Quantum field theory, fall 2023, **Exercise 1.**

1. Derive the inhomogenous Maxwell equations starting from Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu.$$

Show (by use of Noether theorem) that current conservation $\partial_\mu j^\mu = 0$ follows if we require the invariance of the action in a local gauge transform $A_\mu \to A_\mu(x) + \partial_\mu \alpha(x)$.

2. Show that Lagrangian density

$$\mathcal{L} = \left(\partial_\mu \vec{\phi}^*\right) \cdot \left(\partial^\mu \vec{\phi}\right) - m^2 \vec{\phi}^* \cdot \vec{\phi},$$

with complex field $\vec{\phi} \equiv \binom{\phi_1}{\phi_2}$, is invariant under transformation $\phi_i \to \phi_i' = U_{ij}\phi_j$, where U is unitary 2×2-matrix. In general, the matrix U can be written as

$$U = e^{-\frac{i}{2}\alpha} e^{-\frac{i}{2}\theta_k \sigma^k},$$

where α and θ_k are real constants and σ_k's are the Pauli matrices. Furthermore, show that the conserved charges corresponding this transformation are

$$Q = \int d^3x \frac{i}{2} \left(\phi_a^* \pi_a^* - \phi_a \pi_a\right)$$

and

$$Q^k = \int d^3x \frac{i}{2} \left[\phi_a^* (\sigma^k)_{ab} \pi_b^* - \pi_a (\sigma^k)_{ab} \phi_b\right].$$

3. Show in the case of a free Klein-Gordon field that the integral over $T^{0\mu}$ gives a 4-vector whose components are the Hamiltonian- and the momentum density operators:

$$\int d^3x T^{0\mu} = (H, \mathbf{P}).$$

(That is, derive an integral representation for H and P and interpret the results physically.) Now quantize the field $\phi(x)$ and derive expressions for H and $\mathbf{P}$ (normal-ordered) in terms of the creation and annihilation operators $a_\mathbf{p}^\dagger$ and $a_\mathbf{p}$.
Let us now show that the energy-momentum tensor can always be chosen to be symmetric. The key is to first prove that, without changing physics, one can add to $T^{\mu\nu}$ a divergence $\partial_\lambda E^{\lambda\mu\nu}$, where $E^{\lambda\mu\nu}$ is arbitrary apart from being antisymmetric wrt. to its first two indices. (That is, the resulting new $T^{\mu\nu}$ must remain conserved and give rise to the same observed quantities H:lle and $\mathbf{P}$ as the nontranformed one.)

4. Show that

$$\frac{d^3p}{E_p} \qquad \text{and} \qquad E_p\delta^3(\mathbf{p}-\mathbf{p}') \qquad \text{with} \qquad E_p = \sqrt{\mathbf{p}^2+m^2}$$

are invariant under Lorentz transformations.

5. Prove the commutation relations

$$[P^\mu, a_\mathbf{p}^\dagger] = p^\mu a_\mathbf{p}^\dagger\,, \qquad [P^\mu, a_\mathbf{p}] = -p^\mu a_\mathbf{p}$$

and show based on these results that in Heisenberg picture a free particle field operator has a representation

$$\phi(x) = e^{iP\cdot x}\phi(0)e^{-iP\cdot x}.$$