Quantum field theory, fall 2023, **Excercise 10.**

1. Prove the Gaussian identity

$$\int_{-\infty}^{-\infty} \prod_i^N dx_i e^{-x^T A x + Q^T x} = \frac{\pi^{N/2}}{\sqrt{\det A}} e^{\frac{1}{4} Q^T A^{-1} Q},$$

where $x = (x_1, x_2, ..., x_N)$, A is a symmetric $N \times N$-matrix with $\det A \neq 0$, and Q is an arbitrary vector $Q = (q_1, q_2, ..., q_N)$. Now show that the normalization factor N_{KG} in the Klein-Gordon theory partition function $Z[J]$ can be expressed as a functional derivative:

$$\int \mathcal{D}\phi \, e^{i \int d^4x \mathcal{L}_{\mathrm{KG}}} = \mathrm{const} \times [\mathrm{Det}(\partial^2 + m^2)]^{-1/2}.$$

Hint: discretize theory and perform Gaussian integrals over fields ϕ_{x_i}.

2. Show that the tensor function $D^{-1}_{\mu\nu}(k) \equiv -g^{\mu\nu}k^2 + k^\mu k^\nu$ does not have an inverse which satisfies the equation $D^{-1}_{\mu\nu}(k) D^{\nu\rho}(k) = \delta^\rho_\mu$. Show also that $D^{-1}_{\mu\alpha}(k)$ is actually a projection operator. Then show that a tensor

$$D^{-1}_{\mu\nu}(k, \alpha) \equiv -g^{\mu\nu}k^2 + \alpha k^\mu k^\nu$$

does possess an inverse if $\alpha \neq 1$ and compute that inverse.

3. Show that in the Yang-Mills theory the field strength tensor $F_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu]$ obeys the Bianchi identity

$$[F_{\mu\nu}, D_\rho] + [F_{\nu_\rho}, D_\mu] + [F_{\rho\mu}, D_\nu] = 0.$$

For the definition of the covariant derivative see lectures, page 207.

4. Derive the Feynman parametrizations

$$\begin{aligned}
\frac{1}{AB} &= \int_0^1 dx \int_0^1 dy \, \frac{\delta(x+y-1)}{(xA+yB)^2} \\
\frac{1}{AB^n} &= \int_0^1 dx \int_0^1 dy \, \frac{ny^{n-1}\, \delta(x+y-1)}{(xA+yB)^{n+1}} \\
\frac{1}{A_1 A_2 \dots A_n} &= \int_0^1 \prod_{i=1}^n dx_i \, \frac{(n-1)!\; \delta(\sum_i x_i - 1)}{(x_1 A_1 + x_2 A_2 + \cdots + x_n A_n)^n}.
\end{aligned}$$

(This problem is a preparation for the chapter 6 on renormalization.)