^2}\right) = -2(\cancel{Q}+\cancel{k}) - \frac{1}{M_W^2} \cancel{k}(\cancel{Q}+\cancel{k})\cancel{k}$$

$$= -2(\cancel{Q}+\cancel{k}) - \frac{1}{M_W^2}\left(k^2\cancel{k} + 2k\cdot Q\cancel{k} - k^2 \cancel{Q}\right)$$

$$= -2(\cancel{Q}+\cancel{k}) - \frac{1}{M_W^2}\left((k^2 + 2Q\cdot k)\cancel{k} - k^2\cancel{Q}\right) \equiv \cancel{R}_Q$$

Feynman parametrization:

$$\frac{1}{[(Q+k)^2]^2 (k^2-M_W^2)} = \frac{\Gamma(3)}{\Gamma(2)\Gamma(1)} \int dx_1 dx_2\, \delta(x_1+x_2-1) \frac{x_1}{(\underbrace{x_1(Q+k)^2 + x_2(k^2-M_W^2)}_{})^3}$$

$$= k^2 + 2x_1 Q\cdot k + x_1 Q^2 - (1-x_1)M_W^2$$

$$= (k+x_1 Q)^2 + \underbrace{x_1(1-x_1)Q^2 - (1-x_1)M_W^2}_{\equiv a} \equiv l^2 - a^2$$

$l = k + xQ$

$k = l - xQ$

$$\Rightarrow \Sigma_Q = -\frac{ig^2}{8} \sum_i U_{ei}^* U_{\mu i} m_i^2 \cdot 2 \int_0^1 dx\, x \int \frac{d^4l}{(2\pi)^4} \frac{\cancel{R}_Q}{(l^2-a^2)^3} (1-\gamma_5)$$

Using symmetry and picking only leading terms in $\frac{m^2}{M_W^2}$;

$$-2(\not{Q}+\not{k}) \simeq 2\not{Q}(x-1)$$

$$-\frac{1}{M_W^2}\left(k^2(\not{k}-\not{Q}) + 2Q\cdot k\not{k}\right) \simeq -\frac{1}{M_W^2}\left((\ell^2-2x\ell\cdot Q)(\not{\ell}-x\not{Q}-\not{Q}) + 2Q\cdot\ell\not{\ell}\right)$$

$$\simeq -\frac{\ell^2}{M_W^2}\left(-(1+x)\not{Q} + \frac{2}{d}(1-x)\not{Q}\right) =$$

$$\simeq +\frac{\ell^2}{M_W^2}\left(\frac{1}{2}+\frac{3}{2}x\right)\not{Q}$$

$$\Rightarrow \int\frac{d^4\ell}{(2\pi)^4}\frac{\not{R}_a}{(\ell^2-a^2)^3} = \left(2(x-1)\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-a)^3} + \frac{1}{2}(1+3x)\int\frac{d^4\ell}{(2\pi)^4}\frac{\ell^2/M_W^2}{(\ell^2-a)^3}\right)\not{Q}$$

This is actually divergent! Needed to cancel divergences in $\not{\ell}$-terms in T_a.

A nice continuation would be to check that these divergences indeed cancel the one coming to $\not{p}$-terms from S_2 and S_3 ($\ell\cdot\epsilon\,\not{\ell}\,\ell^2$-terms). But not now ;).