Quantum field theory, fall 2023, **Exercise 5.**

These excercises are not much related (apart from fourth) to what we are currently doing in lectures. It is difficult to write excercises on the LSZ-procedure, so these are just problems that give you some skills that come handy later.

1. In many experiments, a polarized beam of electrons can be produced, and it is therefore important to know how to deal with the spin in a covariant language. We will start from the rest frame of the spinor, in which the spin is described by a 3-(unit)vector $\boldsymbol{s}$. In this frame, the Dirac spinors with spin s are eigenstates of the spin operator $(1 \otimes \boldsymbol{\sigma}) \cdot \boldsymbol{s}$

$$(1 \otimes \boldsymbol{\sigma}) \cdot \boldsymbol{s}\, u(0, \boldsymbol{s}) = u(0, \boldsymbol{s}) \qquad (1 \otimes \boldsymbol{\sigma}) \cdot \boldsymbol{s}\, v(0, \boldsymbol{s}) = -v(0, \boldsymbol{s}),$$

and therefore

$$\Sigma(\boldsymbol{s}) \equiv 1 \otimes \frac{1 \pm \boldsymbol{\sigma} \cdot \boldsymbol{s}}{2}$$

with + sign for u-spinor and − sign for v-spinors serves as a *spin projection* operator in the rest frame. That is

$$\Sigma(\boldsymbol{s})\, u(0, \boldsymbol{s}) = u(0, \boldsymbol{s}) \quad \Sigma(\boldsymbol{s})\, v(0, \boldsymbol{s}) = v(0, \boldsymbol{s}) \quad \Sigma(-\boldsymbol{s})\, u(0, \boldsymbol{s}) = \Sigma(-\boldsymbol{s})\, v(0, \boldsymbol{s}) = 0.$$

To obtain a covariant version we promote $\boldsymbol{s}$ to a 4-vector s, that is given by $s^\mu = (0, \boldsymbol{s})$ in the rest frame. What are then the invariants s^2 and $s \cdot p$ where p is the four-momentum of the spinor? By proving that in the rest frame

$$(1 \otimes \boldsymbol{\sigma}) \cdot \boldsymbol{s} = \gamma^5 \not{s} \gamma^0,$$

deduce that the covariant spin projection operator is given by

$$\Sigma(s) = \frac{1 + \gamma^5 \not{s}}{2}.$$

With our definition of spin as a four-vector, and because of the covariant form of the projection operator, we then have

$$\Sigma(s)\, u(p, s) = u(p, s) \quad \Sigma(s)\, v(p, s) = v(p, s) \quad \Sigma(-s)\, u(p, s) = \Sigma(-s)\, v(p, s) = 0.$$

Show that $\Sigma(s)$ satisfies

$$\Sigma(\pm s)\Sigma(\pm s) = \Sigma(\pm s) \qquad \Sigma(\pm s)\Sigma(\mp s) = 0 \qquad \Sigma(\pm s) + \Sigma(\mp s) = 1,$$

as a proper projection operator should.

2. Using the anticommutation properties of γ matrixes prove the following trace theorems

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}\,, \qquad Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\,,$$

where trace is taken over the Dirac indexes. Use these results (and the results on the spin sums of spinors) to calculate the (lepton) tensor

$$L_{\mu\nu} \equiv \sum_{s_1,s_3} \left[\bar{u}(p_3,s_3)\gamma^\mu u(p_1,s_1)\right]\left[\bar{u}(p_3,s_3)\gamma^\nu u(p_1,s_1)\right]^\dagger .$$

3. Show that

$$\mathrm{Tr}[\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}] = 0.$$

$$\mathrm{Tr}[\not{a}_1\cdots\not{a}_n] = \sum_{i=2,n}(-1)^i\, a_1\cdot a_i\,\mathrm{Tr}[\not{a}_2\cdots\hat{\not{a}}_j\cdots\not{a}_n]$$

where notation $\hat{\not{a}}_i$ means that term $\not{a}_i$ is removed from product. Show also that

$$\mathrm{Tr}[\gamma_5\not{a}\not{b}] = 0$$

$$\mathrm{Tr}[\gamma_5\not{a}\not{b}\not{c}\not{d}] = 4i\epsilon_{\mu\nu\rho\sigma}a^\mu a^\nu a^\rho a^\sigma$$

and

$$\gamma_\mu\not{a}\gamma^\mu = -2\not{a}$$

$$\gamma_\mu\not{a}\not{b}\gamma^\mu = 4a\cdot b$$

$$\gamma_\mu\not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a}$$

4. Compute the symmetry factors of the following graphs in the $\lambda\phi^4$-theory: