Quantum field theory, fall 2023, **Exercise 3.**

1. Show that the normalization convention for spinors $\xi_s^\dagger \xi_{s'} \equiv \delta_{ss'}$ implies the results

$$\bar{u}_s(p) u_{s'}(p) = 2m\delta_{ss'} \qquad \bar{v}_s(p) v_{s'}(p) = -2m\delta_{ss'}$$

$$u_s^\dagger(p) u_{s'}(p) = v_s^\dagger(p) v_{s'}(p) = 2E_p \delta_{ss'}$$

Furtheremore prove the following formulas for the spin-sums

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m \qquad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m$$

2. Show the following properties of the Dirac spinors

$$u^*(-\mathbf{p}, -s) = -\gamma^1\gamma^3 u(\mathbf{p}, s)\,,$$
$$v^*(-\mathbf{p}, -s) = +\gamma^1\gamma^3 v(\mathbf{p}, s)$$

and

$$u(\mathbf{p}, s) = -i\gamma^2 v^*(\mathbf{p}, s)\,,$$
$$v(\mathbf{p}, s) = -i\gamma^2 u^*(\mathbf{p}, s)\,.$$

3. Derive the Gordon-identity:

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right] u(p)$$

where $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and $q^\nu \equiv p'^\nu - p^\nu$.

4. Derive a field operatorator representation for angular momentum operator (akin to the Hamiltonian and the momentum operators). First find the transform of the Dirac field under rotation: $\psi \to \Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x)$ and require that the theory remains invariant. Angular momentum operator can now be obtained as a zeroth component of the Noether current corresponding to the rotation symmetry. (Hint. Peskin-Schröder, page 60.)

5. Find the transformation properties of the conserved current of a free charged scalar field

$$j^\mu = i(\partial^\mu\phi^*)\phi - (\partial_\mu\phi)\phi^*)$$

under the discrete tranformations C, P, T and CPT.

6. The Lagrange density for the Quantum Chromodynamics (QCD) can be written as

$$\mathcal{L}_{QCD} = \bar{Q}(i\not{D} - m)Q - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a$$

where Q is a three-component object wrt. $SU(3)$-group: $Q = (q_r, q_b, q_g)^T$ (the components q_i are normal spinor fields), covariant derivative is defined as $D_\mu \equiv \partial_\mu - igT^a A^a_\mu$ and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu.$$

T^a's are the SU(3) generators that obey the Lie algebra $[T^a, T^b] = if^{abc}T^c$. Derive the tranformation properties of the covariant derivative D_μ and of the field $T^a A^a_\mu$ by requiring that $\bar{Q}i\not{D}Q$ is invariant under the local SU(3)-phase transformation

$$Q \to UQ \equiv e^{iT^a\theta^a(x)}Q\,.$$

Now show that the field strength tensor can be constructed as a commutator

$$F_{\mu\nu} \equiv T^a F^a_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu]$$

and hence (the normalization of T^a's in the fundamental representation is $\mathrm{Tr}(T^aT^b) \equiv \frac{1}{2}\delta^{ab}$):

$$-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a = \frac{1}{2g^2}\mathrm{Tr}([D_\mu, D_\nu]^2)\,.$$

Use these results to show that the entire lagrangian $\mathcal{L}_{QCD}$ is invariant. Show that F^2-term contains terms $\sim A^2\partial A$ and $\sim A^4$. What do these terms describe?