Quantum field theory, fall 2023, **Exercise 8.**

1. Derive the form of the helicity spin vector $s_\mu = \frac{h}{m}(|\mathbf{k}|, E\hat{\mathbf{k}})$ from the requirements $s_h \cdot k = 0$, $s_h \cdot s_h = -1$ and $\mathbf{s}_h || \mathbf{k}$. Then show that the projection operator $P_h = \frac{1}{2}(1 + \gamma^5 \not{s})$ obeys $P_h u(p,h') = \delta_{h,h'} u(p,h')$ and $P_h v(p,h') = \delta_{h,h'} v(p,h')$.

2. Let us now treat the issue of the polarization sums rigorously. Show that the vacuum Maxwell equations $\partial_\mu F^{\mu\nu}$ are invariant under the gauge transformation $A_\mu \to A'_\mu + \partial_\mu \alpha$, where α is an arbitrary scalar function. Then note that in the Lorenz-Gauge these equations are replaced by:

$$\Box A_\mu = 0 \qquad\qquad \partial^\mu A_\mu = 0\,.$$

Trying a plane-wave solution $A_\mu = \epsilon_\mu(k)\, e^{-ik\cdot x}$ where ϵ_μ is the *polarization vector*, show that the equations above imply $k^2 = 0$ (photon is massless) and $k^\mu \epsilon_\mu = 0$.

Now show that the Lorenz-gauge does not yet uniquely define the entire gauge potential A_μ. Use the residual gauge degree of freedom to show that we can always require that the polarization vectors of real physical photons are of the form $\epsilon = (0, \boldsymbol{\epsilon})$ with $\boldsymbol{\epsilon}$ transverse to the direction of photon's momentum $\mathbf{k}\cdot\boldsymbol{\epsilon} = 0$. In this way there are always two independent polarization vectors ϵ^1 and ϵ^2. These are space-like vectors, so they are conventionally normalized such that $\epsilon \cdot \epsilon = -1$. Next derive the result for the sum over physical transverse photon polarization states

$$P_{\mu\nu} \equiv \sum_\lambda \epsilon^\lambda_\mu(k) \epsilon^{*\lambda}_\nu(k) = -g_{\mu\nu} + \frac{k_\mu \overline{k}_\nu + k_\nu \overline{k}_\mu}{k \cdot \overline{k}} \qquad , \overline{k} \equiv (k^0, -\mathbf{k})\,.$$

This is, by construction, an idempotent projector operator *onto the physical space* of photon polarizations. Hint: Note that the vectors k, $\overline{k}$ ϵ^1, ϵ^2 form a basis of 4-vectors and thus the second order tensor $P_{\mu\nu}$ can be expanded in terms of $g_{\mu\nu}$, $k_\mu k_\nu$, $\overline{k}_\mu \overline{k}_\nu$, $\overline{k}_\mu k_\nu$ and $k_\mu \overline{k}_\nu$. Finally show, using the Ward identity, that in practical calculations with external photons you can still replace $P_{\mu\nu} \to -g_{\mu\nu}$.

3. Show that the differential cross section for the scattering of an $e^- \, e^+$ -pair into two photons is

$$\left(\frac{d\sigma}{d\cos\theta}\right)_{\mathrm{CM}} = 2\frac{\pi\alpha^2}{s}\left(\frac{E}{p}\right)\frac{E^2 + p^2\cos^2\theta}{m^2 + p^2 \sin^2\theta}.$$

Compute the total cross-section by integrating

$$\sigma_{\mathrm{TOT}} = \int_0^1 d(\cos\theta)\left(\frac{d\sigma}{d\cos\theta}\right)_{\mathrm{CM}},$$

and extract the leading term as $E \gg m_e$.