**Quantum field theory, fall 2023,** **Exercise 4.**

**1.** Let us denote the mutual speed of two particles with $|\boldsymbol{v}_a - \boldsymbol{v}_b| \equiv v_{ab}$. In this case the flux factor $F$ is defined as

$$\frac{1}{F} = \frac{1}{2E_a 2E_b v_{ab}}.$$

Show that if $\boldsymbol{v}_a \parallel \boldsymbol{v}_b$, we have

$$F = 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2},$$

in which $p_a$ and $p_b$ are the four-momenta of particles $a$ and $b$ respectively.
Explain, why our calculation guarantees that the flux factor $F$ is Lorentz invariant under boosts along the direction of the particles motion.

**2.** Denote $n$-particle phase-space element with $d\Gamma_n$, so that

$$d\Gamma_n \equiv \left( \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i} \right) (2\pi)^4 \delta^{(4)}(p_a + p_b - \sum_i p_i).$$

Show that in the center-of-mass frame (CMS) one obtains

$$\int d\Gamma_2 = \int d\Omega \frac{1}{16\pi^2} \frac{|\vec{p}_1|}{E_{\rm CM}},$$

and consequently the differential cross-section for $p_a + p_b \to p_1 + p_2$ scattering can be written as

$$\left( \frac{d\sigma}{d\Omega} \right)_{\rm CM} = \frac{1}{F} \frac{|\vec{p}_1|}{16\pi^2 E_{\rm CM}} |\mathcal{M}(\vec{p}_a, \vec{p}_b \to \vec{p}_1, \vec{p}_2)|^2$$

in the CMS frame.

**3.** Tree level contributions to the scattering amplitudes can be computed directly from the equation of motion and Born expansion. In this problem you shall see how it works for a fermion scattering off a classical source. Starting from QED Lagrange function

$$\mathcal{L}_{\text{QED}} = \bar{\psi}^{(} i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

derive the equation of motion for the field $\psi$:

$$(i\not{\partial} - m)\psi = e\not{A}\psi \,.$$

Show that if $\psi$ satisfies this equation of motion, then the charge conjugated field $\psi^c \equiv C\bar{\psi}^T$, where $C \equiv i\gamma^2\gamma^0$ satisfies the same equation with the opposite charge: $e \to -e$. Moreover, show that

$$\psi(x) \equiv u(p,s)e^{-ip\cdot x} \quad \Rightarrow \quad \psi^c_{p,s}(x) = v(p,s)e^{ip\cdot x} \,.$$

Now show that $\Psi(x)$ has the formal solution

$$\Psi(x) = \psi_0(x) - e\int \mathrm{d}^4x' S(x-x')\not{A}(x')\Psi(x'),$$

where $S(x-x')$ is a Greens function for a free fermion field. Take it to be the Feynman propagator. Now write a formal perturbative solution for $\Psi(x)$ in parameter $e$ (write two lowest order terms). Starting from the momentum space representation (2.92) in the lectures, derive an the following expansion for $S_F$ (remember also the spin sums (2.55) and (2.56)):

$$S_F(x-x') = \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E} \left(\theta(t-t')\psi_{p,s}(x)\bar{\psi}_{p,s}(x') - \theta(t'-t)\psi^c_{p,s}(x)\bar{\psi}^c_{p,s}(x')\right) \,.$$

Interpretation? Now choose the boundary condition to be a free plane wave: $\lim_{t\to\infty}\Psi(x) = \psi_{p1,s1}(x)$ (this is the ÒinÓ-state for the problem). Show that in this case the wave function for the electron in the lowest nontrivial order in the perturbation is

$$\Psi^+_e(x) = \psi_{p_1,s_1}(x) - e\sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E}\psi_{p,s}(x)\int \mathrm{d}^4x'\theta(t-t')\bar{\psi}(x')_{p,s}\not{A}(x')\psi_{p_1,s_1}(x')$$

when $t \to \infty$. With this solution construct the S-matrix element for the process $\psi_{p_1} \to \psi_{p_3}$:

$$\langle p_3, s_3|S|p_1, s_1\rangle \equiv {}_{\text{out}}\langle\psi_{p_3,s_3}|\psi_{p_1,s_1}\rangle_{\text{in}} = \langle p_3, s_3|p_1, s_1\rangle - e\int \mathrm{d}^4x'\bar{\psi}_{p_3,s_3}(x')\not{A}(x')\psi_{p_1,s_1}(x') \,.$$

**4. Rutherford scattering.** Assume that the vector potential appearing in the previous problem describes the static Coulomb potential of a nucleus with charge $-Ze$:

$$A_0 = \frac{-Ze}{4\pi|\mathbf{x}|}\,, \qquad \mathbf{A} = 0\,.$$

Compute the $S$ and $T$-matrices by a direct substitution (you have to know/figure out what is the mometum space representation of $1/|\mathbf{x}|$. Hint: $\mathbf{A}$ is itself essentially a Greens function for a static delta function charge current). From this compute the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$ for the process, (sum over the final state spins and average over the initial state spins) when

$$\mathrm{d}\sigma = \frac{|S_{fi}|^2}{2E_1 v_1 T}\frac{\mathrm{d}^3 p_3}{(2\pi)^3 2E_3}\,.$$

Can you say why $\mathrm{d}\sigma$ here has not been divided by the factor $2E_{\text{nucleus}}V$ in contrast with the generic equation (3.42) derived in the lectures?