3. Standard model, outline

We have now quite thoroughly studied certain simple quantum field theories: $\lambda\phi^4$-model, QED and also the Yukawa theory. All these models turn out to be a part of the standard model for elementary particle physics – one way or the other. In addition SM contains Yang-Mills fields which we encountered in connection with Faddeev-Popov gauge-fixing method.

That is, the basic symmetry group of the SM is

$$\boxed{[SU(2)_L \times U(1)_Y] \times SU(3)_c}$$

- $SU(2)_L$: refers to chirality of SU(2)-gauge interaction
- $U(1)_Y$: Hyper-charge (QED-like gauge-group)
- $SU(3)_c$: Colour-group
  - strong force, hadrons
  - Asymptotic freedom
    RGE: $\beta_{QCD} \to \alpha(Q)$
  - quark matter, URHIC

- Spontaneous symmetry breaking

$$SU(2)_L \times U_Y(1) \longrightarrow U(1)_{em} \quad \text{(QED)}$$

$$\uparrow$$

Higgs $\sim \lambda|\Phi|^4$

- Fermion masses $y\bar{\psi}_L \Phi \psi_R + h.c.$ (Yukawa)

- Renormalizability with SSB
- ABJ-anomalies $\Rightarrow N_c = 3$
- ...

The particle content of the SM is of course familiar from earlier courses:

Matter:

leptons $\begin{pmatrix} \nu_e \\ e \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}$ $\qquad 4\times6 = 24$

quarks $\begin{pmatrix} u \\ d \end{pmatrix}_i, \begin{pmatrix} c \\ s \end{pmatrix}_i, \begin{pmatrix} t \\ b \end{pmatrix}_i$ $\quad i = 1,2,3 \qquad 24\times3 = 72$

"Force":

$$(\gamma, \; W^\pm, Z) \qquad g \qquad\qquad 2 + \overset{(2\times3)}{3\times3} + 2\times8 =$$

photon $\sim U(1)_{em}$, $m_\gamma = 0$

weak bosons $\sim SU(2)_L$, massive

gluons $\sim SU(3)$ strong force. $m_g = 0$

"Higgs" $\quad H = \begin{pmatrix} \Phi^+ \\ \Phi^0 \end{pmatrix}$ complex doublet $\qquad 1 \; (4)$

- MSM contains 96 fermionic and 28 bosonic degrees of freedom.

- There are three independent gauge-couplings:

$$e, \; g, \; g_s$$

the Higgs potential contains 2 parameters

$$\mu^2, \; \lambda$$

and the quark and lepton mass matrices $m_{ij} \sim y_{ij} \bar{\psi} \Phi \psi$ contain

$$10 + 10 + (\text{Majorana d.o.f.:s in } \nu\text{-sector}) = 20+$$

free parameters.

Obviously the large number of parameters and not-so-tight group structure are somewhat less than compelling features. There are still some interesting connections, such as SSB: $SU(2)\times U(1) \to U(1)_{em}$ and the need for 3 quark colours to give anomaly cancellation in the weak sector (couples $SU(3)_c$ to model structure instead of having it just as an add-on). Most importantly however, SM just simply works.

Quite clearly the non-Abelian structures are crucial for SM. Let us now remind/recap the key ingredients of such models.

Yang-Mills theories – SU(N)-symmetry with N > 1.

Consider a Lagrangian

$$\mathcal{L}_{YM} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}(F^a_{\mu\nu})^2 \qquad (3.1)$$

where

$$D_\mu = \partial_\mu + igA^a_\mu t^a$$

$$F^a_{\mu\nu}\cdot t^a = \frac{i}{g}[D_\mu, D_\nu]$$
$$= (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu)\cdot t^a \qquad (3.2)$$

Of course

$$\boxed{[t^a, t^b] = if^{abc}t^c} \qquad (3.3)$$

See Peskin & Schröder 15.1 - 15.2 for an insightful derivation of (3.1) based on the principle of raising the global invariance of the free theory

$$\mathcal{L}_{free} = \bar{\psi}_i \not{\partial} \psi_i + \bar{\psi}_j m_{ji} \psi_i \quad ; \; i \in 1..N$$

under

$$\Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix} \rightarrow e^{i\alpha^a \frac{t^a}{2}} \Psi \qquad \alpha^a = \text{const} \qquad (3.4)$$

to a local one, where $\alpha^a = \alpha^a(x)$. The gauge-symmetry argument, complemented with Wilson's rge-analysis for irrelevant operators and C-and P-invariance* give (3.1) as a unique renormalizable (marginal theory).

The group generators obey normalizations

$$\mathrm{tr}[t_r^a t_r^b] \overset{\text{lie-algebra}}{=} C(r)\,\delta^{ab}$$

$$\underbrace{t_r^a t_r^a}_{t^{a2}\ \text{invariant}} = C_2(r) \cdot \underbrace{1}_{\text{dimension } r \text{ unit matrix.}} \qquad (3.5)$$

We will always put fermions to fundamental representation (dimension N) and gauge-fields to adjoint representation (dimension N^2-1 = number of generators). For these representations (first convention follows from $Tr(t_2^a t_2^a) = \frac{1}{2}\delta^{aa}$ with $t^a = \frac{\sigma^a}{2}$)

$$C(N) \equiv \frac{1}{2} \Rightarrow C_2(N) = \frac{N^2-1}{2N} \qquad (3.6)$$

---

* This can be relaxed, in fact to allow breaking of both P, C, and CP-invariance (first two by interaction structure and CP by mass

and

$$C_2(G) \equiv C(G) = N \qquad (3,7)$$

adjoint reps.

these results suffice for practical calculations in this course.

Box: Some details of Lie algebras for SU(N)-groups.

From the definition

antisymmetric!

$$[t^a, t^b] = i f^{abc} t^c$$

and the identity

$$[t^a,[t^b,t^c]] + [t^b,[t^c,t^a]] + [t^c,[t^a,t^b]] = 0$$

one finds

$$0 = i f^{bcd} \underbrace{[t^a,t^d]}_{if^{ade}t^e} + i f^{cad} \underbrace{[t^b,t^d]}_{if^{bde}t^e} + i f^{abd} \underbrace{[t^c,t^d]}_{if^{cde}t^e} \qquad \forall\, t^e$$

$$\Rightarrow \quad f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0 \qquad \text{(Jacobi)}$$

(3Box.1)

of generators

= # of traceless hermitean matrices in fundamental representation:

$$\psi \to U\psi = e^{i t\cdot\theta}\psi \quad : \quad \begin{array}{l} U^{-1} = U^\dagger \Rightarrow t^a = t^{a\dagger} \\ \det U = 1 \Rightarrow \operatorname{tr} t^a = 0 \end{array}$$

(3Box.2)

N-vector with $\psi_i \in \mathbb{C}$

Unitary $\det U = +1$ $N\times N$-matrix: $2N^2 - N^2 - 1 = N^2 - 1$ d.o.f.'s.

So, quite clearly

$$\underline{\#\text{ of generators in } SU(N) \; = \; N^2-1.}$$

(Prove that for $SO(N)$ # of gen $= N(N-1)/2$.)

Normalizations

Generator matrices merely span a N^2-1-dimensional space such that any U can be expressed in form (3Box.2). This set can be diagonalized and normalized arbitrarily:

$$\mathrm{Tr}\,(t^a t^b) = C(r)\,\delta^{ab} \qquad \text{(3Box.3)}$$

Where $C(r) = \mathrm{Tr}(t^{a2})$ can be chosen differently in different representations. In $SU(N)$ one customarily sets

$$C(N) \equiv \frac{1}{2} \qquad \text{Fundamental reps.} \qquad \text{(3Box.4)}$$

(This follows from the convention with $SU(2)$-spin group, where $t^a \equiv \frac{1}{2}\sigma^a \Rightarrow C(2) = \mathrm{Tr}\left(\frac{1}{2}\sigma^a\right)^2 = \frac{1}{4}\cdot 2 = \frac{1}{2}$.)

Note that given the normalization $C(r)$ and the generators (some suitable set), the structure functions can be computed from

$$f^{abc} = -\frac{i}{C(r)}\,\mathrm{Tr}\left([t^a,t^b]\,t^c\right) \qquad \text{(3Box.5)}$$

These are unique (up to normalization) of course. The structure constants actually define the adjoint representation G

$$(t^b_G)_{ac} \equiv i f^{abc} \qquad \text{(3Box.6)}$$

- These are Hermitean matrices:

$$(t^b)^\dagger_{ac} = (t^b_{ca})^* = -if^{cba} = if^{abc} = (t^b)_{ac},$$

- there are N^2-1 of them, and from the Jacoby-identity (3Box.1) one finds that (after some manipulation of indices)

$$if^{ead}\, if^{dbc} - if^{ebd}\, if^{dac} = +if^{abc}\, if^{edc}$$

$$\Leftrightarrow \left(t_G^a t_G^b - t_G^b t_G^a\right)_{ec} = if^{abd}\left(t_G^d\right)_{ec} \Leftrightarrow [t^a_G, t^b_G] = if^{abc} t^c_G$$

These are the two most important representations for us.

Fermions → Fund. reps. $d_F = N$

Gauge-fields → Adj. reps. $d_G = N^2-1$.

↑ dimension of the representation.

Now back to normalizations. In calculations one often encounters the operator

$$t_r^2 \equiv \sum_a t^a_r t^a_r \qquad \text{(3Box.7)}$$

Clearly $[t_r^2, t^a] = 0$ for all a, so t_r^2 is an invariant. (this is the analog of the total angular momentum operator in SU(N) $N > 2$).
Clearly t^2 must be $\sim \mathbb{1}_r$ so we set

$$t_r^2 = C_2(r)\, \mathbb{1}_r \qquad \text{(3Box.8)}$$

But then, taking the trace we find

$$\mathrm{Tr}\, t_r^2 = C_2(r)\, \dim(r) = \underbrace{\sum_a}_{d_G \text{ elements}} \mathrm{tr}(t^{a2}) = d_G \cdot C(r)$$

This immediately implies the relation

$$C_2(r) = C(r)\,\frac{d_G}{\dim(r)} \qquad (3\text{Box}.9)$$

Thus for our choice (3Box. 4) we find

$$C_2(N) = \frac{N^2-1}{2N} \qquad (3\text{Box}.10)$$

in fundamental representation. This completes derivation of eqn. (3.6). We still need the normalization factors for the Adjoint representation. These can be found by considering product representations:
In general any product can be broken to a direct sum of irreducible reps:

$$r_1 \otimes r_2 = \sum_{\oplus} \tilde{r}_i \qquad (3\text{Box}.11)$$

examples:

$SU(2)$: $2 \otimes 2 = 3 \oplus 1$

$SU(3)$: $3 \otimes \bar{3} = 8 \oplus 1$

$SU(N)$: $N \otimes \bar{N} = (N^2-1) \oplus 1$ (3Box. 12)

For $N \geq 3$ fundamental reps. is complex $\Rightarrow \bar{N} \neq N$.

Then considering the product generator:

$$t^a_{r_1 \otimes r_2} = t^a_{r_1} \otimes 1_{r_2} + 1_{r_1} \otimes t^a_{r_2} \qquad (3.\text{Box}13)$$

and

$$(t^a_{r_1 \otimes r_2})^2 = (t^a_{r_1})^2 \otimes 1_{r_2} + 2\,t^a_{r_1} \otimes t^a_{r_2} + 1_{r_1} \otimes (t^a_{r_2})^2$$

$$\Rightarrow \operatorname{Tr}(t^a_{r_1 \otimes r_2})^2 = (C_2(r_1) + 0 + C_2(r_2))\dim(r_1)\dim(r_2)$$

$$= \sum_{r_i} C_2(\tilde{r}_i) \prod_{r_i} \dim(\tilde{r}_i) \qquad (3\text{Box}.14)$$

Using (3Box.14) on $r_1 = N$, $r_2 = \bar{N}$ one finds (with 3Box.12)

$$2\,\frac{N^2-1}{2N}\cdot N^2 = C_2(G)\cdot(N^2-1) + 0$$

$$\Rightarrow \quad \underline{C_2(G) = N} \qquad (3Box.15)$$

From this it follows that

$$\underline{f^{acd}f^{bcd}} = +if^{acd}\,if^{dcb} = (t^c_G)_{ad}\,(t^c_G)_{db}$$

$$= \sum_c (t^c_G)^2_{ab} \doteq \underline{C_2(G)\,\delta_{ab}} = \underline{N\,\delta_{ab}}. \qquad (3Box.16)$$

Moreover, going backwards with the equation (3Box.9) with the adjoint representation we find that:

$$C(G) = \frac{\dim(G)}{d_G}\,C_2(G) = C_2(G) = N. \qquad (3Box.17)$$

This completes derivation of (3.7). Not surprisingly, written in terms of f^{abc}'s (3Box.17) reduces again to (3Box.16).

Physical space in Y-M- theories.

We've already seen that gauge-fixing procedure in YM- theories introduces new spurious d.o.f's, the ghosts. In QED the Ward identities insured that nonphysical states did not contribute to physical processes (S-matrix). In the case of YM-theories this issue is clearly more complicated. The issue is most easily studied in terms of the BRST- transformation.

BRST- transformation

After Fadeev-Popov gauge fixing the YM-theory (3.1) becomes

$$\mathcal{L} = \underbrace{\bar\Psi(i\not\partial - m)\Psi - \frac{1}{4}(F^a_{\mu\nu})^2}_{\mathcal{L}_{YM,\,classic}} \underbrace{- \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2}_{\mathcal{L}_{gf}} + \underbrace{\bar c^a(\partial^\mu D^{ab}_\mu)c^b}_{\mathcal{L}_{ghost}} \qquad (3.8)$$

where the ghost field derivative is

$$D^{ab}_\mu \equiv \delta^{ab}\partial_\mu - g f^{abc} A^c_\mu . \qquad (3.9)$$

The goal of FP- gauge fixing is to remove the effect of the unphysical gauge d.o.f:s from the physical amplitudes. In QED this removal of contribution from longitudinal polarization modes was effected by the Ward-identity. In QCD the situation is not quite as clear.

The trouble-making $\underbrace{\text{polarization}}_{\text{(light-like, longitudinal)}}$ states can be parametrized as follows

$$\epsilon^+_\mu(k) = \frac{1}{\sqrt{2}|k|}(k^0, \vec k) \; ; \quad \epsilon^-_\mu(k) = \frac{1}{\sqrt{2}|\vec k|}(k^0, -\vec k) \qquad (3.10)$$

"forward" polarization $\qquad$ "backward" polarization

$k^\mu \epsilon^-_\mu = 0$

We thus have the relations, (listed for completeness only)

$$\epsilon^T_i \cdot \epsilon^T_j = -\delta_{ij} \;;\quad \epsilon^{\pm} \cdot \epsilon^T_j = 0$$

$$(\epsilon^+)^2 = (\epsilon^-)^2 = 0 \qquad \epsilon^+ \cdot \epsilon^- = 1$$

and

$$g_{\mu\nu} = \epsilon^-_\mu \epsilon^{+*}_\nu + \epsilon^+_\mu \epsilon^{-*}_\nu - \sum_i \epsilon^T_{i\mu} \epsilon^{T*}_{i\nu} \tag{3.11}$$

Now, it is advantageous to rewrite the gf-term in the Lagrangian as

$$e^{i\int d^4x\left[-\frac{1}{2\xi}(\partial^\mu A^a_\mu)^2\right]} = \int \mathcal{D}B^a \, e^{i\int d^4x\left[\frac{\xi}{2}(B^a)^2 + B^a \partial^\mu A^a_\mu\right]}$$

(auxiliary field)

where B^a is an SU(N) vector but a Lorentz-scalar. (c-number field. Then, one can show that the new Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{YM-classic}} + \mathcal{L}_{\text{ghost}} + \frac{\xi}{2}(B^a)^2 + B^a(\partial^\mu A^a_\mu) \tag{3.12}$$

is invariant under BRST-transformation:

$$\begin{aligned}
\delta A^a_\mu &= \epsilon D^{ab}_\mu c^b \\
\delta \psi &= ig\epsilon c^a t^a \psi \\
\delta c^a &= -\tfrac{1}{2} g\epsilon f^{abc} c^b c^c \\
\delta \bar{c}^a &= \epsilon B^a \\
\delta B^a &= 0
\end{aligned} \tag{3.13}$$

Proof. Exercise.

Now define BRST transformation of a field $\phi = \{A_\mu, \psi, c, \bar{c}, B\}$

$$\delta\phi = \epsilon Q \phi$$

according to (3.13). That is $Q A_\mu^a = D_\mu^{ab} c^b$. It turns out that Q is nilpotent. For example

$$\begin{aligned} Q^2 A_\mu^a &= Q\, D_\mu^{ab} c^b \\ &= -g f^{abc} (Q A_\mu^c) c^b + D_\mu^{ab}\, Q c^b \\ &= \ldots\ldots = 0 \end{aligned} \qquad (3.14)$$

Similarly one can prove that $Q^2\phi = 0$ for all fields.

One can furthermore show that Q commutes with the Hamiltonian. (3.13) is a continuous symmetry* and Q turns out to correspond to the integral of the time component of this current $\Rightarrow$ it is a conserved charge).

Now, a nilpotent operator that commutes with H divides the Hilbert-space of H into three groups

$$\mathcal{H}_1 = \{ |\psi_1\rangle \mid Q|\psi_1\rangle \neq 0 \}$$

$$\mathcal{H}_2 = \{ |\psi_2\rangle \mid |\psi_2\rangle = Q|\psi_1\rangle \}$$ note that $Q|\psi_2\rangle = 0$ then.

$$\mathcal{H}_0 = \{ |\psi_0\rangle \mid Q|\psi_0\rangle = 0,\ |\psi_0\rangle \notin \mathcal{H}_2 \} \qquad (3.15)$$

Now, it can be proven that the physical polarization states

---

* BRST-symmetry is a Quantum generalization of the gauge-symmetry.

span the space $\mathcal{H}_0$ while unphysical polarizations and ghosts belong to $\mathcal{H}_1$ and $\mathcal{H}_2$. (In the single-particle limit, for $g=0$ one can see from (3.13) that $\begin{cases} \epsilon^+ \text{ and } \bar{c}^a \in \mathcal{H}_1 \\ \epsilon^- \text{ and } c \in \mathcal{H}_2. \end{cases}$
It is harder to prove that physical states belong to $\mathcal{H}_0$.)

If we believe that $\mathcal{H}_1 = \{\text{physical states}\}$ then we can prove that the S-matrix, projected onto $\mathcal{H}_0$ is unitary:

$$\langle A, tr | \underbrace{S^+ S}_{=1,\ \text{full space}} | B, tr \rangle \qquad ; \qquad \hat{Q}|A, tr\rangle = 0$$

$$\hat{Q} S |A, tr\rangle = 0$$

↑ $\hat{Q}$ commutes with Hamilton.

$$\Rightarrow S|A, tr\rangle \in \mathcal{H}_0 \oplus \mathcal{H}_2$$

$$= \langle A, tr | S^+ \sum_{C} |C\rangle\langle C| S |B, tr\rangle$$

↖ all states.

$$= \langle A, tr | S^+ \left(\sum_{C} |C, tr\rangle\langle C, tr| + \sum_{C_2} |C_2\rangle \underbrace{\langle C_2|\right) S |B, tr\rangle}$$

$$\sum_{\alpha} \langle C_2 | \overset{\to 0}{\alpha, tr}\rangle + \sum_{\beta_2} \langle C_2 | \beta_2 \rangle$$

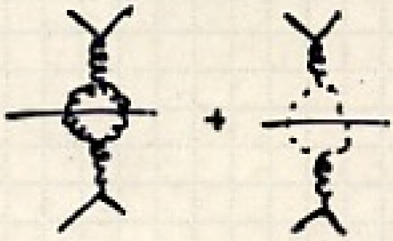
$$= 0 + 0 \ !$$

where I used

$$\langle C_2 | \beta_2 \rangle = \langle C_1 | Q | \beta_2 \rangle = 0 \ !$$

$$= \sum_{C} \langle A, tr | S^+ | C, tr\rangle \langle C, tr | S | B, tr\rangle \qquad (3.16)$$

This result proves that the unphysical polarizations and ghosts cancel each others to all orders in the perturbation theory. For example

 = just physical, to all orders.

BRST - invariance can be used to derive relations between green functions of the theory similar to the Ward- identities in the QED. This is somewhat more tedious however, and we shall not do it here. (See for example Cheng & Li sect. 9,3).

Renormalized Yang-Mills Lagrangean

Starting from bare Lagrangian (3.2):

$$\mathcal{L}_{YM} = -\frac{1}{4}(dA_0^a)^2 - \frac{1}{2}g_0\, dA_0^a [A_0\, A_0]^a - \frac{g_0^2}{4}[A_0\, A_0]^2$$

$$-\frac{1}{2\xi_0}(\partial A_0^a)^2 - \bar{c}_0^a \partial^2 c_0^a - g_0 \bar{c}_0^a \partial^\mu f^{abc} A_\mu^b c^c$$

$$+\bar{\psi}_0 (i\partial\!\!\!/ + m_0 + g_0 A\!\!\!/_0^a \cdot t^a)\psi_0 \qquad (3.17)$$

QED:ssä $gt^a \to -e$. (elektronille)

where I used the shorthand notations

$$dA_0^a \equiv \partial_\mu A_{0\nu}^a - \partial_\nu A_{0\mu}^a$$

$$[A_0 A_0]^a \equiv f^{abc} A_{0\mu}^b A_{0\nu}^c . \qquad (3.18)$$

Defining the renormalized fields

$$\begin{cases} A_0^{a,\mu} = \sqrt{Z_3}\, A^{a,\mu} \\ c_0^a \equiv \sqrt{Z_2^c}\, c^a \\ \psi_0 \equiv \sqrt{Z_2}\, \psi \end{cases} \qquad (3.19)$$

and the renormalized mass $m \equiv Z_2 m_0 + \delta_m$ (3.20)

(3.17) becomes ($\xi_0 Z_3^{-1} \equiv \xi$):

$$\boxed{\begin{aligned} \mathcal{L}_{YM}^{ren} \; & - \frac{1}{4}\underline{\delta_3}(dA)^2 - \underline{\delta_2^c}\,\bar{c}\,\partial^2 c + \bar{\psi}(i\underline{\delta_2}\,\partial\!\!\!/ - \underline{\delta_m})\psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \\ & - \frac{1}{2}\underline{g\delta_1^{3g}}\, dA\,[A\,A] - \frac{1}{4}g^2\underline{\delta_1^{4g}}\,[A\,A]^2 - \underline{g\delta_1^c}\,\partial^\mu f^{abc} A_\mu^b c^c \\ & + \underline{g\delta_1}\,\bar{\psi} A\!\!\!/^a t^a \psi \end{aligned}} \qquad (3.21)$$

Note that in addition to the usual wfr- and electron mass counterterms there is 4 different coupling constant counterterms δ_1, δ_1^c, δ_1^{3g} and δ_1^{4g}. However, if renormalization does not spoil the gauge symmetry, they must all be related. Indeed, if we assume $\mathcal{L}^{ren}_{YM}$ only contains one coupling g (instead of four $g_{\psi\psi A}$, g_{A^3}, g_{A^4} and g_{ccA}) we must have

$$g_0 Z_2 Z_3^{1/2} = Z_1 g \quad \Rightarrow \quad \frac{g_0}{g} = \frac{Z_1}{Z_2\sqrt{Z_3}}$$

$$g_0 Z_3^{3/2} = Z_1^{3g} g \quad \Rightarrow \quad \delta_1^{3g} = \frac{g_0}{g} Z_3^{3/2} - 1$$

$$g_0^2 Z_3^2 = Z_1^{4g} g^2 \quad \Rightarrow \quad \delta^{4g} = \frac{g_0^2}{g^2} Z_3^2 - 1$$

$$g_0 Z_2^c Z_3^{1/2} = Z_1^c g \quad \Rightarrow \quad \delta_1^c = \frac{g_0}{g} Z_2^c Z_3^{1/2} - 1 \qquad (3,22)$$

That is all 8 counter-terms are expressed in terms of 5 divergent quantities arising from (3.19) and (3.20) + (3.21). These relations can be expressed also as follows:

$$\frac{Z_1}{Z_2} = \frac{Z_1^c}{Z_2^c} = \frac{Z_1^{3g}}{Z_3} = \frac{\sqrt{Z_1^{4g}}}{\sqrt{Z_3}} \qquad (3.23)$$

Thes Slavnov-Taylor identities are a generalization of the QED-Ward-identities to the case of nonabelian symmetry. They are special cases of general relations among YM-Green functions and they are most easily derived using the invariance of the classical path integral generating function under a BRST-transformation. Relations (3,23) will guarantee that

The β-functions derived from each of the four vertices separately are identical, which then ensures that each renormalized coupling runs exactly the same way. $\Rightarrow$ universality of couplings is preserved.

β-function in SU(N)

As a result of the ST-identities we can compute the β-function for example from the $\bar{\psi}A\psi$-vertex as in QED. Due to more complicated interaction structure the computation is more tedious however. On the basis of the section (take the massless limit $m_q \to 0$) we expect that the invariant charge is:

$\sim G$

$$\boxed{\begin{aligned} g(Q) &= G(Q, g, \mu) \\ &= G(Q, \bar{g}, \bar{\mu}) \end{aligned}} \qquad (3.24)$$

where $\bar{\mu} = e^t \mu$ and

$$\boxed{\begin{aligned} &\frac{\partial \bar{g}}{\partial t} = \beta(\bar{g}) \\ &\text{with} \quad \beta = \mu \frac{\partial g}{\partial \mu} \end{aligned}} \qquad (3.25)$$

Here $g = g_0 \mu^{-\epsilon/2} \cdot Z_2 \sqrt{Z_3} Z_1^{-1}$

$$\Rightarrow \beta(g) = -\frac{\epsilon}{2} \frac{g}{1 - g\frac{\partial}{\partial g} \log \frac{Z_2\sqrt{Z_3}}{Z_1}} \approx$$

$$\approx -\frac{\epsilon}{2} g \left(1 + g \frac{\partial}{\partial g}\left(\delta_2 - \delta_1 + \frac{1}{2}\delta_3\right)\right) \qquad (3.26)$$

unlike in QED we need all counter terms.
ST-identities merely ensure universality.

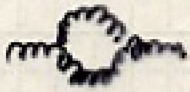
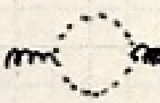
That is, we need to renormalize the fermion and gauge propagators and the $\bar{\psi}A\psi$-vertex. To one loop order:

1) Fermion $(-i\Sigma)$

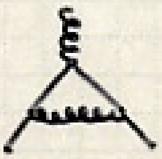
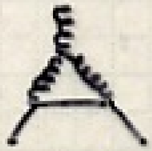
 $+ \; i(\delta_2 \not{p} - \delta_m) \; = \text{finite}$

δ_m: $\emptyset$ in the massless limit.

2) Gauge-boson $(+i\Pi)$

 $+$  $+$  $+$  $\underbrace{}_{-i(g_{\mu\nu}q^2 - q_\mu q_\nu)\delta_3} = \text{finite}$

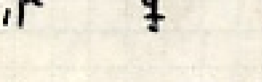
3) Vertex: $(-i\Gamma)$

 $+$  $- ig\frac{t^a}{2}\gamma^\mu \delta_1 \; = \text{finite}$

We will be working in the Feynman gauge and in the $\overline{\text{MS}}$-scheme. Since we are only interested in the β-function here, we only compute the pole terms of each function.

- **SU(N)-feynman rules**

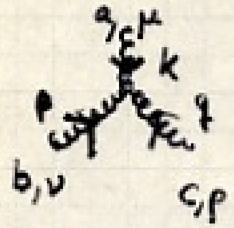
 $-i\frac{\delta^{ab}}{q^2+i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right)$ (ξ: 1 in F-gauge)

 $\frac{i}{\not{p}-m}\delta^{ij}$

 $\frac{i}{p^2+i\epsilon}\delta^{ab}$

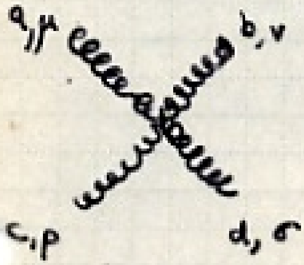
 $\sim +ig\, t^a_{ij}\gamma^\mu$

$$+ \underline{g f^{abc} k_\mu}$$; careful here with conventions!



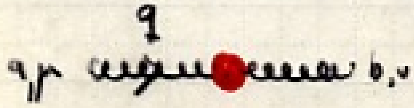
$$g f^{abc} \left((q_\nu - k_\nu) g_{\mu\rho} + (k_\rho - p_\rho) g_{\mu\nu} + (p_\mu - q_\mu) g_{\nu\rho} \right)$$

$$\equiv \underline{g f^{abc} V_{\mu\nu\rho}} (k, p, q)$$

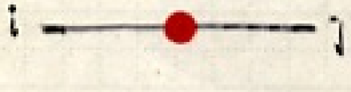


$$= -i g^2 \left[f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \right] \equiv \underline{-i g^2 W^{abcd}_{\mu\nu\rho\sigma}}$$

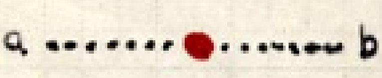
And the counter-terms:



$$-i (g^{\mu\nu} q^2 - q^\mu q^\nu) \delta^{ab} \delta_3$$

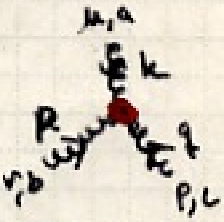


$$i (\not{p} \delta_2 - \delta_m)$$

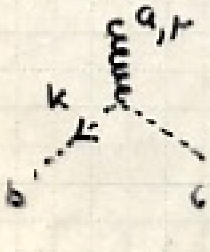


$$i p^2 \delta_2^c$$

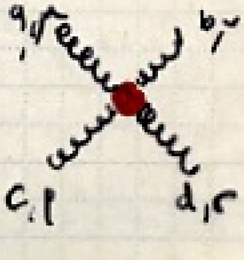
$$i g t^a_{ij} \gamma^\mu \delta_1$$



$$g f^{abc} V_{\mu\nu\rho} \delta_1^{3g}$$

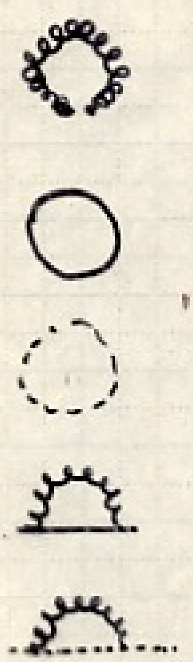


$$g f^{abc} k_\mu \delta_1^c$$



$$-i g^2 W^{abcd}_{\mu\nu\rho\sigma} \delta_1^{4g}$$

In addition one must remember the statistics-sign-rules for the loop integrals:

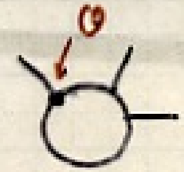


$$+\int\frac{d^dk}{(2\pi)^d}\cdots$$

$$\ominus\int\frac{d^4k}{(2\pi)^4}$$

$$\ominus\int\frac{d^4k}{(2\pi)^4}$$

$$\left.\right\}+\int\frac{d^4k}{(2\pi)^4}$$



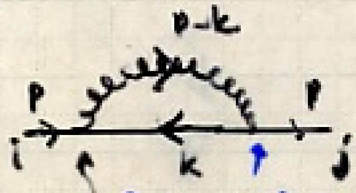
$$\sim \bar\psi_\alpha \mathcal{O}^{\alpha\beta}\overbrace{\psi_\beta \bar\psi_\gamma}^{S_{\beta\gamma}}\mathcal{O}^{\gamma\delta}\cdots\cdots\mathcal{O}^{\delta\epsilon}\psi_\epsilon$$

sum over all chiral indices $\Rightarrow \mathrm{Tr}[S\mathcal{O}\ldots]$

jump over $2n-1$ fields $\Rightarrow (-1)^{2n+1} = -1$

Armed with these rules we can eventually attack the 1-loop diagrams on p. ( ).

1. Fermion



r = dimension of the fermion reps. usually N.

$$-i\Sigma_2 = (ig)^2\, \overbrace{t^a_{ik}t^a_{kj}}^{=C_2(r)\delta_{ij}}\int\frac{d^4k}{(2\pi)^4}\,\overbrace{\gamma^\mu\frac{i\cancel{k}}{k^2}\gamma_\mu}^{-2i\cancel{k}}\,\frac{-i}{(k-p)^2}$$

$$= +\,2g^2C_2(r)\gamma^\mu\underbrace{\int\frac{d^4k}{(2\pi)^4}\frac{k_\mu}{k^2(k-p)^2}}_{=\frac{i}{16\pi^2}\frac{1}{\epsilon_{\overline{MS}}}p_\mu+\text{finite}}$$

(see next page)

$$= \frac{ig^2}{16\pi^2}C_2(r)\,\cancel{p}\,\frac{2}{\epsilon_{\overline{MS}}} + \text{finite} \qquad (3.27)$$

Including the counter-term we then have:

$$-i\Sigma^{full} = -i\Sigma^{loop} + i\,\delta_2^{\overline{MS}}\,\cancel{p}$$

$$= i\not p\left(\frac{g^2 C(r)}{8\pi^2}\frac{1}{\epsilon_{\overline{MS}}} + \delta_2^{\overline{MS}} + \text{finite}\right) \Rightarrow \boxed{\delta_2^{\overline{MS}} = -\frac{g^2}{8\pi^2}C(r)\frac{1}{\epsilon_{\overline{MS}}}} \quad (3.28)$$

It is useful to collect first some specific integrals that show up all the time along the calculation:

Using:

- $$\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^2} = \frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\Delta^{-\epsilon/2} = \frac{i}{16\pi^2}\frac{2}{\epsilon_{\overline{MS}}} + \text{finite} \quad (3.29)$$

- $$\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta)^2} = -\frac{i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(1-\frac{d}{2})}{\Gamma(1)}\Delta^{1-\frac{\epsilon}{2}} = +\frac{i}{16\pi^2}\frac{4}{\epsilon_{\overline{MS}}}\Delta + \text{finite} \quad (3.30)$$

- $$\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell_\mu\ell_\nu}{(\ell^2-\Delta)^2} = +\frac{i}{16\pi^2}\frac{1}{\epsilon_{\overline{MS}}}\Delta\cdot g_{\mu\nu} + \text{finite} \quad (3.31)$$

One can show that

- $$\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k-q)^2} = \int dx\int\frac{d^dk}{(2\pi)^d}\frac{1}{\left((k-qx)^2+x(1-x)q^2+i\epsilon\right)^2} \quad \boxed{;\ \ell \equiv k-qx \quad \Delta \equiv -x(1-x)q^2}$$

$$= \int dx\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^2} = \frac{i}{16\pi^2}\cdot\frac{2}{\epsilon_{\overline{MS}}} + \text{finite} \quad (3.32)$$

- $$\int\frac{d^dk}{(2\pi)^d}\frac{k^2}{k^2(k-q)^2} = 0 \quad (3.33)$$

- $$\int\frac{d^dk}{(2\pi)^d}\frac{k^\mu}{k^2(k-q)^2} = \int dx\int\frac{d^d\ell}{(2\pi)^d}\frac{xq^\mu}{(\ell^2-\Delta)^2} = \frac{i}{16\pi^2}\frac{1}{\epsilon_{\overline{MS}}}q^\mu \quad (3.34)$$

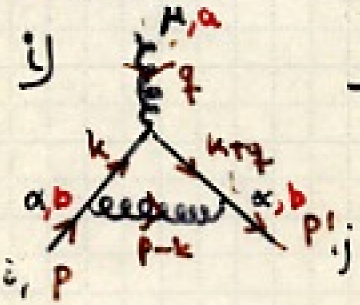
- $$\int\frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu}{k^2(k-q)^2} = \int dx\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu\ell^\nu + x^2q^\mu q^\nu}{(\ell^2-\Delta)^2}$$

$$= \frac{i}{16\pi^2}\frac{1}{\epsilon_{\overline{MS}}}\int dx\left(-x(1-x)q^2 g_{\mu\nu} + 2x^2 q^\mu q^\nu\right) + \text{finite}$$

$$= \frac{i}{16\pi^2}\frac{1}{\epsilon_{\overline{MS}}}\left(-\frac{1}{6}q^2 g_{\mu\nu} + \frac{2}{3}q^\mu q^\nu\right) + \text{finite} \quad (3.35)$$

These integrals turn out to be enough for computing $\beta_{SU(N)}$ at 1 loop

3.5 Vertex



$$-i\Gamma^{(1)} = (ig)^3\int\frac{d^4k}{(2\pi)^4}\frac{-i}{(p-k)^2}\gamma^\alpha t^b_{jn}\frac{i}{\not k+\not q}\gamma^\mu t^a_{nl}\frac{i}{\not k}\gamma_\alpha t^b_{lj}$$

$$= +g^3\left(t^b t^a t^b\right)_{ij}\mu^\epsilon\underbrace{\int\frac{d^dk}{(2\pi)^d}}_{\to i d^dk_E}\frac{\gamma^\alpha\not k\gamma^\mu\not k\gamma_\alpha}{\cancel{k^2}(k+q)^2(p-k)^2}$$

- $\gamma^\alpha\not k\gamma^\mu\not k\gamma_\alpha = -(2-\epsilon)\not k\gamma^\mu\not k = +(2-\epsilon)\left(k^2\gamma^\mu - \overbrace{2k^\mu\not k}^{=\frac{2}{d}k^2\gamma^\mu}\right)$

  $\to \underbrace{\frac{(2-\epsilon)^2}{d}}_{\hookrightarrow 1}\cancel{k^2}\gamma^\mu$

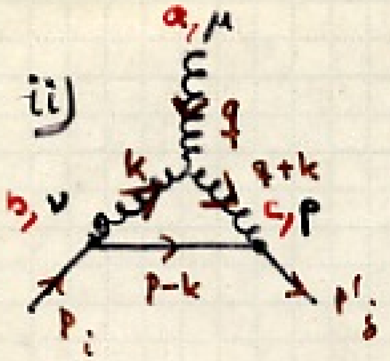
- $t^b t^a t^b = t^b[t^a,t^b] + t^b t^b t^a$

  $= i t^b f^{abc}t^c + C(r)t^a$

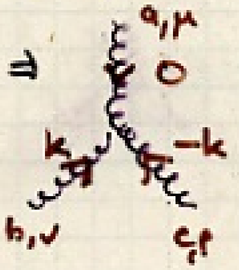
  $= \underbrace{\frac{i^2}{2}f^{abc}f^{bcd}}_{= -\frac{1}{2}C_2(G)\delta^{ad}}t^d + C(r)t^a = \left[C(r) - \frac{1}{2}C_2(G)\right]t^a$

$$\Rightarrow -i\Gamma^{(1)} \approx igt^a\gamma^\mu\left(g^2\left[C(r)-\frac{1}{2}C_2(G)\right]\underbrace{\mu^\epsilon\int\frac{d^dk_E}{(2\pi)^d}\frac{1}{(k+q)^2(p-k)^2}}_{\frac{i}{16\pi^2}\frac{2}{\epsilon_{\overline{MS}}}}\right)$$

$$\approx igt^a\gamma^\mu\left(\frac{-g^2}{16\pi^2}\left(C(r)-\frac{1}{2}C_2(G)\right)\frac{2}{\epsilon_{\overline{MS}}} + \text{finite}\right) \quad (3.36)$$

ii) 

Here neglect all momenta except k in the nominator:

$$\Rightarrow$$  $$\approx g f^{abc}\left(-k^\nu g^{\mu\rho} - k^\rho g^{\mu\nu} + 2k^\mu g^{\nu\rho}\right)$$

$$\Rightarrow -i\Gamma^{(ii)} \approx (ig)^2 g f^{abc}\int \frac{d^4k}{(2\pi)^4}\frac{-i}{k^2(k+q)^2(p-k)^2}\left(-k^\nu g^{\mu\rho} - k^\rho g^{\mu\nu} + 2k^\mu g^{\nu\rho}\right)$$

$$\times \underbrace{\gamma_\nu t^c_{jn}\, i(\not p-\not k)\,\delta_{nk}\,\gamma_\rho t^b_{ki}}_{=-i(t^c t^b)_{ji}\,\gamma_\nu \not k \gamma_\rho}$$

$$+ g^3 \underbrace{f^{abc} t^c t^b}_{\substack{=-\frac{i}{2} f^{abc}f^{abd}t^d \\ =-\frac{i}{2}C_2(G)t^a}} \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k+q)^2(p-k)^2}\underbrace{\left(\gamma_\nu \not k\gamma_\rho\left(2k^\mu g^{\nu\rho} - k^\nu g^{\mu\rho} - k^\rho g^{\mu\nu}\right)\right)}_{\substack{=-4k^\mu \not k - 2\gamma^\mu k^2 \\ \to -(2+\frac{4}{d})\gamma^\mu k^2}}$$

$$\approx +ig^3 t^a \gamma^\mu\left(\frac{3}{2}C_2(G)\int\frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(k+q)^2(p-k)^2}\right)$$

$$\approx +ig t^a\gamma^\mu\left(\frac{g^3}{16\pi^2}\cdot\frac{3}{2}C_2(G)\frac{2}{\epsilon_{\overline{MS}}} + \text{finite}\right) \qquad (3.37)$$

Putting all vertex corrections together then

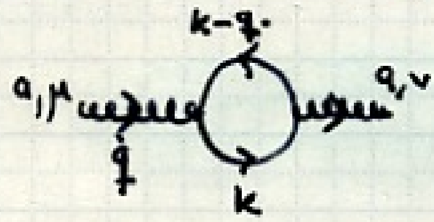
$$-i\Gamma_{full} = +igt^a\gamma^\mu\left(1 + \frac{g^2}{16\pi^2}\left(C(r)\overbrace{-\frac{1}{2}C_2(G) + \frac{3}{2}C_2(G)}^{=+1}\right)\frac{2}{\epsilon_{\overline{MS}}} + \delta_1^{\overline{MS}} + \text{finite}\right)$$

$$\Rightarrow \boxed{\delta_1^{\overline{MS}} = -\frac{g^2}{16\pi}\left(C(r) + C(G)\right)\frac{2}{\epsilon_{\overline{MS}}}} \qquad (3.38)$$

3) Gauge boson polarization

Number of fermion flavours

(I)



$$i\Pi^{(I)} = (-1)(ig)^2 N_f \underbrace{t^a t^a}_{C(r)} \int \frac{d^4k}{(2\pi)^4} \frac{i^2\,\mathrm{Tr}\left(\not k \gamma^\mu (\not k - \not q)\gamma^\nu\right)}{k^2(k-q)^2}$$

$$\bullet\ \mathrm{Tr}(\ ) = 4k^\mu(k-q)^\nu + 4k^\nu(k-q)^\mu - 4g^{\mu\nu}\, k\cdot(k-q)$$

$$= 8k^\mu k^\nu - \underbrace{4(k^\mu q^\nu + k^\nu q^\mu)}_{\text{eff} \to 8k^\mu q^\nu} + 4g^{\mu\nu}(k\cdot q - k^2)$$

$$\Rightarrow i\Pi^{(I)} = -g^2 N_f C(r)\cdot 4 \int \frac{d^4k}{(2\pi)^4} \frac{2k^\mu k^\nu - 2k^\mu q^\nu + g_{\mu\nu}(k\cdot q - k^2)}{k^2(k-q)^2}$$

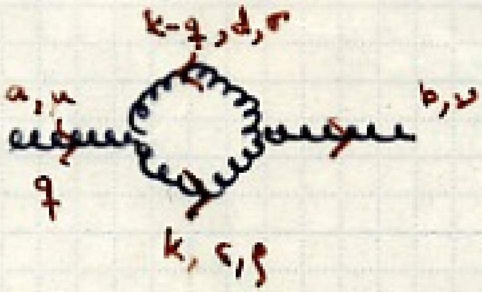
$$= -4g^2 N_f C(r) \frac{i}{16\pi^2} \frac{1}{\epsilon_{\overline{MS}}} \cdot \left(-\frac{1}{3} q^2 g^{\mu\nu} + \frac{4}{3} q^\mu q^\nu - 2q^\mu q^\nu + q^2 g^{\mu\nu}\right)$$

$$= -i\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \frac{g^2}{16\pi^2} \frac{1}{\epsilon_{\overline{MS}}} \cdot \frac{8}{3} N_f C(r) \qquad (3.39)$$

(II)

$= 0$ in dimensional regularization

(III)



$$i\Pi^{(III)} = \frac{g^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i^2 \overbrace{f^{acd} f^{bdc}}^{= -C_2(G)}}{k^2(k-q)^2} \times$$

$$\times V^\mu{}_{\rho\sigma}(q, -k, k-q)\, V^{\nu\sigma\rho}(-q, q-k, k)$$

(3.40)

- $V \cdot V = \left(g_{\rho\sigma}(q-2k)^\mu + g^\mu{}_\rho (q+k)_\sigma + g^\mu{}_\sigma (k-2q)_\rho \right)$

$$\cdot \left(g^{\sigma\rho}(q-2k)^\nu + g^{\nu\sigma}(k-2q)^\rho + g^{\nu\rho}(k+q)^\sigma \right)$$

$$= \overset{\to 3}{(d-1)}\, (q-2k)^\mu (q-2k)^\nu - (k-2q)^\mu (k-2q)^\nu -$$

$$- (q+k)^\mu (q+k)^\nu + g^{\mu\nu}\left((k+q)^2 + (k-2q)^2 \right)$$

$$\overset{d\to 4}{=} -2q^\mu q^\nu - 5(q^\mu k^\nu + q^\nu k^\mu) + 10 k^\mu k^\nu$$

$$+ g^{\mu\nu}\left(2k^2 + 5q^2 - 2k\cdot q \right) \qquad (3.41)$$

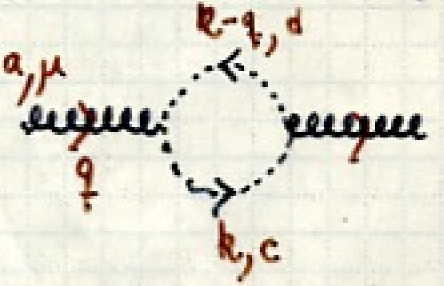
Using results (3.32) - (3.35) and (3.40) this immediately gives

$$i\Pi^{(III)} = \frac{i}{16\pi^2}\frac{1}{\epsilon_{\overline{MS}}}\left(+g^2\frac{C_2(G)}{2}\right)\left(-4q^\mu q^\nu - 10 q^\mu q^\nu + \frac{20}{3}q^\mu q^\nu - \frac{5}{3}q^2 g^{\mu\nu} \right.$$

$$\left. + g^{\mu\nu}\left(0 + 10q^2 - 2q^2\right) \right)$$

$$= +i\frac{g^2}{16\pi^2}\frac{C_2(G)}{\epsilon_{\overline{MS}}}\left(-\frac{11}{3}q^\mu q^\nu + \frac{19}{6}q^2 g^{\mu\nu} \right) \qquad (3.42)$$

Finally there is the Ghost graph:



$$i\Pi^{(IV)} = (-1)\, g^2 f^{acd} f^{bdc} \int \frac{d^4k}{(2\pi)^4} \frac{\overset{=-1}{i^2}\, k_\mu (k-q)_\nu}{k^2 (k-q)^2}$$

$$= +ig^2 C_2(G)\frac{1}{16\pi}\frac{1}{\epsilon_{\overline{MS}}}\left(+\frac{1}{6}q^2 g_{\mu\nu} + \frac{1}{3}q^\mu q^\nu \right)$$

+ finite

$$\Rightarrow i\Pi^{(III)} + i\Pi^{(IV)} = +i\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\frac{g^2}{16\pi^2}\frac{1}{\epsilon_{\overline{MS}}}\cdot\frac{10}{3}C_2(G) \qquad (3.43)$$

Putting all together, with the counterterm $-i\overbrace{(q^2 g^{\mu\nu} - q^\mu q^\nu)}^{\equiv \Delta^{\mu\nu}(q)}\delta_3^{\overline{MS}}$ one finds

$$i\Pi = -i\Delta^{\mu\nu}(q)\left[\frac{g^2}{16\pi^2}\frac{1}{\epsilon_{\overline{MS}}}\left(-\frac{10}{3}C_2(G) + \frac{8}{3}C(r)N_f\right) + \delta_3^{\overline{MS}}\right] + \text{finite}$$

$$\Rightarrow \delta_3^{\overline{MS}} = \frac{g^2}{16\pi^2}\frac{1}{\epsilon_{\overline{MS}}}\left(\frac{10}{3}C_2(G) - \frac{8}{3}C(r)N_f\right) \qquad (3.44)$$

This completes the computation of counter-terms needed for the β-function:

$$\delta_{eff} \equiv \delta_2^{\overline{MS}} - \delta_1^{\overline{MS}} + \frac{1}{2}\delta_3^{\overline{MS}} \overset{\text{1-loop}}{=} \frac{g^2}{16\pi^2}\frac{1}{\epsilon_{\overline{MS}}}\Big(-2\cancel{C(r)} + 2\cancel{C(r)} + 2C_2(G)$$
$$+ \frac{5}{3}C_2(G) - \frac{4}{3}N_f C(r)\Big)$$

$$= \frac{g^2}{16\pi^2}\frac{1}{\epsilon_{\overline{MS}}}\left(\frac{11}{3}C_2(G) - \frac{4}{3}N_f C(r)\right)$$

$$\Rightarrow \beta(g) = -\frac{\epsilon}{2}g\left(1 + g\frac{\partial}{\partial g}\delta_{eff}\right) \approx \boxed{-\frac{g^3}{16\pi^2}\left(\frac{11}{3}C_2(G) - \frac{4}{3}N_f C(r)\right)} \qquad (3.45)$$

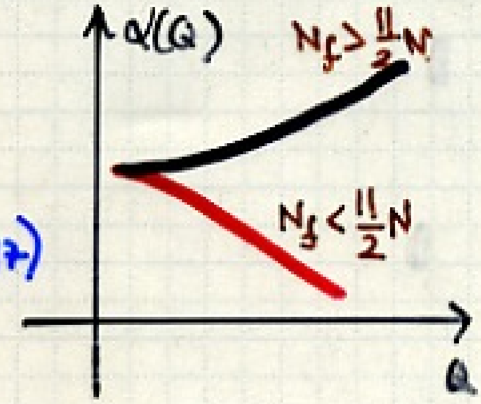
Thus, in Yang-Mills theories the β-function can be either positive or negative depending on the dimensions of the representations and number of fermion flavours. For

$$C_2(G) = N \quad \text{and} \quad C(r) = \frac{1}{2}$$

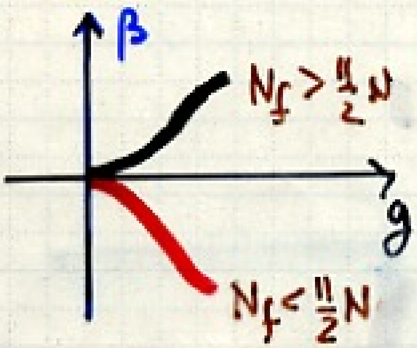
$$\Rightarrow \boxed{\beta(g) = -\frac{g^3}{48\pi^2}\left(11N - 2N_f\right)} \qquad (3.46)$$

That is, for $N_f < \frac{11}{2}N$, the SU(N) YM-theory with N_f fermion flavours is asymtotically free:

$$\alpha(Q) = \frac{\alpha(Q_0)}{1 + \frac{\alpha_0}{12\pi}(11N - 2N_f)\log\left(\frac{Q^2}{Q_0^2}\right)} \qquad (3.47)$$



Asymptotic freedom



Obviously the most famous asymptotically free field theory in physics is the QCD. In QCD we have

$N = 3$ (# of colours)

$N_f = 6$ (# of different types of quarks)
$= 3 \times 2$ ← for QCD each isospin component is a different flavour,

$$\Rightarrow \beta_{QCD} \approx -\frac{g_s^3}{48\pi^2}(33-12) = -\frac{7g_s^3}{16\pi^2} \qquad (3.48)$$

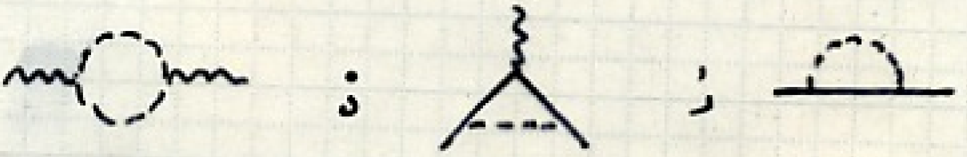
Also the $SU(2)_L$ - YM - theory (weak interactions) in the SM is asymptotically free. Here

$N = 2$ # of isospin components

$N_f = 3 \times 2$ # of families {leptons & quarks}

$$\Rightarrow \beta_{weak}(g) = -\frac{g_w^3}{48\pi^2}(22-10) = -\frac{g_w^3}{4\pi^2} \qquad (3.49)$$

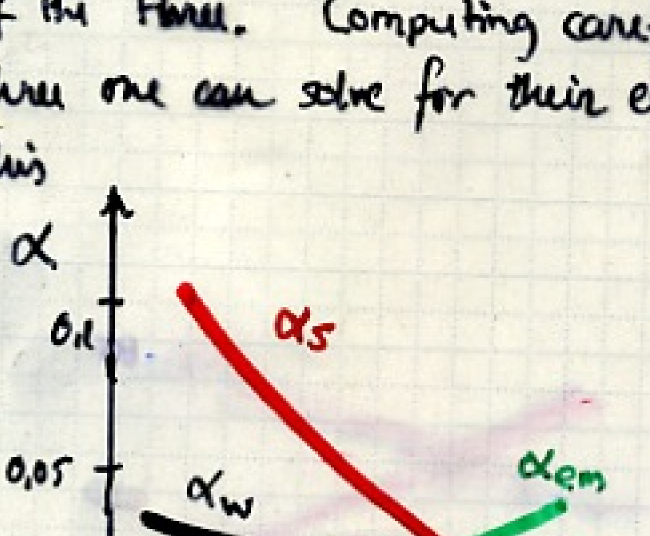
Although $\alpha_{weak} < \alpha_s$ at low energies, for sufficiently large Q α_s becomes smaller of the two, since $|\beta_{QCD}| > |\beta_{weak}|$. In fact this effect is even stronger than what (3.43) and (3.49) would seem to indicate, because (3.49) neglects the contribution to the running of the weak charge due to higgs field:



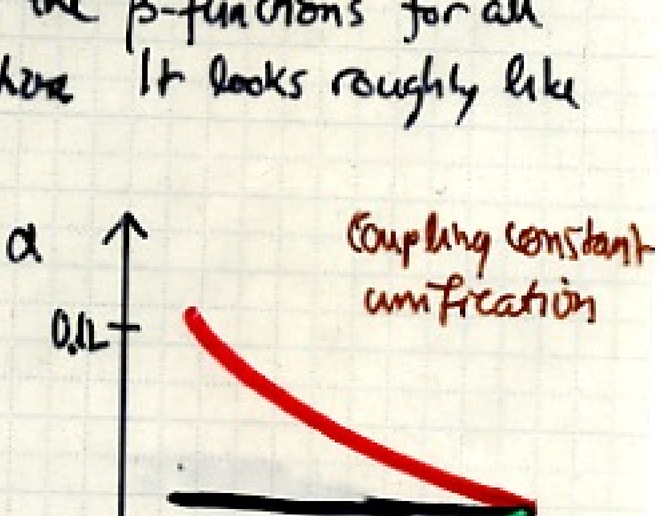
$\Delta\delta_3$; $\Delta\delta_1$; $\Delta\delta_2$

As a result of including these diagrams one eventually finds that $g_s(Q) \approx g_w(Q)$ at $Q \approx 10^{16}$ GeV.

Similarly, the QED coupling is the weakest of the three, but the QED β-function is positive. It will thus increase as a function of Q, eventually becoming the largest of the three. Computing carefully the β-functions for all three one can solve for their evolution. It looks roughly like this



STANDARD MODEL



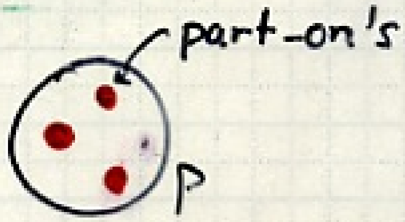
! SUPERSYMMETRY !

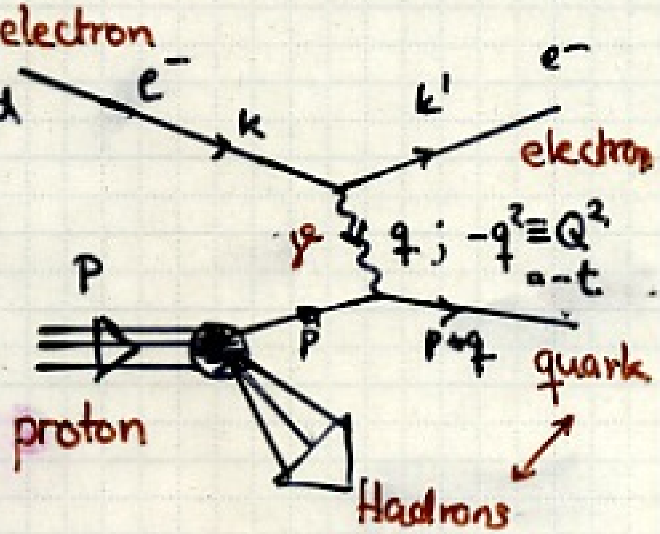
. QUANTUM CHROMODYNAMICS

Parton model

Among first for the correct theory of strong interactions came from observations of Deep inelastic Scattering (DIS) of $e^- p \to e^- +$ hadrons, performed around ~ 70's at MIT-SLAC.

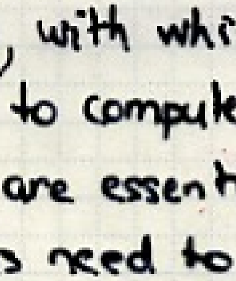
The observed substantial rate of hard scatterings led Björken and Feynman introduce the Parton model for hadrons.





Parton model simply asserts that proton consist of a number of constituent parts (later these came to be called quarks after Gell-Manns rivaling quark model), with which the photons are interacting. One then goes on to compute the ep-scattering assuming that these partons are essentially free! This sounds crazy, because partons need to interact strongly with each other a order to be bound to hadrons! Nevertheless, we just carry on:

- Subprocess $e^- q_i \to e^- q_i$



$$|M|^2 = \frac{e^4 Q_i^2}{4\hat{t}^2} \cdot \overbrace{\mathrm{Tr}(\;)\mathrm{Tr}(\;)}^{16 L_e^{\mu\nu} L_{\mu\nu,q}}$$

$$= \frac{8 e^4 Q_i^2}{\hat{t}^2}\left(p_1\cdot q_1\, p_2\cdot q_2 + p_1\cdot q_2\, p_2\cdot q_1\right)$$

$$= \frac{2 e^4 Q_i^2}{\hat{t}^2}\left(\hat{s}^2 + \hat{u}^2\right)$$

$$\frac{d\sigma}{d\hat{t}} = \frac{4\pi}{\hat{s}}\frac{d\sigma}{d\Omega_{CM}} = \frac{|M|^2}{16\pi\hat{s}^2}$$

$\hat{t} \approx -\frac{\hat{s}}{2}(1-\cos\theta_{CM})$

(neglecting masses)

Here $\hat{s}$, $\hat{u}$ and $\hat{t} \equiv -Q^2$ are the invariant Mandelstam variables referring to the $e^- q_{f\,in}$-subsystem. Putting in the known result from $e\mu$- scattering

$$\frac{d\sigma}{d\hat{t}} = \frac{e^4 Q_i^2}{8\pi \hat{s}^2}\left(\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}\right) = \frac{2\pi\alpha^2}{\hat{s}^2} Q_i^2 \left(\frac{\hat{s}^2+(\hat{s}+\hat{t})^2}{\hat{t}^2}\right) \qquad (3{,}50)$$

Now, assuming that the quark carries a fraction ξ of the proton momentum one has:

$$\hat{s} = (p+k)^2 = 2p\cdot k = 2\xi P\cdot k = \xi s \qquad (3{,}51)$$

Furthermore

$$0 \simeq (p+q)^2 = 2p\cdot q + q^2 = 2\xi P\cdot q - Q^2 \Rightarrow \boxed{\xi = \frac{Q^2}{2P\cdot q}} \qquad (3{,}52)$$

Since $q = k-k'$ can be measured purely from the scattered electron, they provide a very good parameters for the DIS-process, Parton model now makes the following important predictions: Given parton distribution functions $f_i(\xi)$ measured at $\xi = x$, we find the diff. x-section:

$$\frac{d^2\sigma}{dx\,dQ^2} = \underbrace{\sum_i f_i(x)\,Q_i^2}_{\text{QED (and } Q^2\text{-) independent structure of a Hadron.}} \left[\frac{2\pi\alpha^2}{Q^4}\left[1+\left(1-\frac{Q^2}{xs}\right)^2\right]\right] \qquad (3{,}53)$$

all QED-dependence here

QED (and Q^2-) independent structure of a Hadron.

Depends only on x $\hat{=}$ $\boxed{\text{Björken scaling}}$

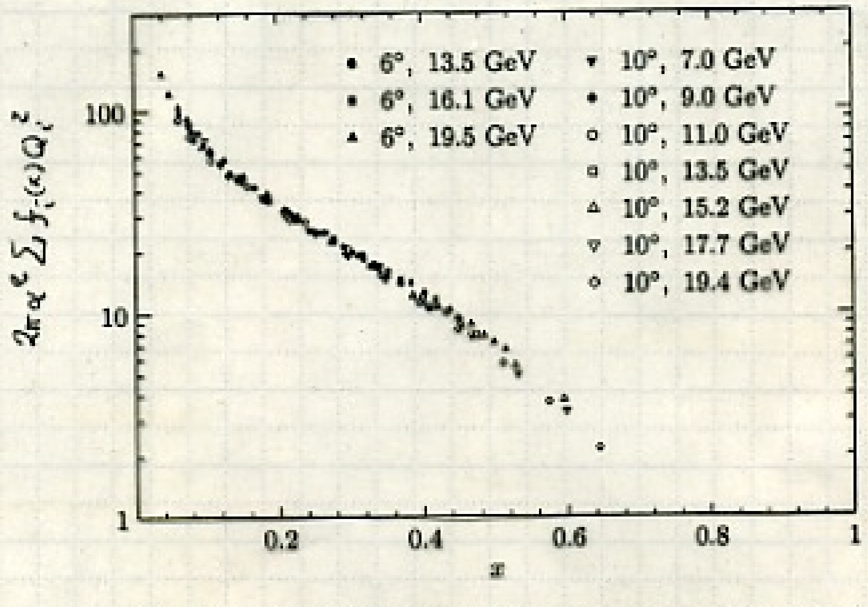
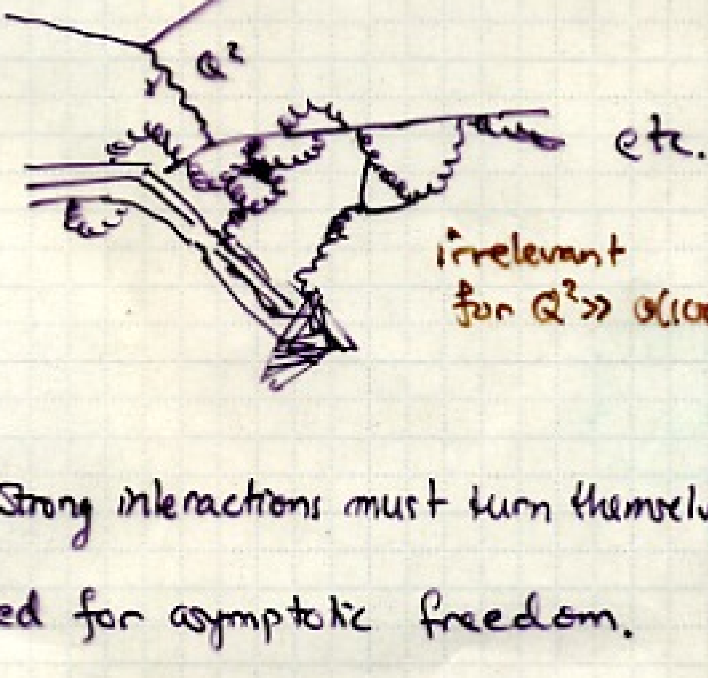


Figure 14.2. Test of Bjorken scaling using the e^-p deep inelastic scattering cross sections measured by the SLAC-MIT experiment, J. S. Poucher, et. al., *Phys. Rev. Lett.* **32**, 118 (1974). We plot $d^2\sigma/dx dQ^2$ divided by the factor (14.9) against x, for the various initial electron energies and scattering angles indicated. The data span the range $1\ \mathrm{GeV}^2 < Q^2 < 8\ \mathrm{GeV}^2$.

The fact that Björken scaling works is quite surprising, given that constituent quarks were supposed to be strongly coupled.

Indeed in the computation we neglected all a'priori important (interaction that confines quarks is <u>strong</u>) interactions as



etc.

irrelevant for $Q^2 \gg O(100)$ MeV

(Initial quark distribution independent of hard scattering)

Somehow the strong interactions must turn themselves off when Q^2 is large.

$\Rightarrow$ Need for asymptotic freedom.

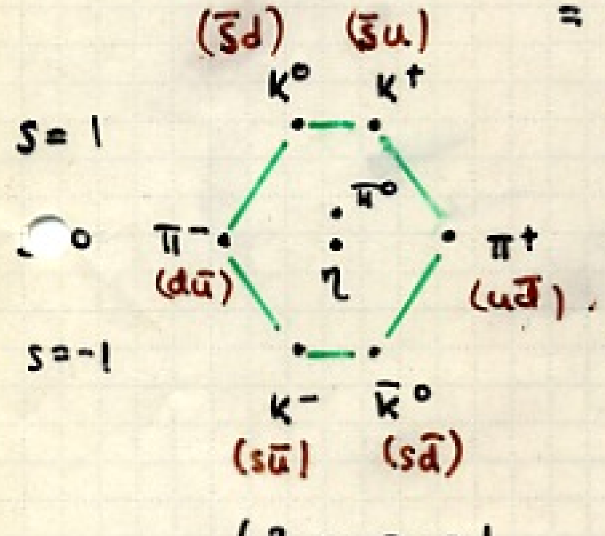
Quark model

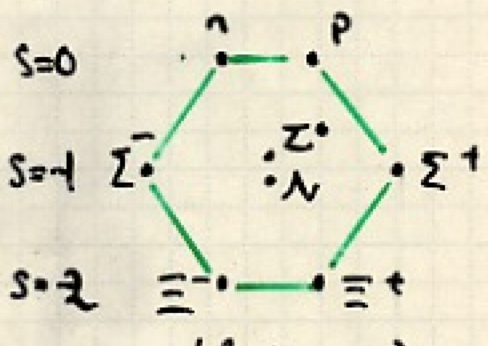
Another step on the way to discovery of QCD involved Gell-Mann's quark model. In order to get some sense to the emerging chaos of new hadronic states Gell-Mann and Zweig proposed that hadrons are composed of three types of quarks: up down and strange, which transform onto themselves according to $SU(3)_{flavour}$. According to quark model all hadrons are of form:

$$\boxed{\begin{array}{ccc} q_i \bar{q}^i & \epsilon^{ijk} q_i q_j q_k & \epsilon_{ijk} \bar{q}^i \bar{q}^j \bar{q}^k \\ \text{mesons} & \text{baryons} & \text{antibaryons} \end{array}}$$

$$3 \times \bar{3} = 1 + 8 \qquad 3 \times 3 \times 3 = 3 \times (\bar{3} + 6) = 1 + 8 + \bar{3} + 10$$

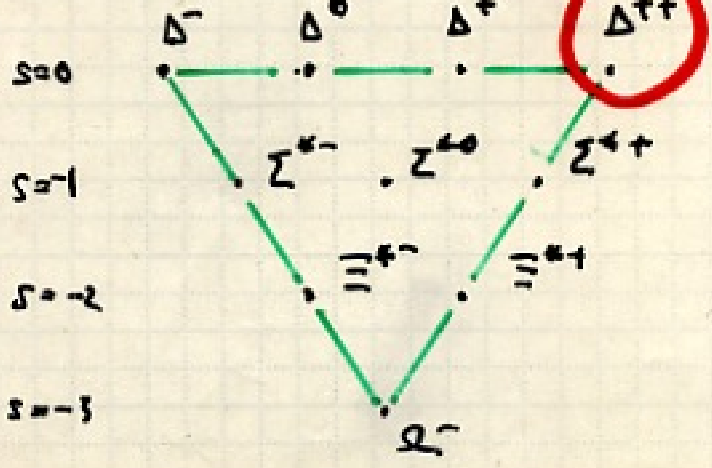
$$q_u = \frac{2}{3} \qquad q_d = q_s = -\frac{1}{3}.$$





(8 baryons)

(8-mesons)



The fact that the spin $\frac{3}{2}$ - fermion state Δ^{++} is characterized as uuu in this scheme led Gell-Mann & others to propose that quarks have an additional quantum number, colour. Then the wave-function

$$\epsilon^{abc} u_a u_b u_c$$

now a colour singlet, can be antisymmetric as required. Clearly at least 3 colours are needed.

That the number of colours is exactly 3 can be seen for example by comparing cross-sections for processes $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to$ hadrons. First:

- $$\sigma_{e^+e^- \to \mu^+\mu^-} = \int_{-s}^{0} dt \, \frac{|M|^2}{16\pi s^2} = \frac{2e^4}{16\pi s^2} \int_{-s}^{0} dt \, \frac{t^2+u^2}{s}$$

  $$= \frac{4\pi\alpha^2}{3s^2} \equiv \sigma_0 \qquad (3,54)$$

- Then with N_c - colours of quarks, the x-section to hadrons is

  over all quarks that can be produced at given s. Threshold - effects.

  $$\sigma_{e^+e^- \to \text{hadrons}} = N_c \cdot \sigma_0 \sum_f Q_f^2$$

  $$= R\sigma_0 \qquad (3,55)$$

$$\Rightarrow R = \frac{\sigma_{e^+e^- \to \text{hadrons}}}{\sigma_{e^+e^- \to \mu^+\mu^-}} = N_c \sum_f Q_f^2 = N_c \cdot \begin{cases} 2\cdot\frac{1}{9} + \frac{4}{9} = \frac{2}{3} & : \sqrt{s} \lesssim 2m_c \\ \frac{2}{3} + \frac{4}{9} = \frac{10}{9} & : \sqrt{s} \lesssim 2m_b \\ \frac{11}{9} & : \sqrt{s} \lesssim 2m_t \end{cases}$$

Peskin & Schröder Fig. 5.3

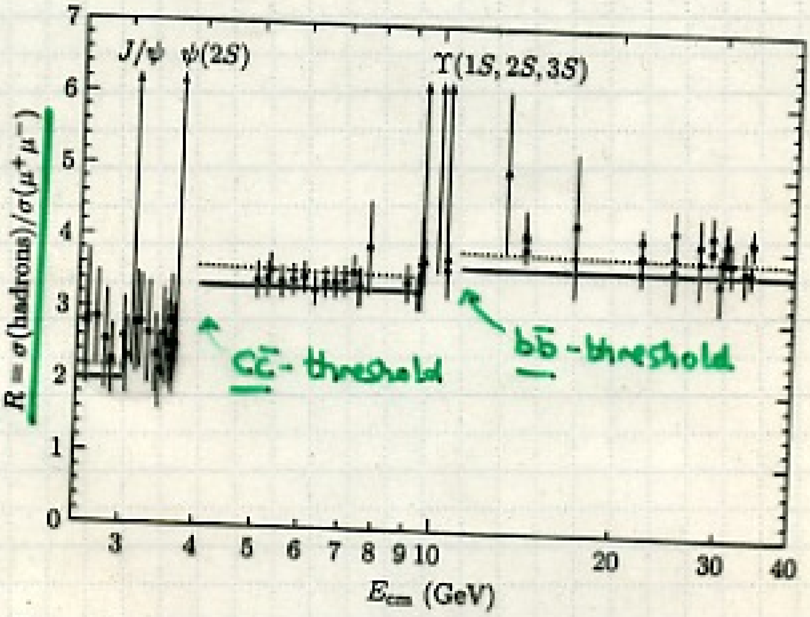
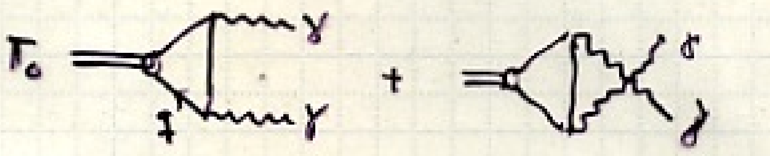


Figure 5.3. Experimental measurements of the total cross section for the reaction $e^+e^- \to$ hadrons, from the data compilation of M. Swartz, *Phys. Rev.* D53, 5268 (1996). Complete references to the various experiments are given there. The measurements are compared to theoretical predictions from Quantum Chromodynamics, as explained in the text. The solid line is the simple prediction (5.16).

Yet another ~~hint~~ evidence for $N_c = 3$ worth mentioning here comes from 2γ-decay of a pion (PCAC):



clearly is $\propto N_c^2$ of quarks. Theoretically (later)

$$\Gamma(\pi^0 \to 2\gamma) = N_c^2 \, (Q_u^2 - Q_d^2)^2 \, \frac{\alpha^2 m_{\pi^0}^3}{64\pi^3 F_\pi}$$

m_{π^0} ~ 134,98 MeV

F_π ~ 91 MeV

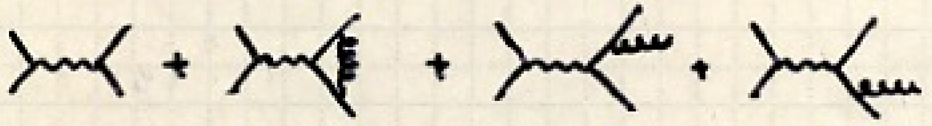
$$\approx N_c^2 \cdot 0{,}84 \text{ eV}$$

$$\approx 7.6 \text{ eV} \quad \text{if } N_c = 3 \qquad (3,56)$$

whereas observationally : $\Gamma(\exp) \simeq 7{,}48 \pm 0{,}33$ eV.

e^+e^-- annihilation to hadrons

To lowest order this process is given by ( ). At next nontrivial order one finds radiative corrections to the vertex and gluon emissions

$\sigma \sim \Big|$  $\Big|^2$

The actual computation is quite involved. The new features on the computation involve collinear singularities with massless quarks along with the usual UV- and IR- singularities

In the d-regularization UV- and IR- singularities mix unless one provides an additional IR-cutoff such as a gluon mass μ_g. In the latter method IR- and collinear singularities still mix providing terms such as:

$$\log \frac{\mu^2}{s} \quad , \quad \left(\log \frac{\mu^2}{s}\right)^2$$

collinear + soft collinear.

Whatever way one approaches the problem, the computation is tedious and involves cancellation of very large number of terms before the final finite, incredibly simple result is found:

$$\sigma = \sigma_0\left(1 + \frac{\alpha_s}{\pi}\right) \Rightarrow R = N_c \sum_f Q_f^2 \left(1 + \frac{\alpha_s}{\pi}\right) \qquad (3{,}50)$$

Running of α_s

Equation ( ) involves the QCD coupling α_s. What is the value of this counstant and how can it be consistently defined? In QED the charge was defined at $q^2=0$ for on-shell electrons. This is not possible in QCD where quarks are not free, and strongly interacting at regime $q^2 \simeq 0$. The way out is to define α_s at some arbitrary high scale where the coupling is weak, and use the RGE to evolve α_s to the energy scale of interest.

Indeed we saw how α_s depends on Q^2. So let us now assume that we define α_s at some scale Q_0. Then evidently

$$\sigma = \sigma(\underbrace{s}_{\text{cm-energy}}, \underbrace{\mu}_{\text{ren. scale}}, \alpha_s) \overset{\text{dim. analysis}}{=} \frac{c}{s} f\left(\frac{s}{\mu^2}, \alpha_s\right) \quad ; \quad \alpha_s(\mu = Q_0) \equiv \alpha_0$$

Because σ is an observable, it obeys an homogenous CS-equation:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g}\right) \sigma(s, \alpha_s) = 0$$

$$\Rightarrow \quad f(e^t s_0, \alpha_s, \mu) = f(s_0, \bar{\alpha}_s, \mu)$$

where $\bar{\alpha}_s = \bar{g}_s^2/4\pi$ with

$$\left.\frac{\partial \bar{g}}{\partial t} = \beta(\bar{g}) \quad ; \quad g_s(Q_0) = g_{s_0} \right\} \Rightarrow \bar{\alpha} = \alpha_s(Q)$$

of eqn (3.51).

and

$$\sigma(s,\alpha_s) = \sigma(e^t s_0,\alpha_s) = \frac{1}{s} f(e^t s_0,\alpha_s)$$

$$= \frac{1}{s} f(s_0,\bar{\alpha})$$

$$= \sigma_0(s)\cdot\left(1+\frac{\bar{\alpha}}{\pi}\right)$$

$$= \frac{4\pi\alpha}{3s} N_c \cdot \sum_f Q_f^2 \left(1+\frac{\alpha_s(\sqrt{s})}{\pi}\right) \qquad (3,52)$$

Now one sees that one can use the very observation of R as the defining measurement of $\alpha_s(Q_0)$. The RGE-improved formula ( ) then tells us how the x-section should depend on varying $\sqrt{s}$.

Dimensional transmutation and Λ_{QCD}.

Just as we did for $\lambda\phi^4$-theory we can dispose of the dimensionless parameter α_s completely by setting $\alpha_s(\Lambda)=\infty$. Unlike for $\lambda\phi^4$-theory and QED this takes place at low energy-scale however:

$$1+\frac{7g_0^2}{8\pi^2}\log\left(\frac{\Lambda}{Q_0}\right)=0 \Rightarrow \log\frac{Q_0}{\Lambda}=\frac{8\pi^2}{7g_0^2}$$

$$\Rightarrow \quad 1+\frac{7g_0^2}{8\pi^2}\log\frac{Q}{Q_0} = \cancel{1}+\frac{7g_0^2}{8\pi^2}\left(\log\frac{Q}{\Lambda}-\cancel{\log\frac{Q_0}{\Lambda}}\right)$$

$$\Rightarrow \alpha_s(Q) = \frac{8\pi^2/7}{\log(Q/\Lambda)}$$

Experimentally $\Lambda \approx 200$ MeV, and hence one should not expect to trust perturbative QCD below energies of order GeV. Indeed from ( ) at 1 GeV (given that $\Lambda = 200$, one finds

$$\alpha_s(\mathrm{GeV}) \approx 0{,}4.$$

Note that the scale

$$1/\Lambda \sim \frac{1}{200\,\mathrm{MeV}} \sim \mathrm{fm}$$

is just the scale of hadrons.

More about QCD in Kari Eskolas Course!