Quantum Field Theory I

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Practicalities:

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- **Schedule:**

  Lectures on Mondays & Wednesdays (FYS3, 10-12), last one on **24.11**

  Exercises on Mondays (FYS3, 12-14), last ones on **29.11**

  Exams: Midterm exams on **22.10** and **3.12** (recommended option)

  Final exam on **14.01.2022** (suicide).

- **Grading:**

  Your final n.o. points will be computed by the function,

$$30 \times \left(\frac{\text{excercise points}}{\text{max. exercise points}} \right) + 30 \times \left(\frac{\text{exam points}}{\text{max. exam points}} \right) \leq 60\,,$$

  rounded up to the next integer. The grade is determined by the table,

Grade	points
1	30 – 35
2	36 – 41
3	42 – 47
4	48 – 53
5	54 – 60

Content:

1. **Quantization of scalar fields**

2. **Quantization of the Dirac field**

3. **Interacting fields**

4. **Feynman rules for Quantum Electrodynamics**

5. **Lehmann-Symanzik-Zimmerman reduction**

6. **Basic QED processes**

7. **Introduction to radiative corrections and renormalization**

The notes follow closely our course book **Peskin & Schroeder: An Introduction to Quantum Field Theory**, but not exactly. In particular, Peskin does not cover the canonical quantization of QED. In several places these notes also take a slightly different view point than how things are presented in Peskin. Most of the notation & conventions in these notes are identical with those of Peskin but not all.

1 Quantization of scalar fields

1.1 Need for quantum fields

[Peskin 2.1]

The necessity of a field theoretical approach in contrast to single-particle quantum mechanics manifests itself in several ways. For a non-relativistic free particle

$$E = \frac{\mathbf{p}^2}{2m} \,. \tag{1.1}$$

Replacing the energy E and momentum $\mathbf{p}$ by the differential operators,

$$E \to i\frac{\partial}{\partial t} \,, \qquad \mathbf{p} \to -i\boldsymbol{\nabla} \,, \tag{1.2}$$

and understanding the resulting equation to act on a complex wavefunction ψ, one finds the standard Schrödinger equation,

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi \,. \tag{1.3}$$

Solutions consistent with Eq. (1.1), are plane waves,

$$\psi = Ne^{-i(Et-\mathbf{p}\cdot\mathbf{x})} \,, \quad E = \mathbf{p}^2/2m \,, \tag{1.4}$$

which fulfill the **continuity equation**,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j} = 0 \,, \tag{1.5}$$

with

$$\rho = |\psi|^2 = |N|^2 \,, \tag{1.6}$$

$$\mathbf{j} = -\frac{i}{2m}\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right) = \frac{\mathbf{p}}{m}|N|^2 \,. \tag{1.7}$$

There are no difficulties in the standard quantum-mechanical interpretation of ρ as the probability density and, the energy eigenvalues are positive,

$$i\frac{\partial \psi}{\partial t} = i(-iE)\psi = E\psi = \frac{\mathbf{p}^2}{2m}\psi \,. \tag{1.8}$$

The problems arise when one tries to implement the relativistic relation,

$$E^2 = \mathbf{p}^2 + m^2 \,, \tag{1.9}$$

and the Lorentz covariance (we'll come back to this in Sect. 2.2). In this case, the replacements of Eq. (1.2) lead to the **Klein-Gordon equation**,

$$\left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 + m^2\right)\psi = \left(\partial_\mu \partial^\mu + m^2\right)\psi = 0 \,. \tag{1.10}$$

Solutions consistent with Eq. (1.9) are again plane waves,

$$\psi = N e^{-i(Et - \mathbf{p}\cdot\mathbf{x})} \,, \quad E = \pm\sqrt{\mathbf{p}^2 + m^2} \,. \tag{1.11}$$

The continuity relation in Eq. (1.5) now holds with,

$$\rho = \frac{i}{2m}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) = 2E|N^2| \,, \tag{1.12}$$

$$\mathbf{j} = -\frac{i}{2m}\left(\psi^* \boldsymbol{\nabla}\psi - \psi \boldsymbol{\nabla}\psi^*\right) = 2\mathbf{p}|N^2| \,. \tag{1.13}$$

As ρ can be negative, it can no longer be interpreted as being the probability density. For charged particles it could be taken as the charge density, but how about electrically neutral particles? Also the energy eigenvalues can be negative,

$$i\frac{\partial \psi}{\partial t} = i(-iE)\psi = E\psi = \left(\pm\sqrt{\mathbf{p}^2 + m^2}\right)\psi \,. \tag{1.14}$$

Similar difficulties arise in the case of Dirac equation: although there one can define a positive-definite probability density ρ, the negative-energy solutions persist. One can wangle around these problems by invoking the concept of "Dirac sea" or interpreting the negative-energy solutions as describing antiparticles (Feynman-Stückelberg). All this is somewhat clumsy. The resolution provided by the quantum field theory is to abandon the interpretation of Klein-Gordon and Dirac equations as being single-particle wave equations. They are taken to be **field equations** instead.

The quantum mechanics also violates causality. Consider the amplitude of a free particle to propagate from $\mathbf{x}_0$ to $\mathbf{x}$ within a time interval t,

$$U(t) = \langle \mathbf{x} | e^{-i\hat{H}t} | \mathbf{x}_0 \rangle \,. \tag{1.15}$$

For a nonrelativistic particle $\hat{H} = \hat{\mathbf{p}}^2/2m$, so

$$\begin{aligned} U(t) &= \langle \mathbf{x} | e^{-it\hat{\mathbf{p}}^2/(2m)} | \mathbf{x}_0 \rangle \\ &= \int d^3p \langle \mathbf{x} | e^{-it\hat{\mathbf{p}}^2/(2m)} | \mathbf{p} \rangle \underbrace{\langle \mathbf{p} | \mathbf{x}_0 \rangle}_{e^{-i\mathbf{p}\cdot\mathbf{x}_0}/(2\pi)^{-3/2}} \\ &= \int d^3p \frac{e^{-i\mathbf{p}\cdot\mathbf{x}_0}}{(2\pi)^{3/2}} e^{-it\mathbf{p}^2/(2m)} \langle \mathbf{x} | \mathbf{p} \rangle = \int \frac{d^3p}{(2\pi)^3} e^{-i\mathbf{p}\cdot(\mathbf{x}_0 - \mathbf{x})} e^{-it\mathbf{p}^2/(2m)} \,. \end{aligned} \tag{1.16}$$

By completing the square,

$$\begin{aligned} &\exp\left[-i\mathbf{p}\cdot(\mathbf{x}_0 - \mathbf{x}) - it\mathbf{p}^2/(2m)\right] \\ =&\exp\left[\frac{im}{2t}(\mathbf{x}_0 - \mathbf{x})^2 - \frac{it}{2m}[\mathbf{p} + (m/t)(\mathbf{x}_0 - \mathbf{x})]^2\right] , \end{aligned} \tag{1.17}$$

so

$$U(t) = e^{im(\mathbf{x}_0 - \mathbf{x})^2/(2t)} \int \frac{d^3p}{(2\pi)^3} e^{-\frac{it}{2m}\mathbf{p}^2} \,.$$

The remaining Gaussian integral is a standard one and can be done by,

$$\int dx e^{-bx^2} = \sqrt{\frac{\pi}{b}} \,, \qquad \mathrm{Re}\, b \geq 0 \,. \tag{1.18}$$

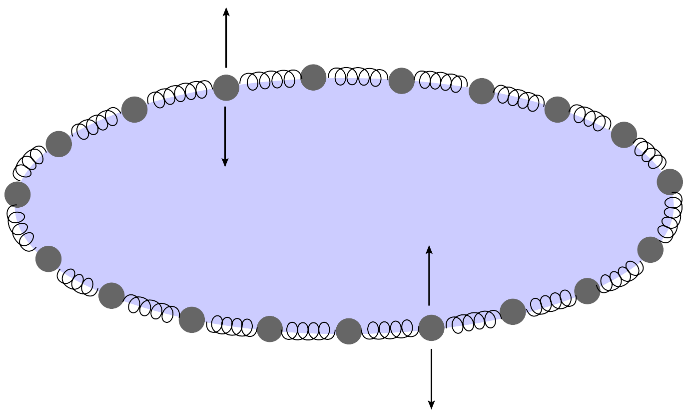
As a result,

$$U(t) = \left(\frac{m}{2\pi it}\right)^{3/2} e^{im(\mathbf{x}_0 - \mathbf{x})^2/(2t)} \,.$$

This is clearly non zero even if $|\mathbf{x}_0 - \mathbf{x}| > t$, i.e. there is a finite probability for the particle to propagate across space-like intervals. The conclusion remains the same even if one uses the relativistic relation $E = \sqrt{\mathbf{p}^2 + m^2}$ (Ex.). Also this type of inconsistency gets resolved in quantized field theory – measurements done at space-like distances will have no mutual effect.

1.2 Notion of a classical field

Let us consider a simple mechanical model with point-like masses attached to each other with light springs. Let the overall length of the system be L, containing N masses each with mass m. The distance between the masses is then $\Delta x = L/N$, and the average density $\rho = Nm/L$.



Let us also suppose that the masses can move only up and downwards (in direction y). The kinetic energy of the system is then

$$T = \sum_{n=1}^{N} \frac{1}{2} m \dot{y}_n^2 \,, \tag{1.19}$$

and we assume that – for small stretches – the potential energy stored in each spring is proportional to the overall strech of that spring ($y_0 = y_N$),

$$U = \sum_{n=1}^{N} \frac{\kappa N}{2L} \left(y_n - y_{n-1}\right)^2 \,, \tag{1.20}$$

where κ is a string-tension constant. The Lagrange function of the system is thus,

$$L = T - U = \sum_{n=1}^{N} \left[\frac{1}{2} m \dot{y}_n^2 - \frac{\kappa N}{2L} \left(y_n - y_{n-1}\right)^2 \right] . \tag{1.21}$$

In the limit of large N, the system becomes eventually continuous. Let us denote by $y_n(t) = \phi(x_n, t)$ the deviation of the nth mass from the equilibrium. Then,

$$\dot{y}_n \to \frac{\partial \phi(x_n, t)}{\partial t} = \dot{\phi}(x_n, t) \tag{1.22}$$

$$\frac{y_n - y_{n-1}}{\Delta x} \to \frac{\partial \phi(x_n, t)}{\partial x_n} = \partial_{x_n} \phi(x_n, t) \,. \tag{1.23}$$

Keeping the density ρ constant, we get

$$L = \sum_{n=1}^{N} \Delta x \left[\frac{1}{2} \rho \dot{\phi}(x_n, t)^2 - \frac{1}{2} \kappa \left(\partial_{x_n} \phi(x_n, t) \right)^2 \right]$$

$$\xrightarrow{N \to \infty} \int_0^L dx \underbrace{\left[\frac{1}{2} \rho \dot{\phi}(x, t)^2 - \frac{1}{2} \kappa \left(\partial_x \phi(x, t) \right)^2 \right]}_{= \textbf{Lagrange density } \mathcal{L}} \,. \tag{1.24}$$

We call the function ϕ as a classical **field variable** (or just a field), and it describes the state of the system in an arbitrary point.

1.3 Classical equations of motion [Peskin 2.2]

As in mechanics, the equations of motion for the fields are determined by finding the critical point of the action,

$$S = \int dt L = \int_V d^4 x \, \mathcal{L}(\phi, \partial_\mu \phi) \,. \tag{1.25}$$

To do this, we compute the **functional derivative** of the action S to "direction" $f(x)$:

$$\int_V d^4 x \frac{\delta S}{\delta \phi(x)} f(x) \tag{1.26}$$

$$\equiv \lim_{\epsilon \to 0} \int_V d^4 x \frac{\mathcal{L}\left[\phi(x) + \epsilon f(x), \partial_\mu \phi(x) + \epsilon \partial_\mu f(x)\right] - \mathcal{L}\left[\phi(x), \partial_\mu \phi(x)\right]}{\epsilon}$$

$$= \int_V d^4 x \left[\frac{\partial \mathcal{L}}{\partial \phi} f(x) + \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \partial_\mu f(x) \right]$$

$$= \int_V d^4 x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} f(x) + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} f(x) \right] - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \right] f(x) \right\}$$

$$= \int_V d^4 x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \right] \right\} f(x) \,,$$

in which the surface term vanishes by the Gauss theorem if we suppose that $f = 0$ in the boundaries of the integration volume. So, we see that when

$$\frac{\delta S}{\delta \phi(x)} \equiv \frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi(x) \right)} \right] = 0 \,, \tag{1.27}$$

the functional derivative is zero to arbitrary direction. Thus, the critical point (minimum) of the actions is given by the condition,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \right] = 0 \,. \tag{1.28}$$

This is the **Euler-Lagrange equation**. If we now substitute the Lagrange density of the spring (1.24) to equation (1.28), we find

$$\left(\frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \phi(x, t) = 0 \,, \quad v = \sqrt{\kappa / \rho} \,. \tag{1.29}$$

As well known, this describes transverse waves propagating with speed v, and its unique solutions are of the form $f(x \pm vt)$.

Hamiltonian formalism:

Instead of the Lagrange technique, we can equally use the Hamiltonian formalism. This is the basis of the canonical quantization. We define the Hamiltonian function H by,

$$H \equiv \int_V d^3x \, \mathcal{H} \equiv \int_V d^3x \, \left[\pi(x) \dot{\phi}(x) - \mathcal{L}(\phi, \partial_\mu \phi) \right] \tag{1.30}$$

$$\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} \,. \tag{1.31}$$

The quantity $\pi(x)$ is called the **conjugate momentum density** of the field $\phi(x)$. By definition, H does not depend explicitly on the time derivative of the field, $\dot{\phi}$. Let us compute the following functional derivatives:

$$\frac{\delta H}{\delta \pi(x)} = \frac{\partial}{\partial \pi(x)} \left[\pi(x)\dot{\phi}(x) - \mathcal{L} \right] = \dot{\phi}(x) \tag{1.32}$$

$$\frac{\delta H}{\delta \phi(x)} = - \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial \nabla \phi(x)} \right] = -\dot{\pi}(x)\,, \tag{1.33}$$

where we used the Euler-Lagrange equation in the last equality. These comprise the **Hamiltonian equations of motion** for classical fields,

$$\frac{\delta H}{\delta \phi(x)} = -\dot{\pi}(x)\,, \quad \frac{\delta H}{\delta \pi(x)} = \dot{\phi}(x)\,. \tag{1.34}$$

Klein-Gordon field:

The non-interacting Klein-Gordon scalar field if defined by the Lagrange density

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu \phi\right) \left(\partial^\mu \phi\right) - \frac{1}{2} m^2 \phi^2\,. \tag{1.35}$$

Using the Euler-Lagrange equation, we get the equation of motion

$$\left(\partial_\mu \partial^\mu + m^2\right) \phi = 0\,, \quad \left(\partial_\mu \partial^\mu = \partial_t^2 - \nabla^2\right), \tag{1.36}$$

known as the Klein-Gordon equation (relativistic Schrödingerin equation). The momentum density is now $\pi(x) = \dot{\phi}(x)$, and the Hamiltonian density reads

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\,. \tag{1.37}$$

1.4 Noether's theorem and conservation laws

[Peskin 2.2]

Let us consider an infinitesimal transformation,

$$\phi(x) \longrightarrow \phi(x) + \alpha \Delta\phi(x), \tag{1.38}$$

in which α is infinitesimal and $\Delta\phi(x)$ some function. If the Lagrange density remains unchanged in this transformation or it changes only by a surface term,

$$\mathcal{L}(x) \longrightarrow \mathcal{L}(x) + \alpha \partial_\mu \mathcal{J}^\mu(x)\,, \tag{1.39}$$

↑— **"surface term"**

the Noether's theorem says that this transformation involves a conserved four-current and a conserved charge. The surface terms are irrelevant as they do not affect the dynamics.

Example 1: Let us consider a Lagrangian containing only the kinetic term,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right)\,, \tag{1.40}$$

and examine a transformation $\phi(x) \longrightarrow \phi(x) + \alpha$, in which α is an infinitesimal constant. The Lagrange density is clearly invariant under this transformation. Let's then calculate the same variation in another way:

$$\Delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\Delta\phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\partial_\mu(\Delta\phi) \quad (\Delta\phi = \alpha) \tag{1.41}$$

$$\overset{\text{E.L.}}{=} \alpha \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\right] = \alpha \partial_\mu \left[\partial^\mu \phi(x)\right] = 0\,. \tag{1.42}$$

$\Longrightarrow$ The transformation involves a **conserved current** $j^\mu(x) = \partial^\mu \phi(x)$.

The conserved charge is obtained by integrating,

$$0 = \int d^3x\, \partial_\mu j^\mu = \int d^3x \left[\partial_0 j^0 + \nabla \cdot \vec{j}\right] = \partial_0 \int d^3x\, j^0 = \partial_0 Q\,, \tag{1.43}$$

which shows that there is a conserved charge $Q = \int d^3x\, j^0$ (meaning that its time derivative is zero).

Example 2: Let us consider a translation

$$x^\mu \longrightarrow x^\mu + \alpha^\mu\,. \tag{1.44}$$

The Lagrange density changes as,

$$\Delta\mathcal{L} = \mathcal{L}(x+\alpha) - \mathcal{L}(x) = \alpha^\mu \partial_\mu \mathcal{L}(x)\,, \tag{1.45}$$

so $\mathcal{L}$ changes only by a surface term. The field itself changes similarly,

$$\phi(x) \to \phi(x+a) = \phi(x) + \alpha^\mu \partial_\mu \phi(x) \,, \tag{1.46}$$

so we mark $\Delta\phi = \alpha^\mu \partial_\mu \phi(x)$. On the other hand,

$$\begin{aligned} \Delta\mathcal{L}(x) &= \frac{\partial \mathcal{L}}{\partial \phi}\Delta\phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \phi\right)}\partial_\nu \Delta\phi \\ &= \alpha^\mu \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \phi\right)}\right]\partial_\mu \phi + \alpha^\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \phi\right)}\partial_\mu\left(\partial_\nu \phi\right) \\ &= \alpha^\mu \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \phi\right)}\partial_\mu \phi\right] \,. \end{aligned} \tag{1.47}$$

This indicates,

$$\alpha^\mu \partial_\mu \mathcal{L} - \alpha^\mu \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \phi\right)}\partial_\mu \phi\right] = 0 \tag{1.48}$$

$$\alpha^\mu \partial_\nu \left[\delta^\nu_\mu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \phi\right)}\partial_\mu \phi\right] = 0 \,. \tag{1.49}$$

The conserved current is consequently

$$T^\nu{}_\mu \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \phi\right)}\partial_\mu \phi - \delta^\nu_\mu \mathcal{L} \,. \tag{1.50}$$

We shall call this the **energy-momentum tensor**, and it entails four conserved currents,

$$P^\mu = \int d^3x \, T^{0\,\mu} \,. \tag{1.51}$$

This corresponds to the field four momentum:

$$T^{0\,0} = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \phi\right)}\partial_0 \phi - \mathcal{L} = \pi(x)\dot\phi(x) - \mathcal{L} \overset{1.30}{=} \mathcal{H} \tag{1.52}$$

$$T^{0\,i} = -\frac{\partial \mathcal{L}}{\partial\left(\partial_0 \phi\right)}\partial_i \phi = -\pi(x)\partial_i \phi(x) \equiv \mathcal{P}^i \tag{1.53}$$

From the first equation we see that $T^{0\,0}$ is the energy density $\mathcal{H}$, so P^0 is the total energy involved. Because P^μ is a four vector, the other three components must correspond to the spatial momenta, and we can interpret

$T^{0\,i}$ naturally as the momentum density. In other words, the independence of the action on the time/coordinate translations implies the conservation of energy/momentum. Similarly, e.g. the invariance under spatial rotations implies the conservation of angular momentum (Ex.).

1.5 Quantization of the Klein-Gordon field [Peskin 2.3]

We will now quantize the Klein-Gordon scalar field. A neat way to do this is to notice first a formal equivalence with the familiar harmonic oscillator whose quantization should be familiar from Quantum Mechanics I course. Let us first write the field $\phi(\vec{x},t)$ as a 3-D Fourier transform, of the momentum-space field $\phi(\vec{p},t)$,

$$\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \phi(\vec{p},t)\,, \tag{1.54}$$

where $\phi^*(\vec{p},t) = \phi(-\vec{p},t)$ since $\phi(\vec{x},t)$ should be real. Substituting to the Klein-Gordon equation (1.36), we get

$$\left(\frac{\partial^2}{\partial t^2} + \mathbf{p}^{\,2} + m^2\right)\phi(\vec{p},t) = 0\,, \tag{1.55}$$

which is formally identical with the equation for a one-dimensional harmonic oscillator

$$m\ddot{x}(t) = -kx(t) \Leftrightarrow \ddot{x}(t) + \omega x(t) = 0 \tag{1.56}$$

when we identify $\omega = \sqrt{\mathbf{p}^{\,2} + m^2}$ as the frequency and $x(t) \leftrightarrow \phi(\vec{p},t)$. Each Fourier mode thus separately obeys the dynamics of an harmonic oscillator.

Harmonic oscillator in Quantum Mechanics

The Hamiltonian operator for the harmonic oscillator is

$$\hat{H} = \frac{\hat{p}^2}{2m_0} + \frac{1}{2}m_0\omega^2\hat{x}^2\,. \tag{1.57}$$

The eigenvalues and eigenstates can be solved by first defining the "ladder operators"

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m_0\omega}{\hbar}}\hat{x} + i\frac{1}{\sqrt{m_0\omega\hbar}}\hat{p}\right) , \tag{1.58}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m_0\omega}{\hbar}}\hat{x} - i\frac{1}{\sqrt{m_0\omega\hbar}}\hat{p}\right) , \tag{1.59}$$

which we can invert to

$$\hat{x} = \sqrt{\frac{\hbar}{m_0\omega}}\left(\frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}\right) , \tag{1.60}$$

$$\hat{p} = \sqrt{m_0\omega\hbar}\left(\frac{\hat{a}-\hat{a}^\dagger}{i\sqrt{2}}\right) . \tag{1.61}$$

The canonical commutation relations $[\hat{x},\hat{x}] = [\hat{p},\hat{p}] = 0$, $[\hat{x},\hat{p}] = i\hbar$ imply the commutation relation $[\hat{a},\hat{a}^\dagger] = 1$ for the ladder operators. In terms of the ladder operators we have,

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right) , \quad \hat{N} \equiv \hat{a}^\dagger\hat{a} , \tag{1.62}$$

so it suffices to find the eigenstates $|n\rangle$ of the operator $\hat{N}$. Assuming that $\hat{N}|n\rangle = n|n\rangle$, we easily find,

$$\hat{N}\hat{a}|n\rangle = (n-1)\hat{a}|n\rangle \tag{1.63}$$

$$\hat{N}\hat{a}^\dagger|n\rangle = (n+1)\hat{a}^\dagger|n\rangle , \tag{1.64}$$

so that acting with $\hat{a}$ and $\hat{a}^\dagger$ we get new eigenstates of $\hat{N}$, whose eigenvalues decrease/increase by one unit. On the other hand, the norm of the states must be positive, so

$$0 \leq |\hat{a}|n\rangle|^2 = \langle n|\hat{a}^\dagger\hat{a}|n\rangle = \langle n|\hat{N}|n\rangle = n\langle n|n\rangle . \tag{1.65}$$

From this we see that n must be a positive integer, as otherwize we would eventually get states with negative norm. This means that there exists a state $|0\rangle$ with the smallest eigenvalue, for which $\hat{a}|0\rangle = 0$. Starting from this ground state we get the entire spectrum of states by operating with $\hat{a}^\dagger$,

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n|0\rangle , \tag{1.66}$$

where the normalization has been selected such that $\langle n|n\rangle = 1$ if we agree that $\langle 0|0\rangle = 1$. The obtained states $|n\rangle$ are thus eigenstates of the Hamiltonian,

$$\hat{H}|n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)|n\rangle\,. \tag{1.67}$$

We are now ready to quantize the Klein-Gordon field. We identify,

$$\hat{\phi}(\mathbf{p}) \longleftrightarrow \hat{x}\,. \tag{1.68}$$

Each momentum value $\mathbf{p}$ has now its own ladder operators $\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{p}}$ so we define (we put here $m_0 = \hbar = 1$),

$$\hat{a}_{\mathbf{p}} = \frac{1}{\sqrt{2}}\left(\sqrt{\omega}\hat{\phi}(\mathbf{p}) + i\frac{1}{\sqrt{\omega}}\hat{\pi}(\mathbf{p})\right)\,, \tag{1.69}$$

$$\hat{a}^\dagger_{\mathbf{p}} = \frac{1}{\sqrt{2}}\left(\sqrt{\omega}\hat{\phi}^\dagger(\mathbf{p}) - i\frac{1}{\sqrt{\omega}}\hat{\pi}^\dagger(\mathbf{p})\right) \tag{1.70}$$

$$= \frac{1}{\sqrt{2}}\left(\sqrt{\omega}\hat{\phi}(-\mathbf{p}) - i\frac{1}{\sqrt{\omega}}\hat{\pi}(-\mathbf{p})\right),$$

where we used the reality conditions $\hat{\phi}^\dagger(\mathbf{p}) = \hat{\phi}(-\mathbf{p})$ and $\hat{\pi}^\dagger(\mathbf{p}) = \hat{\pi}(-\mathbf{p})$. From these equations we get

$$\hat{\phi}(\mathbf{p}) = \frac{1}{\sqrt{2\omega}}\left(\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{-\mathbf{p}}\right) \tag{1.71}$$

$$\hat{\pi}(\mathbf{p}) = -i\sqrt{\frac{\omega}{2}}\left(\hat{a}_{\mathbf{p}} - \hat{a}^\dagger_{-\mathbf{p}}\right)\,. \tag{1.72}$$

By a Fourier transfrom, we get the position-space representation for the field operators $\hat{\phi}(\mathbf{x})$ and $\hat{\pi}(\mathbf{x})$,

$$\hat{\phi}(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_{\mathbf{p}}}}\left[\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{-\mathbf{p}}\right]e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{1.73}$$

$$\hat{\pi}(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{E_{\mathbf{p}}}{2}}\left[\hat{a}_{\mathbf{p}} - \hat{a}^\dagger_{-\mathbf{p}}\right]e^{i\mathbf{p}\cdot\mathbf{x}}\,. \tag{1.74}$$

where we have now written $E_{\mathbf{p}}$ instead of $\omega_{\mathbf{p}}$ as it anyway represents the relativistic energy $E_{\mathbf{p}} = \sqrt{m^2 + \mathbf{p}^2}$. These can also be inverted,

$$\hat{a}_{\mathbf{p}} = \int \frac{d^3x}{\sqrt{2E_{\mathbf{p}}}} \left[i\hat{\pi}(\mathbf{x}) + E_{\mathbf{p}}\hat{\phi}(\mathbf{x}) \right] e^{-i\mathbf{p}\cdot\mathbf{x}} \,, \tag{1.75}$$

$$\hat{a}^\dagger_{\mathbf{p}} = \int \frac{d^3x}{\sqrt{2E_{\mathbf{p}}}} \left[-i\hat{\pi}(\mathbf{x}) + E_{\mathbf{p}}\hat{\phi}(\mathbf{x}) \right] e^{i\mathbf{p}\cdot\mathbf{x}} \,. \tag{1.76}$$

From now on, we no longer drag along the explicit operator designation (e.g. $\hat{\phi}$), but it should be clear from the context whether the fields are operators or ordinary functions. The canonical commutation relations of a harmonic oscillator $[a, a^\dagger] = 1$, $[a, a] = [a^\dagger, a^\dagger] = 0$ generalize to

$$[a_{\mathbf{p}}, a_{\mathbf{k}}] = [a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{k}}] = 0 \,, \tag{1.77}$$

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{k}}] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{k}) \,, \tag{1.78}$$

where the normalization factor $(2\pi)^3$ is a convention. By using these relations and the integral representation of the δ function,

$$\int \frac{d^n x}{(2\pi)^n} e^{ik\cdot x} = \delta^{(n)}(k) \ , \tag{1.79}$$

It is straightforward to show that

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = [\pi(\mathbf{x}), \pi(\mathbf{y})] = 0 \,, \tag{1.80}$$

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \,, \tag{1.81}$$

which correspond to the canonical commutation relations $[\hat{x}, \hat{x}] = [\hat{p}, \hat{p}] = 0$, $[\hat{x}, \hat{p}] = i\hbar$ of the harmonic oscillator. Using our earlier results, (1.37), (1.51) and (1.53), we obtain the momentum operator (Ex.),

$$P^\mu = \int \frac{d^3p}{(2\pi)^3} p^\mu \left(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \frac{1}{2}[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}] \right) \,. \tag{1.82}$$

The last term $\sim \delta^{(3)}(0) = \infty$ is expected, though somewhat disturbing. It just corresponds to the ground-state energy of the harmonic oscillator times ∞ as there are now infinite number of Fourier modes. However, in analogy to e.g. classical physics where the absolute value of the gravitational field is irrelevant, this constant is also unimportant. We will thus forget it and simply define,

$$P^\mu = \int \frac{d^3p}{(2\pi)^3} p^\mu a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \,. \tag{1.83}$$

As in the case of harmonic oscillator, the theory has a ground state $|0\rangle$ for which $a_{\mathbf{p}}|0\rangle = 0$, and the rest are obtained by acting on the ground state with $a_{\mathbf{p}}^\dagger$. Indeed,

$$P^\mu \left[\hat{a}_{\mathbf{p_1}}^\dagger \dots \hat{a}_{\mathbf{p_n}}^\dagger |0\rangle\right] = (p_1^\mu + \cdots p_n^\mu)\, \hat{a}_{\mathbf{p_1}}^\dagger \dots \hat{a}_{\mathbf{p_n}}^\dagger |0\rangle \,, \tag{1.84}$$

so that the obtained states are indeed momentum eigenstates.

1-particle states:

We normalize the 1-particle states as

$$|\mathbf{p}\rangle \equiv \sqrt{2E_{\mathbf{p}}}\, a_{\mathbf{p}}^\dagger |0\rangle \,, \tag{1.85}$$

This leads to

$$\langle \mathbf{k}|\mathbf{p}\rangle = 2E_{\mathbf{p}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{k}) \,, \tag{1.86}$$

which is Lorentz invariant (Ex.). One can easily get convinced that we can express the complete set of 1-particle states as

$$\hat{1} = \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} |\mathbf{p}\rangle\langle\mathbf{p}| \,, \tag{1.87}$$

where the phase-space elemnent $d^3p/E_{\mathbf{p}}$ is also Lorentz invariant (Ex.).

Multi-particle states:

States consisting of several particles are defined in analogy to the 1-particle states,

$$|\mathbf{p}_1 \dots \mathbf{p}_n\rangle \equiv \left[\prod_{i=1}^{n} \sqrt{2E_{\mathbf{p}_i}} a^\dagger_{\mathbf{p}_i}\right] |0\rangle \,. \tag{1.88}$$

Since $[a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{k}}] = 0$, the order of operations if not important here, i.e. $|\mathbf{p}_1\mathbf{p}_2\rangle = |\mathbf{p}_2\mathbf{p}_1\rangle$. In other words, the states are symmetric under interchange of two particles. The Klein-Gordon particles are thus **bosons**, and follow the **Bose-Einstein statistics**. The completeness relation becomes (Ex.),

$$\hat{1} = |0\rangle\langle 0| + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\prod_{i=1}^{n} \int \frac{d^3p_i}{(2\pi)^3 2E_{\mathbf{p}_i}}\right] |\mathbf{p}_1 \dots \mathbf{p}_n\rangle\langle \mathbf{p}_n \dots \mathbf{p}_1| \,. \tag{1.89}$$

Position-space states:

By using Eq. (1.73) we see that

$$\hat{\phi}(\mathbf{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} e^{-i\mathbf{p}\cdot\mathbf{x}} |\mathbf{p}\rangle \,, \tag{1.90}$$

so that

$$\langle \mathbf{k}|\hat{\phi}(\mathbf{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} e^{-i\mathbf{p}\cdot\mathbf{x}} \langle \mathbf{k}|\mathbf{p}\rangle = e^{-i\mathbf{k}\cdot\mathbf{x}} \,. \tag{1.91}$$

In the non-relativistic quantum mechanics the projection of a momentum eigenstate $|\mathbf{k}\rangle$ to position space reads,

$$\langle \mathbf{k}|\mathbf{x}\rangle = \frac{1}{(2\pi)^{3/2}} e^{-i\mathbf{k}\cdot\mathbf{x}} \,, \tag{1.92}$$

so we can interpret the state $\hat{\phi}(\mathbf{x})|0\rangle$ as a position-space eigenstate,

$$\hat{\phi}(\mathbf{x})|0\rangle \hat{=} (2\pi)^{3/2} |\mathbf{x}\rangle \,, \tag{1.93}$$

i.e. $\hat{\phi}(\mathbf{x})$ acting on $|0\rangle$ creates particles at position $\mathbf{x}$.

1.5.1 Klein-Gordon field in the Heisenberg picture [Peskin 2.4]

In the preceding analysis the field operators did not have any time dependence, so they were what we call Schrödinger-picture operators. In relativistic field theory it is more natural to use the Heisenberg picture in which the time is symmetrically involved.

Schrödinger picture:

In the Schrödinger picture the state vectors $|\psi(t)\rangle$ depend on time. The time dependence is dictated by the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle_S = H|\psi(t)\rangle_S\,. \tag{1.94}$$

If the Hamiltonian does not depend explicitly on time, we can solve the time dependence at arbitrary t if we know the state of the system at some initial time t_0,

$$|\psi(t)\rangle_S = \mathcal{U}(t,t_0)|\psi(t_0)\rangle_S\,,\quad \mathcal{U}(t,t_0) \equiv e^{-\frac{i}{\hbar}H(t-t_0)}\,, \tag{1.95}$$

where $\mathcal{U}(t,t_0)$ is a unitary evolution operator.

Heisenberg picture:

The time dependence can also be absorbed into the operators. Starting from a matrix element in the Schrödinger picture,

$$_{\mathrm{S}}\langle\phi(t)|\mathcal{O}_{\mathrm{S}}|\psi(t)\rangle_{\mathrm{S}} = {}_{\mathrm{S}}\langle\phi(0)|\mathcal{U}^\dagger(t)\,\mathcal{O}_{\mathrm{S}}\,\mathcal{U}(t)|\psi(0)\rangle_{\mathrm{S}} = {}_{\mathrm{H}}\langle\phi|\mathcal{O}_{\mathrm{H}}(t)|\psi\rangle_{\mathrm{H}}\,, \tag{1.96}$$

in which we defined the operator in the Heisenberg picture

$$\mathcal{O}_{\mathrm{H}}(t) \equiv \mathcal{U}^\dagger(t)\,\mathcal{O}_{\mathrm{S}}\,\mathcal{U}(t) = e^{\frac{i}{\hbar}Ht}\,\mathcal{O}_{\mathrm{S}}\,e^{-\frac{i}{\hbar}Ht}\,. \tag{1.97}$$

In this viewpoint, the operators are time dependent, not the states. By taking the time derivative,

$$-i\hbar\frac{\partial}{\partial t}\mathcal{O}_{\rm H}(t) = -i\hbar\frac{\partial}{\partial t}\left[e^{\frac{i}{\hbar}Ht}\mathcal{O}_{\rm S}e^{\frac{-i}{\hbar}Ht}\right] = [H, \mathcal{O}_{\rm H}(t)]\ , \tag{1.98}$$

if H does not depend on time. This is the Heisenberg equation of motion. Note that the Hamiltonian H is the same in both Schrödinger and Heisenberg pictures.

The operators in the Heisenberg picture are defined as,

$$\mathcal{O}(x) = \mathcal{O}(\mathbf{x}, t) \equiv e^{iHt}\mathcal{O}(\mathbf{x})e^{-iHt}\,, \tag{1.99}$$

where H is the Hamiltonian operator. Let's now calculate

$$e^{iHt}a_{\mathbf{p}}e^{-iHt} \ \text{ and } \ e^{iHt}a^\dagger_{\mathbf{p}}e^{-iHt}\,, \tag{1.100}$$

required to figure out $\phi(x)$. First we notice that,

$$\begin{aligned} P^\mu a_{\mathbf{p}} &= \int\frac{d^3k}{(2\pi)^3}k^\mu a^\dagger_{\mathbf{k}}a_{\mathbf{k}}a_{\mathbf{p}} = \int\frac{d^3k}{(2\pi)^3}k^\mu a^\dagger_{\mathbf{k}}a_{\mathbf{p}}a_{\mathbf{k}} \\ &= \int\frac{d^3k}{(2\pi)^3}k^\mu\left\{[a^\dagger_{\mathbf{k}}, a_{\mathbf{p}}]a_{\mathbf{k}} + a_{\mathbf{p}}a^\dagger_{\mathbf{k}}a_{\mathbf{k}}\right\} \\ &= \int\frac{d^3k}{(2\pi)^3}k^\mu\left\{-(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{p})a_{\mathbf{k}}\right\} + a_{\mathbf{p}}P^\mu \\ &= -p^\mu a_{\mathbf{p}} + a_{\mathbf{p}}P^\mu = a_{\mathbf{p}}\left(P^\mu - p^\mu\right)\ , \end{aligned} \tag{1.101}$$

so that

$$(P^\mu)^2 a_{\mathbf{p}} = P^\mu\left[a_{\mathbf{p}}\left(P^\mu - p^\mu\right)\right] = a_{\mathbf{p}}\left[\left(P^\mu - p^\mu\right)\right]^2\ , \tag{1.102}$$

and in general,

$$(P^\mu)^n a_{\mathbf{p}} = a_{\mathbf{p}}\left(P^\mu - p^\mu\right)^n\ . \tag{1.103}$$

In the same way,

$$(P^\mu)^n a_{\mathbf{p}}^\dagger = a_{\mathbf{p}}^\dagger \left(P^\mu + p^\mu\right)^n . \tag{1.104}$$

We thus get,

$$e^{iHt} a_{\mathbf{p}}(\mathbf{x}) e^{-iHt} = a_{\mathbf{p}} e^{i(H-E_{\mathbf{p}})t} e^{-iHt} = a_{\mathbf{p}} e^{-iE_{\mathbf{p}}t} , \tag{1.105}$$

and

$$e^{iHt} a_{\mathbf{p}}^\dagger(\mathbf{x}) e^{-iHt} = a_{\mathbf{p}}^\dagger e^{i(H+E_{\mathbf{p}})t} e^{-iHt} = a_{\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t} . \tag{1.106}$$

Applying these identities to the integral representations of $\phi(\mathbf{x})$:n and $\pi(\mathbf{x})$ in Eq. (1.73), we have, in the Heisenberg picture,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{1}{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right] , \tag{1.107}$$

$$\pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\mathbf{p}}}{2}} \left[a_{\mathbf{p}} e^{-ip\cdot x} - a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right] . \tag{1.108}$$

As the plane waves $e^{\pm ip\cdot x}$ are solutions to the Klein-Gordon equation $(\partial_\mu \partial^\mu + m^2)\phi(x) = 0$, we see that the **quantum field obeys its classical equation of motion.**

We can also write the entire x dependence of the field $\phi(x)$ in terms of the momentum operator,

$$e^{iP\cdot x} \phi(0) e^{-iP\cdot x} = e^{iP\cdot x} \left\{ \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{1}{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}} + a_{\mathbf{p}}^\dagger\right] \right\} e^{-iP\cdot x} \tag{1.109}$$

$$= \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{1}{2E_{\mathbf{p}}}} \left[e^{iP\cdot x} a_{\mathbf{p}} e^{-iP\cdot x} + e^{iP\cdot x} a_{\mathbf{p}}^\dagger e^{-iP\cdot x}\right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{1}{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right] = \phi(x) ,$$

so that

$$\phi(x) = e^{iP\cdot x}\phi(0)e^{-iP\cdot x}\,. \tag{1.110}$$

In other words, the translations are generated by the momentum operator.

The commutation relations in the Heisenberg picture are of course different from the Schrödinger picture ones. However, for **equal-time fields** we still have,

$$[\phi(t,\vec{x}),\phi(t,\vec{y})] = [\pi(t,\vec{x}),\pi(t,\vec{y})] = 0\,, \tag{1.111}$$

$$[\phi(t,\vec{x}),\pi(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})\,. \tag{1.112}$$

For two equal-time coordinate-space points, $x = (t,\mathbf{x})$, $y = (t,\mathbf{y})$ we have

$$(x-y)^2 = (0,\mathbf{x}-\mathbf{y})^2 = -(\mathbf{x}-\mathbf{y})^2 < 0 \tag{1.113}$$

i.e. the vector joining the two points is space like. The fact that the above commutators vanish in this kind of distances is necessary to be consistent with the causality principle: if the two field operators are not within each others light cones it should be immaterial in which order the operators act on a given state. The causality principle should of course be fulfilled regardless whether the fields are equal time. In arbitrary space-time points x and y the commutator is,

$$[\phi(x),\phi(y)] = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right]\,. \tag{1.114}$$

Since the integration measure $\mathrm{d}^3p/E_{\mathbf{p}}$ and the dot products $p\cdot(x-y)$ are Lorentz invariant, both terms in the equation above are Lorentz-invariant separately. We can thus make a Lorentz transformation Λ on the vectors x ja y and the integral remains intact. More concretely,

$$\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{ip\cdot(x-y)} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{i(\Lambda p)\cdot\Lambda(x-y)} \tag{1.115}$$

$$= \int \frac{\mathrm{d}^3(\Lambda p)}{(2\pi)^3}\frac{1}{2E_{\mathbf{\Lambda p}}}e^{i(\Lambda p)\cdot\Lambda(x-y)} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{ip\cdot\Lambda(x-y)}\,.$$

When $(x-y)^2 < 0$, there is (Ex.) a Lorentz transformation such that $\Lambda(x-y) = -(x-y)$, which shows that the commutator (1.114) vanishes and the causality principle remains true. For, $(x-y)^2 > 0$ the commutator in Eq. (1.114) is generally non zero.

1.5.2 Klein-Gordon propagators

[Peskin 2.4]

Let us now consider more closely the 2-point function

$$D(x-y) \equiv \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)}\,. \tag{1.116}$$

Using Eq. (1.99) and interpretation (1.93), we have

$$D(x-y)\hat{=}(2\pi)^3\,\langle\mathbf{x}|e^{-iH(x^0-y^0)}|\mathbf{y}\rangle, \tag{1.117}$$

so that $D(x-y)$ corresponds to the quantum amplitude for the particle to propagate from $\mathbf{y}$ to $\mathbf{x}$ in a given time interval x^0-y^0.

The transition amplitudes $D(x-y)$ have a relation to the Green's functions of the Klein-Gordon differential operator ∂^2+m^2. Let us recall that a solution to an inhomogenous Klein-Gordon equation

$$\left(\partial^2+m^2\right)\phi(x) = j(x), \tag{1.118}$$

can be written in a form

$$\phi(x) = \phi_0(x) + i\int d^4y\Delta(x,y)j(y)\ , \tag{1.119}$$

where $\phi_0(x)$ satisfies $(\partial^2+m^2)\phi(x)=0$, and $\Delta(x,y)$ is a Green's function, i.e. it obeys,

$$\left(\partial^2+m^2\right)\Delta(x,y) = -i\delta^{(4)}(x-y)\ . \tag{1.120}$$

In field theory these Green's functions are called **propagators**. Different Green's functions exist, and one that is often encountered is the **retarded Green's function** which propagates the impact of the "source term" $j(y)$ only forward in time, i.e. $D_{\mathrm{R}}(x,y)\propto\theta(x^0-y^0)$. It is straightforward to show that

$$\begin{aligned} D_{\mathrm{R}}(x-y) &= \theta(x^0-y^0)[\phi(x),\phi(y)] = \theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle \\ &= \theta(x^0-y^0)\left[D(x-y)-D(y-x)\right] \end{aligned} \tag{1.121}$$

fulfills (Ex.),

$$\left(\partial^2 + m^2\right) D_{\rm R}(x-y) = -i\delta^{(4)}\left(x-y\right) , \tag{1.122}$$

so it really is a retarded Green's function of the Klein-Gordon differential operator $\partial^2 + m^2$. We have already noted that $[\phi(x), \phi(y)]$, and thereby also $D_{\rm R}(x-y)$ vanishes at space-like distances, $(x-y)^2 < 0$, so $D_{\rm R}(x-y)$ propagates the impact of $j(y)$ only from the past light cone - it's causal. This reinforces our earlier argument that the commutator $[\phi(x), \phi(y)]$ being zero at space-like distances has to do with causality.

Using an integral representation of the Heaviside step funcion θ (Ex.),

$$\theta(x) = \lim_{\epsilon\to 0^+} -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{{\rm d}\tau}{\tau + i\epsilon}e^{-ix\tau} = \lim_{\epsilon\to 0^+}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{{\rm d}\tau}{\tau - i\epsilon}e^{ix\tau} , \tag{1.123}$$

a direct calculation leads to a 4-D integral representation (Ex.),

$$\begin{aligned} D_{\rm R}(x-y) &\equiv \theta(x^0 - y^0)\langle 0|[\phi(x), \phi(y)]|0\rangle \qquad (1.124) \\ &= \int\frac{{\rm d}^4 p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{i}{(p^0 + i\epsilon)^2 - \mathbf{p}^2 - m^2} , \end{aligned}$$

where $i\epsilon$ keeps the poles away from the p^0 integration contour.

The Klein-Gordon operator has also other Green's functions. The most useful will be the **time-ordered 2-point function**,

$$\begin{aligned} D_{\rm F}(x-y) &\equiv \langle 0|T\phi(x)\phi(y)|0\rangle \qquad (1.125) \\ &\equiv \theta(x^0 - y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0 - x^0)\langle 0|\phi(y)\phi(x)|0\rangle \\ &= \int\frac{{\rm d}^4 p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{i}{p^2 - m^2 + i\epsilon} . \end{aligned}$$

Also this is a Green's function of the Klein-Gordon operator and it is called the **Feynman propagator**. The first term clearly propagates the impact forward in time, but the last one would seem to propagate the impact backwards in time.

Propagator by Fourier transform:

A more direct way to solve the general form of the momentum-space propagator is to express the Green's function $D(x-y)$ as a Fourier transform,

$$D(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \tilde{D}(p) \,, \tag{1.126}$$

which puts Eq. (1.122) into a form,

$$(-p^2 + m^2)\tilde{D}(p) = -i \,, \tag{1.127}$$

so that,

$$\tilde{D}(p) = \frac{i}{(p^2 - m^2)} \,, \tag{1.128}$$

and thereby

$$D(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{p^2 - m^2} \,. \tag{1.129}$$

Depending where the $i\epsilon$ is put, different propagators can be obtained.

1.5.3 Complex scalar field

The Lagrangian for a complex scalar field is defined as

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi\phi^* \,, \tag{1.130}$$

where $\phi = \phi(x)$ is complex, i.e.

$$\phi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2) \,. \tag{1.131}$$

Since

$$\mathcal{L} = \frac{1}{2} \sum_{i=1,2} \left[(\partial_\mu \phi_i)(\partial^\mu \phi_i) - m^2 \phi_i^2 \right] \,, \tag{1.132}$$

we see that $\mathcal{L}$ describes two independent Klein-Gordon fields with the same mass. They separately fulfill the Klein-Gordon equation $(\partial^2 + m^2)\phi_i = 0$. The conjugate momenta are now $\pi_i = \partial\mathcal{L}/\partial\dot{\phi}_i = \dot{\phi}_i$, and the Hamiltonian density is, as in Eq. (1.37),

$$\mathcal{H} = \pi_1^2 + \pi_2^2 - \mathcal{L} = \frac{1}{2}\left[\pi_i^2 + (\nabla\phi_i)^2 + m^2\phi_i^2\right] . \tag{1.133}$$

We can use our earlier results (1.107) to write directly,

$$\phi_i(x) = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_{\mathbf{p}}}}\left[a_{\mathbf{p},i}e^{-ip\cdot x} + a^\dagger_{\mathbf{p},i}e^{ip\cdot x}\right], \tag{1.134}$$

$$\pi_i(x) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{E_{\mathbf{p}}}{2}}\left[a_{\mathbf{p},i}e^{-ip\cdot x} - a^\dagger_{\mathbf{p},i}e^{ip\cdot x}\right] , \tag{1.135}$$

where the commutation relations for the creation and annihilation operators are,

$$[a_{\mathbf{p},i}, a_{\mathbf{k},j}] = [a^\dagger_{\mathbf{p},i}, a^\dagger_{\mathbf{k},j}] = 0 , \tag{1.136}$$

$$[a_{\mathbf{p},i}, a^\dagger_{\mathbf{k},j}] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k})\delta_{ij} . \tag{1.137}$$

The Hamiltonian operator is, as earlier,

$$H = \int \frac{d^3p}{(2\pi)^3}E_{\mathbf{p}}\left[a^\dagger_{\mathbf{p},1}a_{\mathbf{p},1} + a^\dagger_{\mathbf{p},2}a_{\mathbf{p},2}\right] . \tag{1.138}$$

The Lagrangian (1.130) is invariant under a global phase shift $\phi \to e^{i\alpha}\phi$, where α is real. The corresponding conserved current and charge are (Ex.),

$$j^\mu = (\partial^\mu\phi^*)\,\phi - (\partial^\mu\phi)\,\phi^* , \tag{1.139}$$

$$Q = \int d^3x\,\left(\partial^0\phi^*\right)\phi - \left(\partial^0\phi\right)\phi^* . \tag{1.140}$$

Substituting here the quantized fields, we have

$$Q = 2i\int \frac{d^3p}{(2\pi)^3}\left[a^\dagger_{\mathbf{p},2}a_{\mathbf{p},1} - a^\dagger_{\mathbf{p},1}a_{\mathbf{p},2}\right] , \tag{1.141}$$

which has no (at least obvious) physical interpretation. However, we can define a new operator basis,

$$a_{\mathbf{p}} = \left(a_{\mathbf{p},1} + ia_{\mathbf{p},2}\right)/\sqrt{2}, \quad a^\dagger_{\mathbf{p}} = \left(a^\dagger_{\mathbf{p},1} - ia^\dagger_{\mathbf{p},2}\right)/\sqrt{2} , \tag{1.142}$$

$$b_{\mathbf{p}} = \left(a_{\mathbf{p},1} - ia_{\mathbf{p},2}\right)/\sqrt{2}, \quad b^\dagger_{\mathbf{p}} = \left(a^\dagger_{\mathbf{p},1} + ia^\dagger_{\mathbf{p},2}\right)/\sqrt{2} . \tag{1.143}$$

From the definition we see that $a^\dagger_{\mathbf{p}}$ and $b^\dagger_{\mathbf{p}}$ are still creation operators (for superpositions of ϕ_1 and ϕ_2 states), and $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$ annihilation operators. The commutation relations are,

$$\left[a_{\mathbf{p}}, a^\dagger_{\mathbf{k}}\right] = \left[b_{\mathbf{p}}, b^\dagger_{\mathbf{k}}\right] = (2\pi)^3\,\delta^{(3)}\left(\mathbf{p}-\mathbf{k}\right)\,, \tag{1.144}$$

and zero for the rest. We easily see that in the new basis both H and Q are diagonal,

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}}\left[a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + b^\dagger_{\mathbf{p}} b_{\mathbf{p}}\right]\,, \tag{1.145}$$

$$Q = \int \frac{d^3p}{(2\pi)^3}\left[b^\dagger_{\mathbf{p}} b_{\mathbf{p}} - a^\dagger_{\mathbf{p}} a_{\mathbf{p}}\right]\,. \tag{1.146}$$

We realize that the states created by $a^\dagger_{\mathbf{p}}$ and $b^\dagger_{\mathbf{p}}$ have a positive energy, but the charges Q are opposite. We will thus interpret the exitations created by $a^\dagger_{\mathbf{p}}$ as **particles** and exitations created by $b^\dagger_{\mathbf{p}}$ as **antiparticles**. Using the decomposition of Eq. (1.131) and the definition of the new operator basis, we get the following representation for the complex field operator $\phi(x)$:

$$\phi(x) = \frac{1}{\sqrt{2}}\left[\phi_1(x) + i\phi_2(x)\right] = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_{\mathbf{p}}}}\left[a_{\mathbf{p}} e^{-ip\cdot x} + b^\dagger_{\mathbf{p}} e^{ip\cdot x}\right]\,.$$

Its conjugate momentum operator is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}^\dagger = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{E_{\mathbf{p}}}{2}}\left[b_{\mathbf{p}} e^{-ip\cdot x} - a^\dagger_{\mathbf{p}} e^{ip\cdot x}\right]\,. \tag{1.147}$$

The propagators of the complex scalar field are essentially the same as earlier. However, now $\langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|\phi^\dagger(x)\phi^\dagger(y)|0\rangle = 0$, and the non-trivial 2-point functions are

$$\langle 0|\phi(x)\phi^\dagger(y)|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)} \propto {}_a\langle\mathbf{x}|e^{-iH(x^0-y^0)}|\mathbf{y}\rangle_a \tag{1.148}$$

$$\langle 0|\phi^\dagger(y)\phi(x)|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(y-x)} \propto {}_b\langle\mathbf{y}|e^{-iH(y^0-x^0)}|\mathbf{x}\rangle_b\,. \tag{1.149}$$

The last proportionalities use similar interpretation as in Eq. (1.117) taking into account that the field operator $\phi(x)$ creates antiparticles (to right) whereas $\phi^\dagger(x)$ creates particles (to right). The Feynman propagator thus corresponds to the definition,

$$\begin{aligned} D_{\mathrm{F}}(x-y) &\equiv \langle 0|T\phi(x)\phi^\dagger(y)|0\rangle \\ &\equiv \theta(x^0-y^0)\langle 0|\phi(x)\phi^\dagger(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi^\dagger(y)\phi(x)|0\rangle \\ &= \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{p^2-m^2+i\epsilon}\,, \end{aligned} \tag{1.150}$$

and it describes the propagation of particles forward in time, and propagation of antiparticles backwards in time. In the case of real scalar field the particle is its own antiparticle and the particle-antiparticle separation is only seen by considering complex field.

In school they often talk about the wave-particle dualism, meaning that particles have some wave-like properties and vice versa. In quantum field theory this is explicit: On one hand, the quantized field fulfills its classical wave equation, and on the other hand it contains operators that create quanta of the field which we call particles.

2 Quantization of the Dirac field [Peskin 3]

The Dirac equation and its features have been widely discussed at the Particle Physics and Quantum Mechanics II courses. Nevertheless, we briefly review the solutions of the Dirac equation and their main properties.

2.1 Dirac equation and its plane-wave solutions

[Peskin 3.2, 3.3]

The Dirac equation is

$$(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0, \tag{2.1}$$

where the γ matrices fulfill the so-called **Clifford algebra**,

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,. \tag{2.2}$$

The γ matrices can be chosen in variety of ways. The usual one – and this convention is what we will use in these lectures – is the **Dirac-Pauli representation**:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{2.3}$$

where I represents a 2×2 unit matrix and σ^i are Pauli spin matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}$$

For us the most important properties of the Pauli matrices are

$$(\sigma^i)^\dagger = \sigma^i \tag{2.5}$$

$$\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k \tag{2.6}$$

$$\sigma^2(\sigma^i)^* = -\sigma^i\sigma^2\,. \tag{2.7}$$

The Dirac equation is solved by a plane-wave ansatz, $\psi(x) = w(p)e^{\pm ip\cdot x}$, where $w(p)$ is a 4-component vector (or spinor), and $p^0 = E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$. Let us consider the case $\mathbf{p} = 0$, so that the Dirac equation (2.1) simplifies to,

$$\left(i\gamma^0\partial_t - m\right)\left[w(0)e^{\pm imt}\right] = -m\left(1 \pm \gamma^0\right)\left[w(0)e^{\pm imt}\right] = 0, \tag{2.8}$$

or in 2×2 "block form",

$$\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} u(0)e^{-imt} = 0\,, \quad \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} v(0)e^{imt} = 0\,, \tag{2.9}$$

where we have marked by u the e^{-imt} spinor and by v the e^{+imt} spinor. The solutions to these equations are clearly of the form

$$u_s(0)e^{-imt} \propto \begin{pmatrix} \xi_s \\ 0 \end{pmatrix} e^{-imt}\,, \quad v(0)_s e^{imt} \propto \begin{pmatrix} 0 \\ \eta_s \end{pmatrix} e^{imt}\,, \tag{2.10}$$

where ξ_s and η_s $(s = 1,2)$ are arbitrary 2-component spinors. We choose the normalization $\xi_s^\dagger \xi_{s'} = \eta_s^\dagger \eta_{s'} = \delta_{ss'}$. In general, the plane-wave ansatz gives,

$$\left(\not{p} - m\right) u(p) = 0\,, \tag{2.11}$$

$$\left(\not{p} + m\right) v(p) = 0\,. \tag{2.12}$$

These are usually referred to as Dirac equations in the momentum space. Now, since

$$\left(\not{p} \pm m\right)\left(\not{p} \mp m\right) = \not{p}\not{p} - m^2 = p^2 - m^2 = 0\,, \tag{2.13}$$

we see that the general solutions are of the form,

$$u_s(p)e^{-ip\cdot x} \propto \left(\not{p} + m\right) u_s(0)e^{-ip\cdot x} \tag{2.14}$$

$$v_s(p)e^{+ip\cdot x} \propto \left(\not{p} - m\right) v_s(0)e^{+ip\cdot x}\,. \tag{2.15}$$

More explicitly,

$$u_s(p) \propto \begin{pmatrix} \xi_s \\ \frac{\vec{\sigma}\cdot\mathbf{p}}{E+m}\xi_s \end{pmatrix}, \quad v_s(p) \propto \begin{pmatrix} \frac{\vec{\sigma}\cdot\mathbf{p}}{E+m}\eta_s \\ \eta_s \end{pmatrix}, \tag{2.16}$$

where $\vec{\sigma}\cdot\mathbf{p} = \sum_{i=1}^{3} \sigma^i p^i$. The normalization of the spinors is a matter of convention. In the standard normalization we define the spinors as,

$$u_s(p) = \sqrt{E_\mathbf{p}+m} \begin{pmatrix} I \\ \frac{\vec{\sigma}\cdot\mathbf{p}}{E+m} \end{pmatrix} \xi_s, \quad v_s(p) = \sqrt{E_\mathbf{p}+m} \begin{pmatrix} \frac{\vec{\sigma}\cdot\mathbf{p}}{E+m} \\ I \end{pmatrix} \eta_s. \tag{2.17}$$

We will often choose $\eta_s = \left(-i\sigma^2\right)\xi_s^*$ for reasons the become clear later on.

These spinors fulfill ($\overline{u}_s \equiv u_s^\dagger \gamma^0$)

$$\overline{u}_s(p) u_{s'}(p) = 2m\delta_{ss'}, \qquad u_s^\dagger(p) u_{s'}(p) = 2E_\mathbf{p}\delta_{ss'}, \tag{2.18}$$

$$\overline{v}_s(p) v_{s'}(p) = -2m\delta_{ss'}, \quad v_s^\dagger(p) v_{s'}(p) = 2E_\mathbf{p}\delta_{ss'}. \tag{2.19}$$

In addition, the following orthogonality relations are obeyed

$$\overline{u}_s(p) v_{s'}(p) = \overline{v}_s(p) u_{s'}(p) = 0, \tag{2.20}$$

$$u_s^\dagger(\mathbf{p}) v_{s'}(-\mathbf{p}) = v_s^\dagger(\mathbf{p}) u_{s'}(-\mathbf{p}) = 0. \tag{2.21}$$

The following **projection operators** are also often needed,

$$\sum_{s=1}^{2} u_s(p)\overline{u}_s(p) = \not{p} + m \tag{2.22}$$

$$\sum_{s=1}^{2} v_s(p)\overline{v}_s(p) = \not{p} - m \tag{2.23}$$

The Dirac representation is particularly useful in the non-relativistic limit $\mathbf{p} \to 0$. At the opposite end, $m \to 0$ the Weyl representation is sometimes useful.

Weyl representation:

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{2.24}$$

$$u_s(p) = \frac{1}{\sqrt{2\,(E_{\mathbf{p}}+m)}} \begin{pmatrix} E+m-\vec{\sigma}\cdot\mathbf{p} \\ E+m+\vec{\sigma}\cdot\mathbf{p} \end{pmatrix} \xi_s \,, \tag{2.25}$$

$$v_s(p) = \frac{-1}{\sqrt{2\,(E_{\mathbf{p}}+m)}} \begin{pmatrix} E+m-\vec{\sigma}\cdot\mathbf{p} \\ -E-m-\vec{\sigma}\cdot\mathbf{p} \end{pmatrix} \eta_s \,. \tag{2.26}$$

In the asymptotic limit $m \to 0$ the Weyl spinors simplify to ($\hat{\mathbf{p}} \equiv \mathbf{p}/|\mathbf{p}|$),

$$u_s(p) \xrightarrow{m\to 0} \sqrt{\frac{|\mathbf{p}|}{2}} \begin{pmatrix} 1-\vec{\sigma}\cdot\hat{\mathbf{p}} \\ 1+\vec{\sigma}\cdot\hat{\mathbf{p}} \end{pmatrix} \xi_s = \sqrt{\frac{|\mathbf{p}|}{2}} \begin{pmatrix} 1-2\hat{h} \\ 1+2\hat{h} \end{pmatrix} \xi_s \,, \tag{2.27}$$

$$v_s(p) \xrightarrow{m\to 0} \sqrt{\frac{|\mathbf{p}|}{2}} \begin{pmatrix} -1+\vec{\sigma}\cdot\hat{\mathbf{p}} \\ 1+\vec{\sigma}\cdot\hat{\mathbf{p}} \end{pmatrix} \eta_s = \sqrt{\frac{|\mathbf{p}|}{2}} \begin{pmatrix} -1+2\hat{h} \\ 1+2\hat{h} \end{pmatrix} \eta_s \,,$$

where we defined the **helicity matrix**

$$\hat{h} \equiv \frac{1}{2}\vec{\sigma}\cdot\hat{\mathbf{p}}. \tag{2.28}$$

If we furthermore choose ξ_s and η_s such that $\hat{h}\xi_\pm = \pm\frac{1}{2}\xi_\pm$ and $\hat{h}\eta_\pm = \pm\frac{1}{2}\eta_\pm$ we have

$$u_+(p) \xrightarrow{m\to 0} \sqrt{2|\mathbf{p}|} \begin{pmatrix} 0 \\ I \end{pmatrix} \xi_+ \,, \quad u_-(p) \xrightarrow{m\to 0} \sqrt{2|\mathbf{p}|} \begin{pmatrix} I \\ 0 \end{pmatrix} \xi_- \,, \tag{2.29}$$

$$v_+(p) \xrightarrow{m\to 0} \sqrt{2|\mathbf{p}|} \begin{pmatrix} 0 \\ I \end{pmatrix} \eta_+ \,, \quad v_-(p) \xrightarrow{m\to 0} \sqrt{2|\mathbf{p}|} \begin{pmatrix} -I \\ 0 \end{pmatrix} \eta_- \,.$$

For example, if we consider a particle moving $+z$ direction, $\hat{h} = \sigma^3/2$, ja $\xi_+ = (1,0)$, $\xi_- = (0,1)$. The helicity $+1/2$ spinors are called **right handed**, and helicity $-1/2$ spinors **left handed**.

Independently of the representation of the spinors we can always choose ξ_s and η_s so that $\hat{h}\xi_\pm = \pm\frac{1}{2}\xi_\pm$ and $\hat{h}\eta_\pm = \pm\frac{1}{2}\eta_\pm$. In this case the u and v spinors are eigenstates of the helicity operator (Ex.)

$$h \equiv \begin{pmatrix} \hat{h} & 0 \\ 0 & \hat{h} \end{pmatrix} \tag{2.30}$$

with eigenvalues $\pm 1/2$.

2.2 Lorentz transformations of fields [Peskin 3.1, 3.2]

In Chapter 1 we already touched upon the Lorentz transformations of the fields but were not very careful about the nature of the transformation. In general, the there are two classes of transformations, active and passive. The situation is particularly transparent in the case of scalar fields:

Passive transformation: The fields do not change, the transformation rotates/boosts the coordinate system,

$$\phi(x) \longrightarrow \phi(x') = \phi(\Lambda x)\,. \tag{2.31}$$

If the field $\phi(x)$ has originally some kind of "hotspot" at $x = x_0$, it is seen at $x = \Lambda^{-1}x_0$ in the new frame.

Active transformation: Fields change in rotations/boosts, coordinate system stays the same.

$$\phi(x) \longrightarrow \phi'(x) = \phi(\Lambda^{-1}x) \tag{2.32}$$

If the original field has a "hotspot" at $x = x_0$, the new field has a "hotspot" at $x = \Lambda x_0$.

The difference is not huge but in calculations one has to know how to deal with e.g. the derivatives ∂_μ. When the transformation is an active one, the coordinate system does not change, so the derivative operator does not change either. In what follows, we will stick to the active transformations i.e. we look how e.g. $\mathcal{L}(x)$ changes in a given coordinate x.

A Lorentz transformation for a 4-vector x can be written, component by component, as

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu \,. \tag{2.33}$$

which corresponds to the matrix form,

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \,. \tag{2.34}$$

For example, boost along the z axis and rotation about the z axis are given by the matrices ($\beta = v/c$, $\gamma = 1/\sqrt{1-\beta^2}$),

$$\begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} , \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \,. \tag{2.35}$$

In our convention $\Lambda^\mu{}_\nu$ correspond to these matrices. By definition, the Λs with indices in other places are always obtained from $\Lambda^\mu{}_\nu$ by the metric tensor. For example, $\Lambda_{\mu\nu} \equiv g_{\mu\rho}\Lambda^\rho{}_\nu$.

The Lagrangian density has to transfrom as a scalar, $\mathcal{L}(x) \to \mathcal{L}(\Lambda^{-1}x)$, which also preserves the action/equations of motion. Let us now check this explicitly for the Klein-Gordon Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - \frac{1}{2}m^2\phi^2 \,. \tag{2.36}$$

Only the derivative part

$$\partial_\mu\phi\left(x\right) \to \partial_\mu\left[\phi\left(\Lambda^{-1}x\right)\right] \,, \tag{2.37}$$

is here nontrivial:

$$\begin{aligned} \partial_\mu\phi\left(x\right) \to \partial_\mu\left[\phi\left(\Lambda^{-1}x\right)\right] &= \frac{\partial}{\partial x^\mu}\left[\phi\left(\Lambda^{-1}x\right)\right] \\ &= \frac{\partial\phi}{\partial x^\nu}\left(\Lambda^{-1}x\right) \times \frac{\partial}{\partial x^\mu}\left[(\Lambda^{-1})^\nu{}_\alpha x^\alpha\right] \\ &= (\Lambda^{-1})^\nu{}_\mu\, \partial_\nu\phi\left(\Lambda^{-1}x\right) \,. \end{aligned} \tag{2.38}$$

Thus,

$$\partial_\mu \phi\,(x)\,\partial^\mu \phi\,(x) \to \partial_\mu\left[\phi\left(\Lambda^{-1}x\right)\right] \times \partial^\mu\left[\phi\left(\Lambda^{-1}x\right)\right] \tag{2.39}$$
$$= (\Lambda^{-1})^\nu{}_\mu\, \partial_\nu \phi\left(\Lambda^{-1}x\right) \times (\Lambda^{-1})_\rho{}^\mu\, \partial^\rho \phi\left(\Lambda^{-1}x\right)$$
$$= \left[(\Lambda^{-1})^\nu{}_\mu\,(\Lambda^{-1})_\rho{}^\mu\right] \partial_\nu \phi\left(\Lambda^{-1}x\right)\; \partial^\rho \phi\left(\Lambda^{-1}x\right)\;.$$

To find out what is the term in square brackets, we need some elementary identities of the Lorentz transformations. First, since the lenght of a 4-vector is invariant,

$$p^2 = p^\mu p_\mu = (\Lambda^\mu{}_\nu p^\nu)\;\left(\Lambda_\mu{}^\rho p_\rho\right) = \Lambda^\mu{}_\nu \Lambda_\mu{}^\rho\, p^\nu p_\rho\,, \tag{2.40}$$

and on the other hand,

$$p^2 = \Lambda^{-1}(\Lambda p)\cdot p = (\Lambda^{-1})^\rho{}_\mu \Lambda^\mu{}_\nu p^\nu p_\rho = \Lambda^\mu{}_\nu (\Lambda^{-1})^\rho{}_\mu\, p^\nu p_\rho\,. \tag{2.41}$$

From these, we can identify,

$$(\Lambda^{-1})^\rho{}_\mu = \Lambda_\mu{}^\rho \tag{2.42}$$
$$\Lambda^\mu{}_\nu \Lambda_\mu{}^\rho = \delta^\rho_\nu\,.$$

Using these two identities,

$$\partial_\mu \phi\,(x)\,\partial^\mu \phi\,(x) \to \left[(\Lambda^{-1})^\nu{}_\mu\,(\Lambda^{-1})_\rho{}^\mu\right] \partial_\nu \phi\left(\Lambda^{-1}x\right)\; \partial^\rho \phi\left(\Lambda^{-1}x\right)\;. \tag{2.43}$$
$$= \left[\Lambda_\mu{}^\nu\, \Lambda^\mu{}_\rho\right] \partial_\nu \phi\left(\Lambda^{-1}x\right)\; \partial^\rho \phi\left(\Lambda^{-1}x\right)$$
$$= \partial_\nu \phi\left(\Lambda^{-1}x\right)\; \partial^\nu \phi\left(\Lambda^{-1}x\right)\;.$$

We thus see that $\mathcal{L}$ indeed transforms as a scalar function,

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu \phi(x)\right]\left[\partial^\mu \phi(x)\right] - \frac{1}{2} m^2 \phi^2(x) \tag{2.44}$$
$$\to \frac{1}{2}\left[\partial_\mu \phi(\Lambda^{-1}x)\right]\left[\partial^\mu \phi(\Lambda^{-1}x)\right] - \frac{1}{2} m^2 \phi^2(\Lambda^{-1}x)\,.$$

Transformations for the Dirac spinors:

The Dirac spinors are 4-component objects so a Lorentz transformation may also suffle these component similarly as Lorentz transformations suffle the components of ordinary 4-vectors. A possible transformation is therefore of the form,

$$\psi(x) \longrightarrow \psi'(x) = S(\Lambda)\,\psi\left(\Lambda^{-1}x\right)\,, \tag{2.45}$$

where $S(\Lambda)$ is a 4×4 matrix. We can find $S(\Lambda)$ by demanding that a Dirac spinor fulfills the Dirac equation also after a Lorentz transformation,

$$(i\gamma^\mu\partial_\mu - m)\,\psi(x) = 0 \longrightarrow (i\gamma^\mu\partial_\mu - m)\,\psi'(x) = 0\,. \tag{2.46}$$

Let's open this:

$$\begin{aligned}
&(i\gamma^\mu\partial_\mu - m)\,S(\Lambda)\,\psi\left(\Lambda^{-1}x\right) \qquad (2.47)\\
=&i\gamma^\mu\partial_\mu S(\Lambda)\,\psi\left(\Lambda^{-1}x\right) - mS(\Lambda)\,\psi\left(\Lambda^{-1}x\right)\\
=&i\gamma^\mu(\Lambda^{-1})^\nu{}_\mu S(\Lambda)\;\partial_\nu\psi\left(\Lambda^{-1}x\right) - mS(\Lambda)\,\psi\left(\Lambda^{-1}x\right)\\
=&S(\Lambda)\left[i(\Lambda^{-1})^\nu{}_\mu S^{-1}(\Lambda)\,\gamma^\mu S(\Lambda)\;\partial_\nu - m\right]\psi\left(\Lambda^{-1}x\right)\,.
\end{aligned}$$

If the matrix S is now such that

$$(\Lambda^{-1})^\nu{}_\mu S^{-1}(\Lambda)\,\gamma^\mu S(\Lambda) = \gamma^\nu\,, \tag{2.48}$$

we get

$$\begin{aligned}
&S(\Lambda)\left[i(\Lambda^{-1})^\nu{}_\mu S^{-1}(\Lambda)\,\gamma^\mu S(\Lambda)\;\partial_\nu - m\right]\psi\left(\Lambda^{-1}x\right) \qquad (2.49)\\
&S(\Lambda)\left[i\gamma^\nu\,\partial_\nu - m\right]\psi\left(\Lambda^{-1}x\right) = 0\,,
\end{aligned}$$

where we get zero since original $\psi(x)$ fulfills the Dirac equation with all arguments. We thus demand,

$$S^{-1}(\Lambda)\,\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu\,. \tag{2.50}$$

We can solve $S\left(\Lambda\right)$ by considering an infinitesimal transformation,

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \delta\omega^{\mu}{}_{\nu}\,. \tag{2.51}$$

By using our earlier result $\Lambda^{\mu}{}_{\nu}\Lambda_{\mu}{}^{\rho} = \delta^{\rho}_{\nu}$, we have

$$\begin{aligned} \delta^{\rho}_{\nu} = \Lambda^{\mu}{}_{\nu}\Lambda_{\mu}{}^{\rho} &= g_{\mu\xi}g^{\rho\alpha}\Lambda^{\mu}{}_{\nu}\Lambda^{\xi}{}_{\alpha} \\ &= g_{\mu\xi}g^{\rho\alpha}\left[\delta^{\mu}_{\nu} + \delta\omega^{\mu}{}_{\nu}\right]\left[\delta^{\xi}_{\alpha} + \delta\omega^{\xi}{}_{\alpha}\right] \\ &= \delta^{\rho}_{\nu} + g^{\rho\alpha}\left[\delta\omega_{\nu\alpha} + \delta\omega_{\alpha\nu}\right] + \mathcal{O}(\delta\omega^2)\,, \end{aligned} \tag{2.52}$$

so $\delta\omega$ must be antisymmetric, $\delta\omega_{\nu\alpha} = -\delta\omega_{\alpha\nu}$. There are thus 6 independent parameters corresponding to 3 boosts and 3 rotations. Let us then expand $S\left(\Lambda\right)$:

$$S\left(\delta\omega\right) = 1 - \frac{i}{4}\delta\omega_{\mu\nu}\sigma^{\mu\nu} + \mathcal{O}(\delta\omega^2)\,, \tag{2.53}$$

where the factor $-i/4$ is merely a convention. By substituting into Eq. (2.50) we can find the $\sigma^{\mu\nu}$ matrices (Ex.),

$$\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right]\,. \tag{2.54}$$

These $\sigma_{\mu\nu}$ are thus the generators of Lorentz transformations in the space of Dirac spinors. The matrices $S(\Lambda)$ form a representation of Lorentz transformation in the spinor space.

Finite transformations are obtained, as usual, by "exponentiating" the infinitesimal transformation. For example, for an infinitesimal boost to the z direction we have $\delta\omega^{0}{}_{3} = \delta\omega^{3}{}_{0} = \delta\eta$, and zero for the rest. For a 4-vector this is

$$1 + \delta\eta\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.55}$$

corresponding to a finite transformation

$$\exp\left[\eta\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}\right]=\begin{pmatrix}\cosh\eta&0&0&\sinh\eta\\0&1&0&0\\0&0&1&0\\\sinh\eta&0&0&\cosh\eta\end{pmatrix}. \tag{2.56}$$

By comparing to the matrix in Eq. (2.35) we see that $\gamma = \cosh\eta$ and $\beta\gamma = \sinh\eta$, from which we can solve,

$$\eta = \frac{1}{2}\log\left(\frac{1+\beta}{1-\beta}\right). \tag{2.57}$$

This corresponds to the definition of **rapidity** which should be familiar from the Particle Physics course.

For a Dirac spinor, the corresponding infinitesimal transformation reads

$$1-\frac{i}{4}\left[\delta\omega_{03}\sigma^{03}+\delta\omega_{30}\sigma^{30}\right]=1+\frac{\delta\eta}{2}\begin{pmatrix}0&\sigma^3\\\sigma^3&0\end{pmatrix}, \tag{2.58}$$

which corresponds to the finite transformation,

$$\exp\left[\frac{\eta}{2}\begin{pmatrix}0&\sigma^3\\\sigma^3&0\end{pmatrix}\right]=\cosh\left(\frac{\eta}{2}\right)+\sinh\left(\frac{\eta}{2}\right)\begin{pmatrix}0&\sigma^3\\\sigma^3&0\end{pmatrix}. \tag{2.59}$$

2.3 Bilinears of Dirac spinors

Since the matrix $S(\Lambda)$ is not unitary, the quantity $\psi^\dagger\psi$ does not transform as a scalar in Lorentz transformations. This kind of bilinear quantities cannot therefore appear in Lagrangian on their own. Instead, $\overline{\psi}\psi$, where $\overline{\psi}\equiv\psi^\dagger\gamma^0$ transforms a a scalar: From,

$$\psi(x)\to S(\Lambda)\psi(\Lambda^{-1}x) \tag{2.60}$$

$$\overline{\psi}(x)\to\left[S(\Lambda)\psi(\Lambda^{-1}x)\right]^\dagger\gamma^0=\psi^\dagger(\Lambda^{-1}x)S^\dagger(\Lambda)\gamma^0=\overline{\psi}(\Lambda^{-1}x)\gamma^0S^\dagger(\Lambda)\gamma^0$$

we see that if the inverse transformation matrix S^{-1} fulfills, $S^{-1}(\Lambda) = \gamma^0 S^\dagger(\Lambda)\gamma^0$, then $\overline{\psi}\psi$ transforms a a scalar function. Indeed, from Eq. (2.53) and using the γ-matrix identity $\gamma_0\gamma_\mu^\dagger\gamma_0 = \gamma_\mu$ one can easily show that,

$$S^{-1}(\Lambda) = \gamma^0 S^\dagger(\Lambda)\gamma^0\,. \tag{2.61}$$

All the Dirac-spinor bilinears and their transformation properties are:

$\overline{\psi}\psi$	**scalar**
$\overline{\psi}\gamma^\mu\psi$	**4 − vector**
$\overline{\psi}\sigma^{\mu\nu}\psi$	**2. rank tensor**
$\overline{\psi}\gamma^5\psi$	**pseudoscalar**
$\overline{\psi}\gamma^5\gamma^\mu\psi$	**pseudo 4 − vector**

In this context the **2. rank tensor** means an object which transform as $F^{\mu\nu}(x) \to \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}(\Lambda^{-1}x)$. In turn, **pseudo scalar** ja **pseudo 4-vector** are objects which transform in continuous Lorentz transformations respectively as a scalar and a 4-vector, but in reflections $\mathbf{x} \to -\mathbf{x}$ they attain a sign change.

The fifth γ matrix appearing in the table above can be defined as

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{2.62}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is a fully antisymmetric object under interchange of indices, with $\epsilon^{0123} = 1$. As in the case of Lorentz transformations, the indices can be up- or downstairs, and the corresponding value is obtained from $\epsilon^{\mu\nu\rho\sigma}$ by $g^{\mu\nu}$. For example, $\epsilon_{0123} = \epsilon^{\mu\nu\rho\sigma} g_{\mu 0} g_{\nu 1} g_{\rho 2} g_{\sigma 3} = -1$. The fifth γ matrix has the following properties:

$$\left(\gamma^5\right)^\dagger = \gamma^5\,, \qquad \left(\gamma^5\right)^2 = 1\,, \qquad \{\gamma^5, \gamma^\mu\} = 0\,. \tag{2.63}$$

The bilinears in the table above form a basis in the sense tha any bilinear quantity $\overline{\psi}\Gamma\psi$, where Γ is a 4×4 matrix, can be expressed with those five. To see that this even can be possible, let us compute the number of independent matrices in the above bilinears,

$$\begin{array}{ccccccccc} I, & & \gamma^5 & & \gamma^\mu & & \sigma^{\mu\nu} & & \gamma^5\gamma^\mu \\ 1 & + & 1 & + & 4 & + & 6 & + & 4 & = & 16\,. \end{array}$$

This matches with the number of independent 4×4 matrices which is also 16. Is thus possible that these form a basis for all 4×4 matrices. It is left as an exercise to show that the above 16 are linearly independent and thus form a basis.

2.4 Quantization of the Dirac field

The free Dirac Lagrangian reads

$$\mathcal{L} = \overline{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi\,, \quad \overline{\psi} = \psi^\dagger\gamma^0\,, \tag{2.64}$$

which transforms as a scalar under Lorentz transformations (as we now know). The field ψ has 4-components and each component is complex, i.e. we can write it as,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_1^a + i\psi_1^b \\ \psi_2^a + i\psi_2^b \\ \psi_3^a + i\psi_3^b \\ \psi_4^a + i\psi_4^b \end{pmatrix}\,, \quad \psi_i^a, \psi_i^b \in \Re\,. \tag{2.65}$$

The Lagrangian in Eq. (2.64) now leads to 8 Euler-Lagrange equations of motion (Ex.), which can be summarized as

$$(i\gamma^\mu\partial_\mu - m)\,\psi = 0 \tag{2.66}$$

$$\overline{\psi}m + i\partial_\mu\overline{\psi}\gamma^\mu = 0\,, \tag{2.67}$$

of which the first one is the Dirac equation and the second one can also be obtained from the first one by taking the Hermitian conjugate and

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2.4 Quantization of the Dirac field [Peskin 3.5]

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of which the first one is the Dirac equation and the second one can also be obtained from the first one by taking the Hermitian conjugate and multiplying by γ^0. The 8 conjugate momentum densities are

$$\pi_k^a = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_k^a} = \left(\overline{\psi} i \gamma^0\right)_k = i\psi_k^\dagger \tag{2.68}$$

$$\pi_k^b = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_k^b} = i\left(\overline{\psi} i \gamma^0\right)_k = -\psi_k^\dagger \tag{2.69}$$

The Hamiltonian density reads,

$$\begin{aligned} H &= \int d^3x \left[\!\!\left[\sum_k \left[\pi_k^a(x)\dot{\psi}_k^a(x) + \pi_k^b(x)\dot{\psi}_k^b(x)\right] - \mathcal{L}(x) \right]\!\!\right] \\ &= \int d^3x \left[\!\!\left[i\psi^\dagger(x)\dot{\psi}(x) - \mathcal{L}(x) \right]\!\!\right], \qquad \pi(x) \equiv i\psi^\dagger(x) \\ &= \int d^3x \overline{\psi}(x) \left[-i\gamma^i\partial_i + m\right] \psi(x) \,. \end{aligned} \tag{2.70}$$

As we saw in the case of scalar fields, the quantum field should obey its wave equation (here Dirac equation) and be a linear combination of all its solutions. The Dirac equation has now 4 independent solutions, $u_s(p)e^{-ip\cdot x}$ and $v_s(p)e^{ip\cdot x}$ ($s = 1, 2$). In the case of complex scalar field the phase factor $e^{-ip\cdot x}$ came with the particle annihilation operator $a_{\mathbf{p}}$, and the phase factor $e^{ip\cdot x}$ came with the antiparticle creation operator $b_{\mathbf{p}}^\dagger$. In analogy, we decompose,

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_{\mathbf{p}}}} \sum_s \left[a_{\mathbf{p},s}u_s(p)e^{-ip\cdot x} + b_{\mathbf{p},s}^\dagger v_s(p)e^{ip\cdot x}\right] \,. \tag{2.71}$$

If we would now proceed as in the case of scalar fields and set the commutation relations for the creation and annihilation operators,

$$\left[a_{\mathbf{p},s}, a_{\mathbf{k},r}^\dagger\right] = \left[b_{\mathbf{p},s}, b_{\mathbf{k},r}^\dagger\right] = (2\pi)^3\,\delta^{(3)}\left(\mathbf{p} - \mathbf{k}\right)\delta_{rs}\,, \tag{2.72}$$

the canonical commutation relation for the fields is not fulfilled,

$$\left[\psi_i(\mathbf{x}, t), \pi_j(\mathbf{y}, t)\right] = \left[\psi_i(\mathbf{x}, t), i\psi_j^\dagger(\mathbf{y}, t)\right] \neq i\delta^{(3)}\left(\mathbf{x} - \mathbf{y}\right)\delta_{ij}\,. \tag{2.73}$$

Here, the indeces i and j refer to the indices of the spinors (components of u_s and v_s). We don't usually explicitly write these indices. Let us now write the left-hand side of (2.73) assuming the commutation realations (2.72):

$$\left[\psi(\mathbf{x},t),\psi^\dagger(\mathbf{y},t)\right] = \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2}\sqrt{\frac{1}{E_\mathbf{p}E_\mathbf{k}}}\sum_{s,s'} \tag{2.74}$$

$$\left[a_{\mathbf{p},s}u_s(p)e^{-ip\cdot x}+b^\dagger_{\mathbf{p},s}v_s(p)e^{ip\cdot x},a^\dagger_{\mathbf{k},s'}u^\dagger_{s'}(k)e^{ik\cdot y}+b_{\mathbf{k},s'}v^\dagger_{s'}(k)e^{-ik\cdot y}\right]$$

$$= \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2}\sqrt{\frac{1}{E_\mathbf{p}E_\mathbf{k}}}\sum_{s,s'}$$

$$\left[a_{\mathbf{p},s},a^\dagger_{\mathbf{k},s'}\right]u_s(p)u^\dagger_{s'}(k)e^{-ip\cdot x+ik\cdot y}+\left[b^\dagger_{\mathbf{p},s},b_{\mathbf{k},s'}\right]v_s(p)v^\dagger_{s'}(k)e^{ip\cdot x-ik\cdot y}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\left[\sum_s u_s(p)u^\dagger_s(p)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}-\sum_s v_s(p)v^\dagger_s(p)e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right]$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\left[\sum_s u_s(p)\overline{u}_s(p)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}-\sum_s v_s(p)\overline{v}_s(p)e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right]\gamma^0$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\left[\left(\not p+m\right)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}-\left(\not p-m\right)e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right]\gamma^0$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\left[\left(\gamma_0E_\mathbf{p}+\gamma_i\mathbf{p}^i+m\right)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}-\left(\gamma_0E_\mathbf{p}+\gamma_i\mathbf{p}^i-m\right)e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right]\gamma^0$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\left[\left(\gamma_0E_\mathbf{p}+\gamma_i\mathbf{p}^i+m\right)-\left(\gamma_0E_\mathbf{p}-\gamma_i\mathbf{p}^i-m\right)\right]\gamma^0$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{E_\mathbf{p}}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\left[\gamma_i\mathbf{p}^i+m\right]\gamma^0\,.$$

This doesn't look good. The problem is in the term

$$\left(\gamma_0E_\mathbf{p}+\gamma_i\mathbf{p}^i+m\right)-\left(\gamma_0E_\mathbf{p}-\gamma_i\mathbf{p}^i-m\right)=2\left(\gamma_i\mathbf{p}^i+m\right)\,.$$

If would have here, instead of a $-$ sign, a $+$ sign between the parenthesis,

$$\left(\gamma_0E_\mathbf{p}+\gamma_i\mathbf{p}^i+m\right)+\left(\gamma_0E_\mathbf{p}-\gamma_i\mathbf{p}^i-m\right)=2\gamma_0E_\mathbf{p}\,,$$

and since $\gamma^0\gamma^0 = 1$, the result of the calculation would be just $\int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} = \delta^{(3)}(\mathbf{x}-\mathbf{y})$, and everything as expected. So how do we reverse the sign? The solution is to postulate, instead of commutators, the **anticommutation rules**,

$$\left\{a_{\mathbf{p},s}, a^\dagger_{\mathbf{k},r}\right\} = \left\{b_{\mathbf{p},s}, b^\dagger_{\mathbf{k},r}\right\} = (2\pi)^3\,\delta^{(3)}\left(\mathbf{p}-\mathbf{k}\right)\delta_{rs}\,. \tag{2.75}$$

In this case the equal-time field operators fulfill,

$$\begin{aligned} \{\psi(\mathbf{x},t), \pi(\mathbf{y},t)\} &= \{\psi(\mathbf{x},t), i\psi^\dagger(\mathbf{y},t)\} = i\delta^{(3)}\left(\mathbf{x}-\mathbf{y}\right)\,, \\ \{\psi(\mathbf{x},t), \psi(\mathbf{y},t)\} &= \{\psi^\dagger(\mathbf{x},t), \psi^\dagger(\mathbf{y},t)\} = 0\,. \end{aligned} \tag{2.76}$$

The anticommutation rules also lead to a sensible Hamiltonian operator. Substituting the decomposition (2.71) to Eq. (2.70) we find (Ex.),

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_s \left[a^\dagger_{\mathbf{p},s} a_{\mathbf{p},s} - b_{\mathbf{p},s} b^\dagger_{\mathbf{p},s}\right]\,, \tag{2.77}$$

where we have not yet used nor commutation nor anticommutation relations. If we would now adopt the commutation relations (2.72), we would get a Hamiltonian,

$$\tilde{H} = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_s \left[a^\dagger_{\mathbf{p},s} a_{\mathbf{p},s} - b^\dagger_{\mathbf{p},s} b_{\mathbf{p},s}\right] + (\text{infinite constant})\,, \tag{2.78}$$

whereas the anticommutation rules (2.75) lead to

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_s \left[a^\dagger_{\mathbf{p},s} a_{\mathbf{p},s} + b^\dagger_{\mathbf{p},s} b_{\mathbf{p},s}\right] + (\text{infinite constant})\,. \tag{2.79}$$

Independently of whether we postulate commutation or anticommutation rules, it is easy to see that if a state $|n\rangle$ is an eigenstate of the number operator,

$$N = \int \frac{d^3p}{(2\pi)^3} \sum_s \left[a^\dagger_{\mathbf{p},s} a_{\mathbf{p},s} + b^\dagger_{\mathbf{p},s} b_{\mathbf{p},s}\right] , \tag{2.80}$$

with eigenvalue n, $N|n\rangle = n|n\rangle$, we have

$$N a_{\mathbf{k},r}|n\rangle = (n-1)\, a_{\mathbf{k},r}|n\rangle\, , \quad N a^\dagger_{\mathbf{k},r}|n\rangle = (n+1)\, a^\dagger_{\mathbf{k},r}|n\rangle\, , \tag{2.81}$$

$$N b_{\mathbf{k},r}|n\rangle = (n-1)\, b_{\mathbf{k},r}|n\rangle\, , \quad N b^\dagger_{\mathbf{k},r}|n\rangle = (n+1)\, b^\dagger_{\mathbf{k},r}|n\rangle\, . \tag{2.82}$$

Both $\tilde{H}$ and H also commute with N (with commutation or anticommutation rules for the ladder operators, respectively) so they have common eigenstates. Requiring that the norm of states remains positive leads to the existense of a vacuum, $a_{\mathbf{k},r}|0\rangle = b_{\mathbf{k},r}|0\rangle = 0$. However, in the case of $\tilde{H}$,

$$\tilde{H} a_{\mathbf{k},r}|n\rangle = (E - E_{\mathbf{k}})\, a_{\mathbf{k},r}|n\rangle\, , \quad \tilde{H} a^\dagger_{\mathbf{k},r}|E\rangle = (E + E_{\mathbf{k}})\, a^\dagger_{\mathbf{k},r}|n\rangle\, , \tag{2.83}$$

$$\tilde{H} b_{\mathbf{k},r}|n\rangle = (E + E_{\mathbf{k}})\, b_{\mathbf{k},r}|n\rangle\, , \quad \tilde{H} b^\dagger_{\mathbf{k},r}|E\rangle = (E - E_{\mathbf{k}})\, b^\dagger_{\mathbf{k},r}|n\rangle\, , \tag{2.84}$$

so the energy of a state could be made negative by creating more antiparticles with $b^\dagger_{\mathbf{k},r}$. Not good. The Hamiltonian (2.79) obtained with the anticommutation relations, however, works logically

$$H a_{\mathbf{k},r}|n\rangle = (E - E_{\mathbf{k}})\, a_{\mathbf{k},r}|n\rangle\, , \quad H a^\dagger_{\mathbf{k},r}|n\rangle = (E + E_{\mathbf{k}})\, a^\dagger_{\mathbf{k},r}|n\rangle\, , \tag{2.85}$$

$$H b_{\mathbf{k},r}|n\rangle = (E - E_{\mathbf{k}})\, b_{\mathbf{k},r}|n\rangle\, , \quad H b^\dagger_{\mathbf{k},r}|n\rangle = (E + E_{\mathbf{k}})\, b^\dagger_{\mathbf{k},r}|n\rangle\, . \tag{2.86}$$

All in all, the anticommutation rules seem to entail a sensible quantization for the Dirac field.

Momentum operator

The momentum operator can be obtained in the same way as the Hamiltonian. Starting from the general result (1.53) we get,

$$\mathbf{P} = \int d^3x\, \psi^\dagger(x) \left(-i\vec{\nabla}\right) \psi(x) = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} \sum_s \left[a^\dagger_{\mathbf{p},s} a_{\mathbf{p},s} + b^\dagger_{\mathbf{p},s} b_{\mathbf{p},s}\right] . \tag{2.87}$$

The momentum operator works as the Hamiltonian, e.g.

$$\mathbf{P}a_{\mathbf{k},r}|n\rangle = (\mathbf{p}_n - \mathbf{k})\, a_{\mathbf{k},r}|n\rangle\,, \quad \mathbf{P}a^\dagger_{\mathbf{k},r}|n\rangle = (\mathbf{p}_n + \mathbf{k})\, a^\dagger_{\mathbf{k},r}|n\rangle\,, \tag{2.88}$$

$$\mathbf{P}b_{\mathbf{k},r}|n\rangle = (\mathbf{p}_n - \mathbf{k})\, b_{\mathbf{k},r}|n\rangle\,, \quad \mathbf{P}b^\dagger_{\mathbf{k},r}|n\rangle = (\mathbf{p}_n + \mathbf{k})\, b^\dagger_{\mathbf{k},r}|n\rangle\,, \tag{2.89}$$

Charge operator

As in the case of complex scalar field, the symmetry of the Dirac Lagrangian under a global phase shift $\psi(x) \to e^{i\alpha}\psi(x)$ entails a conserved Noether current $j^\mu = \overline{\psi}\gamma^\mu\psi$ and a conserved charge,

$$Q = \int d^3x j^0(x) = \int d^3x \psi^\dagger(x)\psi(x) \tag{2.90}$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s \left[a^\dagger_{\mathbf{p},s} a_{\mathbf{p},s} - b^\dagger_{\mathbf{p},s} b_{\mathbf{p},s}\right] + \text{infinite constant}\,.$$

So also in the Dirac case the particles and antiparticles have the same charge but of opposite sign.

Angular momentum and spin

The spin-$\frac{1}{2}$ property of the Dirac particles becomes explicit when we consider rotations. Under an infinitesimal rotation (2.35) about the z axis, the non-zero components of the Lorentz-transformation are $\delta\omega^1{}_2 = \theta$, $\delta\omega^2{}_1 = -\theta$. This corresponds to the spinor transformation matrix,

$$S = 1 - \frac{i}{4}\delta\omega_{\mu\nu}\sigma^{\mu\nu} = 1 + \frac{i}{2}\theta\sigma^{12} \tag{2.91}$$

$$\sigma^{12} = \frac{i}{2}\left[\gamma^1, \gamma^2\right] = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} = \Sigma^3 \tag{2.92}$$

$$\Longrightarrow S = 1 - \frac{i}{4}\delta_{\mu\nu}\sigma^{\mu\nu} = 1 + \frac{i}{2}\theta\Sigma^3\,. \tag{2.93}$$

The transformed spinor is thus,

$$\psi'(x) = S\psi(\Lambda^{-1}x) = \left(1 + \frac{i}{2}\theta\Sigma^3\right)\psi(t, x - \theta y, y + \theta x, z) \tag{2.94}$$

$$= \left(1 + \frac{i}{2}\theta\Sigma^3\right)[1 - \theta y\partial_1 + \theta x\partial_2]\,\psi(x)\,, \tag{2.95}$$

so the field transforms as

$$\delta\psi(x) = \psi'(x) - \psi(x) = \theta\left[-y\partial_1 + x\partial_2 + \frac{i}{2}\Sigma^3\right]\psi(x)\,. \tag{2.96}$$

The Lagrange density transforms as a scalar,

$$\Delta\mathcal{L} = \mathcal{L}(\Lambda^{-1}x) - \mathcal{L}(x) = \theta\,[-y\partial_1 + x\partial_2]\,\mathcal{L}(x) = \theta\,\partial_\mu J^\mu\,, \tag{2.97}$$

$$J^\mu \equiv (0, -y, x, 0)\ \mathcal{L}\,. \tag{2.98}$$

On the other hand, by the Euler-Lagrange equations,

$$\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\partial_\mu\left(\delta\psi\right) = \left[\partial_\mu\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\right]\delta\psi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\partial_\mu\left(\delta\psi\right)$$

$$= \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\delta\psi\right] = \theta\partial_\mu\left[\overline{\psi}i\gamma^\mu\left(-y\partial_1 + x\partial_2 + \frac{i}{2}\Sigma^3\right)\psi(x)\right]. \tag{2.99}$$

The conserved current in a rotation about the z axis is thus,

$$j^\mu = \overline{\psi}i\gamma^\mu\left(-y\partial_1 + x\partial_2 + \frac{i}{2}\Sigma^3\right)\psi(x) - J^\mu\,, \tag{2.100}$$

entailing a conserved "charge",

$$\int d^3x j^0 = \int d^3x\,\left\{(-i)\psi^\dagger(x)\,[-y\partial_1 + x\partial_2]\,\psi(x) + \psi^\dagger\left(\frac{1}{2}\Sigma^3\right)\psi(x)\right\}. \tag{2.101}$$

The rotations about the x and y axes generate their own conserved charges. The three thus constitute a 3-vector

$$\mathbf{J} \equiv \int d^3x\,\psi^\dagger\left[\mathbf{x}\times(-i\nabla) + \frac{1}{2}\mathbf{\Sigma}\right]\psi\,. \tag{2.102}$$

From Eq. (2.87) we see that the term $\psi^\dagger(-i\nabla)\psi$ corresponds to the momentum density, so we interpret the first term in Eq. (2.102) as the orbital angular mometum. The last $\mathbf{\Sigma}/2$ term has no counterpart in scalar theory, so it therefore is purely related to the inner properties of the Dirac field that behave as angular momentum. This kind of property we call the **spin**.

In section 2.1 we mentioned that it is always possible to choose ξ_s and η_s in the Dirac spinors such that they are eigenspinors of the helicity operator with eigenvalues $\pm 1/2$,

$$\left(\frac{1}{2}\mathbf{\Sigma}\cdot\hat{\mathbf{p}}\right)u_\pm(p) = \pm\frac{1}{2}u_\pm(p)\ , \tag{2.103}$$

$$\left(\frac{1}{2}\mathbf{\Sigma}\cdot\hat{\mathbf{p}}\right)v_\pm(p) = \pm\frac{1}{2}v_\pm(p)\ . \tag{2.104}$$

By expressing the angular-momentum operator $\mathbf{J}$ in terms of creation and annihilation operators we can show (Ex.), that with this choice of spinors,

$$(\mathbf{J}\cdot\hat{\mathbf{p}})\ a^\dagger_{\mathbf{p},\pm}|0\rangle = \pm\frac{1}{2}\,a^\dagger_{\mathbf{p},\pm}|0\rangle \tag{2.105}$$

$$(\mathbf{J}\cdot\hat{\mathbf{p}})\ b^\dagger_{\mathbf{p},\pm}|0\rangle = \mp\frac{1}{2}\,b^\dagger_{\mathbf{p},\pm}|0\rangle\ . \tag{2.106}$$

In other words, **independently of the magnitude of the momentum, the projection of the angular momentum to the direction of motion is always $\pm\frac{1}{2}$.** This shows at quantum level why the Dirac field corresponds to spin-$\frac{1}{2}$ particles. The above result also justifies why the choice $\eta_s = (-i\sigma^2)\xi_s^*$ for the antiparticle spinor makes sense: with this choice both $a^\dagger_{\mathbf{p},s}$ and $b^\dagger_{\mathbf{p},s}$ create exitations that have the same physical helicity (when we choose $\hat{h}\xi_\pm = \pm\frac{1}{2}\xi_\pm$).

Statistics and the Pauli exclusion principle

The anticommutation relations dictate the behaviour of the multiparticle states:

$$|(\mathbf{p},s);(\mathbf{k},r)\rangle \equiv \sqrt{2E_{\mathbf{p}}}\sqrt{2E_{\mathbf{k}}}a^{\dagger}_{\mathbf{p},s}a^{\dagger}_{\mathbf{k},r}|0\rangle = -\sqrt{2E_{\mathbf{p}}}\sqrt{2E_{\mathbf{k}}}a^{\dagger}_{\mathbf{k},r}a^{\dagger}_{\mathbf{p},s}|0\rangle$$

$$= -|(\mathbf{k},r);(\mathbf{p},s)\rangle\,, \tag{2.107}$$

$$\langle(\mathbf{p},s);(\mathbf{k},r)| \equiv \langle 0|a_{\mathbf{k},r}a_{\mathbf{p},s}\sqrt{2E_{\mathbf{p}}}\sqrt{2E_{\mathbf{k}}} = -\langle 0|a_{\mathbf{p},s}a_{\mathbf{k},r}\sqrt{2E_{\mathbf{p}}}\sqrt{2E_{\mathbf{k}}}$$

$$= -\langle(\mathbf{k},r);(\mathbf{p},s)|\,. \tag{2.108}$$

This shows that the states are antisymmetric under an interchange of two particles. Particles that behave like this are called **fermions** and the obey the Fermi-Dirac statistics. Since $a^{\dagger}_{\mathbf{p},s}a^{\dagger}_{\mathbf{p},s} = 0$, we cannot create a state that contains two particles with the same spin and momentum. This is the **Pauli exclusion principle**.

2.5 Dirac propagator

As in the case a scalar field, we can find the Green's functions of the Dirac operator $(i\gamma^{\mu}\partial_{\mu} - m)$ which we call propagators. The most important turns out to be the Feynman propagator which we define in a similar manner as in Eq. (1.150). However, in the Dirac case we include one minus sign in the definition of the time-ordered product,

$$S_{\mathrm{F}}(x-y) \equiv \langle 0|T\psi(x)\overline{\psi}(y)|0\rangle \tag{2.109}$$

$$\equiv \theta(x^0-y^0)\langle 0|\psi(x)\overline{\psi}(y)|0\rangle - \theta(y^0-x^0)\langle 0|\overline{\psi}(y)\psi(x)|0\rangle$$

$$= \int \frac{\mathrm{d}^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\frac{i\left(\not{p}+m\right)}{p^2-m^2+i\epsilon}\,.$$

The latter integral form is again a result of a straightforward calculation substituting the field operators (2.71), using the anticommutation relations (2.75) and the integral representation of the θ function (1.123).

2.6 Discrete symmetries of the Dirac field [Peskin 3.6]

We will now consider the following discrete transformations:

Reflection: $(t, \mathbf{x}) \xrightarrow{P} (t, -\mathbf{x})$

Time reversal: $(t, \mathbf{x}) \xrightarrow{T} (-t, \mathbf{x})$

Charge conjugation: particle $\xleftrightarrow{C}$ antiparticle

The first two can be considered as Lorentz transformations as the length of a 4-vector is clearly invariant under these operations. However, we cannot usually parametrize them with continuous boosts/rotations and that's why they are called discrete transformations. The charge conjugation, in turn, is not a space-time transformation at all.

Reflection of space – parity transformation:

The reflection of the spatial components of a 4-vector corresponds to the Lorentz-transformation matrix,

$$(\Lambda_{\mathrm{P}})^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = (\Lambda_{\mathrm{P}}^{-1})^{\mu}{}_{\nu}\,. \tag{2.110}$$

We can use our earlier result $\gamma^{\mu} S(\Lambda) = \Lambda^{\mu}{}_{\nu} S(\Lambda)\gamma^{\nu}$, for the spinor transformation matrices S. The solution is,

$$S_{\mathrm{P}} = \eta_P \gamma^0\,, \tag{2.111}$$

where η_P is a constant. The inverse transformation is $S_{\mathrm{P}}^{-1} = \gamma^0 S_{\mathrm{P}}^{\dagger}\gamma^0 = \eta_P^{*}\gamma^0$, so requiring $S_{\mathrm{P}} S_{\mathrm{P}}^{-1} = 1$ implies that η_P is just a phase, $\eta_P = e^{i\phi}$. Let's now see how the plane-wave solutions to the Dirac equations behave under $\psi(t,\mathbf{x}) \to S_{\mathrm{P}}\psi(t,-\mathbf{x})$:

$$u_s(p)e^{-ip\cdot x} \longrightarrow \eta_P\gamma^0 u_s(p)e^{-i(Et+\mathbf{p}\cdot\mathbf{x})} \tag{2.112}$$

$$v_s(p)e^{ip\cdot x} \longrightarrow \eta_P\gamma^0 v_s(p)e^{i(Et+\mathbf{p}\cdot\mathbf{x})}\,. \tag{2.113}$$

Using the identities (Ex.),

$$\gamma^0 u_s(E_{\mathbf{p}},-\mathbf{p}) = u_s(E_{\mathbf{p}},\mathbf{p})\,, \tag{2.114}$$

$$\gamma^0 v_s(E_{\mathbf{p}},-\mathbf{p}) = -v_s(E_{\mathbf{p}},\mathbf{p})\,, \tag{2.115}$$

we find

$$u_s(p)e^{-ip\cdot x} \longrightarrow \eta_P u_s(E_{\mathbf{p}},-\mathbf{p})e^{-i(Et+\mathbf{p}\cdot\mathbf{x})} \tag{2.116}$$

$$v_s(p)e^{ip\cdot x} \longrightarrow -\eta_P v_s(E_{\mathbf{p}},-\mathbf{p})e^{i(Et+\mathbf{p}\cdot\mathbf{x})}\,. \tag{2.117}$$

From this we can conclude that the parity transformation flips the spatial momentum of the particles. For the creation and annihilation operators we thus expect,

$$Pa_{\mathbf{p},s}P^\dagger = \eta_a a_{-\mathbf{p},s}\,, \qquad Pb_{\mathbf{p},s}P^\dagger = \eta_b b_{-\mathbf{p},s}\,, \tag{2.118}$$

$$Pa^\dagger_{\mathbf{p},s}P^\dagger = \eta_a^* a^\dagger_{-\mathbf{p},s}\,, \qquad Pb^\dagger_{\mathbf{p},s}P^\dagger = \eta_b^* b^\dagger_{-\mathbf{p},s}\,, \tag{2.119}$$

where P is unitary and $\eta_{a,b}$ are possible phase factors.

Let's now do this transform to quantum fields:

$$P\psi(x)P^\dagger = \int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_{\mathbf{p}}}}\sum_s\left[\eta_a a_{-\mathbf{p},s}u_s(p)e^{-ip\cdot x} + \eta_b^* b^\dagger_{-\mathbf{p},s}v_s(p)e^{ip\cdot x}\right]$$

$$= \gamma^0\int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_{\mathbf{p}}}}\sum_s\left[\eta_a a_{-\mathbf{p},s}u_s(E_{\mathbf{p}},-\mathbf{p})e^{-ip\cdot x} - \eta_b^* b^\dagger_{-\mathbf{p},s}v_s(E_{\mathbf{p}},-\mathbf{p})e^{ip\cdot x}\right]$$

$$= \gamma^0\int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_{\mathbf{p}}}}\sum_s\left[\eta_a a_{\mathbf{p},s}u_s(p)e^{-i(Et+\mathbf{p}\cdot\mathbf{x})} - \eta_b^* b^\dagger_{\mathbf{p},s}v_s(p)e^{i(Et+\mathbf{p}\cdot\mathbf{x})}\right]\,.$$

If the phase factors are related by $\eta_b^* = -\eta_a$, we get

$$P\psi(x)P^\dagger = \eta_a\gamma^0\psi(t,-\mathbf{x})\,. \tag{2.120}$$

which is identical with the transformation law of non-quantized fields. From this we easily get the transformation property for the conjugated field,

$$P\overline{\psi}(x)P^\dagger = (P\psi(x)P)^\dagger \gamma^0 = \left(\eta_a\gamma^0\psi(t,-\mathbf{x})\right)^\dagger \gamma^0 \qquad (2.121)$$
$$= \eta_a^*\psi^\dagger(t,-\mathbf{x})\gamma^0\gamma^0 = \eta_a^*\overline{\psi}(t,-\mathbf{x})\gamma^0\,.$$

With (2.120) ja (2.121) we can find how different bilinears transform. For example, $\overline{\psi}(x)\psi(x)$ transforms as a scalar,

$$P\overline{\psi}(x)\psi(x)P^\dagger = P\overline{\psi}(x)P^\dagger P\psi(x)P^\dagger = \eta_a^*\overline{\psi}(t,-\mathbf{x})\gamma^0\eta_a\gamma^0\psi(t,-\mathbf{x})$$
$$= \overline{\psi}(t,-\mathbf{x})\psi(t,-\mathbf{x})\,, \qquad (2.122)$$

as we could expect. Since all the bilinears contain one ψ and one $\overline{\psi}$ field, the phase factor η_a has no practical effect. An important result, however, is the phase difference (-1) between the transformation of Dirac particles and antiparticles. This is necessary for the parity to be a symmetry of the theory. We say, that the Dirac particles and antiparticles have an opposite **intrinsic parity**. For this reason, a state consisting of a fermion and an antifermion flips its sign under a parity transformation:

$$Pa^\dagger_{\mathbf{p},s}b^\dagger_{\mathbf{p},s}|0\rangle = Pa^\dagger_{\mathbf{p},s}P^\dagger Pb^\dagger_{\mathbf{p},s}P^\dagger P|0\rangle \qquad (2.123)$$
$$= \eta_a^* a^\dagger_{-\mathbf{p},s}\eta_b^* b^\dagger_{-\mathbf{p},s}|0\rangle = -(\eta_a^*\eta_a)a^\dagger_{-\mathbf{p},s}b^\dagger_{-\mathbf{p},s}|0\rangle = -a^\dagger_{-\mathbf{p},s}b^\dagger_{-\mathbf{p},s}|0\rangle\,.$$

Below is the complete table of how the Dirac bilinears trasform under parity:

$$P\overline{\psi}(x)\psi(x)P^\dagger = \overline{\psi}(t,-\mathbf{x})\psi(t,-\mathbf{x})$$
$$Pi\overline{\psi}(x)\gamma^5\psi(x)P^\dagger = -i\overline{\psi}(t,-\mathbf{x})\gamma^5\psi(t,-\mathbf{x})$$
$$P\overline{\psi}(x)\gamma^\mu\psi(x)P^\dagger = (-1)^\mu \times \overline{\psi}(t,-\mathbf{x})\gamma^\mu\psi(t,-\mathbf{x}) \qquad (2.124)$$
$$P\overline{\psi}(x)\gamma^\mu\gamma^5\psi(x)P^\dagger = -(-1)^\mu \times \overline{\psi}(t,-\mathbf{x})\gamma^\mu\gamma^5\psi(t,-\mathbf{x})$$
$$P\overline{\psi}(x)\sigma^{\mu\nu}\psi(x)P^\dagger = (-1)^\mu \times (-1)^\nu \times \overline{\psi}(t,-\mathbf{x})\sigma^{\mu\nu}\psi(t,-\mathbf{x})$$

where

$$(-1)^{\mu} \equiv \begin{cases} 1 & \text{if} \quad \mu = 0 \\ -1 & \text{if} \quad \mu \neq 0 \end{cases} \tag{2.125}$$

Time reversal:

The time reversal for 4-vectors is implemented by the matrix

$$(\Lambda_{\mathrm{T}})^{\mu}{}_{\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (\Lambda_{\mathrm{T}}^{-1})^{\mu}{}_{\nu} \,. \tag{2.126}$$

If we try to solve for the spinor transformation matrix from $S^{-1}(\Lambda)\,\gamma^{\mu} S(\Lambda) = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}$ like before, we find no solution (Ex.). Thus the time reversal in the form $\psi(t, \mathbf{x}) \to S(\Lambda_{\mathrm{T}})\psi(-t, \mathbf{x})$ does not seem to work out. However, if we include also complex conjugation,

$$\psi(t, \mathbf{x}) \to S(\Lambda_{\mathrm{T}})\psi^{*}(-t, \mathbf{x}) \,, \tag{2.127}$$

the Dirac equation transforms as [see (2.47)]

$$\begin{aligned} &(i\gamma^{\mu}\partial_{\mu} - m)\,\psi(x) = 0 \longrightarrow (i\gamma^{\mu}\partial_{\mu} - m)\,S(\Lambda_{\mathrm{T}})\psi^{*}(-t, \mathbf{x}) \\ &= S(\Lambda_{\mathrm{T}})\left[i(\Lambda_{\mathrm{T}}^{-1})^{\nu}{}_{\mu}S^{-1}(\Lambda_{\mathrm{T}})\,\gamma^{\mu}S(\Lambda_{\mathrm{T}})\;\partial_{\nu} - m\right]\psi^{*}(-t, \mathbf{x}) \,. \end{aligned}$$

If there exist a matrix $S(\Lambda_{\mathrm{T}})$, which fulfills the condition

$$(\Lambda_{\mathrm{T}}^{-1})^{\nu}{}_{\mu}S^{-1}(\Lambda_{\mathrm{T}})\,\gamma^{\mu}S(\Lambda_{\mathrm{T}}) = -\gamma^{\nu *} \,, \tag{2.128}$$

we get

$$(i\gamma^{\mu}\partial_{\mu} - m)\,S(\Lambda_{\mathrm{T}})\psi^{*}(-t, \mathbf{x}) = S(\Lambda_{\mathrm{T}})\left[-i\gamma^{\nu *}\,\partial_{\nu} - m\right]\psi^{*}(-t, \mathbf{x}) = 0 \,.$$

where we get zero since the obtained form is the complex-conjugated version of the Dirac equation $(i\gamma^{\nu}\,\partial_{\nu} - m)\,\psi(x) = 0$ which is true with all arguments x of the field. A sensible time reversal thus required that equation

$$\gamma^{\mu}S(\Lambda_{\mathrm{T}}) = -(\Lambda_{\mathrm{T}})^{\mu}{}_{\nu}S(\Lambda_{\mathrm{T}})\,\gamma^{\nu *} \,, \tag{2.129}$$

has a solution. It is straightforward to show that

$$S(\Lambda_{\rm T}) = \eta_{\rm T}\gamma^1\gamma^3 \tag{2.130}$$

fulfills Eq. (2.129), so $\gamma^1\gamma^3\psi^*(-t,\mathbf{x})$ is a solution to the Dirac equation. Since $(\gamma^1\gamma^3)^* = \gamma^1\gamma^3$,

$$\psi(t,\mathbf{x}) \xrightarrow{\rm T} \eta_{\rm T}\gamma^1\gamma^3\psi^*(-t,\mathbf{x}) \tag{2.131}$$

$$\overline{\psi}(t,\mathbf{x}) \xrightarrow{\rm T} \eta_{\rm T}^*\overline{\psi}^*(-t,\mathbf{x})\gamma^3\gamma^1\,, \tag{2.132}$$

and we can easily verify that the Lagrange density is covariant under time reversal (provided that $\eta_{\rm T}\eta_{\rm T}^* = 1$):

$$\begin{aligned}
&\overline{\psi}(x)\,(i\gamma^\mu\partial_\mu - m)\,\psi(x) \longrightarrow |\eta_{\rm T}|^2\overline{\psi}^*(x)\,[-i\gamma^{\mu*}\,\partial_\mu - m]\,\psi^*(-t,\mathbf{x})\\
&= \left[\overline{\psi}(-t,\mathbf{x})\,(i\gamma^\mu\,\partial_\mu - m)\,\psi(-t,\mathbf{x})\right]^* \\
&= \overline{\psi}(-t,\mathbf{x})\,(i\gamma^\mu\,\partial_\mu - m)\,\psi(-t,\mathbf{x})\,,
\end{aligned} \tag{2.133}$$

where the last equality follows from the reality of the Lagrangian with all arguments of the field. The action and the equations of motion thus remain intact and the time reversal is a symmetry of the Dirac theory. What is the physics content of this transformation? We can plainly see this if we again find out how the plane-wave solutions behave:

$$u_s(p)e^{-ip\cdot x} \longrightarrow \eta_{\rm T}\gamma^1\gamma^3u_s^*(p)e^{i(-Et-\mathbf{p}\cdot\mathbf{x})} \tag{2.134}$$

$$v_s(p)e^{ip\cdot x} \longrightarrow \eta_{\rm T}\gamma^1\gamma^3v_s^*(p)e^{-i(-Et-\mathbf{p}\cdot\mathbf{x})}\,. \tag{2.135}$$

We now use the identities (Ex.),

$$u_{-s}(E_{\mathbf{p}},-\mathbf{p}) = -\gamma^1\gamma^3u_s^*(E_{\mathbf{p}},\mathbf{p}) \tag{2.136}$$

$$v_{-s}(E_{\mathbf{p}},-\mathbf{p}) = -\gamma^1\gamma^3v_s^*(E_{\mathbf{p}},\mathbf{p})\,, \tag{2.137}$$

where the negative spin index $-s$ refers to a flip of the spin part ξ_s,η_s of the spinors,

$$\xi_{-s} \equiv -i\sigma^2(\xi_s)^*\,, \qquad \eta_{-s} \equiv -i\sigma^2(\eta_s)^*\,. \tag{2.138}$$

For example, if we choose $\xi_1 = (1,0)$ ja $\xi_2 = (0,1)$, then

$$\xi_{-1} = -i\sigma^2(\xi_1)^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \xi_2 \tag{2.139}$$

$$\xi_{-2} = -i\sigma^2(\xi_2)^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\xi_1 \,. \tag{2.140}$$

This shows that the projection of the spin in the z direction reverses.

The above identities imply,

$$u_s(p)e^{-ip\cdot x} \longrightarrow -\eta_{\mathrm{T}} u_{-s}(E_{\mathbf{p}}, -\mathbf{p})e^{i(-Et-\mathbf{p}\cdot\mathbf{x})} \tag{2.141}$$

$$v_s(p)e^{ip\cdot x} \longrightarrow -\eta_{\mathrm{T}} v_{-s}(E_{\mathbf{p}}, -\mathbf{p})e^{-i(-Et-\mathbf{p}\cdot\mathbf{x})} \,. \tag{2.142}$$

From this we see that the time reversal flips the direction of the 3-momentum and the spin (but not the helicity).

We now know that the time reversal should reverse the momentum and the spin, so we must have,

$$T a_{\mathbf{p},s} T^\dagger = \eta_a a_{-\mathbf{p},-s} \,, \qquad T b_{\mathbf{p},s} T^\dagger = \eta_b b_{-\mathbf{p},-s} \,, \tag{2.143}$$

where we define $a_{-\mathbf{p},-s}$ similarly as for the spinors,

$$a_{\mathbf{p},-1} \equiv a_{\mathbf{p},2} \,, \qquad a_{\mathbf{p},-2} \equiv -a_{\mathbf{p},1} \tag{2.144}$$

$$b_{\mathbf{p},-1} \equiv b_{\mathbf{p},2} \,, \qquad b_{\mathbf{p},-2} \equiv -b_{\mathbf{p},1} \,. \tag{2.145}$$

If T would be a unitary operator, we would not reach an acceptable transformation since $T\psi(x)T^\dagger$ does not lead to a transformation that would be a symmetry of the (quantized) Lagrangian. A sensible result is obtained if we take T to be an **antilinear** and **antiunitary**:

antilinearity: $T(a\psi + b\phi) = a^* T\psi + b^* T\phi$

antiunitarity: $\langle T\psi | T\phi\rangle = \langle\psi|T^\dagger T|\phi\rangle^* = \langle\psi|\phi\rangle^* = \langle\phi|\psi\rangle$

If we accept this, e.g. the combo $a_{\mathbf{p},s}u_s(p)e^{-ip\cdot x}$ transforms as,

$$T\left[a_{\mathbf{p},s}u_s(p)e^{-ip\cdot x}\right]T^\dagger = \eta_a a_{-\mathbf{p},-s}u_s^*(p)e^{ip\cdot x}\,. \tag{2.146}$$

Lets see now what is $T\psi(x)T^\dagger$:

$$T\psi(x)T^\dagger \to \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_\mathbf{p}}}\sum_s \left[\eta_a a_{-\mathbf{p},-s}u_s^*(p)e^{ip\cdot x} + \eta_b^* b^\dagger_{-\mathbf{p},-s}v_s^*(p)e^{-ip\cdot x}\right]$$

$$= \gamma^1\gamma^3 \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_\mathbf{p}}}\sum_s \left[\eta_a a_{-\mathbf{p},-s}u_{-s}(E_\mathbf{p},-\mathbf{p})e^{ip\cdot x}\right.$$

$$\left. + \eta_b^* b^\dagger_{-\mathbf{p},-s}v_{-s}(E_\mathbf{p},-\mathbf{p})e^{-ip\cdot x}\right]$$

$$= \gamma^1\gamma^3 \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_\mathbf{p}}}\sum_s \left[\eta_a a_{\mathbf{p},s}u_s(p)e^{i(Et+\mathbf{p}\cdot\mathbf{x})} + \eta_b^* b^\dagger_{\mathbf{p},s}v_s(p)e^{-i(Et+\mathbf{p}\cdot\mathbf{x})}\right]$$

$$= \gamma^1\gamma^3 \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_\mathbf{p}}}\sum_s \left[\eta_a a_{\mathbf{p},s}u_s(p)e^{-i(-Et-\mathbf{p}\cdot\mathbf{x})} + \eta_b^* b^\dagger_{\mathbf{p},s}v_s(p)e^{i(-Et-\mathbf{p}\cdot\mathbf{x})}\right]$$

If now $\eta_a = \eta_b^* = \eta_\mathrm{T}$, we get at the operator level,

$$T\psi(t,\mathbf{x})T = \eta_\mathrm{T}\gamma^1\gamma^3\psi(-t,\mathbf{x})\,, \tag{2.147}$$

which closely resembles the spinor transformation law (2.131). The transformation properties of different bilinears follow from this. Since $\overline{\psi}(x)$ transforms as

$$T\overline{\psi}(x)T^\dagger = T\psi^\dagger(x)T^\dagger\gamma^0 = \left(T\psi(x)T^\dagger\right)^\dagger\gamma^0 = \eta_\mathrm{T}^*\left(\gamma^1\gamma^3\psi\left(-t,\mathbf{x}\right)\right)^\dagger\gamma^0$$

$$= \eta_\mathrm{T}^*\psi^\dagger\left(-t,\mathbf{x}\right)\gamma^{3\dagger}\gamma^{1\dagger}\gamma^0 = \eta_\mathrm{T}^*\overline{\psi}\left(-t,\mathbf{x}\right)\gamma^3\gamma^1\,, \tag{2.148}$$

we have, for the bilinear $\overline{\psi}\psi$,

$$T\overline{\psi}(x)\psi(x)T^\dagger = \left[T\overline{\psi}(x)T^\dagger\right]\left[T\psi(x)T^\dagger\right] \tag{2.149}$$

$$= \overline{\psi}\left(-t,\mathbf{x}\right)\gamma^3\gamma^1\gamma^1\gamma^3\psi\left(-t,\mathbf{x}\right)$$

$$= \overline{\psi}\left(-t,\mathbf{x}\right)\psi\left(-t,\mathbf{x}\right)\,.$$

Again, the phase factor η_{T} does not seem to play a role.

As a curiosity, if we do to consecutive time reversals,

$$\begin{aligned} T\left(T\psi(x)T^{\dagger}\right)T^{\dagger} = T\left[\eta_{\mathrm{T}}\gamma^{1}\gamma^{3}\psi(-t,\mathbf{x})\right]T^{\dagger} &= \eta_{\mathrm{T}}^{*}\gamma^{1*}\gamma^{3*}T\psi(-t,\mathbf{x})T^{\dagger} \\ &= \eta_{\mathrm{T}}^{*}\gamma^{1}\gamma^{3}\eta_{\mathrm{T}}\gamma^{1}\gamma^{3}\psi(x) = -\psi(x)\,, \end{aligned} \tag{2.150}$$

so the double time reversal reverses the sign even if $(\Lambda_{\mathrm{T}})^{\mu}{}_{\alpha}(\Lambda_{\mathrm{T}})^{\alpha}{}_{\nu} = \delta^{\mu}_{\nu}$. We say that T is a **projective representation** of the Lorentz transformation T. The fact the two consecutive time reversals inevitably reverses the sign of $\psi(x)$ makes the representation **intrinsically projective**.

Below is the complete table of how the Dirac bilinears trasform under time reversal:

$$\begin{aligned} T\overline{\psi}(x)\psi(x)T^{\dagger} &= \overline{\psi}(-t,\mathbf{x})\psi(-t,\mathbf{x}) \\ Ti\overline{\psi}(x)\gamma^{5}\psi(x)T^{\dagger} &= -i\overline{\psi}(-t,\mathbf{x})\gamma^{5}\psi(-t,\mathbf{x}) \\ T\overline{\psi}(x)\gamma^{\mu}\psi(x)T^{\dagger} &= (-1)^{\mu}\times\overline{\psi}(-t,\mathbf{x})\gamma^{\mu}\psi(-t,\mathbf{x}) \\ T\overline{\psi}(x)\gamma^{\mu}\gamma^{5}\psi(x)T^{\dagger} &= (-1)^{\mu}\times\overline{\psi}(-t,\mathbf{x})\gamma^{\mu}\gamma^{5}\psi(-t,\mathbf{x}) \\ T\overline{\psi}(x)\sigma^{\mu\nu}\psi(x)T^{\dagger} &= -(-1)^{\mu}\times(-1)^{\nu}\times\overline{\psi}(-t,\mathbf{x})\sigma^{\mu\nu}\psi(-t,\mathbf{x}) \end{aligned} \tag{2.151}$$

Charge conjugation:

The free Dirac theory is also symmetric in transformation

$$\psi(x) \xrightarrow{C} S(C)\psi^{*}(x)\,, \tag{2.152}$$

where $S(C)$ is a 4×4 matrix. This is different than the time reversal since the argument of the field x remains unchanged. The Dirac equation transforms as

$$0 = (i\gamma^\mu \partial_\mu - m)\,\psi(x) \xrightarrow{C} (i\gamma^\mu \partial_\mu - m)\,S(C)\psi^*(x) \tag{2.153}$$
$$= \left[(-i\gamma^{\mu *}\partial_\mu - m)\,S^*(C)\psi(x)\right]^*$$
$$= \left[S^*(C)\left(-iS^{*-1}(C)\gamma^{\mu *}S^*(C)\partial_\mu - m\right)\psi(x)\right]^* .$$

If we can find a matrix $S^*(C)$ such that

$$-S^{*-1}(C)\gamma^{\mu *}S^*(C) = \gamma^\mu\,, \tag{2.154}$$

we get

$$(i\gamma^\mu \partial_\mu - m)\,\psi(x) \xrightarrow{C} \left[S^*(C)\,(i\gamma^\mu \partial_\mu - m)\,\psi(x)\right]^* = 0\,, \tag{2.155}$$

where the nullity (suomeksi nolluus) follows from the original Dirac equation. The matrix $S(C)$ has then to fulfill,

$$\gamma^\mu S(C) = -S(C)\gamma^{\mu *}\,. \tag{2.156}$$

One can easily verify that a possible solution is

$$S(C) = i\gamma^2\,, \tag{2.157}$$

where the front factor i is a choice. Thus the transformation (2.152) is a symmetry of the Dirac theory. Let's see how the plane-wave solutions behave under this transformation:

$$u_s(p)e^{-ip\cdot x} \to (i\gamma^2)u_s^*(p)e^{ip\cdot x} \tag{2.158}$$

$$v_s(p)e^{ip\cdot x} \to (i\gamma^2)v_s^*(p)e^{-ip\cdot x}\,. \tag{2.159}$$

We will now use the spinor identities (Ex.)

$$v_s(p) = i\gamma^2 u_s^*(p)\,, \quad u_s(p) = i\gamma^2 v_s^*(p)\,, \tag{2.160}$$

which hold when we define the spin part of the v spinors as $\eta_s = (-i\sigma^2)\xi_s^*$. We end up with

$$u_s(p)e^{-ip\cdot x} \to (i\gamma^2)i\gamma^2 v_s(p)e^{ip\cdot x} = v_s(p)e^{ip\cdot x} \tag{2.161}$$

$$v_s(p)e^{ip\cdot x} \to (i\gamma^2)i\gamma^2 u_s(p)e^{-ip\cdot x} = u_s(p)e^{-ip\cdot x}\,. \tag{2.162}$$

The transformation (2.152) thus turns particles into antiparticles and vice versa. At the operator level we can thus expect,

$$C a_{\mathbf{p},s} C^\dagger = b_{\mathbf{p},s}\,, \qquad C b_{\mathbf{p},s} C^\dagger = a_{\mathbf{p},s}\,, \tag{2.163}$$

$$C a^\dagger_{\mathbf{p},s} C^\dagger = b^\dagger_{\mathbf{p},s}\,, \qquad C b^\dagger_{\mathbf{p},s} C^\dagger = a^\dagger_{\mathbf{p},s}\,. \tag{2.164}$$

Let's check how the quantized field $\psi(x)$ behaves assuming that C is unitary:

$$\begin{aligned}
C\psi(t,\mathbf{x})C^\dagger &= \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_\mathbf{p}}}\sum_s \left[b_{\mathbf{p},s} u_s(p) e^{-ip\cdot x} + a^\dagger_{\mathbf{p},s} v_s(p) e^{ip\cdot x}\right] \\
&= i\gamma^2 \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_\mathbf{p}}}\sum_s \left[b_{\mathbf{p},s} v^*_s(p) e^{-ip\cdot x} + a^\dagger_{\mathbf{p},s} u^*_s(p) e^{ip\cdot x}\right] \\
&= i\gamma^2 \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_\mathbf{p}}}\sum_s \left[b_{\mathbf{p},s} v^\dagger_s(p) e^{-ip\cdot x} + a^\dagger_{\mathbf{p},s} u^\dagger_s(p) e^{ip\cdot x}\right]^T \\
&= i\gamma^2 \left(\psi^\dagger\right)^T = i\gamma^2 \left(\overline{\psi}\gamma^0\right)^T = i\left(\overline{\psi}\gamma^0\gamma^2\right)^T \,.
\end{aligned}$$

Thus, for quantum field

$$C\psi C^\dagger = i\left(\overline{\psi}\gamma^0\gamma^2\right)^T \,. \tag{2.165}$$

The corresponding transformation for the conjugated spinor reads,

$$\begin{aligned}
C\overline{\psi}C^\dagger &= C\psi^\dagger\gamma^0 C^\dagger = (C\psi C^\dagger)^\dagger\gamma^0 = \left[i\left(\overline{\psi}\gamma^0\gamma^2\right)^T\right]^\dagger\gamma^0 \qquad (2.166) \\
&= -i\left[\left(\psi^\dagger\gamma^0\gamma^0\gamma^2\right)^T\right]^\dagger\gamma^0 = -i\left[\gamma^{2T}\psi^{\dagger T}\right]^\dagger\gamma^0 \\
&= -i\psi^T\gamma^{2*}\gamma^0 = i\psi^T\gamma^2\gamma^0 = i\psi^T\gamma^{2T}\gamma^{0T} \\
&= i\left(\gamma^0\gamma^2\psi\right)^T \,.
\end{aligned}$$

Below is the complete table of how the Dirac bilinears trasform under charge conjugation:

$$
\begin{aligned}
C\overline{\psi}(x)\psi(x)C^\dagger &= \overline{\psi}(x)\psi(x) \\
Ci\overline{\psi}(x)\gamma^5\psi(x)C^\dagger &= i\overline{\psi}(x)\gamma^5\psi(x) \\
C\overline{\psi}(x)\gamma^\mu\psi(x)C^\dagger &= -\overline{\psi}(x)\gamma^\mu\psi(x) \\
C\overline{\psi}(x)\gamma^\mu\gamma^5\psi(x)C^\dagger &= \overline{\psi}(x)\gamma^\mu\gamma^5\psi(x) \\
C\overline{\psi}(x)\sigma^{\mu\nu}\psi(x)C^\dagger &= -\overline{\psi}(x)\sigma^{\mu\nu}\psi(x)
\end{aligned}
\tag{2.167}
$$

3 Interacting fields [Peskin 4]

In the preceding two sections we considered non-interacting fields, i.e. there could not be any momentum exchange between the eigenstates of the Hamiltonian and the particle number could not change,

$$\langle \mathbf{k}_1 \mathbf{k}_2 | \mathbf{p}_1 \dots \mathbf{p}_n \rangle \neq 0 \text{ only if } n = 2, \text{ and } \mathbf{p}_1 = \mathbf{k}_{1,2}\,, \mathbf{p}_2 = \mathbf{k}_{2,1}\,.$$

To facilitate momentum exchange and particle creation the Lagrangian needs to contain terms which are higher than quadratic in fields. In this case we can (usually) no longer solve the spectrum of the theory exactly as we did in the preceding sections. In this chapter we will develop a perturbative method to deal with these higher-order terms.

3.1 Pictures in quantum mechanics

The different pictures of quantum mechanics have been discussed in Quantum Mechanics II course (and touched upon in Sect. 1.5.1). Let us recap them here.

Schrödinger picture:

In the Schrödinger picture the state vectors $|\psi(t)\rangle$ depend on time. The time dependence is dictated by the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle_S = H |\psi(t)\rangle_S\,. \tag{3.1}$$

If the Hamiltonian does not depend explicitly on time, we can solve the time dependence at arbitrary t if we know the state of the system at some initial time t_0,

$$|\psi(t)\rangle_S = \mathcal{U}(t,t_0)|\psi(t_0)\rangle_S\,, \quad \mathcal{U}(t,t_0) \equiv e^{-\frac{i}{\hbar}H(t-t_0)}\,, \tag{3.2}$$

where $\mathcal{U}(t,t_0)$ is a unitary evolution operator.

Heisenberg picture:

The time dependence can also be absorbed to the operators. Starting from a matrix element in the Schrödinger picture ($t_0 = 0$),

$$
{}_{\mathrm{S}}\langle\phi(t)|\mathcal{O}_{\mathrm{S}}|\psi(t)\rangle_{\mathrm{S}} = {}_{\mathrm{S}}\langle\phi(0)|\mathcal{U}^\dagger(t)\,\mathcal{O}_{\mathrm{S}}\,\mathcal{U}(t)|\psi(0)\rangle_{\mathrm{S}} = {}_{\mathrm{H}}\langle\phi|\mathcal{O}_{\mathrm{H}}(t)|\psi\rangle_{\mathrm{H}}\,, \tag{3.3}
$$

in which we defined the operator in the Heisenberg picture

$$
\mathcal{O}_{\mathrm{H}}(t) \equiv \mathcal{U}^\dagger(t)\,\mathcal{O}_{\mathrm{S}}\,\mathcal{U}(t) = e^{\frac{i}{\hbar}Ht}\,\mathcal{O}_{\mathrm{S}}\,e^{-\frac{i}{\hbar}Ht}\,. \tag{3.4}
$$

In this viewpoint, the operators are time dependent, not the states. By taking the time derivative,

$$
-i\hbar\frac{\partial}{\partial t}\mathcal{O}_{\mathrm{H}}(t) = -i\hbar\frac{\partial}{\partial t}\left[e^{\frac{i}{\hbar}Ht}\mathcal{O}_{\mathrm{S}}e^{\frac{-i}{\hbar}Ht}\right] = \left[H, \mathcal{O}_{\mathrm{H}}(t)\right]\,, \tag{3.5}
$$

if H does not depend on time. This is the Heisenberg equation of motion. Note that the Hamiltonian H is the same in both Schrödinger and Heisenberg pictures.

Interaction picture a.k.a Dirac picture:

In the interaction picture we split the Hamiltonian of the system into two pieces,

$$
H = H_0 + H_{\mathrm{int}}\,. \tag{3.6}
$$

In practice, H_0 will be the Hamiltonian of the free theory and H_{int} will contain all the interactions between the fields. The state-vectors and operators in the interaction picture are defined as,

$$
|\psi(t)\rangle_I \equiv e^{\frac{i}{\hbar}H_0 t}|\psi(t)\rangle_S \quad \xrightarrow{H_{\mathrm{int}}\to 0} |\psi(0)\rangle_S = |\psi\rangle_H\,, \tag{3.7}
$$

$$
\mathcal{O}_I(t) \equiv e^{\frac{i}{\hbar}H_0 t}\mathcal{O}_S e^{\frac{-i}{\hbar}H_0 t} \xrightarrow{H_{\mathrm{int}}\to 0} \mathcal{O}_H\,. \tag{3.8}
$$

We see that if the effect of H_{int} is "small," the interaction picture is "close" to the Heisenberg picture. The matrix elements remain the same,

$$
\begin{aligned}
{}_S\langle\phi(t)|\mathcal{O}_{\mathrm{S}}|\psi(t)\rangle_S =&\ {}_S\langle\phi(t)|e^{\frac{-i}{\hbar}H_0 t}e^{\frac{i}{\hbar}H_0 t}\mathcal{O}_{\mathrm{S}}e^{\frac{-i}{\hbar}H_0 t}e^{\frac{i}{\hbar}H_0 t}|\psi(t)\rangle_S \qquad (3.9)\\
=&\ {}_I\langle\phi(t)|\mathcal{O}_{\mathrm{I}}(t)|\psi(t)\rangle_I\,.
\end{aligned}
$$

By taking the time derivatives we get the equations of motion,

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle_I = H_I(t)|\psi(t)\rangle_I\,, \tag{3.10}$$

$$-i\hbar\frac{\partial}{\partial t}\mathcal{O}_{\mathrm{I}}(t) = [H_0, \mathcal{O}_{\mathrm{I}}(t)]\ . \tag{3.11}$$

The time dependence of the states is thus dicated by H_I which is H_{int} in the interaction picture, $H_I(t) = e^{\frac{i}{\hbar}H_0 t}H_{\text{int}}e^{\frac{-i}{\hbar}H_0 t}$. The time dependence of the operators is given by H_0. If we are interested in how the interactions change the eigenstates of the Hamiltonian, the nontrivial part is the time dependence of the state vectors.

3.2 Perturbative expansion of correlation functions

[Peskin 4.2]

In the case of non-interacting fields we already considered time-ordered 2-point functions $\langle 0|T\phi(x)\phi(y)|0\rangle$ which, as we saw, are essentially Green's functions of the free-theory differential operators. In this section we introduce a perturbative method to calculate similar objects in the interacting theory,

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle, \tag{3.12}$$

Here, $|\Omega\rangle$ is the ground state of the interacting theory, and the fields are in the Heisenberg picture.

For simplicity, we will consider again a real scalar field supplementing the free Klein-Gordon Lagrangian with an interaction term,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\,, \tag{3.13}$$

which defines the so-called **ϕ^4 theory**. This leads to a Hamiltonian,

$$H = \underbrace{\int d^3x\left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right]}_{H_0} + \underbrace{\int d^3x\frac{\lambda}{4!}\phi^4}_{H_{\text{int}}}\ . \tag{3.14}$$

The theory is quantized by postulating the canonical equal-time commutation relations,

$$[\phi(t,\vec{x}),\phi(t,\vec{y})] = [\pi(t,\vec{x}),\pi(t,\vec{y})] = 0\,, \tag{3.15}$$

$$[\phi(t,\vec{x}),\pi(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})\,. \tag{3.16}$$

Here the fields are in the Heisenberg picture, but the (equal time) commutation relations remain the same in all pictures. The Heisenberg and interaction picture fields are defined as above,

$$\phi(x) = \phi_H(x) = e^{iHt}\phi(\mathbf{x})e^{-iHt} \tag{3.17}$$

$$\phi_I(x) = e^{iH_0t}\phi(\mathbf{x})e^{-iH_0t}\,, \tag{3.18}$$

and the field operator $\phi_I(t,\mathbf{x})$ in the interaction picture obeys,

$$-i\frac{\partial}{\partial t}\phi_I(x) = [H_0,\phi_I(x)]\ . \tag{3.19}$$

From this and the corresponding equation for the conjugated momentum $\pi = \partial\mathcal{L}/\partial\dot{\phi}$, we find (Ex.),

$$\left(\Box + m^2\right)\phi_I(x) = 0, \tag{3.20}$$

which says that $\phi_I(x)$ fulfills the standard free-theory Klein-Gordon equation. Thus, **all our earlier results for the free scalar theory hold as such for $\phi_\mathbf{I}(x)$**. The operator equations of the interaction picture (3.11) therefore indeed reduce to the free-theory results which we already know.

The time evolution of the state vectors is non-trivial. Integrating both sides of Eq. (3.10) with $t > t_0$,

$$\int_{t_0}^{t} dt'\frac{\partial}{\partial t'}|\psi(t')\rangle_I = |\psi(t)\rangle_I - |\psi(t_0)\rangle_I = -i\int_{t_0}^{t} dt' H_I(t')|\psi(t')\rangle_I$$

$$\Longrightarrow |\psi(t)\rangle_I = |\psi(t_0)\rangle_I - i\int_{t_0}^{t} dt' H_I(t')|\psi(t')\rangle_I \tag{3.21}$$

This type of equation can be solved iteratively, i.e. substituting the above form for $|\psi(t)\rangle_I$ to the right-hand side of the equation repeatedly:

$$|\psi(t)\rangle_I = |\psi(t_0)\rangle_I - i\int_{t_0}^{t} dt' H_I(t') \left[|\psi(t_0)\rangle_I + (-i)\int_{t_0}^{t'} dt'' H_I(t'')|\psi(t'')\rangle_I \right]$$

$$= \left[1 + (-i)\int_{t_0}^{t} dt' H_I(t')\right] |\psi(t_0)\rangle_I \tag{3.22}$$

$$+ (-i)^2 \int_{t_0}^{t} dt' H_I(t') \int_{t_0}^{t'} dt'' H_I(t'')|\psi(t'')\rangle_I$$

$$= \left[1 + (-i)\int_{t_0}^{t} dt_1 H_I(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 H_I(t_1) \int_{t_0}^{t_1} dt_2 H_I(t_2)\right] |\psi(t_0)\rangle_I$$

$$+ \cdots$$

where $t \geq t_1 \geq t_2 \geq \ldots$. Continuing the iteration, we can write $|\psi(t)\rangle_I$ as an infinite **Dyson's series**,

$$|\psi(t)\rangle_I = U(t, t_0)|\psi(t_0)\rangle_I \tag{3.23}$$

$$U(t, t_0) = 1 + \sum_{n=1}^{\infty}(-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n H_I(t_1) \cdots H_I(t_n)\,,$$

where $t \geq t_1 \geq t_2 \geq \cdots \geq t_n$. The time ordering is here most essential and we must keep $H_I(t_i)$s in the correct order as $H_I(t_i)$s at different times do not generally commute. By using the definition of the time-ordered product, we can write the series in a shorter form: The nth term of the series is of the form,

$$(-i)^n U_n = \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n H_I(t_1) \cdots H_I(t_n) \tag{3.24}$$

$$= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n T\left\{H_I(t_1) \cdots H_I(t_n)\right\}\,,$$

where we didn't yet do anything. By extending all the integrals from t_0 to t we get another integral,

$$(-i)^n S_n \equiv \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n T\left\{H_I(t_1) \cdots H_I(t_n)\right\}\,. \tag{3.25}$$

There are n integration variables in this integral so we can split the integration domain into separate parts,

$$
\begin{aligned}
&t_1 \geq t_2 \geq t_3 \geq \cdots \\
&t_1 \geq t_3 \geq t_2 \geq \cdots \\
&t_2 \geq t_1 \geq t_3 \geq \cdots \\
&t_2 \geq t_3 \geq t_1 \geq \cdots \\
&t_3 \geq t_1 \geq t_2 \geq \cdots \\
&t_3 \geq t_2 \geq t_1 \geq \cdots \\
&\vdots
\end{aligned}
$$

There are clearly $n!$ separate regions like this (e.g. $3! = 6$ as above), as n objects can be ordered in $n!$ different ways. Because of the time ordering in the integrand, $T\{H_I(t_1)\cdots H_I(t_n)\}$, all the integration domains give the same result. For example, in the $n = 3$ case the domain $t_3 \geq t_2 \geq t_1$ reduces to the part $t_1 \geq t_2 \geq t_3$:

$$
\begin{aligned}
&\int_{t_0}^{t} dt_3 \int_{t_0}^{t_3} dt_2 \int_{t_0}^{t_2} dt_1 T\{H_I(t_1)H_I(t_2)H_I(t_3)\} \qquad (3.26)\\
&\int_{t_0}^{t} dt_3 \int_{t_0}^{t_3} dt_2 \int_{t_0}^{t_2} dt_1 T\{H_I(t_3)H_I(t_2)H_I(t_1)\} \\
&= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 T\{H_I(t_1)H_I(t_2)H_I(t_3)\} \ ,
\end{aligned}
$$

where, in the last step, we just renamed the integration variables. Thus,

$$
S_n = n!\, U_n \, , \tag{3.27}
$$

and the original time-evolution operator can be written in a shorter form,

$$
\begin{aligned}
U(t,t_0) &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n T\left\{H_I(t_1)\cdots H_I(t_n)\right\} \\
&= T\left\{\exp\left[-i\int_{t_0}^{t} dt' H_I(t')\right]\right\} . \qquad (3.28)
\end{aligned}
$$

Now back to the 2-point function (3.12). From Eqs. (3.17) and (3.18) we get a relation between the Heisenberg- and interaction-picture operators,

$$\phi(x) = \tilde{U}^\dagger(t)\phi_I(x)\tilde{U}(t) \qquad (3.29)$$

$$\tilde{U}(t) = e^{iH_0t}e^{-iHt} . \qquad (3.30)$$

By differentiating $\tilde{U}(t)$ with respect to time, we find an evolution equation,

$$
\begin{aligned}
i\frac{\partial}{\partial t}\tilde{U}(t) &= e^{iH_0t}\left(H - H_0\right)e^{-iHt} \qquad (3.31) \\
&= e^{iH_0t}\, H_{\text{int}}\, e^{-iH_0t}e^{iH_0t}e^{-iHt} \\
&= H_I(t)\tilde{U}(t) ,
\end{aligned}
$$

where $H_I(t)$ is H_{int} in the interaction picture,

$$H_I(t) = e^{iH_0t}\, H_{\text{int}}\, e^{-iH_0t} = \int d^3x \frac{\lambda}{4!}\phi_I^4 . \qquad (3.32)$$

This differential equation is formally the same as what we had for the interaction-picture states $|\psi(t)\rangle_I$. Thus, the solution is also the same,

$$\tilde{U}(t) = U(t,t_0)\tilde{U}(t_0) . \qquad (3.33)$$

By using the definition (3.30) we get an explicit representation for $U(t,t_0)$,

$$U(t,t_0) = \tilde{U}(t)\tilde{U}^\dagger(t_0) = e^{iH_0t}e^{-iH(t-t_0)}e^{-iH_0t_0} , \qquad (3.34)$$

which is clearly unitary. Also, $\tilde{U}(t) = U(t,0)$. From these we easily find the following properties,

$$U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3) \qquad (3.35)$$

$$U^\dagger(t_1,t_2) = U(t_2,t_1) . \qquad (3.36)$$

In the case of free theory, the requirement of positive-norm states implied the existence of the vacuum $|0\rangle$. As already noted, the fields in the interaction picture ϕ_I fulfill the Klein-Gordon equation so we can still write H_0 in the form $H_0 = \int d^3p/(2\pi)^3 E_{\mathbf{p}} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$, and $H_0|0\rangle = 0$. Using this vacuum state it is possible to "project out" the ground state $|\Omega\rangle$ of the interacting theory by expanding $e^{-iHT}|0\rangle$ in terms of the spectral representation of H,

$$e^{-iHT}|0\rangle = e^{-iE_nT}\sum_n |n\rangle\langle n|0\rangle \tag{3.37}$$

$$= e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + e^{-iE_nT}\sum_{n\neq\Omega}|n\rangle\langle n|0\rangle\,,$$

where $|n\rangle$ are eigenstates of the full Hamiltonian including the interactions. Thus,

$$|\Omega\rangle = \frac{e^{+iE_0T}}{\langle\Omega|0\rangle}e^{-iHT}|0\rangle - \frac{e^{-iT(E_n-E_0)}}{\langle\Omega|0\rangle}\sum_{n\neq\Omega}|n\rangle\langle n|0\rangle\,. \tag{3.38}$$

Since the energy of the ground state is the smallest, $E_n - E_0 > 0$, the last term vanishes in the limit $T \to \infty(1-i\epsilon)$:

$$\lim_{T\to\infty(1-i\epsilon)} e^{-iT(E_n-E_0)} = \lim_{T\to\infty} e^{-iT(1-i\epsilon)(E_n-E_0)} \tag{3.39}$$

$$= \lim_{T\to\infty} e^{-iT(E_n-E_0)}e^{-T\epsilon(E_n-E_0)} = 0\,,$$

provided $\epsilon > 0$. Thus,

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)}\frac{e^{+iE_0T}}{\langle\Omega|0\rangle}e^{-iHT}|0\rangle = \lim_{T\to\infty(1-i\epsilon)}\frac{e^{+iE_0T}}{\langle\Omega|0\rangle}e^{-iHT}e^{iH_0T}e^{-iH_0T}|0\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)}\frac{e^{+iE_0T}}{\langle\Omega|0\rangle}e^{-iHT}e^{iH_0T}|0\rangle = \lim_{T\to\infty(1-i\epsilon)}\frac{e^{+iE_0T}}{\langle\Omega|0\rangle}\tilde{U}^\dagger(-T)|0\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)}\frac{e^{+iE_0T}}{\langle\Omega|0\rangle}U(0,-T)|0\rangle\,, \tag{3.40}$$

where we used $H_0|0\rangle = 0$. In the same way,

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)}\frac{e^{+iE_0T}}{\langle 0|\Omega\rangle}\langle 0|U(T,0)\,. \tag{3.41}$$

We can now finally have an expression for the interacting-theory 2-point function:

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{e^{+iE_0T}}{\langle 0|\Omega\rangle}\frac{e^{+iE_0T}}{\langle\Omega|0\rangle} \tag{3.42}$$

$$\langle 0|U(T,0)\left[U^\dagger(x_0,0)\phi_I(x)U(x_0,0)\right]\left[U^\dagger(y^0,0)\phi_I(y)U(y^0,0)\right]U(0,-T)|0\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{e^{+2iE_0T}}{|\langle 0|\Omega\rangle|^2}\langle 0|U(T,x^0)\phi_I(x)\,U(x^0,y^0)\,\phi_I(y)U(y^0,-T)|0\rangle \ .$$

The corresponding result without the field operators is clearly,

$$1 = \langle\Omega|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{e^{+2iE_0T}}{|\langle 0|\Omega\rangle|^2}\langle 0|U(T,0)U(0,-T)|0\rangle \ , \tag{3.43}$$

so

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \tag{3.44}$$

$$\lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|U(T,x^0)\phi_I(x)\,U(x^0,y^0)\,\phi_I(y)U(y^0,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle} \ .$$

If $x^0 > y^0$ this also corresponds to the time-ordered expectation value $\langle\Omega|T\left\{\phi(x)\phi(y)\right\}|\Omega\rangle$. Note that the terms in the numerator are also in this time order. By staring this for awhile, we realize that it can be written formally as:

$$\langle\Omega|T\left\{\phi(x)\phi(y)\right\}|\Omega\rangle = \tag{3.45}$$

$$\lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T}dtH_I(t)\right]\right\}|0\rangle}{\langle 0|T\left\{\exp\left[-i\int_{-T}^{T}dtH_I(t)\right]\right\}|0\rangle} \ .$$

This formula is the basis of the whole perturbation theory. Everything inside the time-ordered product has been expressed in terms of the interaction-picture operators ϕ_I which behave exactly as the free-theory operators. Furthermore, $|0\rangle$ is the free-theory vacuum. By writing the exponential as a Taylor series we get terms which go in powers of the coupling constant λ forming a perturbative series.

3.3 Wick's theorem [Peskin 4.3]

By expanding the exponential (3.45) we get piles of time-ordered n-point functions,

$$\langle 0| T\left[\phi_I(x_1)\phi_I(x_2)\ldots\phi_I(x_n)\right]|0\rangle \ . \tag{3.46}$$

Since ϕ_I behaves like a free field, we know how to compute these: simply substitute the expansion (1.107) and use the properties of creation and annihilation operators. The Wick's theorem simplifies this process.

Let us split the field operator in the interaction picture into two parts,

$$\phi_I(x) = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_{\mathbf{p}}}}\left[a_{\mathbf{p}}e^{-ip\cdot x} + a^\dagger_{\mathbf{p}}e^{ip\cdot x}\right] = \phi_I^+(x) + \phi_I^-(x) \tag{3.47}$$

$$\phi_I^+(x) \equiv \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_{\mathbf{p}}}}a_{\mathbf{p}}e^{-ip\cdot x} \tag{3.48}$$

$$\phi_I^-(x) \equiv \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_{\mathbf{p}}}}a^\dagger_{\mathbf{p}}e^{ip\cdot x} \, . \tag{3.49}$$

With this definition,

$$\phi_I^+(x)|0\rangle = 0\,, \quad \langle 0|\phi_I^-(x) = 0\,. \tag{3.50}$$

Let us now consider the product $T\left[\phi_I(x)\phi_I(y)\right]$, and set first $x^0 > y^0$. Then

$$T\left[\phi_I(x)\phi_I(y)\right] = \phi_I(x)\phi_I(y) \tag{3.51}$$

$$= \phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) + \phi_I^+(x)\phi_I^-(y) + \phi_I^-(x)\phi_I^+(y)\,.$$

Writing now $\phi_I^+(x)\phi_I^-(y) = [\phi_I^+(x), \phi_I^-(y)] + \phi_I^-(y)\phi_I^+(x)$, this becomes

$$T\left[\phi_I(x)\phi_I(y)\right] = \phi_I(x)\phi_I(y) = [\phi_I^+(x), \phi_I^-(y)] \tag{3.52}$$

$$+ \phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^+(y)\,.$$

All the terms in the last line are now organized such that the creation operators are on the left and annihilation operators on the right. A product like this is said to be in **normal order**. We will denote the normal-ordered product by an N-operator, so with this notation

$$T\left[\phi_I(x)\phi_I(y)\right] = \phi_I(x)\phi_I(y) = [\phi_I^+(x), \phi_I^-(y)] + N\left[\phi_I(x)\phi_I(y)\right] \,.$$

In the opposite time order $y^0 > x^0$,

$$\begin{aligned} T\left[\phi_I(x)\phi_I(y)\right] = \phi_I(y)\phi_I(x) = [\phi_I^+(y), \phi_I^-(x)] \\ + \phi_I^+(y)\phi_I^+(x) + \phi_I^-(y)\phi_I^-(x) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) \\ =[\phi_I^+(y), \phi_I^-(x)] + \phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^+(y) \\ =[\phi_I^+(y), \phi_I^-(x)] + N\left[\phi_I(x)\phi_I(y)\right] \,. \end{aligned} \tag{3.53}$$

In the third line we used $[a_{\mathbf{p}}, a_{\mathbf{k}}] = [a_{\mathbf{p}}^\dagger, a_{\mathbf{k}}^\dagger] = 0$. Combining the two time orderings, we get in total

$$T\left[\phi_I(x)\phi_I(y)\right] = N\left[\phi_I(x)\phi_I(y)\right] + \overline{\phi_I(x)\phi_I}(y) \,, \tag{3.54}$$

where we have used the **contraction notation**:

$$\overline{\phi_I(x)\phi_I}(y) \equiv \begin{cases} [\phi_I^+(x), \phi_I^-(y)] \,, & \text{if } x^0 > y^0 \\ [\phi_I^+(y), \phi_I^-(x)] \,, & \text{if } y^0 > x^0 \end{cases} \tag{3.55}$$

By comparing to the results of Sect. 1.5.2 we easily see that this contraction is nothing else than the Feynman propagator,

$$\overline{\phi_I(x)\phi_I}(y) = D_F\left(x-y\right) \,. \tag{3.56}$$

To get convinced about how this generalizes to n-point functions we will still consider the 4-point function $T\left[\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\right]$ explicitly. To speed up the notation we write this as $T\left[\phi_1\phi_2\phi_3\phi_4\right]$. Let's first suppose $x_1^0, x_2^0 > x_3^0, x_4^0$. Then,

$$T\left[\phi_1\phi_2\phi_3\phi_4\right] = T\left[\phi_1\phi_2\right]\, T\left[\phi_3\phi_4\right] = \left[N(\phi_1\phi_2) + \overbracket{\phi_1\phi_2}\right]\left[N(\phi_3\phi_4) + \overbracket{\phi_3\phi_4}\right]$$

$$= N(\phi_1\phi_2)\, N(\phi_3\phi_4) + N(\phi_1\phi_2)\, \overbracket{\phi_3\phi_4} + N(\phi_3\phi_4)\, \overbracket{\phi_1\phi_2} + \overbracket{\phi_1\phi_2}\overbracket{\phi_3\phi_4}\,.$$

Let's open $N(\phi_1\phi_2)\, N(\phi_3\phi_4)$:

$$\begin{aligned}
&N(\phi_1\phi_2)\, N(\phi_3\phi_4) = \qquad (3.57)\\
&\left[\phi_1^+\phi_2^+ + \phi_1^-\phi_2^- + \phi_2^-\phi_1^+ + \phi_1^-\phi_2^+\right]\left[\phi_3^+\phi_4^+ + \phi_3^-\phi_4^- + \phi_4^-\phi_3^+ + \phi_3^-\phi_4^+\right]\\
&= \phi_1^-\phi_2^-\phi_3^+\phi_4^+ + \phi_1^-\phi_2^-\phi_3^-\phi_4^- + \phi_1^-\phi_2^-\phi_4^-\phi_3^+ + \phi_1^-\phi_2^-\phi_3^-\phi_4^+ +\\
&\quad \phi_1^+\phi_2^+\phi_3^+\phi_4^+ + \phi_1^+\phi_2^+\phi_3^-\phi_4^- + \phi_1^+\phi_2^+\phi_4^-\phi_3^+ + \phi_1^+\phi_2^+\phi_3^-\phi_4^+ +\\
&\quad \phi_2^-\phi_1^+\phi_3^+\phi_4^+ + \phi_2^-\phi_1^+\phi_3^-\phi_4^- + \phi_2^-\phi_1^+\phi_4^-\phi_3^+ + \phi_2^-\phi_1^+\phi_3^-\phi_4^+ +\\
&\quad \phi_1^-\phi_2^+\phi_3^+\phi_4^+ + \phi_1^-\phi_2^+\phi_3^-\phi_4^- + \phi_1^-\phi_2^+\phi_4^-\phi_3^+ + \phi_1^-\phi_2^+\phi_3^-\phi_4^+
\end{aligned}$$

Now 7 of the terms are automatically in normal order, but the 9 terms with colored background are not yet normally ordered. With some effort, we can rewrite the above stack of terms as

$$\begin{aligned}
N(\phi_1\phi_2)\, N(\phi_3\phi_4) &= N(\phi_1\phi_2\phi_3\phi_4) + [\phi_1^+, \phi_3^-][\phi_2^+, \phi_4^-] + [\phi_1^+, \phi_4^-][\phi_2^+, \phi_3^-]\\
&\quad + [\phi_1^+, \phi_3^-]N(\phi_2\phi_4) + [\phi_1^+, \phi_4^-]N(\phi_2\phi_3) \qquad (3.58)\\
&\quad + [\phi_2^+, \phi_3^-]N(\phi_1\phi_4) + [\phi_2^+, \phi_4^-]N(\phi_1\phi_3)\,,
\end{aligned}$$

so the time-ordered product reads $(x_1^0, x_2^0 > x_3^0, x_4^0)$,

$$
\begin{aligned}
T\left[\phi_1\phi_2\phi_3\phi_4\right] = {} & N(\phi_1\phi_2)\,\overbrace{\phi_3\phi_4} + N(\phi_3\phi_4)\,\overbrace{\phi_1\phi_2} + \overbrace{\phi_1\phi_2}\overbrace{\phi_3\phi_4} + N(\phi_1\phi_2\phi_3\phi_4) \\
& + [\phi_1^+, \phi_3^-][\phi_2^+, \phi_4^-] + [\phi_1^+, \phi_4^-][\phi_2^+, \phi_3^-] \\
& + [\phi_1^+, \phi_3^-]N(\phi_2\phi_4) + [\phi_1^+, \phi_4^-]N(\phi_2\phi_3) \qquad (3.59) \\
& + [\phi_2^+, \phi_3^-]N(\phi_1\phi_4) + [\phi_2^+, \phi_4^-]N(\phi_1\phi_3)\,.
\end{aligned}
$$

In the opposite time ordering $x_3^0, x_4^0 > x_1^0, x_2^0$

$$
\begin{aligned}
T\left[\phi_1\phi_2\phi_3\phi_4\right] = {} & N(\phi_3\phi_4)\,\overbrace{\phi_1\phi_2} + N(\phi_1\phi_2)\,\overbrace{\phi_3\phi_4} + \overbrace{\phi_3\phi_4}\overbrace{\phi_1\phi_2} + N(\phi_3\phi_4\phi_1\phi_2) \\
& + [\phi_3^+, \phi_1^-][\phi_4^+, \phi_2^-] + [\phi_3^+, \phi_2^-][\phi_4^+, \phi_1^-] \\
& + [\phi_3^+, \phi_1^-]N(\phi_4\phi_2) + [\phi_3^+, \phi_2^-]N(\phi_4\phi_1) \qquad (3.60) \\
& + [\phi_4^+, \phi_1^-]N(\phi_3\phi_2) + [\phi_4^+, \phi_2^-]N(\phi_3\phi_1)\,.
\end{aligned}
$$

By using the definition of the contraction (3.55), we can again combine the two different time orderings to a single expression $(x_1^0, x_2^0 > x_3^0, x_4^0$ or $x_3^0, x_4^0 > x_1^0, x_2^0)$

$$
\begin{aligned}
T\left[\phi_1\phi_2\phi_3\phi_4\right] = {} & N(\phi_1\phi_2\phi_3\phi_4) + \overbrace{\phi_1\phi_2}\overbrace{\phi_3\phi_4} + \overbrace{\phi_1\phi_3}\overbrace{\phi_2\phi_4} + \overbrace{\phi_1\phi_4}\overbrace{\phi_2\phi_3} \\
& + N(\phi_1\phi_2)\,\overbrace{\phi_3\phi_4} + N(\phi_3\phi_4)\,\overbrace{\phi_1\phi_2} + \overbrace{\phi_1\phi_3}N(\phi_2\phi_4) \\
& + \overbrace{\phi_1\phi_4}N(\phi_2\phi_3) + \overbrace{\phi_2\phi_3}N(\phi_1\phi_4) + \overbrace{\phi_2\phi_4}N(\phi_1\phi_3)\,. \\
= {} & N(\phi_1\phi_2\phi_3\phi_4) + \sum_{\text{perm.}} \overbrace{\phi_i\phi_j}\overbrace{\phi_k\phi_\ell} + \sum_{\text{perm.}} N(\phi_i\phi_j)\overbrace{\phi_k\phi_\ell}\,,
\end{aligned}
$$

where $\sum_{\text{perm.}}$ denotes a sum over different permutations. The other pair-wize time orderings can be obtained from the above expression by interchanging the indices, but the result is trivially the same since the order of fields inside the normal-ordered products or in contractions is immaterial. Thus, the above result is the final one.

Having now explicitly checked how the normal ordering works, we can just declare the general result known as the **Wick's theorem:**

$$
\begin{aligned}
T\left[\phi_1\cdots\phi_n\right] = N(\phi_1\cdots\phi_n) &+ \sum_{\text{perm.}} \overbrace{\phi_{i_1}\phi_{i_2}} N(\phi_{i_3}\cdots\phi_{i_n}) \\
&+ \sum_{\text{perm.}} \overbrace{\phi_{i_1}\phi_{i_2}}\overbrace{\phi_{i_3}\phi_{i_4}} N(\phi_{i_5}\cdots\phi_{i_n}) \\
&\vdots \\
&+ \sum_{\text{perm.}} \overbrace{\phi_{i_1}\phi_{i_2}}\cdots\overbrace{\phi_{i_{n-1}}\phi_{i_n}}
\end{aligned}
\tag{3.61}
$$

If n is odd, the last line is of the form,

$$
\sum_{\text{perm.}} \overbrace{\phi_{i_1}\phi_{i_2}}\cdots\overbrace{\phi_{i_{n-2}}\phi_{i_{n-1}}} N(\phi_{i_n})
\tag{3.62}
$$

The power of the Wick's theorem is that for normal-ordered products,

$$
\langle 0|N(\phi_I(x)\phi_I(y)\cdots)|0\rangle = 0\,.
\tag{3.63}
$$

It follows that for vacuum-expectation values

$$
\begin{aligned}
\langle 0|T\left[\phi_I(x_1)\phi_I(x_2)\ldots\phi_I(x_n)\right]|0\rangle &= \langle 0|\sum_{\text{perm.}} \overbrace{\phi_{i_1}\phi_{i_2}}\cdots\overbrace{\phi_{i_{n-1}}\phi_{i_n}}|0\rangle \\
&= \sum_{\text{perm.}} D_F(x_{i_1}-x_{i_2})\cdots D_F(x_{i_{n-1}}-x_{i_n})\,,
\end{aligned}
\tag{3.64}
$$

so only terms that have been fully contracted give a non-zero contributions ($\sim$ free theory propagators). By definition, we can also express the contractions as

$$
\langle 0|T\left[\phi_1\phi_2\phi_3\phi_4\right]|0\rangle = \overbrace{\phi_1\phi_2}\overbrace{\phi_3\phi_4} + \contraction{}{\phi_1}{\phi_2}{\phi_3}\contraction[2ex]{\phi_1}{\phi_2}{\phi_3}{\phi_4}\phi_1\phi_2\phi_3\phi_4 + \overbrace{\phi_1\overbrace{\phi_2\phi_3}\phi_4}\,,
\tag{3.65}
$$

which is the same thing as if the contracted fields were always next to each other. With this notation it is easier to visually keep track of the different possible contractions.

3.4 Feynman diagrams and symmetry factors

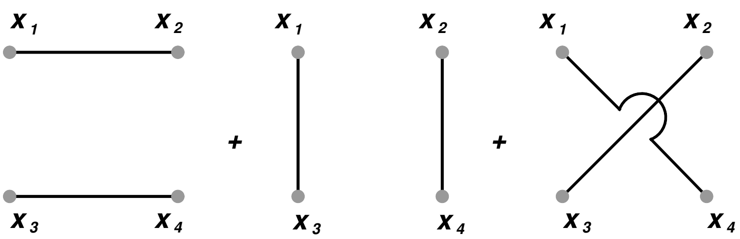
[Peskin 4.4]

The Feynman diagrams/graphs comprise a handy tool to visualize and classify the above-defined contractions. We denote the propagator $D_F(x-y)$ simply as a line between the points x and y:

$$D_F(x-y) =$$

The order of x and y is immaterial since $D_F(x-y) = D_F(y-x)$. We can thus represent the 4-point function in Eq. (3.65) as

$$\langle 0|T\left[\phi_1\phi_2\phi_3\phi_4\right]|0\rangle \;=$$



$$= D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_3)D_F(x_2-x_4)$$
$$+ D_F(x_1-x_4)D_F(x_2-x_3)$$

The same space-time point can (and usually will) appear several times. For example, upon expanding the exponential in the 2-point function in Eq. (3.45), the first term is of the form,

$$\frac{-i\lambda}{4!}\int d^4z\langle 0|T\left[\phi_I(x)\,\phi_I(y)\,\phi_I(z)\,\phi_I(z)\,\phi_I(z)\,\phi_I(z)\right]|0\rangle \qquad (3.66)$$

This produces in total 15 different fully contracted terms:

- First can contract $\phi_I(x)$ to 5 different places

- From the remaining 4, one can be contracted to 3 different places
- The remaining 2 should be contracted to each other

$\Longrightarrow$ In total $5 \times 3 = 15$ terms, but <u>only 2 are different</u>:

$$(i) \qquad \phi_x\phi_y \left[\phi_z\phi_z\phi_z\phi_z \;+\; \phi_z\phi_z\phi_z\phi_z \;+\; \phi_z\phi_z\phi_z\phi_z\,, \right]$$

First 4 way $x \to z$ finally this

$$(ii) \qquad \phi_x\ \phi_y\ \phi_z\ \phi_z\ \phi_z\ \phi_z \tag{3.67}$$

Then 3 ways $y \to z$

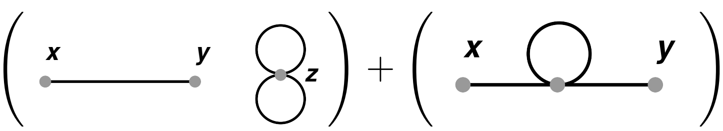
There are 3 terms of type (i) and $4 \times 3 = 12$ terms of type (ii). In total 15.

$$\begin{aligned}
&\frac{-i\lambda}{4!}\int d^4z\langle 0|T\left[\phi_x\,\phi_y\,\phi_z\,\phi_z\,\phi_z\,\phi_z\right]|0\rangle \\
&= \frac{-i\lambda}{4!} \times \mathbf{3} \times D_F(x-y)\int d^4z D_F(z-z)\,D_F(z-z) \\
&+ \frac{-i\lambda}{4!} \times \mathbf{12} \times \int d^4z D_F(x-z)D_F(y-z)\,D_F(z-z)
\end{aligned}$$

Diagrammatically we would represent this as

$$\frac{-i\lambda}{4!}\int d^4z\langle 0|T\left[\phi_x\,\phi_y\,\phi_z\,\phi_z\,\phi_z\,\phi_z\right]|0\rangle \tag{3.68}$$

$=$



The point in which the propagator lines meet (z above) is called a **vertex**. As we saw, different diagrams come with a weight factor,

$$\textbf{weight} = \frac{1}{n!}\left(\frac{1}{4!}\right)^n \times (\textbf{combinatorical factor})\,. \tag{3.69}$$

Here, the factor n comes from the Taylor expansion of the exponential, which in the above example was $n = 1$. The combinatorical factors were 3 and 12 for the above diagrams. The weight can also be expressed in terms of a **symmetry factor**,

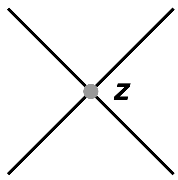
$$\textbf{weight} = \frac{1}{\textbf{symmetry factor}} \,. \tag{3.70}$$

In the example above, the symmetry factor was $\left(\frac{3}{4!}\right)^{-1} = 8$ for the diagram (i) and $\left(\frac{12}{4!}\right)^{-1} = 2$ for the diagram (ii).

The Feynman rules are instructions how to obtain the mathematical expression from a given diagram. On the basis of the above example, we can write down the Feynman rules in the position space:

1. Lines 

$= D_F(x - y)$

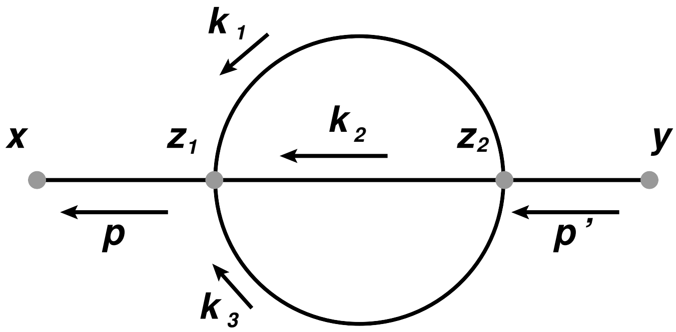
2. Vertices 

$= -i\lambda \int d^4 z$

3. Compute the weight factor

Physically, we may interpret the "lines" as probability densities for the particles to move from one space-time point to another, and the vertices as propability densities for interactions. The vertices can be anywhere in the space time — we always integrate over their positions. This should not be taken too literally, however.

We can also express the Feynman rules in the momentum space. Let us consider the following diagram:



Based on the position-space Feynman rules this corresponds to

$$(-i\lambda)^2 \frac{1}{6} \int d^4z_1 d^4z_2 D_F(x-z_1)\ [D_F(z_1-z_2)]^3\ D_F(z_2-y)\,. \quad (3.71)$$

Let us now substitute the integral representation of the propagator,

$$D_F\,(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}, \quad (3.72)$$

$\Longrightarrow$

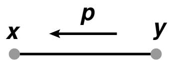
$$(-i\lambda)^2 \int d^4z_1 d^4z_2 \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-z_1)} \quad (3.73)$$

$$\times \left[\prod_{i=1}^{3} \int \frac{d^4k_i}{(2\pi)^4} \frac{i}{k_i^2-m^2+i\epsilon} e^{-ik_i\cdot(z_1-z_2)}\right] \times \int \frac{d^4p'}{(2\pi)^4} \frac{i}{p'^2-m^2+i\epsilon} e^{-ip'\cdot(z_2-y)}$$

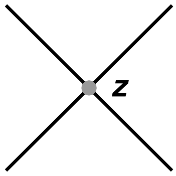
$$= (-i\lambda)^2 \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} \left[\prod_{i=1}^{3} \int \frac{d^4k_i}{(2\pi)^4} \frac{i}{k_i^2-m^2+i\epsilon}\right] \int \frac{d^4p'}{(2\pi)^4} \frac{i}{p'^2-m^2+i\epsilon}$$

$$\times\, e^{-ip\cdot x} \underbrace{\left[\int d^4z_1 e^{iz_1\cdot(p-k_1-k_2-k_3)}\right]}_{(2\pi)^4\delta^{(4)}(p-k_1-k_2-k_3)} \underbrace{\left[\int d^4z_2 e^{iz_2\cdot(-p'+k_1+k_2+k_3)}\right]}_{(2\pi)^4\delta^{(4)}(-p'+k_1+k_2+k_3)} e^{ip'\cdot y}$$

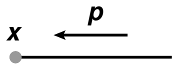
We see that for each line there is the momentum-space propagator and the two δ functions force the momentum conservation in the vertices z_1 and z_2. We can thus deduce the Feynman rules in the momentum space:

1. Each line 

$= \frac{i}{p^2 - m^2 + i\epsilon}$

2. Each vertex 

$= -i\lambda$

3. External leg 

$= e^{-ip\cdot x}$

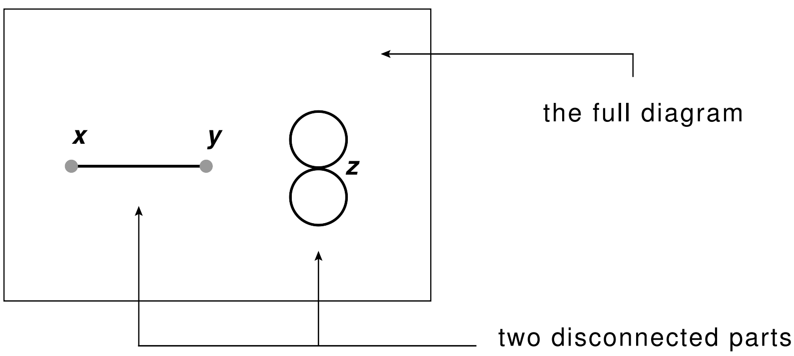
4. Choose 4-momenta such that the momentum is conserved in vertices

5. Integrate over undetermined momenta with weight $\int \frac{d^4p}{(2\pi)^4}$

6. Compute the weight factor

3.4.1 Disconnected diagrams

We call **unattached** or **disconnected** such diagrams that consist of parts that are not attached to each other by any line. We already encountered this case:



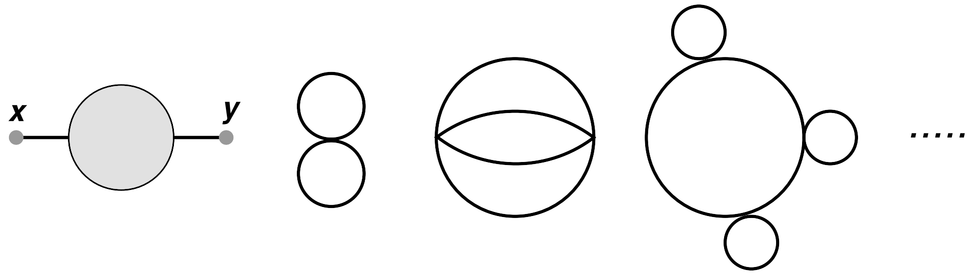
which had an expression

$$D_F(x-y) \times \left[\frac{-i\lambda}{4!} \times 3 \int d^4z D_F(z-z)\, D_F(z-z) \right] ,$$

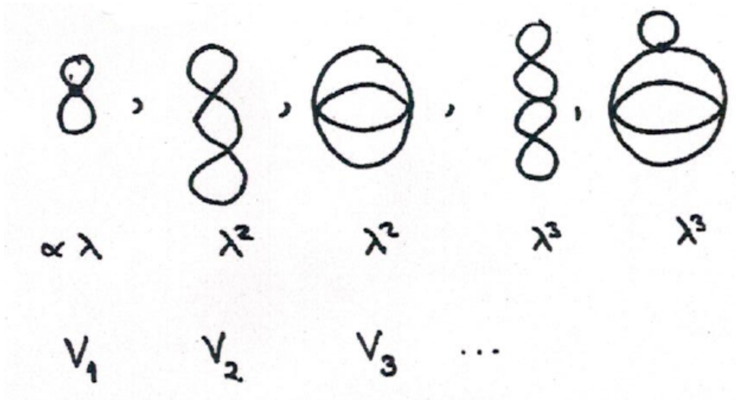
where the term in square brackets corresponds to the latter diagram. We see from here that the contributions of unattached diagrams factor into separate multiplicative parts. They can thus be computed separately. The opposite to a disconnected diagram is a **connected diagram** in which all parts are attached to each other. When computing the 2-point expectation value,

$$\langle 0|T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T}dtH_I(t)\right]\right\}|0\rangle \tag{3.74}$$

the factorizable part that contains ϕ_x contains also ϕ_y:



We can list all the unattached diagrams and name them:



By definition, V_i is here also the value of the diagram. For example,

$$V_1 = \frac{-i\lambda}{4!}\int d^4z\left[\phi_z\phi_z\phi_z\phi_z \;+\; \phi_z\phi_z\phi_z\phi_z \;+\; \phi_z\phi_z\phi_z\phi_z\,,\right].$$

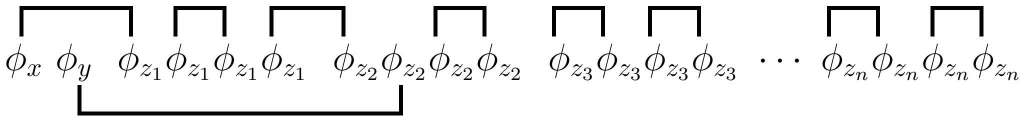
In general, we can write the value of an arbitrary diagram as

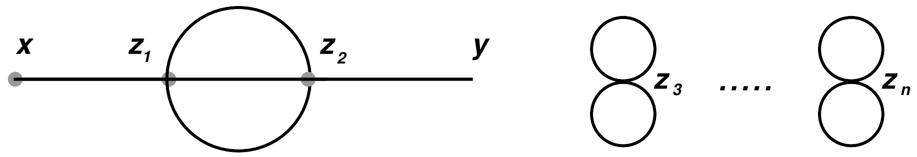
$$[\textbf{value of the connected part}]\times[\textbf{weight}]\times\prod_i (V_i)^{n_i}\,, \tag{3.75}$$

where n_i the number of V_is in the diagram. The factor [**weight**] is again combinatorical. Let us consider a term of the nth order,

$$\frac{1}{n!}\langle 0|T\left[\phi_x\,\phi_y\left(\frac{-i}{4!}\int d^4z_1\phi^4_{z_1}\right)\cdots\left(\frac{-i}{4!}\int d^4z_n\phi^4_{z_n}\right)\right]|0\rangle \tag{3.76}$$

It contains, for example, a contraction like this:





There are, however, other contractions that give the same result, e.g. one in which we contract ϕ_x to ϕ_{z_2} and ϕ_y to ϕ_{z_1}. When the order of the term is n, there are $n!$ ways to order the fields. These all lead to the same result and cancel the factor $1/n!$ from the Taylor expansion. However, all permutations do not give new terms. For example, interchanging z_3 and z_n does not yield new terms whereas interchanging z_1 and z_2 does. In general, if a self-connected term V_i appears n_i times in the diagram, by permutating all indices would overcount the number of terms by a factor of $n_i!$. By this reasoning,

$$[\textbf{weight}] = \frac{1}{n!}\left[\frac{n!}{\prod_i n_i!}\right] = \frac{1}{\prod_i n_i!}\,. \tag{3.77}$$

In our example with $n = 4$ we would first have 4 combos (of 4 fields) to choose where to contract ϕ_x, and after that 3 combos (of 4 fields) where to contract ϕ_y. The rest of the remaining two combos are contracted to themselves. From this we get, $[\textbf{weight}] = (4 \times 3)/4! = 1/2$ which agrees with the above general result. The value of a specific diagram thus reads,

$$[\textbf{value of the connected piece}] \times \prod_i \frac{1}{n_i!}\,(V_i)^{n_i}\,. \tag{3.78}$$

Let's now suppose that the term V_i appears in the nth order expansion term just once. This graph thus comes with a weight,

$$[\textbf{rest of the diagram}] \times \frac{1}{1!}\,(V_i)^1\,. \tag{3.79}$$

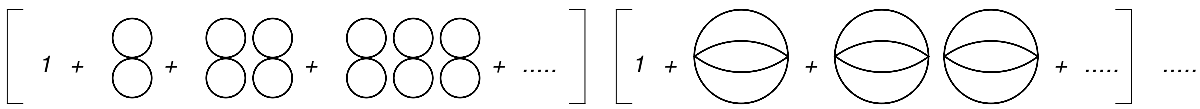
If the order of V_i in coupling is k (that is, $V_i \sim \lambda^k$), in the order $n+k$ there is a term which is otherwize identical with the previous one, but contains V_i twice. This contribution comes with a weight,

$$[\textbf{rest of the diagram}] \times \frac{1}{2!}\left(V_i\right)^2 . \tag{3.80}$$

When we sum over all such terms we see that the contribution of V_i exponentiates,

$$[\textbf{rest of the diagram}] \times e^{V_i}. \tag{3.81}$$

All combinations of unattached graphs is naturally a product of the form,



in which each part exponentiates as above. Thus,

$$\begin{aligned}
&\langle 0|T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T} dtH_I(t)\right]\right\}|0\rangle \qquad (3.82)\\
&= \sum_k (\textbf{connected})_k \times \prod_i e^{V_i}\\
&= \sum_k (\textbf{connected})_k \times \exp\left[\sum_i e^{V_i}\right] .
\end{aligned}$$

We call this as the **exponentiation of the "vacuum bubbles"**. Without the external fields $\phi_I(x)$ and $\phi_I(y)$ the expectation value is obviously,

$$\langle 0|T\left\{\exp\left[-i\int_{-T}^{T} dtH_I(t)\right]\right\}|0\rangle \; = \exp\left[\sum_i e^{V_i}\right] .$$

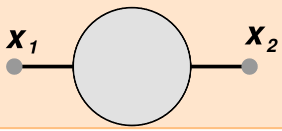
By putting these results together, Eq. (3.45) simplifies to the form,

$$\langle\Omega|T\left\{\phi(x)\phi(y)\right\}|\Omega\rangle \tag{3.83}$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T} dt H_I(t)\right]\right\}|0\rangle}{\langle 0|T\left\{\exp\left[-i\int_{-T}^{T} dt H_I(t)\right]\right\}|0\rangle}$$

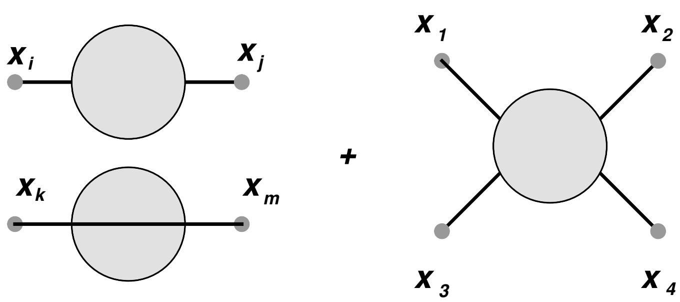
$$= \frac{\sum_k (\textbf{connected})_k \times \exp\left[\sum_i e^{V_i}\right]}{\exp\left[\sum_i e^{V_i}\right]}$$

$$= \sum_k (\textbf{connected})_k \ .$$

$=$ 

In other words, **the contribution of the vacuum bubbles disappears when computing the ground-state expectation values of the interacting theory.** This generalizes directly to the correlation functions with more than two external legs. The difference with the 2-point function is, however, that the part containing the external legs is not necessarily fully connected but it can contain several disconnected pieces. For example, the 4-point function contains two classes of contributions:

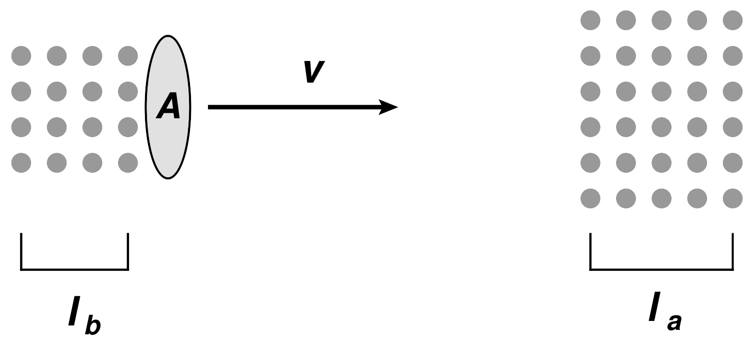
$$\langle\Omega|T\left[\prod_{i=1}^{4}\phi(x_i)\right]|\Omega\rangle =$$



Later on, we will see that only the fully connected piece will contribute to the scattering matrix elements.

3.5 Cross section [Peskin 4.5]

By the term "cross section" we essentially mean the probability of a specific process to take place in a scattering of two particles. Let us consider a collision of two bunches of particles with relative speed v:



We will consider the density of particles within the bunches, ρ_a and ρ_b, to be constants. We denote the lenghts of the bunches by ℓ_a and ℓ_b. The more there are particles that have a chance to collide the higher is the probability to observe a specific particle in the final state. The cross section should be independent of such experimental conditions and we therefore define the cross section σ by,

$$\sigma \equiv \frac{\text{number of specific final-state particles}}{\rho_a \ell_a \, \rho_b \ell_b \, A} . \tag{3.84}$$

The **denominator reflects the amount of colliding matter** and it can also be expressed as

$$\frac{N_a \times N_b}{A} = N_a \times n_b = N_b \times n_a , \tag{3.85}$$

in which $N_a = \rho_a \ell_a A$ and $N_b = \rho_b \ell_b A$ denote the particles within the overlapping area A, and $n_{a,b}$ are the particle densities per area. We can call the denominator as **luminosity** (per bunch crossing) $\mathcal{L}$, and it has the dimension of $[\text{length}]^{-2}$. Usually we are mostly interested in the momentum or angular distributions of some final-state particles. In this case we talk about **differential cross section**,

$$\frac{d\sigma}{d^3\mathbf{p}_i \cdots d^3\mathbf{p}_n} \equiv \lim_{\Delta \mathbf{p}_i \to 0} \frac{N\left(\mathbf{p}_i \in \Delta\mathbf{p}_i, \ldots, \mathbf{p}_n \in \Delta\mathbf{p}_n\right)}{\left(\Delta\mathbf{p}_1 \cdots \Delta\mathbf{p}_n\right)} \frac{1}{\mathcal{L}} . \tag{3.86}$$

in which $N\left(\mathbf{p}_i \in \Delta\mathbf{p}_i, \ldots, \mathbf{p}_n \in \Delta\mathbf{p}_n\right)$ is the number of particles whose momenta are within $\Delta\mathbf{p}_i \cdots \Delta\mathbf{p}_n$. In reality, the "bins" $\Delta\mathbf{p}_i$ are, of course, of finite size and the truly differential cross section is a theoretical limit.

3.5.1 Scattering matrix

We will describe the initial- and final-state particles of a scattering process with wave packets,

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\phi(\mathbf{k})|\mathbf{k}\rangle\,, \tag{3.87}$$

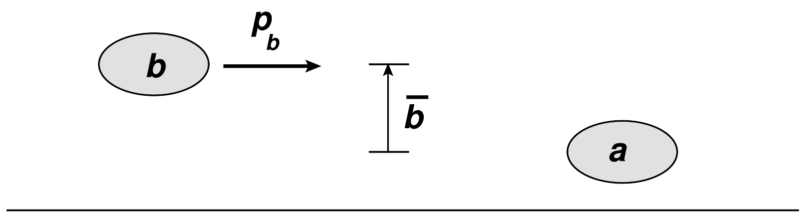
in which $|\mathbf{k}\rangle$s are 1-particle states of the interacting theory (momentum eigenstates). In a real scattering experiment, particularly the initial-state particles are well separated and localizable before the collision so a wave-packet treatment should simulate well the experimental conditions. At this moment we don't know much about the states of the interacting theory but we will choose to normalize them as in the free theory,

$$\langle\mathbf{k}'|\mathbf{k}\rangle = 2E_{\mathbf{k}}\,(2\pi)^3\,\delta^{(3)}\left(\mathbf{k}-\mathbf{k}'\right)\,. \tag{3.88}$$

Then $\langle\phi|\phi\rangle = 1$, if the wave-packet functions have been normalized as

$$\int \frac{d^3k}{(2\pi)^3}|\phi(\mathbf{k})|^2 = 1\,. \tag{3.89}$$

We will suppose that the functions $\phi(\mathbf{k})$ are concentrated around some initial- and final-state momenta. Let us first consider the initial state of one a and one b particle.



We choose the coordinate system such that the particle a is on the z axis. As in the figure, the particle b is not necessarily on the same line but can be shifted in the transverse plane by amount given by the **impact parameter** $\mathbf{b}$. In some distant moment before the collision the particles are also well separated in the z direction, e.g. $x_a^3 = -x_b^3 = z_0$. We can write a state vector fulfilling these requirements as,

$$|\phi_A\phi_B\rangle \overset{T\lll 0}{=} \int \frac{d^3k_A}{(2\pi)^3}\int \frac{d^3k_B}{(2\pi)^3}\frac{e^{-i\mathbf{b}\cdot\mathbf{k}_B}e^{+iz_0(k_A^3-k_B^3)}}{2\sqrt{E_AE_B}}\phi(\mathbf{k}_A)\phi(\mathbf{k}_B)|\mathbf{k}_A\mathbf{k}_B\rangle_{\text{in}}\,. \tag{3.90}$$

In principle, this represents two approaching wave packets but the states are in the Heisenberg picture so the time dependence is not explicit here. The initial state will look like this at some time $T \ll 0$ before the collision. We use the notation $|\mathbf{k}_A\mathbf{k}_B\rangle_{\rm in}$ to remind us that only at the limit $T \ll 0$ the incoming particles can be considered as 1-particle states. At later times they will certainly not be 1-particle states if some collision takes place. Similarly, we write the final state as a wave packet,

$$|f_n\rangle \overset{T\gg 0}{=} \left[\prod_{i=1}^{n}\int \frac{d^3k_i}{(2\pi)^3}\frac{e^{-i\mathbf{x_i}\cdot\mathbf{k}_i}}{\sqrt{2E_i}}\phi(\mathbf{k}_i)\right]|\mathbf{k}_1\cdots\mathbf{k}_n\rangle_{\rm out}\,. \tag{3.91}$$

We will consider that the initial and final states are of this form at some distant past/future.

The differential transition probability from state $|\phi_A\phi_B\rangle$ to state $|f_n\rangle$ is defined as

$$d\mathcal{P}(AB\to f_n) = \frac{1}{n!}\left[\prod_{i=1}^{n}\frac{d^3p_{f_i}}{(2\pi)^3}\right]|\langle f_n|\phi_A\phi_B\rangle|^2\,. \tag{3.92}$$

To see that this makes sense, let's compute the total probability for the initial state to become whatever where ever:

$$\begin{aligned}
P &= \sum_{n=1}^{\infty}\frac{1}{n!}\int\left[\prod_{i=1}^{n}d^3x_i\frac{d^3p_{f_i}}{(2\pi)^3}\right]|\langle f_n|\phi_A\phi_B\rangle|^2 \qquad (3.93)\\
&= \sum_{n=1}^{\infty}\frac{1}{n!}\int\left[\prod_{i=1}^{n}d^3x_i\frac{d^3p_{f_i}}{(2\pi)^3}\right]\left[\prod_{i=1}^{n}\int\frac{d^3k_id^3k_i'}{(2\pi)^6}\frac{e^{-i\mathbf{x_i}\cdot(\mathbf{k}_i-\mathbf{k}_i')}}{\sqrt{2E_i}\sqrt{2E_i'}}\phi(\mathbf{k}_i)\phi^*(\mathbf{k}_i')\right]\\
&\quad\times\langle\phi_B\phi_A|\mathbf{k}_1\cdots\mathbf{k}_n\rangle_{\rm out\ out}\langle\mathbf{k'}_n\cdots\mathbf{k'}_1|\phi_A\phi_B\rangle\\
&= \sum_{n=1}^{\infty}\frac{1}{n!}\int\left[\prod_{i=1}^{n}\frac{d^3p_{f_i}}{(2\pi)^3}\right]\left[\prod_{i=1}^{n}\int\frac{d^3k_i}{(2\pi)^3}\frac{1}{2E_i}|\phi(\mathbf{k}_i)|^2\right]\\
&\quad\times\langle\phi_B\phi_A|\mathbf{k}_1\cdots\mathbf{k}_n\rangle_{\rm out\ out}\langle\mathbf{k}_n\cdots\mathbf{k}_1|\phi_A\phi_B\rangle
\end{aligned}$$

We will consider that $|\phi(\mathbf{k}_i)|^2$ are reasonably concetrated around definite final-state momenta $\mathbf{p}_{f_i}$ – as a limiting case $|\phi(\mathbf{k}_i)|^2 \to (2\pi)^3\delta^{(3)}(\mathbf{p}_{k_i}-\mathbf{p}_{f_i})$

– so that we can replace $\mathbf{k}_i \to \mathbf{p}_{f_i}$ in all other places than $\phi(\mathbf{k}_i)$. Then,

$$P = \langle \phi_B \phi_A | \left[\sum_{n=1}^{\infty} \frac{1}{n!} \int \left(\prod_{i=1}^{n} \frac{d^3 p_{f_i}}{(2\pi)^3 2E_{\mathbf{p}_{f_i}}} \right) |\mathbf{p}_1 \cdots \mathbf{p}_n\rangle_{\text{out out}}\langle \mathbf{p}_n \cdots \mathbf{p}_1 | \right] |\phi_A \phi_B\rangle$$
$$= \langle \phi_B \phi_A | \phi_A \phi_B \rangle = 1 \,, \tag{3.94}$$

where we used the completeness relation (1.89), and suppose that the normalization has been chosen as in the free-field case. Thus the probability to find the initial state at some state after the collision is unity. Makes sense.

When we write the inner products $\langle f_n | \phi_A \phi_B \rangle$ by substituting Eqs. (3.90) and (3.91) we end up with inner products of the form,

$$_{\text{out}}\langle \mathbf{k}_1 \cdots \mathbf{k}_n | \mathbf{k}_A \mathbf{k}_B \rangle_{\text{in}} \,. \tag{3.95}$$

Since the "in" and "out" states have been defined at different reference times (far past, far future) their overlap is non trivial. Nevertheless they belong to the same space of state vectors so there should be an operator S such that,

$$|\mathbf{k}_A \mathbf{k}_B\rangle_{\text{out}} = S^\dagger |\mathbf{k}_A \mathbf{k}_B\rangle_{\text{in}} \,. \tag{3.96}$$

By using this in Eq. (3.94) above and requiring that we still get $P = 1$ indicates that the S operator should be unitary. Thus,

$$_{\text{out}}\langle \mathbf{k}_1 \cdots \mathbf{k}_n | \mathbf{k}_A \mathbf{k}_B \rangle_{\text{in}} = {}_{\text{out}}\langle \mathbf{k}_1 \cdots \mathbf{k}_n | S | \mathbf{k}_A \mathbf{k}_B \rangle_{\text{out}} \tag{3.97}$$

In this expression, the states have been defined at the same reference time and we can forget about the "out" tag. The unitary S operator is called the **scattering matrix**, or just shortly **S matrix**. It will be useful to split it into two pieces,

$$S = 1 + iT \,. \tag{3.98}$$

The unit operator represents the case that nothing happens in the scattering (no scattering at all) and the actual interactions are contained in the T operator. We define the **invariant matrix element $\mathcal{M}$** as,

$$\langle \mathbf{k}_1 \cdots \mathbf{k}_n | iT | \mathbf{k}_A \mathbf{k}_B \rangle = (2\pi)^4 \, \delta^{(4)} \left(k_A + k_B - \sum_i k_i \right) \, i\mathcal{M} \left(k_A, k_B \to k_f \right) \tag{3.99}$$

We have here separated an overall momentum-conserving δ function as it will turn out that $\langle \mathbf{k}_1 \cdots \mathbf{k}_n | iT | \mathbf{k}_A \mathbf{k}_B \rangle$ is always proportional to this object.

The (differential) number of observations is obtained by folding the probabilities in single collisions $d\mathcal{P}$ with the densities of the particle bunches and integrating over the positions of the final state particles,

$$dN = \int d^2\mathbf{x}_a \, n_a(\mathbf{x}_a) \int d^2\mathbf{x}_\mathbf{b} \, n_b(\mathbf{x}_\mathbf{b}) \int \left(\prod_{i=1}^{n} d^3 x_i \right) \, d\mathcal{P}(\mathbf{b}, \mathbf{x}_i) \, ,$$

where the impact parameter is now $\mathbf{b} = \mathbf{x}_a - \mathbf{x}_b$. In the simplest case the particle densities n_a and n_b are constants within the overlapping area A, so they can be taken outside the integrals. In this case,

$$dN = N_a n_b \int d^2\mathbf{b} \, \int \left(\prod_{i=1}^{n} d^3 x_i \right) \, d\mathcal{P}(\mathbf{b}, \mathbf{x}_i) \, , \tag{3.100}$$

or

$$\int d^2\mathbf{b} \int \left(\prod_{i=1}^{n} d^3 x_i \right) \, d\mathcal{P}(\mathbf{b}, \mathbf{x}_i) = \frac{dN}{N_A \, n_b} = d\sigma \, , \tag{3.101}$$

according to Eq. (3.84). By combining the formulae of the previous couple of pages we have the following expression for the differential cross section:

$$d\sigma = \int d^2\mathbf{b} \left(\prod_i \int d^3\mathbf{x}_i \right) \frac{1}{n!} \left[\prod_{i=1}^{n} \frac{d^3 p_{f_i}}{(2\pi)^3} \right] \tag{3.102}$$

$$\left[\prod_{i=1}^{n} \int \frac{d^3 k_i}{(2\pi)^3} \frac{e^{-i\mathbf{x_i}\cdot\mathbf{k}_i}}{\sqrt{2E_i}} \phi(\mathbf{k}_i) \right] \left[\prod_{i=1}^{n} \int \frac{d^3 k_i'}{(2\pi)^3} \frac{e^{+i\mathbf{x_i}\cdot\mathbf{k'}_i}}{\sqrt{2E_i'}} \phi^*(\mathbf{k'}_i) \right]$$

$$\int \frac{d^3 k_A}{(2\pi)^3} \int \frac{d^3 k_B}{(2\pi)^3} \frac{e^{-i\mathbf{b}\cdot\mathbf{k}_B} e^{+iz_0(k_A^3 - k_B^3)}}{2\sqrt{E_A E_B}} \phi(\mathbf{k}_A) \phi(\mathbf{k}_B)$$

$$\int \frac{d^3 k_A'}{(2\pi)^3} \int \frac{d^3 k_B'}{(2\pi)^3} \frac{e^{+i\mathbf{b}\cdot\mathbf{k}_B'} e^{-iz_0(k_A'^3 - k_B'^3)}}{2\sqrt{E_A' E_B'}} \phi^*(\mathbf{k'}_A) \phi^*(\mathbf{k'}_B)$$

$$(2\pi)^4 \, \delta^{(4)} \left(k_A + k_B - \sum_i k_i \right) \, i\mathcal{M} \left(k_A, k_B \to k_f \right)$$

$$(2\pi)^4 \, \delta^{(4)} \left(k_A' + k_B' - \sum_i k_i' \right) \, i\mathcal{M} \left(k_A', k_B' \to k_f' \right) \, .$$

This can be simplified by first noting that

$$\left(\prod_i^n \int d^3\mathbf{x}_i\right) e^{-i\mathbf{x_i}\cdot(\mathbf{k}_i-\mathbf{k}'_i)} = \prod_i^n \left[(2\pi)^3\delta^{(3)}\left(\mathbf{k}_i - \mathbf{k}'_i\right)\right] \qquad (3.103)$$

$$\int d^2\mathbf{b} e^{-i\mathbf{b}\cdot(\mathbf{k}_B-\mathbf{k}'_B)} = (2\pi)^2\delta^{(2)}\left(\mathbf{k}_{\perp B} - \mathbf{k}'_{\perp B}\right) \qquad (3.104)$$

By using these δ functions we are able to do the d^3k' and $d^2\mathbf{k}'_{\perp B}$ integrals. In addition, we can perform the $d^2\mathbf{k}'_{\perp A}$ integral by using the lowermost δ function in Eq. (3.102). What remains is,

$$\sigma = \frac{1}{n!}\left[\prod_{i=1}^n \frac{d^3p_{f_i}}{(2\pi)^3}\right]\left[\prod_{i=1}^n \int \frac{d^3k_i}{(2\pi)^3}\frac{1}{2E_i}\phi(\mathbf{k}_i)\phi^*(\mathbf{k}_i)\right] \qquad (3.105)$$

$$\int \frac{d^3k_A}{(2\pi)^3}\int \frac{d^3k_B}{(2\pi)^3}\frac{e^{+iz_0(k_A^3-k_B^3)}}{2\sqrt{E_AE_B}}\phi(\mathbf{k}_A)\phi(\mathbf{k}_B)\frac{e^{-iz_0(k_A'^3-k_B'^3)}}{2\sqrt{E'_AE'_B}}\phi^*(\mathbf{k}'_A)\phi^*(\mathbf{k}'_B)$$

$$(2\pi)^4\,\delta^{(4)}\left(k_A + k_B - \sum_i k_i\right)\, i\mathcal{M}\left(k_A, k_B \to k_f\right)(-i)\mathcal{M}^*\left(k'_A, k'_B \to k_f\right)$$

$$\int dk'_{z,A}dk'_{z,B}\delta\left(E'_A + E'_B - \sum_i E_i\right)\delta\left(k'_{z,A} + k'_{z,B} - \sum_i k_{z,i}\right),$$

where now $\mathbf{k}'_{\perp A} = \mathbf{k}_{\perp A}$ and $\mathbf{k}'_{\perp B} = \mathbf{k}_{\perp B}$. The last integral can be done (Ex.),

$$\int dk'_{z,A}dk'_{z,B}\delta(E'_A + E'_B - \sum_i E_i)\delta(k'_{z,A} + k'_{z,B} - \sum_i k_{z,i}) = \frac{1}{|\mathbf{v_A} - \mathbf{v_B}|},$$

where $|\mathbf{v}_A - \mathbf{v}_B| = v_{ab}$ is the relative speed between the initial-state particles. Our expression for the cross section thus simplifies to

$$\sigma = \frac{1}{n!}\left[\prod_{i=1}^n \frac{d^3p_{f_i}}{(2\pi)^3}\right]\left[\prod_{i=1}^n \int \frac{d^3k_i}{(2\pi)^3}\frac{1}{2E_i}|\phi(\mathbf{k}_i)|^2\right] \qquad (3.106)$$

$$\int \frac{d^3k_A}{(2\pi)^3}\int \frac{d^3k_B}{(2\pi)^3}\frac{1}{4E_AE_Bv_{ab}}|\phi(\mathbf{k}_A)|^2|\phi(\mathbf{k}_B)|^2$$

$$(2\pi)^4\,\delta^{(4)}\left(k_A + k_B - \sum_i k_i\right)\,|\mathcal{M}\left(k_A, k_B \to k_f\right)|^2.$$

Considering again that $|\phi(\mathbf{k}_i)|^2$ are concentrated around $\mathbf{p}_{f_i}$ and that $|\phi(\mathbf{k}_A)|^2$ and $\phi(\mathbf{k}_B)|^2$ are peaked around $\mathbf{p}_A$ and $\mathbf{p}_B$, we have

$$d\sigma = \frac{1}{n!}\left[\prod_{i=1}^{n}\frac{d^3p_{f_i}}{(2\pi)^3}\right]\left[\prod_{i=1}^{n}\int\frac{d^3k_i}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}_{f_i}}}|\phi(\mathbf{k}_i)|^2\right] \tag{3.107}$$

$$\int\frac{d^3k_A}{(2\pi)^3}\int\frac{d^3k_B}{(2\pi)^3}\frac{1}{4E_{\mathbf{p}_A}E_{\mathbf{p}_A}v_{ab}}|\phi(\mathbf{k}_A)|^2|\phi(\mathbf{k}_B)|^2$$

$$(2\pi)^4\,\delta^{(4)}\left(p_A+p_B-\sum_i p_{f_i}\right)\,|\mathcal{M}\left(p_A,p_B\to p_{f_i}\right)|^2$$

$$= \frac{1}{4E_{\mathbf{p}_A}E_{\mathbf{p}_B}v_{ab}}\frac{1}{n!}\left[\prod_{i=1}^{n}\frac{d^3p_{f_i}}{(2\pi)^3 2E_{\mathbf{p}_{f_i}}}\right] \tag{3.108}$$

$$\times\,(2\pi)^4\,\delta^{(4)}\left(p_A+p_B-\sum_i p_{f_i}\right)\,|\mathcal{M}\left(p_A,p_B\to p_{f_i}\right)|^2$$

We have thus derived the following result for the differential cross section:

$$d\sigma(p_A,p_B\to p_i,\ldots,p_n) = \frac{1}{F}\frac{d\Gamma_n}{n!}\,|\mathcal{M}\left(p_A,p_B\to p_i\right)|^2 \tag{3.109}$$

$$F \equiv 4E_{\mathbf{p}_A}E_{\mathbf{p}_B}v_{ab}$$

$$d\Gamma_n \equiv \left[\prod_{i=1}^{n}\frac{d^3p_i}{(2\pi)^3 2E_{\mathbf{p}_i}}\right](2\pi)^4\,\delta^{(4)}\left(p_A+p_B-\sum_i p_{f_i}\right)$$

The factor $1/n!$ is inherited from the completeness of Klein-Gordon states. More generally, **if there are n identical particles in the final state, the cross section has to be divided by $n!$** (or alternatively the phase space limited). When the initial-state particles are collinear, $\mathbf{v}_A||\mathbf{v}_B$, the **flux factor** $F = 4E_{\mathbf{p}_A}E_{\mathbf{p}_B}v_{ab}$ can be written in a Lorentz-invariant form (Ex.),

$$4E_{\mathbf{p}_A}E_{\mathbf{p}_B}v_{ab} = 4\sqrt{(p_a\cdot p_b)^2 - m_a^2m_b^2}\,. \tag{3.110}$$

If there are only two final-state particles, the phase-space element can be expressed in the center-of-mass frame as (Ex.),

$$\Gamma_2 = \int d\Omega \frac{|\mathbf{p}_{1,\mathrm{cm}}|}{16\pi^2\sqrt{s}} \tag{3.111}$$

$$\Omega = d\phi \sin\theta d\theta = d\phi d\cos\theta\,,$$

where $\sqrt{s}$ is the center-of-mass energy $s = (p_\mathrm{A} + p_\mathrm{B})^2$, and the angular variables θ and ϕ refer to either of the final-state particles in some fixed frame of reference. If the masses of all particles are identical, $m_\mathrm{A} = m_\mathrm{B} = m_1 = m_2$, we get a particularly simple result (Ex.):

Identical final-state particles:

$$\left(\frac{d\sigma}{d\Omega}\right)_\mathrm{cm} = \frac{1}{2!}\frac{|\mathcal{M}(\theta,\phi)|^2 + |\mathcal{M}(\pi-\theta,\pi+\phi)|^2}{64\pi^2 s}\,. \tag{3.112}$$

Non-identical final-state particles:

$$\left(\frac{d\sigma}{d\Omega}\right)_\mathrm{cm} = \frac{|\mathcal{M}(\theta,\phi)|^2}{64\pi^2 s}\,. \tag{3.113}$$

3.5.2 Relation of the S matrix and the ground-state expectation value

It turns out that there is a relation between the time-ordered ground-state expectation values and the iT part of the S matrix,

$$\langle\Omega|T\left[\phi(x_1)\cdots\phi(x_{n+2})\right]\Omega\rangle \Leftrightarrow \langle\mathbf{k}_1\cdots\mathbf{k}_n|iT|\mathbf{k}_A\mathbf{k}_B\rangle$$

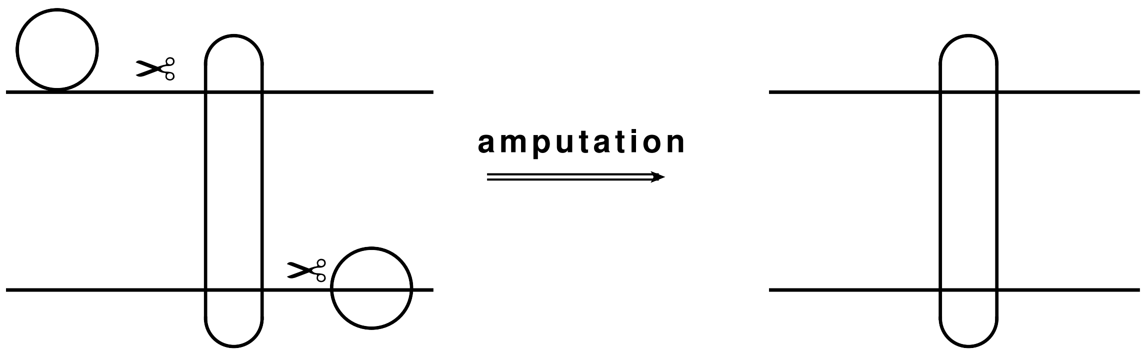
To establish the exact relation is complicated by the fact that the 1-particle states of the interacting theory that appear in the S matrix are not 1-particle states in a sense that they would only contain this one single particle. In an interacting theory a particle is always surrounded by a cloud of virtual particles. It will be somewhat easier to understand how the exact relation comes about after first attaining some experince in how it works. The relation is:

$$\langle \mathbf{k}_1 \cdots \mathbf{k}_n | iT | \mathbf{k}_A \mathbf{k}_B \rangle = \left[\sqrt{Z_A} \sqrt{Z_B} \prod_{i=1}^{n} \sqrt{Z_i} \right] \tag{3.114}$$
$$\times \left[{}_I\langle \mathbf{k}_1 \cdots \mathbf{k}_n | T \left\{ \exp \left[-i \int dt H_I(x) \right\} \right] | \mathbf{k}_A \mathbf{k}_B \rangle_I \right]_{\substack{\text{connected} \\ \text{amputated}}}$$

In the above formula the factors $\sqrt{Z_i}$ are related to the virtual correction to the 1-particle states, and the subscript I refers to the interaction picture. We recall that the states in the interaction picture behave as the free-theory states so, for example,

$$|\mathbf{k}_A \mathbf{k}_B\rangle_I = \left[\sqrt{2E_{\mathbf{k}_A}} a^\dagger_{\mathbf{k}_A} \right] \left[\sqrt{2E_{\mathbf{k}_B}} a^\dagger_{\mathbf{k}_B} \right] |0\rangle \,. \tag{3.115}$$

The word **connected** refers to – as earlier – to the fact that all the parts of the diagram should be attached to each other. The term **amputated** means that diagrams from which we get acceptable connected diagrams by "cutting" or "amputating" single external lines, are not taken into account:



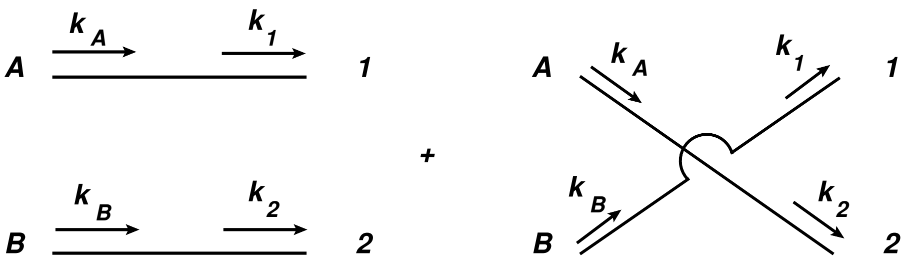
So the left-hand diagram would not do. The contributions from the external lines that we "amputate away" are later on accounted for in the $\sqrt{Z_i}$ factors.

Example 1:

Let us first consider an elastic $2 \to 2$ process. In the lowest order in the Taylor expansion we have, simply, $_I\langle \mathbf{k}_1\mathbf{k}_2|\mathbf{k}_A\mathbf{k}_B\rangle_I$. This is easy to evaluate using the commutation relations $[a_{\mathbf{p}}, a^\dagger_{\mathbf{k}}] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{p})$:

$$\begin{aligned} _I\langle \mathbf{k}_1\mathbf{k}_2|\mathbf{k}_A\mathbf{k}_B\rangle_I &= \sqrt{2E_{\mathbf{k}_A}}\sqrt{2E_{\mathbf{k}_B}}\sqrt{2E_{\mathbf{k}_1}}\sqrt{2E_{\mathbf{k}_2}}\langle 0|a_{\mathbf{k}_1}a_{\mathbf{k}_2}a^\dagger_{\mathbf{k}_A}a^\dagger_{\mathbf{k}_B}|0\rangle \\ &= (2\pi)^6 2E_{\mathbf{k}_A}2E_{\mathbf{k}_B}\Big[\delta^{(3)}(\mathbf{k}_2-\mathbf{k}_A)\delta^{(3)}(\mathbf{k}_1-\mathbf{k}_B) \\ &\quad +\delta^{(3)}(\mathbf{k}_1-\mathbf{k}_A)\delta^{(3)}(\mathbf{k}_2-\mathbf{k}_B)\Big] \end{aligned} \tag{3.116}$$

Diagrammatically this corresponds to



The external lines refer to the initial- and final-state particles. This contribution corresponds to the "1" in the S matrix $S = 1 + iT$ so it's not very interesting. Nor are the diagrams connected so these are not considered in Eq. (3.114).

Example 2:

The first non-trivial term obtained when expanding the time-ordered exponential is

$$\frac{-i\lambda}{4!}\,{}_I\langle \mathbf{k}_1\mathbf{k}_2|T\left[\int d^4x\phi_I^4(x)\right]|\mathbf{k}_A\mathbf{k}_B\rangle_I\,. \tag{3.117}$$

According to the Wick's theorem,

$$T\left[\int d^4x\phi_I^4(x)\right] = N(\phi_x^4) + 3N(\phi_x^2)\,\overbrace{\phi_x\phi_x} + 3\,\overbrace{\phi_x\phi_x}\,\overbrace{\phi_x\phi_x} \tag{3.118}$$

In the case of vacuum expectation value the term $N(\phi_x^4)$ gave zero. We will now open $N(\phi_x^4)$ – it contains 16 terms:

$$N(\phi_x^4) = 1\,\phi_x^-\phi_x^-\phi_x^-\phi_x^- + 4\,\phi_x^-\phi_x^-\phi_x^-\phi_x^+ + 6\,\phi_x^-\phi_x^-\phi_x^+\phi_x^+ \\ + 4\,\phi_x^-\phi_x^+\phi_x^+\phi_x^+ + 1\,\phi_x^+\phi_x^+\phi_x^+\phi_x^+ \,. \tag{3.119}$$

Let's see what happens when ϕ_x^+ hits $|\mathbf{k}_A\mathbf{k}_B\rangle_I$:

$$\phi_x^+|\mathbf{k}_A\mathbf{k}_B\rangle_I = \sqrt{2E_{\mathbf{k}_A}}\sqrt{2E_{\mathbf{k}_B}}\int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_{\mathbf{p}}}}e^{-ip\cdot x}a_{\mathbf{p}}a^\dagger_{\mathbf{k}_A}a^\dagger_{\mathbf{k}_B}|0\rangle \\ = e^{-ik_A\cdot x}|\mathbf{k}_B\rangle_I + e^{-ik_B\cdot x}|\mathbf{k}_A\rangle_I \,. \tag{3.120}$$

In the contraction language, this would be:

$$\phi_x^+|\mathbf{k}_A\mathbf{k}_B\rangle_I + \phi_x^+|\mathbf{k}_A\mathbf{k}_B\rangle_I = e^{-ik_A\cdot x}|\mathbf{k}_B\rangle_I + e^{-ik_B\cdot x}|\mathbf{k}_A\rangle_I \,.$$

If ϕ_x^+ hits this again then

$$\phi_x^+\phi_x^+|\mathbf{k}_A\mathbf{k}_B\rangle_I = e^{-ik_A\cdot x}\phi_x^+|\mathbf{k}_B\rangle_I + e^{-ik_B\cdot x}\phi_x^+|\mathbf{k}_A\rangle_I \\ = 2e^{-i(k_A+k_B)\cdot x}|0\rangle \,, \tag{3.121}$$

or in the contraction sense,

$$\phi_x^+\phi_x^+|\mathbf{k}_A\mathbf{k}_B\rangle_I + \phi_x^+\phi_x^+|\mathbf{k}_A\mathbf{k}_B\rangle_I = 2e^{-i(k_A+k_B)\cdot x}|0\rangle \,, \tag{3.122}$$

A third ϕ_x^+ would then give zero. The same thing happens when ϕ_x^-:n operates on the left:

$${}_I\langle\mathbf{k}_1\mathbf{k}_2|\phi_x^- = {}_I\langle\mathbf{k}_1|e^{+ik_2\cdot x} + {}_I\langle\mathbf{k}_2|e^{+ik_1\cdot x} \tag{3.123}$$

$${}_I\langle\mathbf{k}_1\mathbf{k}_2|\phi_x^-\phi_x^- = 2\langle 0|e^{+i(k_1+k_2)\cdot x} \,. \tag{3.124}$$

We see that when computing the expectation value of $N(\phi_x^4)$, in Eq. (3.119) only the term $6\,\phi_x^-\phi_x^-\phi_x^+\phi_x^+$ gives something non-zero:

$$\frac{-i\lambda}{4!}{}_I\langle\mathbf{k}_1\mathbf{k}_2|N(\phi_x^4)|\mathbf{k}_A\mathbf{k}_B\rangle_I = 6\frac{-i\lambda}{4!}{}_I\langle\mathbf{k}_1\mathbf{k}_2|\phi_x^-\phi_x^-\phi_x^+\phi_x^+|\mathbf{k}_A\mathbf{k}_B\rangle_I \\ = 6\frac{-i\lambda}{4!}\times 2\times 2\times e^{+i(k_1+k_2-k_A-k_B)\cdot x} \\ = -i\lambda\, e^{+i(k_1+k_2-k_A-k_B)\cdot x} \,. \tag{3.125}$$

We can thus "contract" the fields to the external states,

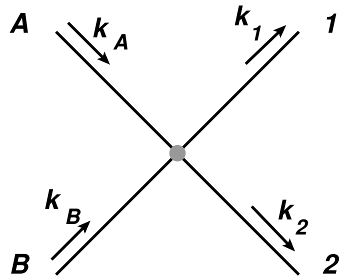
$$\frac{-i\lambda}{4!}{}_I\langle \mathbf{k}_1\mathbf{k}_2|N(\phi_x^4)|\mathbf{k}_A\mathbf{k}_B\rangle_I = \frac{-i\lambda}{4!}{}_I\langle \mathbf{k}_1\mathbf{k}_2|\phi_x\phi_x\phi_x\phi_x|\mathbf{k}_A\mathbf{k}_B\rangle_I \quad (3.126)$$

$$+\,23 \text{ other permutations}$$

with each contraction to the right giving a factor $e^{-ix\cdot k_i}$ and each contraction to the left giving a factor $e^{+ix\cdot k_i}$. The contracted field does not have to be next to the state (similarly as in the case of Wick's theorem two contracted fields didn't have to be next to each other). The $\int d^4x$ integral turns the exponential into a δ-function so that

$$\int d^4x \frac{-i\lambda}{4!}{}_I\langle \mathbf{k}_1\mathbf{k}_2|N(\phi_x^4)|\mathbf{k}_A\mathbf{k}_B\rangle_I = -i\lambda\,(2\pi)^4\delta^{(4)}\,(k_1+k_2-k_A-k_B)\ . \quad (3.127)$$

This contribution corresponds to a diagram,



which is connected so it will be consider when computing the S matrix. By definition,

$$\langle \mathbf{k}_1\cdots\mathbf{k}_n|iT|\mathbf{k}_A\mathbf{k}_B\rangle = (2\pi)^4\,\delta^{(4)}\left(k_A+k_B-\sum_i k_i\right)\,i\mathcal{M}\,(k_A,k_B\to k_f)\ , \quad (3.128)$$

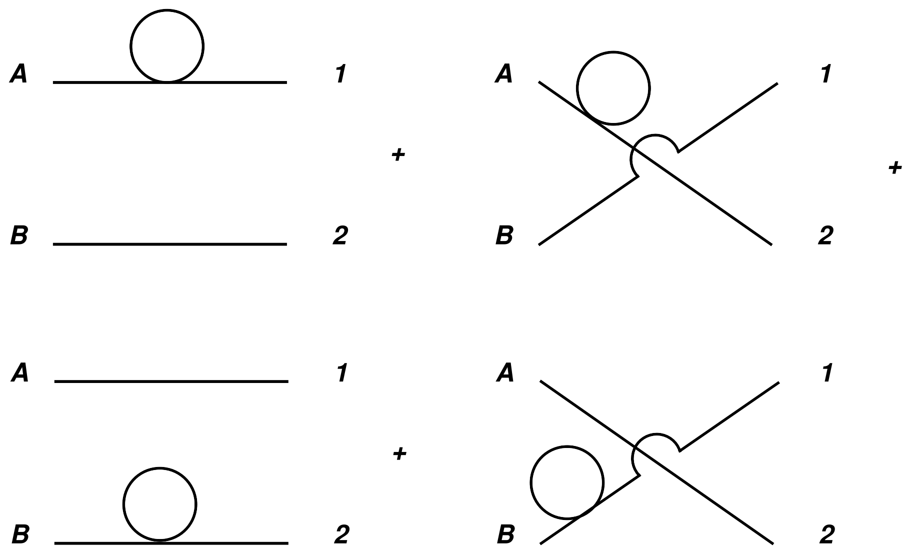
so the matrix-element $\mathcal{M}$ corresponding to the above diagram is simply $-\lambda$.

The remaining term is Eq. (3.118) are handled similarly. The non-zero contributions of the term $3N(\phi_x^2)\,\overbrace{\phi_x\phi_x}$ are

$$\frac{-i\lambda}{4!}\,{}_I\langle\mathbf{k}_1\mathbf{k}_2|3N(\phi_x^2)\,\overbrace{\phi_x\phi_x}|\mathbf{k}_A\mathbf{k}_B\rangle_I = 3\,\overbrace{\phi_x\phi_x}\frac{-i\lambda}{4!}2\left[{}_I\langle\overbrace{\mathbf{k}_1\mathbf{k}_2|\phi_x}\overbrace{\phi_x|\mathbf{k}_A}\mathbf{k}_B\rangle_I\right.$$

$$\left.+\,{}_I\langle\mathbf{k}_1\overbrace{\mathbf{k}_2|\phi_x}\overbrace{\phi_x|\mathbf{k}_A}\mathbf{k}_B\rangle_I + {}_I\langle\mathbf{k}_1\overbrace{\mathbf{k}_2|\phi_x}\overbrace{\phi_x|\mathbf{k}_A\mathbf{k}_B}\rangle_I \;+ {}_I\langle\overbrace{\mathbf{k}_1\mathbf{k}_2|\phi_x}\overbrace{\phi_x|\mathbf{k}_A\mathbf{k}_B}\rangle_I\right]$$

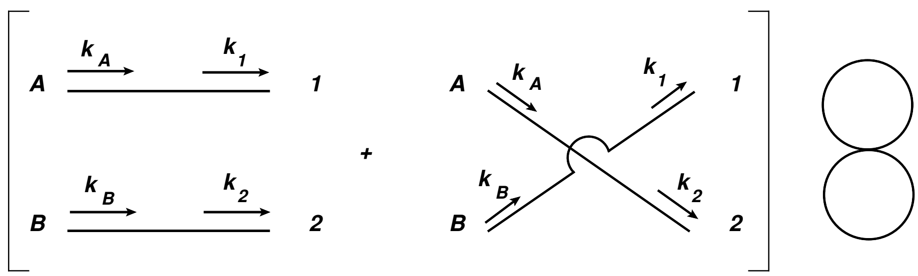
They correspond to the diagrams,



of which none is connected so they get thrown away. We still have the case, $3\,\overbrace{\phi_x\phi_x}\overbrace{\phi_x\phi_x}$, which corresponds to an expression

$$3\,\overbrace{\phi_x\phi_x}\overbrace{\phi_x\phi_x}\frac{-i\lambda}{4!}\,{}_I\langle\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_A\mathbf{k}_B\rangle_I\,. \tag{3.129}$$

Diagrammatically,



so it contains a vacuumb bubble and is not connected – to the trash bin it goes. All in all, only one diagram survives at this order of coupling constant λ, and

$$\mathcal{M}\,(k_A, k_B \to k_1, k_2) = -\lambda + \mathcal{O}(\lambda^2) \tag{3.130}$$

$$\implies \left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{\lambda^2}{64\pi^2 s} + \mathcal{O}(\lambda^3)\,. \tag{3.131}$$

The cross section is **isotropical** so it does not have any angular dependence. Thus, the angular integration in total cross section gives just a factor of 4π,

$$\sigma_{\text{total}} = \frac{1}{2!}\int d\Omega \left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = 2\pi \left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{\lambda^2}{32\pi s}\,. \tag{3.132}$$

Note that the total cross section is not $\int d\Omega\,(d\sigma/d\Omega)_{\text{CM}}$ since we have two identical particels in the final state! It is better to call the quantity $\int d\Omega\,(d\sigma/d\Omega)_{\text{CM}}$ as an **integrated cross section.**

Feynman rules for S matrix in position space:

In summary, the contribution of a given diagram to the S matrix

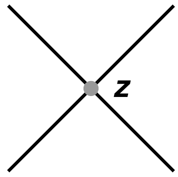
$$\langle \mathbf{k}_1 \cdots \mathbf{k}_n | iT | \mathbf{k}_A \mathbf{k}_B \rangle = (2\pi)^4 \, \delta^{(4)} \left(k_A + k_B - \sum_i k_i \right) i\mathcal{M} \left(k_A, k_B \to k_f \right) ,$$

is found by the following Feynman rules:

1. Each line

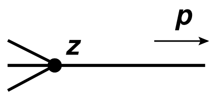
$= D_F(x-y)$

2. Each vertex

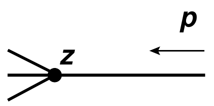


$= i\lambda \int d^4 z$

3. External legs



$= e^{+ip\cdot z}$



$= e^{-ip\cdot z}$

4. Compute the weight

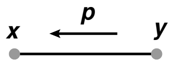
Only fully connected and amputated diagrams should be considered.

Feynman rules for S matrix in momentum space:

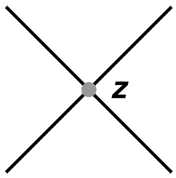
The contribution of a given diagram to the matrix element

$$i\mathcal{M}\,(k_A, k_B \to k_f)\ ,$$

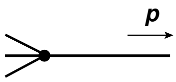
is found by the following Feynman rules:

1. Each line 

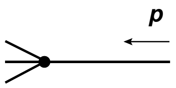
$= \frac{i}{p^2 - m^2 + i\epsilon}$

2. Each vertex 

$= -i\lambda$

3. External legs 

$= 1$



$= 1$

4. Choose the 4-momenta such that the momentum is conserved in vertices

5. Integrate over the undetermined momenta by $\int \frac{d^4p}{(2\pi)^4}$

6. Compute the weight factor

Only consider fully connected and amputated diagrams.

3.6 Feynman rules involving fermions

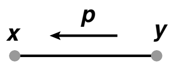
The formula (3.114) to compute the scattering matrix is completely general, but the Feynman rules depend on the content of the Hamiltonian. For fermionic fields the anticommutation relations cause some changes in how the time-ordered product and normal-ordered product are defined.

Feynman rules for S matrix in momentum space:

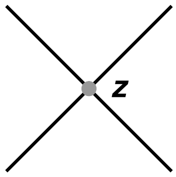
The contribution of a given diagram to the matrix element

$$i\mathcal{M}\left(k_A, k_B \to k_f\right) \; ,$$

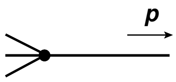
is found by the following Feynman rules:

1. Each line 

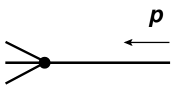
$= \frac{i}{p^2 - m^2 + i\epsilon}$

2. Each vertex 

$= -i\lambda$

3. External legs 

$= 1$



$= 1$

4. Choose the 4-momenta such that the momentum is conserved in vertices

5. Integrate over the undetermined momenta by $\int \frac{d^4p}{(2\pi)^4}$

6. Compute the weight factor

Only consider fully connected and amputated diagrams.

3.6 Feynman rules involving fermions [Peskin 4.7]

The formula (3.114) to compute the scattering matrix is completely general, but the Feynman rules depend on the content of the Hamiltonian. For fermionic fields the anticommutation relations cause some changes in how the time-ordered product and normal-ordered product are defined.

We already defined the time-ordered product in the case of two field operators,

$$T\left[\psi_\alpha(x)\overline{\psi}_\beta(y)\right] = \begin{cases} \psi_\alpha(x)\overline{\psi}_\beta(y)\,, & x^0 > y^0 \\ -\overline{\psi}_\beta(y)\psi_\alpha(x)\,, & y^0 > x^0 \end{cases} \tag{3.133}$$

where α and β now refer to the spinor indices. This vacuum expectation value corresponds to the Feynman propagator,

$$\begin{aligned} S_{\mathrm{F}}(x-y) &= \langle 0|T\psi(x)\overline{\psi}(y)|0\rangle \\ &\equiv \theta(x^0-y^0)\langle 0|\psi(x)\overline{\psi}(y)|0\rangle - \theta(y^0-x^0)\langle 0|\overline{\psi}(y)\psi(x)|0\rangle \\ &= \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i\left(\not{p}+m\right)}{p^2-m^2+i\epsilon}\,. \end{aligned}$$

In general, the time-ordered product for fermionic fields is defined as follows:

When $x^0_{i_1} > x^0_{i_2} > \cdots > x^0_{i_n}$

$$T\left[\psi(x_1)\psi(x_2)\cdots\psi(x_n)\right] \equiv (-1)^{N_p}\,\psi(x_{i_1})\psi(x_{i_2})\cdots\psi(x_{i_n}) \tag{3.134}$$

where N_p is the number of anticommutations that is needed to bring the fields into the correct time order. Here $\psi = \psi, \overline{\psi}$ (either ones or mixed). For example, if $x^0_3 > x^0_2 > x^0_1$, then

$$T\left[\psi(x_1)\psi(x_2)\psi(x_3)\right] = (-1)^3\psi(x_3)\psi(x_2)\psi(x_1)\,.$$

The normal-ordered product is equipped with a similar sign convention,

$$N\left[a_{\mathbf{p}_1}a_{\mathbf{p}_2}a^\dagger_{\mathbf{p}_3}\right] = (-1)^2 a^\dagger_{\mathbf{p}_3}a_{\mathbf{p}_1}a_{\mathbf{p}_2} = (-1)^3 a^\dagger_{\mathbf{p}_3}a_{\mathbf{p}_2}a_{\mathbf{p}_1}\,. \tag{3.135}$$

It follows that

$$N\left[\psi(x_1)\psi(x_2)\cdots\psi(x_n)\right] \equiv (-1)^{N_p}\,N\left[\psi(x_{i_1})\psi(x_{i_2})\cdots\psi(x_{i_n})\right] \tag{3.136}$$

where N_p is again the number of anticommutations required to bring the fields from the order $\psi(x_1)\psi(x_2)\cdots\psi(x_n)$ to the order $\psi(x_{i_1})\psi(x_{i_2})\cdots\psi(x_{i_n})$.

Let us split the Dirac quantum field

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_{\mathbf{p}}}}\sum_s \left[a_{\mathbf{p},s}u_s(p)e^{-ip\cdot x} + b^\dagger_{\mathbf{p},s}v_s(p)e^{ip\cdot x}\right] , \quad (3.137)$$

into two parts $\psi(x) = \psi^+(x) + \psi^-(x)$ and $\overline{\psi}(x) = \overline{\psi}^+(x) + \overline{\psi}^-(x)$, where

$$\psi^+(x) = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_{\mathbf{p}}}}\sum_s a_{\mathbf{p},s}u_s(p)e^{-ip\cdot x} \quad (3.138)$$

$$\psi^-(x) = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_{\mathbf{p}}}}\sum_s b^\dagger_{\mathbf{p},s}v_s(p)e^{ip\cdot x}$$

$$\overline{\psi}^+(x) = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_{\mathbf{p}}}}\sum_s b_{\mathbf{p},s}\overline{v}_s(p)e^{-ip\cdot x}$$

$$\overline{\psi}^-(x) = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_{\mathbf{p}}}}\sum_s a^\dagger_{\mathbf{p},s}\overline{u}_s(p)e^{ip\cdot x} .$$

Using the definitions of time- and normal-ordered products it is easy to see that (Ex.)

$$T\left[\psi(x)\overline{\psi}(y)\right] = N\left[\psi(x)\overline{\psi}(y)\right] + \contraction{}{\psi}{(x)}{\overline{\psi}}\psi(x)\overline{\psi}(y) , \quad (3.139)$$

where the contraction between two spinor field is

$$\contraction{}{\psi}{(x)}{\overline{\psi}}\psi(x)\overline{\psi}(y) \equiv \begin{cases} \left\{\psi^+(x)\,,\overline{\psi}^-(y)\right\} , & x^0 > y^0 \\ -\left\{\overline{\psi}^+(y)\,,\psi^-(x)\right\} , & y^0 > x^0 \end{cases} \quad (3.140)$$

$$= S_F(x-y) ,$$

so the contraction again corresponds to the Feynman propagator. In addition,

$$T\left[\psi(x)\psi(y)\right] = N\left[\psi(x)\psi(y)\right] \quad (3.141)$$

$$T\left[\overline{\psi}(x)\overline{\psi}(y)\right] = N\left[\overline{\psi}(x)\overline{\psi}(y)\right] ,$$

so that

$$\contraction{}{\psi}{(x)}{\psi}\psi(x)\psi(y) = \contraction{}{\overline{\psi}}{(x)}{\overline{\psi}}\overline{\psi}(x)\overline{\psi}(y) = 0\,. \tag{3.142}$$

The Wick's theorem is almost identical as in the bosonic case:

$$\begin{aligned} T\left[\psi_1\cdots\psi_n\right] &= N(\psi_1\cdots\psi_n) + \sum_{\text{perm.}}(-1)^{N_p}\contraction{}{\psi}{_{i_1}}{\psi}\psi_{i_1}\psi_{i_2}N(\psi_{i_3}\cdots\psi_{i_n}) \\ &+ \sum_{\text{perm.}}(-1)^{N_p}\contraction{}{\psi}{_{i_1}}{\psi}\psi_{i_1}\psi_{i_2}\contraction{}{\psi}{_{i_3}}{\psi}\psi_{i_3}\psi_{i_4}N(\psi_{i_5}\cdots\psi_{i_n}) \\ &\vdots \\ &+ \sum_{\text{perm.}}(-1)^{N_p}\contraction{}{\psi}{_{i_1}}{\psi}\psi_{i_1}\psi_{i_2}\cdots\contraction{}{\psi}{_{i_{n-1}}}{\psi}\psi_{i_{n-1}}\psi_{i_n}\,, \end{aligned} \tag{3.143}$$

where N_p is again the number of anticommutatios required to bring the fields from the order $\psi_1\cdots\psi_n$ to the order in which they apperar in each permutation.

3.6.1 The Yukawa theory

Historically, the Yukawa theory was used to model interactions between pions and nucleons. In the Standard Model, the Higgs boson couples to fermions via Yukawa coupling.

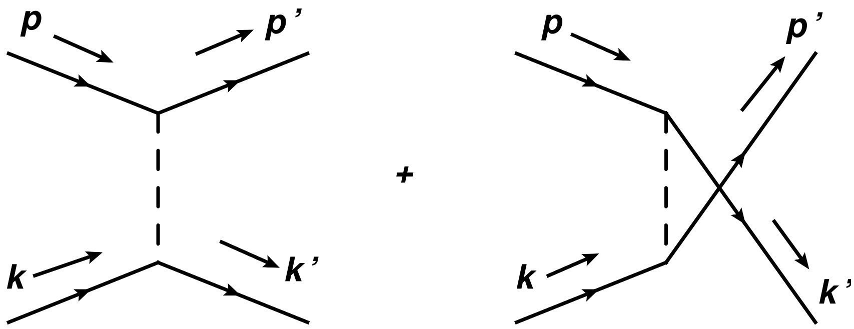
The Hamiltonian for the Yukawa theory is of the form,

$$H = H_{\text{Dirac}} + H_{\text{Klein-Gordon}} + g\int d^3x\overline{\psi}(x)\psi(x)\phi(x)\,, \tag{3.144}$$

where H_{Dirac} and $H_{\text{Klein-Gordon}}$ are the free-field Hamiltonians of Dirac and (real) Klein-Gordon particles. Let us consider a fermion-fermion scattering:

$$f(p,s_p) + f(k,s_k) \longrightarrow f(p',s_{p'}) + f(k',s_{k'})\,,$$

where we have indicated the momenta and spins of initial- and final-state fermions. As diagrams, the lowest-order graphs are:



We will draw the fermions as lines including the so-called particle arrow, and the scalar particles as dashed lines. The values for these (and many other) diagrams stem from the expression

$$\langle(\mathbf{p}', s_{p'}); (\mathbf{k}', s_{k'})|iT|(\mathbf{p}, s_p); (\mathbf{k}, s_k)\rangle \tag{3.145}$$

$$= \frac{1}{2!}\langle(\mathbf{p}', s_{p'}); (\mathbf{k}', s_{k'})|T\left[\left(-ig\int d^4x\overline{\psi}_x\psi_x\phi_x\right)\left(-ig\int d^4y\overline{\psi}_y\psi_y\phi_y\right)\right]$$

$$|(\mathbf{p}, s_p); (\mathbf{k}, s_k)\rangle$$

We use the Wick's theorem,

$$T\left[\phi_x\phi_y\right] = N\left[\phi_x\phi_y\right] + \overbracket{\phi_x\phi_y} \tag{3.146}$$

$$T\left[\overline{\psi}_x\psi_x\overline{\psi}_y\psi_y\right] = N\left[\overline{\psi}_x\psi_x\overline{\psi}_y\psi_y\right] + \ldots \tag{3.147}$$

In the case of fermion fields, the rest of the term will not produce connected diagrams so we don't need them. Also, in the case of scalar fields the normally-ordered term yields zero (no external scalar particles now). The relevant part thus shortens to

$$\frac{-g^2}{2!}\int d^4xd^4y\,\overbracket{\phi_x\phi_y}\,\langle(\mathbf{p}', s_{p'}); (\mathbf{k}', s_{k'})|N\left[\overline{\psi}_x\psi_x\overline{\psi}_y\psi_y\right]|(\mathbf{p}, s_p); (\mathbf{k}, s_k)\rangle\,.$$

To continue, we open the normal-ordered product:

$$
\begin{aligned}
N\left[\overline{\psi}_x\psi_x\overline{\psi}_y\psi_y\right] &= N\left[\left(\overline{\psi}_x^+ + \overline{\psi}_x^-\right)\left(\psi_x^+ + \psi_x^-\right)\left(\overline{\psi}_y^+ + \overline{\psi}_y^-\right)\left(\psi_y^+ + \psi_y^-\right)\right] \\
&= \overline{\psi}_x^+\psi_x^+\overline{\psi}_y^+\psi_y^+ - \psi_y^-\overline{\psi}_x^+\psi_x^+\overline{\psi}_y^+ + \overline{\psi}_y^-\overline{\psi}_x^+\psi_x^+\psi_y^+ \\
&+ \overline{\psi}_y^-\psi_y^-\overline{\psi}_x^+\psi_x^+ - \psi_x^-\overline{\psi}_x^+\overline{\psi}_y^+\psi_y^+ - \psi_x^-\psi_y^-\overline{\psi}_x^+\overline{\psi}_y^+ \\
&+ \psi_x^-\overline{\psi}_y^-\overline{\psi}_x^+\psi_y^+ - \psi_x^-\overline{\psi}_y^-\psi_y^-\overline{\psi}_x^+ + \overline{\psi}_x^-\psi_x^+\overline{\psi}_y^+\psi_y^+ \\
&+ \overline{\psi}_x^-\psi_y^-\psi_x^+\overline{\psi}_y^+ - \underline{\overline{\psi}_x^-\overline{\psi}_y^-\psi_x^+\psi_y^+} + \overline{\psi}_x^-\overline{\psi}_y^-\psi_y^-\psi_x^+ \\
&+ \overline{\psi}_x^-\psi_x^-\overline{\psi}_y^+\psi_y^+ - \overline{\psi}_x^-\psi_x^-\psi_y^-\overline{\psi}_y^+ + \overline{\psi}_x^-\psi_x^-\overline{\psi}_y^-\psi_y^+ \\
&+ \overline{\psi}_x^-\psi_x^-\overline{\psi}_y^-\psi_y^-
\end{aligned}
$$

Looks like a big mess. Well, let's see what is $\psi_y^+|(\mathbf{p}, s_p);(\mathbf{k}, s_k)\rangle$:

$$
\begin{aligned}
(\psi_y^+)_\beta|(\mathbf{p}, s_p);(\mathbf{k}, s_k)\rangle &= (\psi_y^+)_\beta\sqrt{2E_\mathbf{p}}\sqrt{2E_\mathbf{k}}a^\dagger_{\mathbf{p},s_p}a^\dagger_{\mathbf{k},s_k}|0\rangle \qquad (3.148) \\
&= \int\frac{d^3p'}{(2\pi)^3}\sqrt{\frac{1}{2E_{\mathbf{p}'}}}\sum_s a_{\mathbf{p}',s}\left[u_s(p')\right]_\beta e^{-ip'\cdot y}\sqrt{2E_\mathbf{p}}\sqrt{2E_\mathbf{k}}a^\dagger_{\mathbf{p},s_p}a^\dagger_{\mathbf{k},s_k}|0\rangle \\
&= \left[u_{s_p}(p)\right]_\beta e^{-ip\cdot y}|(\mathbf{k}, s_k)\rangle - \left[u_{s_k}(k)\right]_\beta e^{-ik\cdot y}|(\mathbf{p}, s_p)\rangle\,.
\end{aligned}
$$

With the contraction notation this would be,

$$
(\psi_y^+)_\beta|(\mathbf{p}, s_p);(\mathbf{k}, s_k)\rangle = \left[u_{s_p}(p)\right]_\beta e^{-ip\cdot y}|(\mathbf{k}, s_k)\rangle \qquad (3.149)
$$

$$
(\psi_y^+)_\beta|(\mathbf{p}, s_p);(\mathbf{k}, s_k)\rangle = -\left[u_{s_k}(k)\right]_\beta e^{-ik\cdot y}|(\mathbf{p}, s_p)\rangle\,, \qquad (3.150)
$$

i.e. if we "jump" over one fermion in the contraction, we get a minus sign.

Let's then hit this state with ψ_x^+:

$$
\begin{aligned}
(\psi_x^+)_\alpha(\psi_y^+)_\beta|(\mathbf{p}, s_p);(\mathbf{k}, s_k)\rangle &= \left[u_{s_k}(k)\right]_\alpha\left[u_{s_p}(p)\right]_\beta e^{-ip\cdot y}e^{-ik\cdot x}|0\rangle \\
&- \left[u_{s_p}(p)\right]_\alpha\left[u_{s_k}(k)\right]_\beta e^{-ik\cdot y}e^{-ip\cdot x}|0\rangle\,.
\end{aligned}
$$

And the same in the contraction notation:

$$(\psi_x^+)_\alpha(\psi_y^+)_\beta|(\mathbf{p},s_p);(\mathbf{k},s_k)\rangle \;=[u_{s_k}(k)]_\alpha\left[u_{s_p}(p)\right]_\beta e^{-ip\cdot y}e^{-ik\cdot x}|0\rangle$$

$$(\psi_x^+)_\alpha(\psi_y^+)_\beta|(\mathbf{p},s_p);(\mathbf{k},s_k)\rangle \;=-\left[u_{s_p}(p)\right]_\alpha[u_{s_k}(k)]_\beta e^{-ik\cdot y}e^{-ip\cdot x}|0\rangle\,.$$

Because this was a fermion state (no antifermions), $\overline{\psi}_y^+|(\mathbf{p},s_p);(\mathbf{k},s_k)\rangle = 0$. In the same way,

$$\begin{aligned}\langle(\mathbf{p}',s_{p'});(\mathbf{k}',s_{k'})|(\overline{\psi}_x^-)_\alpha(\overline{\psi}_y^-)_\beta &= \langle 0|\left[\overline{u}_{s_{p'}}(p')\right]_\alpha\left[\overline{u}_{s_{k'}}(k')\right]_\beta e^{ik'\cdot y}e^{ip'\cdot x}\\ &\quad -\langle 0|\left[\overline{u}_{s_{k'}}(k')\right]_\alpha\left[\overline{u}_{s_{p'}}(p')\right]_\beta e^{ip'\cdot y}e^{ik'\cdot x}\,,\end{aligned}$$

or,

$$\langle(\mathbf{p}',s_{p'});(\mathbf{k}',s_{k'})|(\overline{\psi}_x^-)_\alpha(\overline{\psi}_y^-)_\beta \;=\langle 0|\left[\overline{u}_{s_{p'}}(p')\right]_\alpha\left[\overline{u}_{s_{k'}}(k')\right]_\beta e^{ik'\cdot y}e^{ip'\cdot x}$$

$$\langle(\mathbf{p}',s_{p'});(\mathbf{k}',s_{k'})|(\overline{\psi}_x^-)_\alpha(\overline{\psi}_y^-)_\beta \;=-\langle 0|\left[\overline{u}_{s_{k'}}(k')\right]_\alpha\left[\overline{u}_{s_{p'}}(p')\right]_\beta e^{ip'\cdot y}e^{ik'\cdot x}\,.$$

Note the signs which follow from our convention (2.108) for multi-particle Dirac states. Now $\langle(\mathbf{p}',s_{p'});(\mathbf{k}',s_{k'})|\psi_x^- = 0$, because we have only fermions in our final state. Thus, from the 16 terms in the normally ordered product, only the term $-\overline{\psi}_x^-\overline{\psi}_y^-\psi_x^+\psi_y^+$ gives something non zero. Our scattering amplitude goes now into the form,

$$
\frac{g^2}{2!}\int d^4x d^4y\, \overline{\phi_x \phi_y}\, \langle(\mathbf{p}', s_{p'});(\mathbf{k}', s_{k'})|(\overline{\psi}_x^-)_\alpha(\overline{\psi}_y^-)_\beta(\psi_x^+)_\alpha(\psi_y^+)_\beta|(\mathbf{p}, s_p);(\mathbf{k}, s_k)\rangle
$$

$$
= \frac{g^2}{2!}\int d^4x d^4y \int \frac{\mathrm{d}^4 q}{(2\pi)^4} e^{-iq\cdot(x-y)} \frac{i}{q^2 - m_\phi^2 + i\epsilon} \tag{3.151}
$$

$$
\Big[\left[\overline{u}_{s_{p'}}(p')\right]_\alpha \left[\overline{u}_{s_{k'}}(k')\right]_\beta e^{ik'\cdot y} e^{ip'\cdot x} - \left[\overline{u}_{s_{k'}}(k')\right]_\alpha \left[\overline{u}_{s_{p'}}(p')\right]_\beta e^{ip'\cdot y} e^{ik'\cdot x}\Big]
$$

$$
\Big[\left[u_{s_k}(k)\right]_\alpha \left[u_{s_p}(p)\right]_\beta e^{-ip\cdot y} e^{-ik\cdot x} - \left[u_{s_p}(p)\right]_\alpha \left[u_{s_k}(k)\right]_\beta e^{-ik\cdot y} e^{-ip\cdot x}\Big]
$$

$$
= \frac{g^2}{2!}\int d^4x d^4y \int \frac{\mathrm{d}^4 q}{(2\pi)^4} e^{-iq\cdot(x-y)} \frac{i}{q^2 - m_\phi^2 + i\epsilon}
$$

$$
\Big\{- \left[\overline{u}_{s_{k'}}(k') u_{s_k}(k)\right] \left[\overline{u}_{s_{p'}}(p') u_{s_p}(p)\right] e^{ik'\cdot y} e^{ip'\cdot x} e^{-ik\cdot y} e^{-ip\cdot x}
$$

$$
- \left[\overline{u}_{s_{p'}}(p') u_{s_p}(p)\right] \left[\overline{u}_{s_{k'}}(k') u_{s_k}(k)\right] e^{ip'\cdot y} e^{ik'\cdot x} e^{-ip\cdot y} e^{-ik\cdot x}
$$

$$
+ \left[\overline{u}_{s_{k'}}(k') u_{s_p}(p)\right] \left[\overline{u}_{s_{p'}}(p') u_{s_k}(k)\right] e^{ip'\cdot y} e^{ik'\cdot x} e^{-ik\cdot y} e^{-ip\cdot x}
$$

$$
+ \left[\overline{u}_{s_{p'}}(p') u_{s_k}(k)\right] \left[\overline{u}_{s_{k'}}(k') u_{s_p}(p)\right] e^{ik'\cdot y} e^{ip'\cdot x} e^{-ip\cdot y} e^{-ik\cdot x}\Big\}
$$

$$
= \frac{g^2}{2!}\int d^4x d^4y \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{i}{q^2 - m_\phi^2 + i\epsilon}
$$

$$
\Big\{- \left[\overline{u}_{s_{k'}}(k') u_{s_k}(k)\right] \left[\overline{u}_{s_{p'}}(p') u_{s_p}(p)\right] e^{i(k'-k+q)\cdot y} e^{i(p'-p-q)\cdot x}
$$

$$
- \left[\overline{u}_{s_{p'}}(p') u_{s_p}(p)\right] \left[\overline{u}_{s_{k'}}(k') u_{s_k}(k)\right] e^{i(p'-p+q)\cdot y} e^{i(k'-k-q)\cdot x}
$$

$$
+ \left[\overline{u}_{s_{k'}}(k') u_{s_p}(p)\right] \left[\overline{u}_{s_{p'}}(p') u_{s_k}(k)\right] e^{i(p'-k+q)\cdot y} e^{i(k'-p-q)\cdot x}
$$

$$
+ \left[\overline{u}_{s_{p'}}(p') u_{s_k}(k)\right] \left[\overline{u}_{s_{k'}}(k') u_{s_p}(p)\right] e^{i(k'-p+q)\cdot y} e^{i(p'-k-q)\cdot x}\Big\} \, .
$$

In the contraction notation, the first term is

$$
\frac{(-ig)^2}{2!}\int d^4x d^4y \langle(\mathbf{p}', s_{p'});(\mathbf{k}', s_{k'})|\left(\overline{\psi}_x \psi_x \phi_x\right)\left(\overline{\psi}_y \psi_y \phi_y\right)|(\mathbf{p}, s_p);(\mathbf{k}, s_k)\rangle
$$

It takes two fermionic anticommutations to bring the fields that are contracted next to each other, so the overall sign is $(-i)^2 = -1$. Doing the x- ja y integrals we get δ-functions,

$$\frac{g^2}{2!}(2\pi)^8 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{i}{q^2 - m_\phi^2 + i\epsilon}$$

$$\Big\{ - \left[\overline{u}_{s_{k'}}(k') u_{s_k}(k)\right] \left[\overline{u}_{s_{p'}}(p') u_{s_p}(p)\right] \delta^{(4)}(k' - k + q) \delta^{(4)}(p' - p - q)$$

$$- \left[\overline{u}_{s_{p'}}(p') u_{s_p}(p)\right] \left[\overline{u}_{s_{k'}}(k') u_{s_k}(k)\right] \delta^{(4)}(p' - p + q) \delta^{(4)}(k' - k - q)$$

$$+ \left[\overline{u}_{s_{k'}}(k') u_{s_p}(p)\right] \left[\overline{u}_{s_{p'}}(p') u_{s_k}(k)\right] \delta^{(4)}(p' - k + q) \delta^{(4)}(k' - p - q)$$

$$+ \left[\overline{u}_{s_{p'}}(p') u_{s_k}(k)\right] \left[\overline{u}_{s_{k'}}(k') u_{s_p}(p)\right] \delta^{(4)}(k' - p + q) \delta^{(4)}(p' - k - q) \Big\}$$

$$= g^2 \, (2\pi)^4 \delta^{(4)}(k + p - k' - p')$$

$$\left\{ \frac{-i}{(k' - k)^2 - m_\phi^2 + i\epsilon} \left[\overline{u}_{s_{k'}}(k') u_{s_k}(k)\right] \left[\overline{u}_{s_{p'}}(p') u_{s_p}(p)\right] \right. \tag{3.152}$$

$$\left. + \frac{i}{(k' - p)^2 - m_\phi^2 + i\epsilon} \left[\overline{u}_{s_{k'}}(k') u_{s_p}(p)\right] \left[\overline{u}_{s_{p'}}(p') u_{s_k}(k)\right] \right\}$$

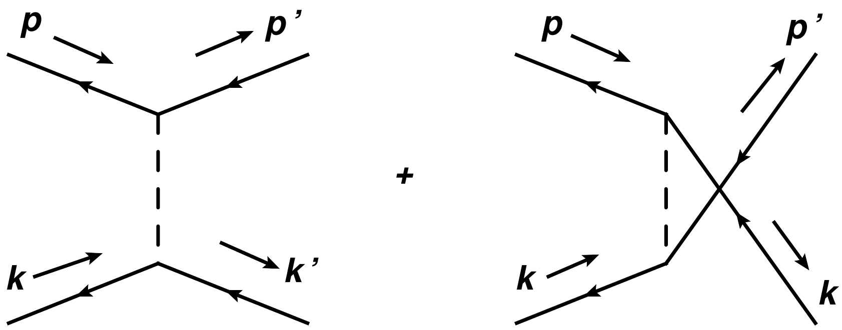
The δ function that conserves the overall momentum again appears as a multiplicative front factor, and we can identify our final matrix element,

$$i\mathcal{M}_{ff \to ff} = g^2 \left\{ \frac{-i}{(k' - k)^2 - m_\phi^2} \left[\overline{u}_{s_{k'}}(k') u_{s_k}(k)\right] \left[\overline{u}_{s_{p'}}(p') u_{s_p}(p)\right] \right.$$
$$\left. + \frac{i}{(k' - p)^2 - m_\phi^2} \left[\overline{u}_{s_{k'}}(k') u_{s_p}(p)\right] \left[\overline{u}_{s_{p'}}(p') u_{s_k}(k)\right] \right\} .$$

The first term corresponds to the "direct" diagram and the second one to the one in which the final-state fermions are "crossed". The relative minus sign is important and is a reflection of the anticommutation relations for fermions. In general, if two given diagrams differ only by an interchange of two identical fermions, there is a relative sign difference between the two diagrams.

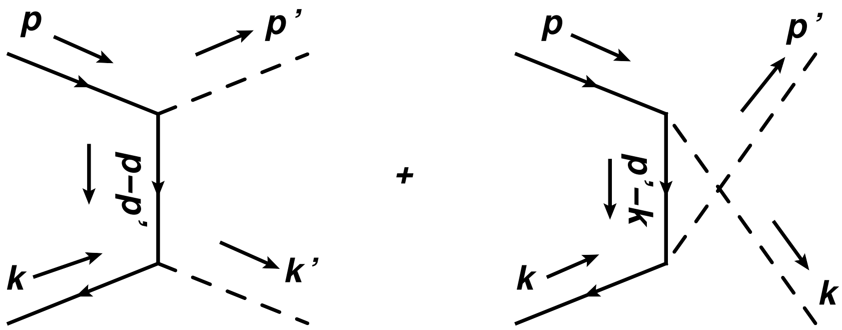
The same calculation in the case of antifermion scattering gives a similar result,

$$i\mathcal{M}_{\overline{f}\overline{f}\to\overline{f}\overline{f}} = g^2 \left\{ \frac{-i}{(k'-k)^2 - m_\phi^2} \left[\overline{v}_{s_k}(k) v_{s_{k'}}(k')\right] \left[\overline{v}_{s_p}(p) v_{s_{p'}}(p')\right] \right.$$
$$\left. + \frac{i}{(k'-p)^2 - m_\phi^2} \left[\overline{v}_{s_p}(p) v_{s_{k'}}(k')\right] \left[\overline{v}_{s_k}(k) v_{s_{p'}}(p')\right] \right\}.$$



We see that external fermion legs are represented in matrix elements by u and v spinors, each vertex yields $-ig$, and internal scalar lines correspond to Klein-Gordon propagators. An internal fermion line yields analogously a fermion propagator. To get convinced about this, we can consider a process,

$$f(p, s_p) + \overline{f}(k, s_k) \longrightarrow \phi(p') + \phi(k').$$



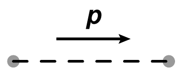
It's left as an exercise to shows that these diagrams correspond to an expression:

$$i\mathcal{M}_{f\overline{f}\to\phi\phi} = -g^2\, \overline{v}_{s_k}(k) \left[\frac{i\left(\not{p} - \not{p}' + m\right)}{(p-p')^2 - m_f^2} + \frac{i\left(\not{p}' - \not{k} + m\right)}{(p'-k)^2 - m_f^2} \right] u_{s_p}(p). \tag{3.153}$$

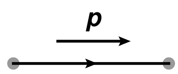
The momentum appearing in the Fermion propagator is always in the direction of the particle line. Along with these explicit calculations, we can write down the Feynman rules for the Yukawa theory (in momentum space):

Feynman rules for Yukawa theory:

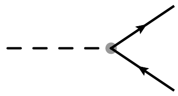
The matrix element $i\mathcal{M}(k_A, k_B \to k_f)$, of a given diagram is obtained by:

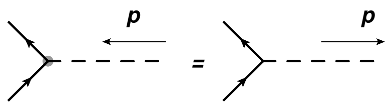
1. Each scalar line 

$= \frac{i}{p^2 - m^2 + i\epsilon}$

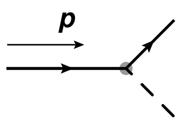
2. Each fermion line 

$= \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$

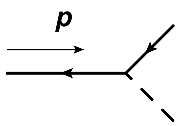
3. Each vertex 
$= -ig$

4. External scalar leg 

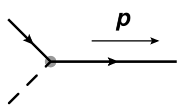
$= 1$

5. Initial-state fermions 

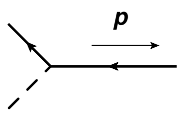
$= u_s(p)$



$= \overline{v}_s(p)$

6. Final-state fermions 

$= \overline{u}_s(p)$



$= v_s(p)$

7. Choose the 4-momenta such that the momentum is conserved in vertices

8. Integrate over undetermined momenta with weight $\int \frac{d^4p}{(2\pi)^4}$

9. Compute the weight factor (including signs)

Yukawa potential:

In the non-relativistic limit $|\mathbf{p}| \ll m$ the inter-fermion interaction can also be described with a time-independent potential. For simplicity, we consider here scattering between two distinguishable fermions:

$$f_A(p, s_p) + f_B(k, s_k) \longrightarrow f_A(p', s_{p'}) + f_B(k', s_{k'}) .$$

The Yukawa interaction part of the Hamiltonian is thus of the form,

$$H_{\text{int}} = g \int d^3x \left[\overline{\psi}_A(x)\psi_A(x) + \overline{\psi}_B(x)\psi_B(x)\right] \phi(x) . \tag{3.154}$$

In the lowest order, the S-matrix element contains only one term which we can read off from Eq. (3.152),

$$\langle \mathbf{p'_A}; \mathbf{k'_B}|S|\mathbf{p_A}; \mathbf{k_B}\rangle = g^2\, (2\pi)^4 \delta^{(4)}(k + p - k' - p')$$
$$\left\{ \frac{-i}{(k'-k)^2 - m_\phi^2 + i\epsilon} \left[\overline{u}^B_{s_{k'}}(k') u^B_{s_k}(k)\right] \left[\overline{u}^A_{s_{p'}}(p') u^A_{s_p}(p)\right] \right\} . \tag{3.155}$$

In the non-relativistic limit the spinors in the Dirac representation are particularly simple,

$$u_s(p) \approx \sqrt{2m} \begin{pmatrix} I \\ 0 \end{pmatrix} \xi_s \tag{3.156}$$

$$\overline{u}_s(p) = u_s^\dagger(p)\gamma^0 \approx \sqrt{2m}\xi_s^\dagger \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \sqrt{2m}\xi_s^\dagger \begin{pmatrix} I & 0 \end{pmatrix} .$$

In addition, if the A and B particles have the same mass, $m_A = m_B = m$, then

$$(k' - k)^2 \approx -|\mathbf{k}' - \mathbf{k}|^2 . \tag{3.157}$$

In total,

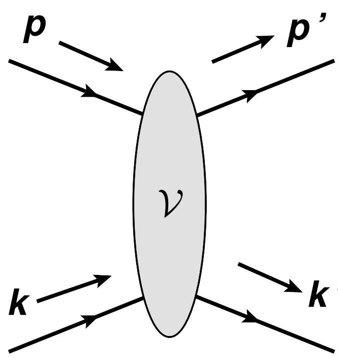
$$\langle \mathbf{p'_A}; \mathbf{k'_B}|S|\mathbf{p_A}; \mathbf{k_B}\rangle \approx (2\pi)^4 \delta^{(4)}(k + p - k' - p')$$
$$\left[\frac{ig^2}{|\mathbf{k}' - \mathbf{k}|^2 + m_\phi^2} (2m)\delta_{s_k, s_{k'}} (2m)\delta_{s_p, s_{p'}} \right] . \tag{3.158}$$

From this we see that at the extreme non-relativistic limit the spin states of the fermions do not change.

When modeling the interaction with a time-independent potential $\mathcal{V}(\mathbf{x})$, the strength of the interaction depends only of the mutual spatial distance between the particles. The interaction term is thus of the general form,

$$H_{\text{int,V}} = \int d^3x d^3y \, \left[\overline{\psi}_A(t,\mathbf{x})\psi_A(t,\mathbf{x})\right] \left[\overline{\psi}_B(t,\mathbf{y})\psi_B(t,\mathbf{y})\right] \mathcal{V}(\mathbf{x}-\mathbf{y}) \tag{3.159}$$

If $\mathcal{V}(|\mathbf{x}-\mathbf{y}|) \propto \delta^{(3)}(\mathbf{x}-\mathbf{y})$ this would be a local interaction term for 4 fermions – in other cases the range of the interaction is broader. We will draw this as,



In the lowest order the scattering amplitude is then

$$\langle \mathbf{p'_A}; \mathbf{k'_B}|iT|\mathbf{p_A}; \mathbf{k_B}\rangle = -i \int d^4x d^4y \, \delta\left(x^0 - y^0\right) \tag{3.160}$$

$$\langle \mathbf{p'_A}; \mathbf{k'_B}|T\left\{ \left[\overline{\psi}_A(x)\psi_A(x)\right] \left[\overline{\psi}_B(y)\psi_B(y)\right] \mathcal{V}(\mathbf{x}-\mathbf{y})\right\} |\mathbf{p_A}; \mathbf{k_B}\rangle \,.$$

Only one contraction is possible:

$$\langle \mathbf{p'_A}; \mathbf{k'_B}| \left[\overline{\psi}_A(x)\psi_A(x)\right] \left[\overline{\psi}_B(y)\psi_B(y)\right] |\mathbf{p_A}; \mathbf{k_B}\rangle$$

$$= \left[\overline{u}_{s_{k'}}(k')u_{s_k}(k)\right] \left[\overline{u}_{s_{p'}}(p')u_{s_p}(p)\right] e^{iy\cdot k'} e^{ix\cdot p'} e^{-iy\cdot k} e^{-ix\cdot p} \,. \tag{3.161}$$

Writing the δ function as an integral representation,

$$\delta\left(x^0 - y^0\right) = \int \frac{dq^0}{2\pi} e^{-iq^0(x^0-y^0)} \,, \tag{3.162}$$

and expressing the potential $\mathcal{V}$ as its Fourer transform,

$$\mathcal{V}(\mathbf{x}-\mathbf{y}) = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}\mathcal{V}(\mathbf{q})\,, \tag{3.163}$$

we get the scattering amplitude into a form

$$\langle \mathbf{p}'_{\mathbf{A}}; \mathbf{k}'_{\mathbf{B}}|iT|\mathbf{p}_{\mathbf{A}}; \mathbf{k}_{\mathbf{B}}\rangle = -i\int \frac{d^4q}{(2\pi)^4} d^4x d^4y\, \mathcal{V}(\mathbf{q}) \tag{3.164}$$

$$\times \left[\overline{u}_{s_{k'}}(k')u_{s_k}(k)\right] \left[\overline{u}_{s_{p'}}(p')u_{s_p}(p)\right] e^{iy\cdot(k'-k+q)}e^{ix\cdot(p'-p-q)}$$

$$= \left[\overline{u}_{s_{k'}}(k')u_{s_k}(k)\right] \left[\overline{u}_{s_{p'}}(p')u_{s_p}(p)\right] \left[-i\mathcal{V}(\mathbf{k}-\mathbf{k}')\right] (2\pi)^4\delta^{(4)}\left(k+p-k'-p'\right)$$

$$\approx (2m)\delta_{s_k,s_{k'}}(2m)\delta_{s_p,s_{p'}}\left[-i\mathcal{V}(\mathbf{k}-\mathbf{k}')\right](2\pi)^4\delta^{(4)}\left(k+p-k'-p'\right)$$

Comparing this to Eq. (3.158), we find a correspondence

$$\mathcal{V}(\mathbf{q}) = \frac{-g^2}{|\mathbf{q}|^2 + m_\phi^2}\,. \tag{3.165}$$

We can get back to the position space by making an inverse Fourier transformation (Ex.),

$$\mathcal{V}(\mathbf{r}) = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}}\mathcal{V}(\mathbf{q}) = \cdots = \frac{-g^2}{4\pi}\frac{e^{-m_\phi|\mathbf{r}|}}{|\mathbf{r}|}\,. \tag{3.166}$$

From the minus sign we can conclude that the obtained potential is **confining/attractive**. In fact, the Yukawa potential is attractive for all, fermion-fermion, fermion-antifermion and antifermion-antifermion interactions. Because of the exponential factor, the range of the Yukawa interaction can be very short and at some point it was a candidate for the theory of the strong interaction.

4 Quantization of Electrodynamics [Peskin 4.8]

The Quantum Electrodynamics – QED – quantizes the electromagnetic field. We begin the discussion here by briefly recalling the Maxwell's equations.

4.1 Maxwell's equations in covariant form

The classical electromagnetic field obeys the **Maxwell equations**:

$$\nabla \cdot \mathbf{E} = \rho \tag{4.1}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{4.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4.3}$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}\,, \tag{4.4}$$

where ρ is the charge density and $\mathbf{j}$ the current density. The electric and magnetic fields ($\mathbf{E}$ and $\mathbf{B}$) have both three components, so 6 components in total. However, Eqs. (4.2) and (4.3) both set one condition on the fields so the the number of independent components is reduced to 4. Thus the $\mathbf{E}$ and $\mathbf{B}$ fields have in total 4 independent degrees of freedom. We can thus pack all the information into a 4-potential $A^\mu \equiv (A^0, \mathbf{A})$ from which the electric and magnetic fields are obtained as,

$$\mathbf{E} = -\nabla A^0 - \frac{\partial \mathbf{A}}{\partial t} \tag{4.5}$$

$$\mathbf{B} = \nabla \times \mathbf{A}\,. \tag{4.6}$$

If we still define a 4-current $j^\mu \equiv (\rho, \mathbf{j})$, the Maxwell equations can be written as

$$\partial_\nu \partial^\nu A^\mu - \partial^\mu (\partial_\nu A^\nu) = j^\mu\,. \tag{4.7}$$

Still more compact form is reached by defining the antisymmetric **field-strength tensor**

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} . \tag{4.8}$$

Using this object we can express the Maxwell's equations in a very short form,

$$\partial_\mu F^{\mu\nu} = j^\nu \,. \tag{4.9}$$

We can also derive the Maxwell's equations as Euler-Lagrange equations starting from the Lagrange density,

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \,. \tag{4.10}$$

Demanding that the Lagrangian transforms as a scalar function or – equivalently – that the Maxwell's equations are covariant under Lorentz transformation, one can show that the A^μ and j^μ must transform as four vectors. That is, we can raise and lower the indices of A^μ, j^μ and $F^{\mu\nu}$ with the metric tensor $g^{\mu\nu}$.

4.1.1 Gauge freedom

A given 4-potential A^μ uniquely sets the values of electric and magnetic fields by mappings (4.5) and (4.6). However, these mapping are not injective so more than one 4-potential A^μ corresponds to the same $\mathbf{E}$ and $\mathbf{B}$ fields. Indeed, it is easy to show that the $\mathbf{E}$ and $\mathbf{B}$ fields, as well as the Maxwell's equations remain unchanged under a **gauge transformation**,

$$A^\mu(x) \longrightarrow A'^\mu(x) = A^\mu(x) + \partial^\mu \chi(x) \,, \tag{4.11}$$

where $\chi(x)$ is an arbitrary function of x. The exact way we choose $\chi(x)$ **sets the gauge**. For example the choice

$$\chi(x) = \int dx'^3 \theta(x^3 - x'^3) A^3(x^0, x^1, x^2, x'^3) \,, \tag{4.12}$$

is equivalent with a **gauge condition**

$$A'^3(x) = 0 \,, \tag{4.13}$$

which is one of the so-called **axial gauges**. Since a particular direction is preferred, this is not a Lorentz-invariant gauge condition. A Lorentz-invariant gauge condition is

$$\partial_\mu A^\mu = 0\,, \tag{4.14}$$

which sets the **Lorenz gauge** (no letter "t" here). In practical calculations we always have to choose the gauge. If no gauge is chosen, difficulties (=infinities) are met which stem from counting repeatedly physically equivalent configurations of the A^μ field. In principle, all gauge choices lead to the same result for physical observables.

Let's now solve the Maxwell's equations (4.7) in the case of free field, $j^\mu = 0$. We write (a Fourier transform),

$$A^\mu(x) = \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot x}\left[\epsilon^\mu_q + \epsilon^{*\mu}_{-q}\right]\,, \quad A^{*\mu}(x) = A^\mu(x) \tag{4.15}$$

where ϵ^μ_q is called a **polarization vector**. This leads to a condition,

$$-q^2\,\epsilon^\mu_q + q^\mu(q\cdot\epsilon_q) = 0\,, \tag{4.16}$$

so $q^2 = 0$ and $q\cdot\epsilon = 0$. Since $q^2 = 0$, the quantum of the Maxwell's field – photon – is massless. If we now choose the **Coulomb gauge**, $\nabla\cdot\mathbf{A} = 0$, we get an extra condition,

$$\mathbf{q}\cdot\boldsymbol{\epsilon}_q = 0. \tag{4.17}$$

From Eq. (4.17) we see that the **polarization vector is transverse to the photon momentum**. There are thus 2 independent polarization vectors, $\epsilon_{\mathbf{q},1}$, $\epsilon_{\mathbf{q},2}$. By convention, they have been normalized such that $\epsilon_{\mathbf{q},\lambda}\cdot\epsilon^*_{\mathbf{q},\lambda} = -1$. If the photon travels into the z direction, we can choose e.g.

$$\epsilon_1 = (0,1,0,0)\,,\ \epsilon_2 = (0,0,1,0)\,. \tag{4.18}$$

they correspond to the **linear polarization**. From the viewpoint of spin properties a useful choice is

$$\epsilon_{\rm R} = \frac{1}{\sqrt{2}}(0,1,i,0)\,,\ \epsilon_{\rm L} = \frac{1}{\sqrt{2}}(0,1,-i,0)\,. \tag{4.19}$$

which correspond to right- and left-handed **circular polarization**.

Often it is not necessary to use explicit polarization vectors in calculations but the polarization sum can be used instead (Ex.),

$$\sum_{\lambda=1,2} \epsilon^{\mu}_{\mathbf{k},\lambda}\epsilon^{*\nu}_{\mathbf{k},\lambda} = -g^{\mu\nu} + \frac{k^\mu \overline{k}^\nu + k^\nu \overline{k}^\mu}{k \cdot \overline{k}} \tag{4.20}$$

$$\overline{k} = (k^0, -\mathbf{k})\,.$$

4.2 Quantization of free photon field in Coulomb gauge

Let us stick to the Coulomb gauge and try to quantize the free electromagnetic field. The Lagrange density which leads to the Maxwell's equations (4.9) is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) = \frac{1}{2}\left(\dot{\mathbf{A}}^2 - \mathbf{B}^2\right), \tag{4.21}$$

where in the last equality we used the fact that $A^0 = 0$ in the Coulomb gauge, $E^i = -\dot{A}^i - \partial^i A^0 = -\dot{A}^i$. The conjugated momentum densities corresponding to A^μ are

$$\pi^0(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0 \tag{4.22}$$

$$\pi^i(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \dot{A}^i\,,$$

so we get the Hamiltonian function,

$$H = \int d^3x \left(\pi^i \dot{A}^i - \mathcal{L}\right) = \int d^3x \,\frac{1}{2}\left(\dot{\mathbf{A}}^2 + \mathbf{B}^2\right)\,. \tag{4.23}$$

Based on our earlier experience, we write the quantized free fields directly as a linear combination of the plane-wave solutions:

$$A^\mu(x) = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_\mathbf{p}}}\sum_{\lambda=1}^{2}\left[a_{\mathbf{p},\lambda}\epsilon^\mu_{\mathbf{p},\lambda}e^{-ip\cdot x} + a^\dagger_{\mathbf{p},\lambda}\epsilon^{\mu*}_{\mathbf{p},\lambda}e^{ip\cdot x}\right] \tag{4.24}$$

$$\dot{A}^\mu(x) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{E_\mathbf{p}}{2}}\sum_{\lambda=1}^{2}\left[a_{\mathbf{p},\lambda}\epsilon^\mu_{\mathbf{p},\lambda}e^{-ip\cdot x} - a^\dagger_{\mathbf{p},\lambda}\epsilon^{\mu*}_{\mathbf{p},\lambda}e^{ip\cdot x}\right], \tag{4.25}$$

where $A^0 = \dot{A}^0(x) = 0$. Note that, by construction, the A^μ field is Hermitean as it should be. We postulate the commutation relations for the creation and annihilations operators,

$$[a_{\mathbf{p},i}, a_{\mathbf{k},j}] = [a^\dagger_{\mathbf{p},i}, a^\dagger_{\mathbf{k},j}] = 0\,, \tag{4.26}$$

$$[a_{\mathbf{p},i}, a^\dagger_{\mathbf{k},j}] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k})\delta_{ij}\,, \tag{4.27}$$

as in the Klein-Gordon case. Using these commutation relations it is straightforward to compute the field commutators (Ex.),

$$[A^\mu(t,\mathbf{x}), A^\nu(t,\mathbf{y})] = \left[\dot{A}^\mu(t,\mathbf{x}), \dot{A}^\nu(t,\mathbf{y})\right] = 0 \tag{4.28}$$

$$\left[A^\mu(t,\mathbf{x}), \dot{A}^\nu(t,\mathbf{y})\right] = i\int \frac{d^3p}{(2\pi)^3}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\left(-g^{\mu\nu} + \frac{p^\mu\overline{p}^\nu + p^\nu\overline{p}^\mu}{p\cdot\overline{p}}\right).$$

We see that the commutation relation is now a bit different than in the case of scalar fields. Without the tensor structure the result would be just $i\delta^{(3)}(\mathbf{x}-\mathbf{y})$. However, to fullfill the Coulomb condition $\nabla\cdot\mathbf{A} = 0$ and that the right-hand side is zero if either $\mu = 0$ or $\nu = 0$, the tensor structure is necessary. Note also that $[A^i(t,\mathbf{x}), \dot{A}^j(t,\mathbf{y})] \neq 0$.

With the expansions (4.24) and (4.25) in Eq. (4.23) we get the Hamiltonian operator (Ex.),

$$H = \sum_{\lambda=1,2}\int \frac{d^3p}{(2\pi)^3}E_\mathbf{p}\, a^\dagger_{\mathbf{p},\lambda}a_{\mathbf{p},\lambda}\,. \tag{4.29}$$

This is of the same form as in the real Klein-Gordon field. As earlier with scalar fields, we can deduce that the theory again has a vacuum $|0\rangle$, which $a_{\mathbf{p},\lambda}$ annihilates, $a_{\mathbf{p},\lambda}|0\rangle = 0$. The other momentum eigenstates can be obtained by operating on the vacuum with $a^\dagger_{\mathbf{p},\lambda}$.

The Feynman propagator of the photon field is defined as a time-ordered product (Ex.):

$$
\begin{aligned}
D^{\text{Coulomb}}_{\text{F},\mu\nu}(x-y) &\equiv \langle 0|T\left[A_\mu(x)A_\nu(y)\right]|0\rangle \qquad (4.30)\\
&\equiv \theta(x^0-y^0)\langle 0|A_\mu(x)A_\nu(y)|0\rangle + \theta(y^0-x^0)\langle 0|A_\nu(y)A_\mu(x)|0\rangle\\
&= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2+i\epsilon}e^{-ip\cdot(x-y)}\\
&\quad \left[-g_{\mu\nu} + \frac{p_\mu\overline{p}_\nu + p_\nu\overline{p}_\mu}{p\cdot\overline{p}-p^2} - \frac{(p_\mu+\overline{p}_\mu)(p_\nu+\overline{p}_\nu)}{p\cdot\overline{p}-p^2}\,\frac{p^2}{p\cdot\overline{p}+p^2}\right].
\end{aligned}
$$

The polarization part look a bit more complicated as in Eq. (4.20). The difference is superficial and stems from the fact that in (4.20) the photons are on mass shell, $p^2 = 0$, but in the integral above $p^2 \neq 0$. Component by component the polarization part is, however, exactly the same as in Eq. (4.20) it has just been written in a different way. Speaking of which, an alternative way to write the polarization part is,

$$
-g_{\mu\nu} - \frac{p_\mu p_\nu}{(p\cdot n)^2-p^2} + \frac{(p\cdot n)(p_\mu n_\nu + p_\nu n_\mu)}{(p\cdot n)^2-p^2} - \frac{p^2 n_\mu n_\nu}{(p\cdot n)^2-p^2}\,, \qquad (4.31)
$$

where $n = (1,0,0,0)$. We will meet this form later on.

All in all, we get quite sensible quantization for the electromagnetic field in the Coulomb gauge. The only difficulty is that the Coulomb gauge is not explicitly Lorentz invariant, but certain directions are special ($A^0 = 0$). This will cause some trouble when we next couple the photon field with fermions.

4.3 QED in the Coulomb gauge

We define the QED by the Lagrange density,

$$\mathcal{L}_{\rm QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi - e\overline{\psi}\gamma^\mu\psi A_\mu\,. \tag{4.32}$$

Here we have first the free photon and Dirac field Lagrangians, and then the interaction term with coupling constant e. The last two terms are often written together with the aid of **covariant derivative**,

$$\overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi - e\overline{\psi}\gamma^\mu\psi A_\mu = \overline{\psi}\left(iD\!\!\!\!/ - m\right)\psi \tag{4.33}$$

$$D_\mu = \partial_\mu + ieA_\mu\,. \tag{4.34}$$

For $\mathcal{L}_{\rm QED}$ to be gauge invariant (similarly as the free-photon Lagrangian is), also the fermion field must change in under a gauge transformation. If we extend the gauge transformation to the fermion fields in the form,

$$A^\mu(x) \longrightarrow A'^\mu(x) = A^\mu(x) + \partial^\mu\chi(x) \tag{4.35}$$

$$\psi(x) \longrightarrow \psi'(x) = e^{-ie\chi(x)}\psi(x) \tag{4.36}$$

$$\overline{\psi}(x) \longrightarrow \overline{\psi}'(x) = e^{+ie\chi(x)}\overline{\psi}(x)\,, \tag{4.37}$$

we see that $D\!\!\!\!/\psi$ transforms as ψ:

$$D_\mu\psi(x) = \left(\partial_\mu + ieA_\mu\right)\psi(x) \tag{4.38}$$

$$\longrightarrow \left[\partial_\mu + ieA_\mu + ie\partial^\mu\chi(x)\right]e^{-ie\chi(x)}\psi(x)$$

$$= e^{-ie\chi(x)}\left[\partial_\mu + ieA_\mu + ie\partial^\mu\chi(x)\right]\psi(x) - iee^{-ie\chi(x)}\partial^\mu\chi(x)\psi(x)$$

$$= e^{-ie\chi(x)}\left[\partial_\mu + ieA_\mu\right]\psi(x) = e^{-ie\chi(x)}D_\mu\psi(x)\,.$$

Thus the gauge transformation as defined above is a symmetry of the Lagrangian density (4.32).

Comparing the Lagrangian (4.32) to the more general form (4.10), we have the QED equations of motion

$$\partial_\mu F^{\mu\nu} = j^\nu\,, \qquad j^\mu = +e\overline{\psi}\gamma^\mu\psi\,. \tag{4.39}$$

from the Maxwell's equation we know that j^0 represents the charge density, so the coupling constant e clearly represents the electric charge of the fermion.

As in the free field case, the conjugated momentum density for the zeroth component of the A field is zero, $\partial\mathcal{L}_{\text{QED}}/\partial\dot{A}_0 = 0$. Thus it's not a dynamical component and can be solved form the equation of motion, $\partial_\mu F^{\mu 0} = j^0$:

$$\begin{aligned}\partial_\mu F^{\mu 0} &= \partial_\mu\left(\partial^\mu A^0 - \partial^0 A^\mu\right) && (4.40)\\ &= \partial_0\partial^0 A^0 + \partial_i\partial^i A^0 - \partial^0\partial_0 A^0 - \partial^0\partial_i A^i \\ &= \partial_i\partial^i A^0 = -\nabla^2 A^0 \\ \Longrightarrow -\nabla^2 A^0 &= e\overline{\psi}\gamma^0\psi = e\psi^\dagger\psi\,, && (4.41)\end{aligned}$$

where we used the Coulomb-gauge condition $\nabla\cdot\mathbf{A} = 0$. Using the δ function identity

$$\nabla_x^2\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right) = -4\pi\delta^{(3)}\left(\mathbf{x}-\mathbf{y}\right)\,, \tag{4.42}$$

we have the following solution for A^0,

$$A^0(x) = e\int d^3x'\frac{\psi^\dagger(x^0,\mathbf{x}')\psi(x^0,\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|}. \tag{4.43}$$

Let's now massage a little bit the free-field part (4.21) of the QED Lagrangian,

$$\int d^3x\frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) = \int d^3x\frac{1}{2}\left[\left(\nabla A^0 + \frac{\partial\mathbf{A}}{\partial t}\right)\cdot\left(\nabla A^0 + \frac{\partial\mathbf{A}}{\partial t}\right) - \mathbf{B}^2\right]$$

$$= \int d^3x\frac{1}{2}\left[\nabla A^0\cdot\nabla A^0 + \dot{\mathbf{A}}\cdot\dot{\mathbf{A}} + 2\nabla A^0\cdot\dot{\mathbf{A}} - \mathbf{B}^2\right]\,.$$

The term $2\nabla A^0 \cdot \dot{\mathbf{A}}$ vanishes in the Coulomb gauge by partial integration. Also the first term can be integrated by parts:

$$\int d^3x \partial_i A^0 \partial_i A^0 = \int d^3x \left[\partial_i \left(A^0 \partial_i A^0\right) - A^0 \partial_i \partial_i A^0\right] \tag{4.44}$$
$$= \int d^3x \left[-A^0 \nabla^2 A^0\right] .$$

Substituting here our solution for the A^0 component we eventually get,

$$\int d^3x \frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) \tag{4.45}$$
$$= \int d^3x \frac{1}{2}\left[\dot{\mathbf{A}} \cdot \dot{\mathbf{A}} - \mathbf{B}^2 + e^2 \int d^3x' \frac{\psi^\dagger(x)\psi(x)\psi^\dagger(x^0, \mathbf{x}')\psi(x^0, \mathbf{x}')}{4\pi|\mathbf{x} - \mathbf{x}'|}\right] .$$

The full QED Lagrange function takes now the form,

$$L_{\mathrm{QED}} = \int d^3x \left[\frac{1}{2}\left(\dot{\mathbf{A}} \cdot \dot{\mathbf{A}} - \mathbf{B}^2\right) + \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi - e\overline{\psi}\gamma^i\psi A_i \right. \tag{4.46}$$
$$\left. - \frac{e^2}{2} \int d^3x' \frac{\psi^\dagger(x)\psi(x)\psi^\dagger(x^0, \mathbf{x}')\psi(x^0, \mathbf{x}')}{4\pi|\mathbf{x} - \mathbf{x}'|}\right] .$$

As a result of this drudgery, the first part $(\dot{\mathbf{A}} \cdot \dot{\mathbf{A}} - \mathbf{B}^2)/2$ is exactly the same as in the case of free photon field (4.21), but the interaction terms look awkward. The last term corresponds to a **non-local interaction** since two different space-time points appear in the same term of the Lagrangian. The original Lagrangian (4.32) was completely local so the non-local term is just an illusion and originates from the gauge choice.

The conjugated momentum densities for the fields A^μ and ψ are now,

$$\pi^0(x) = \frac{\partial \mathcal{L}_{\mathrm{QED}}}{\partial \dot{A}_0} = 0 \tag{4.47}$$
$$\pi^i(x) = \frac{\partial \mathcal{L}_{\mathrm{QED}}}{\partial \dot{A}_i} = \dot{A}^i$$
$$\pi^\psi(x) = \frac{\partial \mathcal{L}_{\mathrm{QED}}}{\partial \dot{\psi}} = i\psi^\dagger ,$$

which are clearly the same as in the free-field case, see (4.22) and (2.68). The Hamiltonian function is therefore,

$$H = \int d^3x \left[\pi^i \dot{A}^i + \pi^\psi \dot{\psi} - \mathcal{L}_{\rm QED}\right] \tag{4.48}$$

$$= \int d^3x \left[\dot{\mathbf{A}}^2 + i\psi^\dagger\partial_0\psi - \frac{1}{2}\left(\dot{\mathbf{A}}\cdot\dot{\mathbf{A}} - \mathbf{B}^2\right) - \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi\right.$$
$$\left. + e\overline{\psi}\gamma^i\psi A_i + \frac{e^2}{2}\int d^3x' \frac{\psi^\dagger(x)\psi(x)\psi^\dagger(x^0,\mathbf{x}')\psi(x^0,\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|}\right]$$

$$= \int d^3x \left[\frac{1}{2}\left(\dot{\mathbf{A}}\cdot\dot{\mathbf{A}} + \mathbf{B}^2\right) - \overline{\psi}\left(i\gamma^i\partial_i - m\right)\psi\right.$$
$$\left. + e\overline{\psi}\gamma^i\psi A_i + \frac{e^2}{2}\int d^3x' \frac{\psi^\dagger(x)\psi(x)\psi^\dagger(x^0,\mathbf{x}')\psi(x^0,\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|}\right] .$$

Comparing to formulae (4.23) and (2.70), the Hamiltonian splits into following pieces,

$$H = H_{\rm Maxwell} + H_{\rm Dirac} + H_{\rm int} \tag{4.49}$$

$$H_{\rm int} = \int d^3x \left[e\overline{\psi}\gamma^i\psi A_i + \frac{e^2}{2}\int d^3x' \frac{\psi^\dagger(x)\psi(x)\psi^\dagger(x^0,\mathbf{x}')\psi(x^0,\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|}\right] .$$

The Feynman rules can be obtained by considering some specific scattering process. Let's concentrate on the case (which we already considered in the Yukawa case),

$$f(p,s_p) + f(k,s_k) \longrightarrow f(p',s_{p'}) + f(k',s_{k'})\,.$$

Using $H_{\rm int}$ above, we can expand the S matrix up to order e^2:

$$\langle(\mathbf{p}', s_{p'}); (\mathbf{k}', s_{k'})|S|(\mathbf{p}, s_p); (\mathbf{k}, s_k)\rangle \tag{4.50}$$

$$= \langle(\mathbf{p}', s_{p'}); (\mathbf{k}', s_{k'})|T\exp\left[-i\int dt H_{\rm I}\right]|(\mathbf{p}, s_p); (\mathbf{k}, s_k)\rangle$$

$$= \langle(\mathbf{p}', s_{p'}); (\mathbf{k}', s_{k'})|T\left[\frac{1}{1!}\frac{-ie^2}{2}\int d^4x d^3x' \frac{\psi^\dagger(x)\psi(x)\psi^\dagger(x^0,\mathbf{x}')\psi(x^0,\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|}\right.$$

$$\left. + \frac{1}{2!}(-ie)^2\int d^4x d^4y \left(\overline{\psi}_x\gamma^i\psi_x A_i(x)\right)\left(\overline{\psi}_y\gamma^i\psi_y A_i(y)\right)\right]|(\mathbf{p}, s_p); (\mathbf{k}, s_k)\rangle$$

$$+ \mathcal{O}(e^3)\,.$$

Let's now work on the first term. First, trivially,

$$\int d^3x' \frac{\psi^\dagger(x^0,\mathbf{x}')\psi(x^0,\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|} = \int d^4x' \frac{\psi^\dagger(x')\psi(x')}{4\pi|\mathbf{x}-\mathbf{x}'|}\delta\left(x^0 - x'^0\right)\,. \tag{4.51}$$

Using the Fourier transformation (in distribution sense),

$$\frac{1}{4\pi|\mathbf{x}|} = \int \frac{d^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{|\mathbf{p}|^2}\,. \tag{4.52}$$

we end up with,

$$\int d^3x' \frac{\psi^\dagger(x^0,\mathbf{x}')\psi(x^0,\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|} = \int d^4x' \int \frac{d^4q}{(2\pi)^4}\frac{e^{-iq\cdot(x-x')}}{|\mathbf{q}|^2}\psi^\dagger(x')\psi(x')\,.$$

Using now the vector n defined earlier, $n = (1, 0, 0, 0)$, the first term of the S matrix can be written as,

$$K_1 = T\left[\frac{1}{1!}\frac{-ie^2}{2}\int d^4x d^3x' \frac{\psi^\dagger(x)\psi(x)\psi^\dagger(x^0,\mathbf{x}')\psi(x^0,\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|}\right] \tag{4.53}$$

$$= T\left\{\frac{(-ie)^2}{2!}\int d^4x d^4x' \left[\overline{\psi}(x)\gamma^0\psi(x)\right]\,\left[\overline{\psi}(x')\gamma^0\psi(x')\right]\right\}$$

$$\times \int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2+i\epsilon}e^{-iq\cdot(x-x')}\left[\frac{q^2 n_0 n_0}{(q\cdot n)^2 - q^2}\right]$$

$$= T\left\{\frac{(-ie)^2}{2!}\int d^4x d^4x' \left[\overline{\psi}(x)\gamma^\mu\psi(x)\right]\,\left[\overline{\psi}(x')\gamma^\nu\psi(x')\right]\right\}$$

$$\times \int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2+i\epsilon}e^{-iq\cdot(x-x')}\left[\frac{q^2 n_\mu n_\nu}{(q\cdot n)^2 - q^2}\right]\,,$$

in the last step we simply added zero since all but the $\mu = \nu = 0$ case vanish. Let's then open the second term in the scattering matrix:

$$
\begin{aligned}
K_2 =& T\left\{\frac{1}{2!}(-ie)^2 \int d^4x d^4y \left[\overline{\psi}_x \gamma^i \psi_x A_i(x)\right] \left[\overline{\psi}_y \gamma^j \psi_y A_j(y)\right]\right\} \qquad (4.54)\\
=& T\left\{\frac{1}{2!}(-ie)^2 \int d^4x d^4y \left[\overline{\psi}_x \gamma^i \psi_x\right] \left[\overline{\psi}_y \gamma^j \psi_y\right]\right\} \overbrace{A_i(x) A_j(y)}\\
=& T\left\{\frac{1}{2!}(-ie)^2 \int d^4x d^4y \left[\overline{\psi}_x \gamma^i \psi_x\right] \left[\overline{\psi}_y \gamma^i \psi_y\right]\right\} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{i}{q^2 + i\epsilon} e^{-iq\cdot(x-y)}\\
&\times \left[-g_{ij} - \frac{q_i q_j}{(q\cdot n)^2 - q^2} + \frac{(q\cdot n)(q_i n_j + q_j n_i)}{(q\cdot n)^2 - q^2} - \frac{q^2 n_i n_j}{(q\cdot n)^2 - q^2}\right],\\
=& T\left\{\frac{1}{2!}(-ie)^2 \int d^4x d^4y \left[\overline{\psi}_x \gamma^\mu \psi_x\right] \left[\overline{\psi}_y \gamma^\nu \psi_y\right]\right\} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{i}{q^2 + i\epsilon} e^{-iq\cdot(x-y)}\\
&\times \left[-g_{\mu\nu} - \frac{q_\mu q_\nu}{(q\cdot n)^2 - q^2} + \frac{(q\cdot n)(q_\mu n_\nu + q_\nu n_\mu)}{(q\cdot n)^2 - q^2} - \frac{q^2 n_\mu n_\nu}{(q\cdot n)^2 - q^2}\right].
\end{aligned}
$$

In the last step we again added zero since the polarization part vanishes if either $\mu = 0$ or $\nu = 0$. Astonishingly, both contributions are of the same form and we can thus sum them together,

$$
\begin{aligned}
K_1 + K_2 =& T\left\{\frac{1}{2!}(-ie)^2 \int d^4x d^4y \left[\overline{\psi}_x \gamma^\mu \psi_x\right] \left[\overline{\psi}_y \gamma^\nu \psi_y\right]\right\} \qquad (4.55)\\
\times& \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{i}{q^2 + i\epsilon} e^{-iq\cdot(x-y)} \left[-g_{\mu\nu} - \frac{q_\mu q_\nu}{(q\cdot n)^2 - q^2} + \frac{(q\cdot n)(q_\mu n_\nu + q_\nu n_\mu)}{(q\cdot n)^2 - q^2}\right].
\end{aligned}
$$

Effectively on the last part of the Coulomb propagator dropped away. Thus, effectively,

$$
D^{\text{Coulomb}}_{\mathrm{F},\mu\nu}(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip\cdot(x-y)} D^{\text{Coulomb}}_{\mathrm{F},\mu\nu}(p) \qquad (4.56)
$$

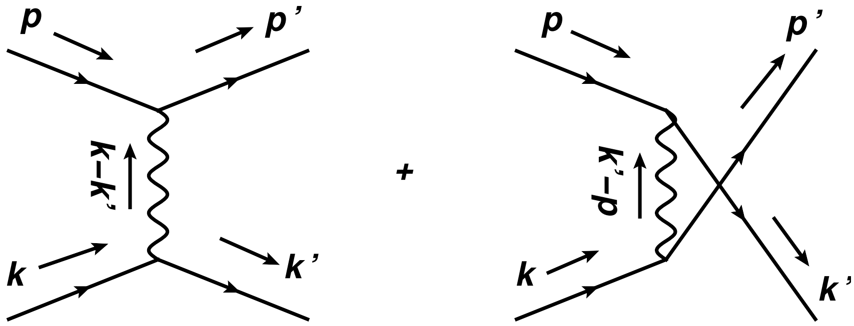
$$
D^{\text{Coulomb}}_{\mathrm{F},\mu\nu}(p) = \frac{i}{p^2 + i\epsilon}\left[-g_{\mu\nu} - \frac{p_\mu p_\nu}{(p\cdot n)^2 - p^2} + \frac{(p\cdot n)(p_\mu n_\nu + p_\nu n_\mu)}{(p\cdot n)^2 - p^2}\right]
$$

From here, the calculation goes almost identically as in the Yukawa case.

The final matrix elements reads

$$i\mathcal{M}_{ff\to ff} = e^2 \left\{ -D^{\text{Coulomb}}_{\mu\nu}(k'-k) \left[\overline{u}_{s_{k'}}(k')\gamma^\nu u_{s_k}(k)\right] \left[\overline{u}_{s_{p'}}(p')\gamma^\mu u_{s_p}(p)\right] \right.$$
$$\left. + D^{\text{Coulomb}}_{\mu\nu}(k'-p) \left[\overline{u}_{s_{k'}}(k')\gamma^\mu u_{s_p}(p)\right] \left[\overline{u}_{s_{p'}}(p')\gamma^\nu u_{s_k}(k)\right] \right\} ,$$

which correspond to the diagrams,



In the case of external photons,

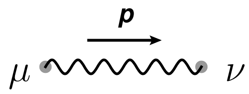
$$[A^\mu(x)]^+ \left|(\mathbf{k},\lambda_\mathbf{k})\right\rangle = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_\mathbf{p}}}\sum_{\lambda=1}^{2} a_{\mathbf{p},\lambda}\epsilon^\mu_{\mathbf{p},\lambda}e^{-ip\cdot x}\sqrt{2E_\mathbf{k}}a^\dagger_{\mathbf{k},\lambda_\mathbf{k}}|0\rangle$$
$$= e^{-ik\cdot x}\,\epsilon^\mu_{\mathbf{k},\lambda_\mathbf{k}}|0\rangle , \tag{4.57}$$

$$\left\langle(\mathbf{k},\lambda_\mathbf{k})\right| [A^\mu(x)]^- = \langle 0|a_{\mathbf{k},\lambda_\mathbf{k}}\sqrt{2E_\mathbf{k}} \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_\mathbf{p}}}\sum_{\lambda=1}^{2} a^\dagger_{\mathbf{p},\lambda}(\epsilon^\mu_{\mathbf{p},\lambda})^* e^{ip\cdot x}$$
$$= \langle 0|e^{ik\cdot x}\,(\epsilon^\mu_{\mathbf{k},\lambda_\mathbf{k}})^* , \tag{4.58}$$

so the extrenal photon legs give polarization vectors, $\epsilon^\mu_{\mathbf{p},\lambda}$ from the initial state and $\epsilon^{*\mu}_{\mathbf{p},\lambda}$ from the final state.

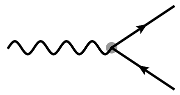
We can deduce the **Coulomb-gauge Feynman rules for QED:**

- Photon propagator



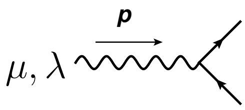
$$= \frac{i}{p^2+i\epsilon}\left[-g_{\mu\nu} - \frac{p_\mu p_\nu}{(p\cdot n)^2-p^2} + \frac{(p\cdot n)(p_\mu n_\nu + p_\nu n_\mu)}{(p\cdot n)^2-p^2}\right]$$

- Vertex



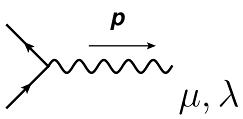
$$= -ie\gamma^\mu$$

- Initial-state photon



$$= \epsilon_{\mu,\lambda}$$

- Final-state photon



$$= \epsilon^*_{\mu,\lambda}$$

In the case of Klein-Gordon and Dirac fields we saw that the propagators are essentially Green's functions of the differential operator appearing in the free-field equation of motion. After all the shuffling we just did it may not be completely clear whether such a simple principle still works. But it does. To see the correspondence we must impose the boundary condition $\nabla\cdot\mathbf{A}=0$ by the method of **Lagrange multipliers**. In the case of Coulomb gauge we would define the Lagrangian as,

$$\mathcal{L}_{\text{Coulomb}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}\left(\nabla\cdot\mathbf{A}\right)^2 , \tag{4.59}$$

where the last term restricts (as one also demands $\partial\mathcal{L}/\partial(1/\alpha)=0$) the solutions of the Euler-Lagrange equations to the cases in which $\nabla\cdot\mathbf{A}=0$. The factor $1/2\alpha$ is formally a Lagrange multiplier. We first rewrite the above Lagrangian as

$$\mathcal{L}_{\text{Coulomb}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}\left[\left(\partial_\mu - n_\mu n\cdot\partial\right)A^\mu\right]^2 , \tag{4.60}$$

where we again used the vector $n=(1,0,0,0)$. From this we get the Euler-Lagrange equations of motion

$$\left\{\partial_\beta\partial^\beta g^{\alpha\mu} - \partial^\mu\partial^\alpha + \frac{1}{\alpha}\left[\partial^\alpha - n^\alpha(n\cdot\partial)\right]\left[\partial^\mu - n^\mu(n\cdot\partial)\right]\right\}A_\mu(x) = 0 . \tag{4.61}$$

The Green's function of the differential operator appearing in the equation above can be solved from,

$$\left\{\partial_\beta\partial^\beta g^{\alpha\mu} - \partial^\mu\partial^\alpha + \frac{1}{\alpha}\left[\partial^\alpha - n^\alpha(n\cdot\partial)\right]\left[\partial^\mu - n^\mu(n\cdot\partial)\right]\right\} D_{\mu\nu}(x-y)$$

$$= i\delta^\alpha_\nu \delta^{(4)}(x-y) \ . \tag{4.62}$$

We first write $D_{\mu\nu}(x-y)$ as a Fourier transformation,

$$D_{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} D_{\mu\nu}(p) \,, \tag{4.63}$$

so that we need to solve

$$\left\{-p^2 g^{\alpha\mu} + p^\mu p^\alpha - \frac{1}{\alpha}\left[p^\alpha - n^\alpha(n\cdot p)\right]\left[p^\mu - n^\mu(n\cdot p)\right]\right\} D_{\mu\nu}(p) = i\delta^\alpha_\nu \, .$$

The solution is

$$D_{\mu\nu}(p) = \frac{i}{p^2}\left[-g_{\mu\nu} - \frac{p_\mu p_\nu}{(p\cdot n)^2 - p^2} + \frac{(p\cdot n)\left(p_\mu n_\nu + p_\nu n_\mu\right)}{(p\cdot n)^2 - p^2}\right.$$

$$\left. -\alpha \frac{p^2 p_\mu p_\nu}{\left[(p\cdot n)^2 - p^2\right]^2}\right] \, . \tag{4.64}$$

In the limit $\alpha \to 0$, the expression agrees exactly (modulo the $+i\epsilon$ prescription) with the effective Coulomb-gauge propagator (4.56) we found earlier. In principle the factor α – **the gauge parameter** as we call it in this context – can be freely chosen. The scattering matrices will not depend on the value of α we choose.

4.4 Coulomb potential

Let us now return for a moment to the elastic fermion-fermion scattering in QED. The matrix elements for scattering of two distinguishable fermions of **the same charge** is

$$i\mathcal{M}_{f_A f_B \to f_A f_B} = \tag{4.65}$$

$$e^2 \left\{-D^{\text{Coulomb}}_{\mu\nu}(k'-k)\left[\overline{u}^A_{s_{p'}}(p')\gamma^\mu u^A_{s_p}(p)\right]\left[\overline{u}^B_{s_{k'}}(k')\gamma^\nu u^B_{s_k}(k)\right]\right\} \, .$$

The Green's function of the differential operator appearing in the equation above can be solved from,

$$\left\{\partial_\beta\partial^\beta g^{\alpha\mu} - \partial^\mu\partial^\alpha + \frac{1}{\alpha}\left[\partial^\alpha - n^\alpha(n\cdot\partial)\right]\left[\partial^\mu - n^\mu(n\cdot\partial)\right]\right\} D_{\mu\nu}(x-y)$$

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so that we need to solve

$$\left\{-p^2 g^{\alpha\mu} + p^\mu p^\alpha - \frac{1}{\alpha}\left[p^\alpha - n^\alpha(n\cdot p)\right]\left[p^\mu - n^\mu(n\cdot p)\right]\right\} D_{\mu\nu}(p) = i\delta^\alpha_\nu\,.$$

The solution is

$$D_{\mu\nu}(p) = \frac{i}{p^2}\left[-g_{\mu\nu} - \frac{p_\mu p_\nu}{(p\cdot n)^2 - p^2} + \frac{(p\cdot n)\,(p_\mu n_\nu + p_\nu n_\mu)}{(p\cdot n)^2 - p^2}\right.$$

$$\left. -\alpha \frac{p^2 p_\mu p_\nu}{\left[(p\cdot n)^2 - p^2\right]^2}\right]\ . \tag{4.64}$$

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$$i\mathcal{M}_{f_A f_B \to f_A f_B} = \tag{4.65}$$

$$e^2\left\{-D^{\rm Coulomb}_{\mu\nu}(k'-k)\left[\overline{u}^A_{s_{p'}}(p')\gamma^\mu u^A_{s_p}(p)\right]\left[\overline{u}^B_{s_{k'}}(k')\gamma^\nu u^B_{s_k}(k)\right]\right\}\ .$$

The strucure of the matrix element is now a bit more complicated than in the corresponding Yukawa calculation. However, we can easily check that the terms proportional to the factor

$$(k'_\mu - k_\mu) = (p_\mu - p'_\mu)\,,$$

in the propagator $D^{\rm Coulomb}_{\mu\nu}(k'-k)$ can be thrown away. Indeed,

$$\overline{u}^A_{s_{p'}}(p')\left[\not p' - \not p\right] u^A_{s_p}(p) = \overline{u}^A_{s_{p'}}(p')\left[m - m\right] u^A_{s_p}(p) = 0\,, \tag{4.66}$$

where we used the Dirac equation (2.11) in the momentum space,

$$\left(\not p - m\right) u^A_{s_p}(p) = 0\,, \tag{4.67}$$

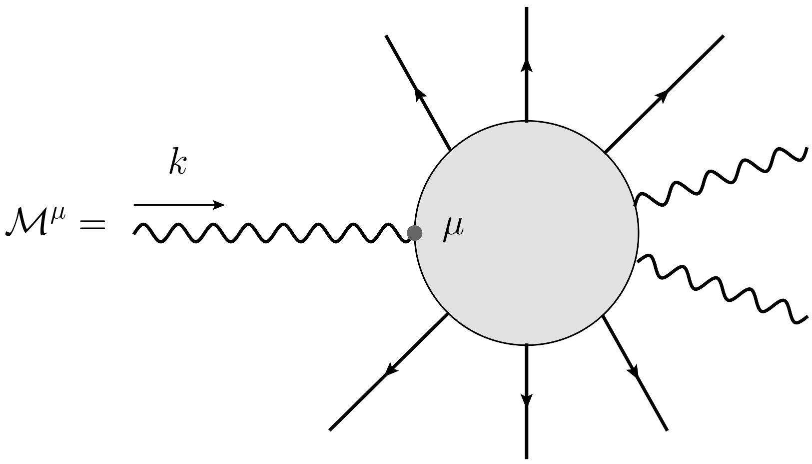
$$\overline{u}^A_{s_p}(p)\left(\not p - m\right) = 0\,. \tag{4.68}$$

Thus, only the first term in the propagator $D^{\rm Coulomb}_{\mu\nu}$ is relevant,

$$i\mathcal{M}_{f_A f_B \to f_A f_B} = \tag{4.69}$$

$$e^2\left\{\frac{+ig_{\mu\nu}}{(k'-k)^2 + i\epsilon}\left[\overline{u}^A_{s_{p'}}(p')\gamma^\mu u^A_{s_p}(p)\right]\left[\overline{u}^B_{s_{k'}}(k')\gamma^\nu u^B_{s_k}(k)\right]\right\}\,.$$

The fact that only the $g_{\mu\nu}$ part of the photon propagator gives a non-zero contribution to the scattering matrix is a special case of a more general property which is called the **Ward identity** (derived in QFTII): Denote by $\mathcal{M}^\mu$ a sum of (amputated & connected) diagrams with fixed number of on-shell external electrons and photons,



The gray blob here denotes a summation over all possible diagrams (the photon with momentum k cannot bend back to the diagram). The Ward identity states that,

$$k_\mu \mathcal{M}^\mu = 0\,. \tag{4.70}$$

In the non-relativistic limit,

$$u_s(p) \approx \sqrt{2m}\begin{pmatrix} I \\ 0 \end{pmatrix}\xi_s \qquad \overline{u}_s(p) = \sqrt{2m}\xi_s^\dagger \begin{pmatrix} I & 0 \end{pmatrix},$$

so

$$\overline{u}^A_{s_{p'}}(p')\gamma^0 u^A_{s_p}(p) \approx 2m\xi^\dagger_{s'_p}\begin{pmatrix} I & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} I \\ 0 \end{pmatrix}\xi_{s_p} = 2m\delta_{s_p,s'_p}$$

$$\overline{u}^A_{s_{p'}}(p')\gamma^i u^A_{s_p}(p) \approx 2m\xi^\dagger_{s'_p}\begin{pmatrix} I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} I \\ 0 \end{pmatrix}\xi_{s_p} = 0\,,$$

and only the case $\mu = \nu = 0$ contributes. Thus,

$$i\mathcal{M}_{f_A f_B \to f_A f_B} \approx \frac{-ie^2}{|\mathbf{k}' - \mathbf{k}|^2} 2m\delta_{s_p,s'_p} 2m\delta_{s_k,s'_k}\,. \tag{4.71}$$

Comparing this to our general result for potential scattering (3.164), we can identify

$$\mathcal{V}_{\text{Coulomb}}(\mathbf{q}) = \frac{+e^2}{|\mathbf{k}' - \mathbf{k}|^2}\,, \tag{4.72}$$

which corresponds to the position-space potential, see (3.166),

$$\mathcal{V}_{\text{Coulomb}}(\mathbf{r}) = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} \mathcal{V}_{\text{Coulomb}}(\mathbf{q}) = \frac{+e^2}{4\pi|\mathbf{r}|} = \frac{\alpha}{|\mathbf{r}|}\,, \tag{4.73}$$

where we identified **the fine-structure constant** $\alpha \equiv +e^2/4\pi \approx 1/137$. Since the potential comes with a $+$ sign, this is a **repelling potential**. This is the case for two fermions of a same sign. If we replace one of the fermion with an antifermion, the potential turns into an attractive one. The antifermion-antrifermion potential is again repelling. Thus, we have established a field-theoretical justification for the rather fundamental fact that **like-sign charges repel while unlike charges attract each other.**

4.5 Gupta-Bleuler quantization

The advantage of the Coulomb gauge is its physicality – the gauge fields are written through the two physical polarizations. In simple QED-calculations the Coulomb gauge causes no trouble. However, the fact that the Coulomb gauge is not Lorentz invariant causes difficulties in higher-order loop calculations. If we don't wish to break the Lorentz invariance at the level of quantization, we will need to do the so-called Gupta-Bleuler quantization. In this case we adopt the Lorenz gauge condition $\partial_\mu A^\mu = 0$. The free-field Lagrangian with this condition reads,

$$\mathcal{L}_{\text{Lorenz}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} \left(\partial^\mu A^\nu\right) \left(\partial_\mu A_\nu\right) . \tag{4.74}$$

The conjugate momentum densities are

$$\pi^\mu(x) = \frac{\partial \mathcal{L}_{\text{Lorenz}}}{\partial \dot{A}_\mu} = -\dot{A}^\mu ,$$

so now also the zeroth component is non zero. The Hamiltonian becomes,

$$\begin{aligned} H = \int d^3x \left(\pi^\mu \dot{A}_\mu - \mathcal{L}\right) &= \int d^3x \frac{1}{2} \left[-\dot{A}^\mu \dot{A}_\mu + \left(\partial^i A^\nu\right) \left(\partial_i A_\nu\right)\right] \\ &= \int d^3x \frac{1}{2} \left[-\dot{A}^\mu \dot{A}_\mu + A^\mu \ddot{A}_\mu\right] . \end{aligned} \tag{4.75}$$

In principle, the gauge condition $\partial_\mu A^\mu = 0$ reduces the number of independent field components from 4 to 3 but since we don't want to break the Lorentz invariance we still quantize all the 4 polarization degrees of freedom:

$$A^\mu(x) = \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{1}{2E_{\mathbf{p}}}} \sum_{\lambda=0}^{3} \left[a_{\mathbf{p},\lambda} \epsilon^\mu_{\mathbf{p},\lambda} e^{-ip\cdot x} + a^\dagger_{\mathbf{p},\lambda} \epsilon^{\mu *}_{\mathbf{p},\lambda} e^{ip\cdot x}\right] \tag{4.76}$$

$$\pi^\mu(x) = \int \frac{d^3p}{(2\pi)^3} (+i) \sqrt{\frac{E_{\mathbf{p}}}{2}} \sum_{\lambda=0}^{3} \left[a_{\mathbf{p},\lambda} \epsilon^\mu_{\mathbf{p},\lambda} e^{-ip\cdot x} - a^\dagger_{\mathbf{p},\lambda} \epsilon^{\mu *}_{\mathbf{p},\lambda} e^{ip\cdot x}\right] . \tag{4.77}$$

So the number of polarization vectors $\epsilon^\mu_{\mathbf{p},\lambda}$ is now four, $\epsilon^\mu_{\mathbf{p},0}, \epsilon^\mu_{\mathbf{p},1}, \epsilon^\mu_{\mathbf{p},2}, \epsilon^\mu_{\mathbf{p},3}$. We normalize these as,

$$\epsilon_{\mathbf{p},\lambda} \cdot \epsilon^*_{\mathbf{p},\lambda'} = g_{\lambda\lambda'} , \tag{4.78}$$

which leads to a completeness relation,

$$\sum_{\lambda=0}^{3} g^{\lambda\lambda} \epsilon^\mu_{\mathbf{p},\lambda} \epsilon^{*\nu}_{\mathbf{p},\lambda} = g^{\mu\nu} . \tag{4.79}$$

We can choose these polarization vectors as follows: If the momentum p is along the z axis, $p \propto (1,0,0,1)$, then

$$\epsilon_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \epsilon_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ \epsilon_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ \epsilon_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.80}$$

The following relations clearly hold,

$$\epsilon_{\mathbf{p},0} \cdot p = -\epsilon_{\mathbf{p},3} \cdot p \tag{4.81}$$

$$\epsilon_{\mathbf{p},1} \cdot p = \epsilon_{\mathbf{p},2} \cdot p = 0\,. \tag{4.82}$$

For all other momentum directions we can find the polarization vectors $\epsilon^{\mu}_{\mathbf{p},\lambda}$ from the above ones by making an appropriate Lorentz transformation.

How to postulate commutation relations that are Lorentz symmetric? It is straightforward to show that if we set the following commutation relations for the creation and annihilation operators,

$$\begin{aligned} [a_{\mathbf{p},\lambda}, a_{\mathbf{k},\lambda'}] &= [a^{\dagger}_{\mathbf{p},\lambda}, a^{\dagger}_{\mathbf{k},\lambda'}] = 0\,, \\ [a_{\mathbf{p},\lambda}, a^{\dagger}_{\mathbf{k},\lambda'}] &= -g_{\lambda\lambda'}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k})\,, \end{aligned} \tag{4.83}$$

the field operators obey,

$$[A^{\mu}(t,\mathbf{x}), A^{\nu}(t,\mathbf{y})] = [\pi^{\mu}(t,\mathbf{x}), \pi^{\nu}(t,\mathbf{y})] = 0 \tag{4.84}$$

$$[A^{\mu}(t,\mathbf{x}), \pi^{\nu}(t,\mathbf{y})] = i g^{\mu\nu}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,.$$

This is all Lorentz symmetric. If we now calculate the Hamiltonian operator corresponding to (4.75) we get the result,

$$H_{\text{Lorenz}} = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left[\sum_{\lambda=1,2,3} a^{\dagger}_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda} - a^{\dagger}_{\mathbf{p},0} a_{\mathbf{p},0} \right]. \tag{4.85}$$

This is not an acceptable Hamiltonian as the expectation value for the state $|(\mathbf{k},0)\rangle = \sqrt{2E_\mathbf{k}}a^\dagger_{\mathbf{k},0}|0\rangle$ is negative:

$$\begin{aligned}\langle(\mathbf{k},0)|H_{\rm Lorenz}|(\mathbf{k},0)\rangle &= -2E_\mathbf{k}\int\frac{d^3p}{(2\pi)^3}E_\mathbf{p}\,\langle 0|a_{\mathbf{k},0}a^\dagger_{\mathbf{p},0}a_{\mathbf{p},0}a^\dagger_{\mathbf{k},0}|0\rangle\\ &= 2E^2_\mathbf{k}\langle 0|a_{\mathbf{k},0}a^\dagger_{\mathbf{k},0}|0\rangle = E_\mathbf{k}\langle(\mathbf{k},0)|(\mathbf{k},0)\rangle\,,\end{aligned}$$

which is negative since the norm $\langle(\mathbf{k},0)|(\mathbf{k},0)\rangle < 0$:

$$|(\mathbf{p},0)\rangle|^2 \propto |a^\dagger_{\mathbf{p},0}|0\rangle|^2 = \lim_{\mathbf{q}\to\mathbf{p}}\langle 0|a_{\mathbf{q},0}a^\dagger_{\mathbf{p},0}|0\rangle = \lim_{\mathbf{q}\to\mathbf{p}} -(2\pi)^3\delta^{(3)}\left(\mathbf{p}-\mathbf{q}\right)\,.$$

Doesn't look good. The reason for all this is that we have quantized 4 polarization state whereas we know that only two are physical. We clearly need an extra condition which eliminates two degrees of freedom. In the Gupta-Bleuler formalism this is achieved by setting the Lorenz condition for the physical, acceptable states,

$$\partial^\mu A^+_\mu|\psi\rangle = 0 \implies \langle\psi|\partial^\mu A_\mu|\psi\rangle = 0 \tag{4.86}$$

Let's write this explicitly:

$$\begin{aligned}\partial^\mu A^+_\mu|\psi\rangle &= \partial_\mu\int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_\mathbf{p}}}\sum_{\lambda=0}^{3}a_{\mathbf{p},\lambda}\epsilon^\mu_{\mathbf{p},\lambda}e^{-ip\cdot x}|\psi\rangle\\ &= -i\int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_\mathbf{p}}}e^{-ip\cdot x} \qquad (4.87)\\ &\left[(p\cdot\epsilon_{\mathbf{p},0})a_{\mathbf{p},0} + (p\cdot\epsilon_{\mathbf{p},1})a_{\mathbf{p},1} + (p\cdot\epsilon_{\mathbf{p},2})a_{\mathbf{p},2} + (p\cdot\epsilon_{\mathbf{p},3})a_{\mathbf{p},3}\right]|\psi\rangle\,.\end{aligned}$$

Using (4.81) and (4.82),

$$\partial^\mu A^+_\mu|\psi\rangle = -i\int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_\mathbf{p}}}e^{-ip\cdot x}(p\cdot\epsilon_{\mathbf{p},0})\left[a_{\mathbf{p},0}-a_{\mathbf{p},3}\right]|\psi\rangle = 0\,,$$

so,

$$L_\mathbf{p}|\psi\rangle = \langle\psi|L^\dagger_\mathbf{p} = 0 \tag{4.88}$$

$$L_\mathbf{p} \equiv a_{\mathbf{p},0} - a_{\mathbf{p},3}\,. \tag{4.89}$$

What kind of states fulfill this condition? Clearly the vacuum $|0\rangle$ and states $|\psi_T\rangle$ which contain only transversally polarized photons are OK. Also the scalar and longitudinal polarization may occur if they appear in combinations created by $L^\dagger_{\mathbf{p}} \equiv a^\dagger_{\mathbf{p},0} - a^\dagger_{\mathbf{p},3}$. Since $[L^\dagger_{\mathbf{p}}, L_{\mathbf{k}}] = 0$, the states obtained do fulfill the Gupta-Bleuler condition. These states, however, do not affect physical observables. For example, the expectation value of the Hamiltonian depends only on the transverse photons:

$$\begin{aligned}
\langle\psi|H_{\text{Lorenz}}|\psi\rangle &= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \langle\psi| \left[\sum_{\lambda=1,2,3} a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda} - a^\dagger_{\mathbf{p},0} a_{\mathbf{p},0}\right] |\psi\rangle \\
&= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \langle\psi| \left[\sum_{\lambda=1,2,3} a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda} - a^\dagger_{\mathbf{p},3} a_{\mathbf{p},3}\right] |\psi\rangle \\
&= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \langle\psi| \sum_{\lambda=1,2} a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda} |\psi\rangle \,. \qquad (4.90)
\end{aligned}$$

All in all, the Gupta-Bleuler condition thus leads to a Hamiltonian which effectively counts only transverse photons and the norms of the states are non-negative.

Using the quantum fields (4.76), commutation relations (4.83), and the completeness (4.79), computing the propagator is a straightforward task. The result is,

$$\begin{aligned}
D^{\text{Lorentz}}_{\text{F},\mu\nu}(x-y) &\equiv \langle 0|T\left[A_\mu(x)A_\nu(y)\right]|0\rangle \\
&\equiv \theta(x^0-y^0)\langle 0|A_\mu(x)A_\nu(y)|0\rangle + \theta(y^0-x^0)\langle 0|A_\nu(y)A_\mu(x)|0\rangle \\
&= \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{i}{p^2+i\epsilon} e^{-ip\cdot(x-y)} \left[-g_{\mu\nu}\right] . \qquad (4.91)
\end{aligned}$$

The propagator is evidently Lorentz symmetric and a lot more simpler than in the Coulomb gauge. The derivation of the Feynman rules for QED goes essentially as in the Coulomb-gauge case but in the Lorenz gauge the procedure is a lot easier as the A^0 field is not treated separately (as was done in the Coulomb case). The Feynman rules differ only with respect to the propagator.

The Lorenz-gauge propagator can be obtained by using the Lagrange multipliers. In the Lorenz gauge we can write the Lagrangian as

$$\mathcal{L}_{\text{Lorentz}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \, . \tag{4.92}$$

from which we get the Euler-Lagrange equation

$$\left[\partial_\beta \partial^\beta g^{\alpha\mu} - \partial^\mu \partial^\alpha \left(1 - \frac{1}{\alpha} \right) \right] A_\mu(x) = 0 \, . \tag{4.93}$$

Following the steps made in the Coulomb case, we can solve the Green's function of the appearing differential operator from,

$$\left[\partial_\beta \partial^\beta g^{\alpha\mu} - \partial^\mu \partial^\alpha \left(1 - \frac{1}{\alpha} \right) \right] D_{\mu\nu}(x - y) = i \delta^\alpha_\nu \delta^{(4)} (x - y) \, . \tag{4.94}$$

The solution in the momentum space is

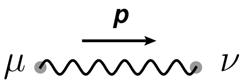
$$D_{\mu\nu}(p) = \frac{i}{p^2} \left[-g_{\mu\nu} + (1 - \alpha) \frac{p_\mu p_\nu}{p^2} \right] \, . \tag{4.95}$$

In the limit $\alpha \to 1$, the expression is exactly the same (modulo the $+i\epsilon$ prescription), as we found above (4.91). The special case $\alpha = 1$ is known as the **Feynman gauge**. Some other choices also have a name:

- $\alpha = 1$: **Feynman gauge**
- $\alpha = 0$: **Landau gauge**
- $\alpha = 3$: **Yennie gauge**

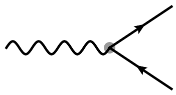
Feynman rules in the Lorenz gauge:

- Photon propagator



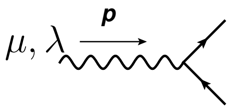
$$= \frac{i}{p^2+i\epsilon}\left[-g_{\mu\nu} + (1-\alpha)\frac{p_\mu p_\nu}{p^2}\right]$$

- Vertices



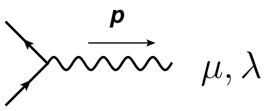
$$= -ie\gamma^\mu$$

- Initial-state photon



$$= \epsilon_{\mu,\lambda}$$

- Final-state photon



$$= \epsilon^*_{\mu,\lambda}$$

5 Lehmann-Symanzik-Zimmerman reduction

Let us now return to the question of how to relate the time-ordered ground-state expectation values and S-matrix elements,

these are what we need

$$\langle\Omega|T\left[\phi(x_1)\cdots\phi(x_{n+2})\right]\Omega\rangle \Leftrightarrow \langle\mathbf{k}_1\cdots\mathbf{k}_n|S|\mathbf{k}_A\mathbf{k}_B\rangle$$

these we know how to do

5.1 Analytical structure of the 2-point function

[Peskin 7.1]

The time-ordered two-point function in the free Klein-Gordon theory is rather simple,

$$\int d^4x e^{ip\cdot x}\langle 0|T\phi(x)\phi(0)|0\rangle = \int d^4x e^{ip\cdot x}\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\frac{i}{k^2-m_0^2+i\epsilon}$$
$$= \frac{i}{p^2-m_0^2+i\epsilon}\,. \tag{5.1}$$

Here, we have already written m_0 instead of m as it turns out that the physical mass m will be different from the mass parameter m_0 that appears in the Lagrangian. For interacting theory the corresponding object

$$\langle\Omega|T\left[\phi(x)\phi(y)\right]|\Omega\rangle\,,$$

will have much richer structure. Let's now process this a bit by inserting a complete set of states in the form,

$$\hat{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda\int\frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}(\lambda)}|\lambda_{\mathbf{p}}\rangle\langle\lambda_{\mathbf{p}}|\,, \tag{5.2}$$

which is similar as in the case of free theory (1.89). The states $|\lambda_{\mathbf{p}}\rangle$ appearing here are eigenstates of the 4-momentum:

$$\hat{H}|\lambda_{\mathbf{p}}\rangle = \sqrt{m_\lambda^2 + \mathbf{p}^2}|\lambda_{\mathbf{p}}\rangle = E_{\mathbf{p}}(\lambda)|\lambda_{\mathbf{p}}\rangle \tag{5.3}$$

$$\hat{\mathbf{P}}|\lambda_{\mathbf{p}}\rangle = \mathbf{p}|\lambda_{\mathbf{p}}\rangle \tag{5.4}$$

$$|\lambda_{\mathbf{p}}\rangle = U\left[\Lambda(\mathbf{p})\right]|\lambda_0\rangle \tag{5.5}$$

$$\hat{H}|\lambda_0\rangle = m_\lambda|\lambda_0\rangle \tag{5.6}$$

$$\hat{\mathbf{P}}|\lambda_0\rangle = \mathbf{0}|\lambda_0\rangle = 0\,. \tag{5.7}$$

Here $U\left[\Lambda(\mathbf{p})\right]$ is the unitary operator that implements the Lorentz transformation $\mathbf{0} \to \mathbf{p}$. The parameter λ indexes the states having different quantum numbers including e.g. 1-particle sates, 2-particle states etc... Only for 1-particle states the energy m_λ of the ground state $|\lambda_0\rangle$ corresponds to the physical mass of the particle. For other states it's merely the invariant mass of the multi-particle state.

Let us now first assume that $x^0 > y^0$. Then,

$$\begin{aligned}\langle\Omega|T\left[\phi(x)\phi(y)\right]|\Omega\rangle &= \langle\Omega|\phi(x)|\Omega\rangle\langle\Omega|\phi(y)|\Omega\rangle \\ &\quad + \sum_\lambda \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}(\lambda)}\langle\Omega|\phi(x)|\lambda_{\mathbf{p}}\rangle\langle\lambda_{\mathbf{p}}|\phi(y)|\Omega\rangle\,.\end{aligned} \tag{5.8}$$

In the case of free theory we verified explicitly how the momentum operator generates the translations , $\phi(x) = e^{i\hat{P}\cdot x}\phi(0)e^{-i\hat{P}\cdot x}$. The result is, however, completely general. By this relation,

$$\langle\Omega|\phi(x)|\Omega\rangle = \langle\Omega|e^{i\hat{P}\cdot x}\phi(0)e^{-i\hat{P}\cdot x}|\Omega\rangle = \langle\Omega|\phi(0)|\Omega\rangle = \text{constant}\,, \tag{5.9}$$

since the aggregate momentum of the ground state is zero. Thus the first term in (5.8) is just a constant. In the same spirit,

$$\begin{aligned}\langle\Omega|\phi(x)|\lambda_{\mathbf{p}}\rangle &= \langle\Omega|e^{i\hat{P}\cdot x}\phi(0)e^{-i\hat{P}\cdot x}|\lambda_{\mathbf{p}}\rangle \\ &= \langle\Omega|\phi(0)e^{-ip\cdot x}|\lambda_{\mathbf{p}}\rangle \\ &= \langle\Omega|U^{\dagger}\left[\Lambda(\mathbf{p})\right]\phi(0)U\left[\Lambda(\mathbf{p})\right]|\lambda_0\rangle e^{-ip\cdot x} \\ &= \langle\Omega|\phi\Big[\Lambda(\mathbf{p})0\Big]|\lambda_0\rangle e^{-ip\cdot x} \\ &= \langle\Omega|\phi(0)|\lambda_0\rangle e^{-ip\cdot x}\,.\end{aligned} \tag{5.10}$$

Substitute this into (5.8):

$$\langle\Omega|T\left[\phi(x)\phi(y)\right]|\Omega\rangle = \sum_{\lambda}\int\frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}(\lambda)}e^{-ip\cdot(x-y)}|\langle\Omega|\phi(0)|\lambda_0\rangle|^2\Big|_{p^0=E_{\mathbf{p}}(\lambda)}\,.$$

Here we still have $x^0 > y^0$. We can turn this into a 4-D integral by the result,

$$\int\frac{dp^0}{2\pi}\frac{i}{p^2-m_{\lambda}^2+i\epsilon}e^{-ip\cdot(x-y)} \overset{x^0\geq y^0}{=} \frac{1}{2E_{\mathbf{p}}(\lambda)}e^{-ip\cdot(x-y)}\Big|_{p^0=E_{\mathbf{p}}(\lambda)}\,, \tag{5.11}$$

so in total,

$$\langle\Omega|T\left[\phi(x)\phi(y)\right]|\Omega\rangle \overset{x^0\geq y^0}{=} \sum_{\lambda}\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m_{\lambda}^2+i\epsilon}e^{-ip\cdot(x-y)}|\langle\Omega|\phi(0)|\lambda_0\rangle|^2\,.$$

In the opposite time-ordering $x^0 < y^0$ the result is identical with this so,

$$\begin{aligned}\langle\Omega|T\left[\phi(x)\phi(y)\right]|\Omega\rangle &= \sum_{\lambda}\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m_{\lambda}^2+i\epsilon}e^{-ip\cdot(x-y)}|\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \\ &= \sum_{\lambda}D_F(x-y;m_{\lambda})|\langle\Omega|\phi(0)|\lambda_0\rangle|^2\,,\end{aligned}$$

where $D_F(x-y;m_{\lambda}^2)$ is nothing else than the Feynman propagator with mass m_{λ}. We can write this as a **spectral representation**,

$$\langle\Omega|T\left[\phi(x)\phi(y)\right]|\Omega\rangle = \int_0^{\infty}\frac{dM^2}{2\pi}\rho(M^2)D_F(x-y;M^2) \tag{5.12}$$

where the **spectral density** $\rho(M^2)$ is

$$\rho(M^2) \equiv \sum_\lambda (2\pi)\delta\left(M^2 - m_\lambda^2\right) |\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \,. \tag{5.13}$$

In the case of non-interacting theory the spectral density is simply,

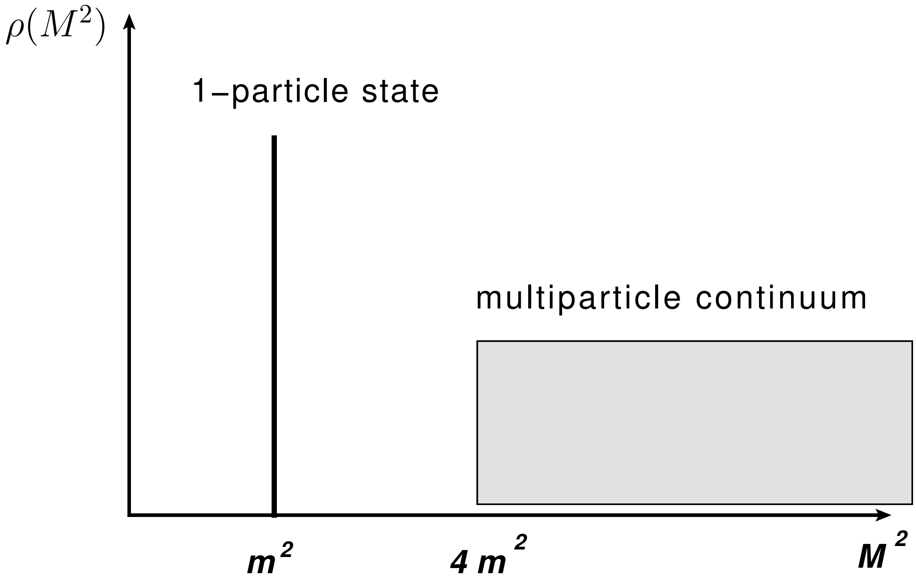
$$\rho^{\text{free}}(M^2) \equiv (2\pi)\delta\left(M^2 - m_0^2\right) \tag{5.14}$$

where m_0 is the mass parameter that appears in the Lagrangian. In interacting theory the 1-particle state (the one with lowest energy) is still of the same form

$$\rho^{1-\text{particle}}(M^2) = (2\pi)\delta\left(M^2 - m^2\right) Z \tag{5.15}$$

$$Z \equiv |\langle\Omega|\phi(0)|(m, \mathbf{p} = 0)\rangle|^2 \,, \tag{5.16}$$

but where m corresponds to the physical mass. Because of the virtual interaction the physical (measurable) mass differs from the one that appears in the Lagrangian, m_0. The parameter m_0 is often referred to as **bare mass**. In the expression for $\rho(M^2)$ we sum over all eigenstates of the Hamiltonian and this include also multi-particle states with invariant mass $M^2 \geq (2m)^2$. Although $\hat{\mathbf{P}}|\lambda_0\rangle = 0$ for all the states that appear in $\rho(M^2)$ the particles in them can still have non-zero mutual momenta, so the spectrum of $\rho(M^2)$ will be continuous above $M^2 = (2m)^2$.



In the beginning of this section we noted that in the non-interacting case,

$$\int d^4x e^{ip\cdot x}\langle 0|T\phi(x)\phi(0)|0\rangle = \frac{i}{p^2 - m_0^2 + i\epsilon}\,. \tag{5.17}$$

We can now write the same also in the interacting case,

$$\int d^4x e^{ip\cdot x}\langle \Omega|T\phi(x)\phi(0)|\Omega\rangle = \int d^4x e^{ip\cdot x}\int_0^\infty \frac{dM^2}{2\pi}\rho(M^2)D_F(x;M^2)$$

$$= \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{4m^2}^\infty \frac{dM^2}{2\pi}\frac{i\rho(M^2)}{p^2 - M^2 + i\epsilon}\,. \tag{5.18}$$

The most important result from here is.

$$\int d^4x e^{ip\cdot x}\langle \Omega|T\phi(x)\phi(0)|\Omega\rangle \xrightarrow{p^2\sim m^2} \frac{iZ}{p^2 - m^2 + i\epsilon}\,. \tag{5.19}$$

This says that the 2-point function of the interacting theory has a single pole at $p^2 = m^2 - i\epsilon$. Thus, in the neighborhood of this point the 2-point function of the interacting case resembles closely that of the free theory. We call the factor $Z = |\langle\Omega|\phi(0)|(m, \mathbf{p} = 0)\rangle|^2$ as **renormalization constant**. We can interpret it as the probability for the operator $\phi(0)$ to create a 1-particle state in the vacuum or, equivalently, as the probability for the operator $\phi(0)$ to annihilate the 1-particle state. Based on our general argument about the presence of a multiparticle continuum in the spectral density, the operator $\phi(0)$ can also create/annihilate several field exitations. Remember that in free theories, the field operators can create or annihilate only a single particle.

5.2 LSZ reduction theorem

That was the 2-point function. Let us now consider a more general n-point function:

$$\langle\Omega|T\left[\phi(x_1)\phi(x_2)\phi(x_3)\dots\phi(x_n)\right]|\Omega\rangle$$

Let's denote $x = x_1$, as above, and Fourier transform with respect to x,

$$\int d^4x e^{ip\cdot x}\langle\Omega|T\left[\phi(x)\phi(x_2)\phi(x_3)\dots\phi(x_n)\right]|\Omega\rangle\,. \tag{5.20}$$

For $n = 2$ this function has simple poles at $p^0 = \pm E_{\mathbf{p}} \mp i\epsilon$. What if $n > 2$? We first split the x^0 integral into three separate regions:

$$\textbf{Region I}: \quad T_+ < x^0 < \infty$$

$$\textbf{Region II}: \quad T_- < x^0 < T_+$$

$$\textbf{Region III}: \quad -\infty < x^0 < T_-$$

where T_- and T_+ have bee chosen such that

$$T_+ > x_2^0, x_3^0, \dots, x_n^0 \tag{5.21}$$

$$T_- < x_2^0, x_3^0, \dots, x_n^0\,. \tag{5.22}$$

We will now inspect all these regions separately:

Region I:

Because T_+ is the latest time, $\phi(x)$ is the leftmost in the time-ordered product,

$$I_+ \equiv \int_{T_+}^{\infty} dx^0 e^{ip^0x^0} \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}}\langle\Omega|\phi(x)\,T\left[\phi(x_2)\phi(x_3)\dots\phi(x_n)\right]|\Omega\rangle\,. \tag{5.23}$$

We insert again a unit operator,

$$I_+ = \int_{T_+}^{\infty} dx^0 e^{ip^0x^0} \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} \sum_\lambda \int \frac{d^3q}{(2\pi)^3 2E_\mathbf{q}(\lambda)} \langle\Omega|\phi(x)|\lambda_\mathbf{q}\rangle$$

$$\langle\lambda_\mathbf{q}| T\left[\phi(x_2)\phi(x_3)\dots\phi(x_n)\right]|\Omega\rangle\,, \tag{5.24}$$

and as earlier,

$$\langle\Omega|\phi(x)|\lambda_\mathbf{q}\rangle = \langle\Omega|\phi(0)|\lambda_0\rangle e^{-iq\cdot x}\Big|_{q^0=E_\mathbf{q}(\lambda)}\,,$$

so that

$$I_+ = \int_{T_+}^{\infty} dx^0 e^{ip^0x^0} \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} \sum_\lambda \int \frac{d^3q}{(2\pi)^3 2E_\mathbf{q}(\lambda)} \langle\Omega|\phi(0)|\lambda_0\rangle e^{-iq\cdot x}\Big|_{q^0=E_\mathbf{q}(\lambda)}$$

$$\langle\lambda_\mathbf{q}| T\left[\phi(x_2)\phi(x_3)\dots\phi(x_n)\right]|\Omega\rangle$$

$$= \int_{T_+}^{\infty} dx^0 e^{ix^0(p^0-E_\mathbf{q}(\lambda))} \int d^3x e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{q})} \sum_\lambda \int \frac{d^3q}{(2\pi)^3 2E_\mathbf{q}(\lambda)} \langle\Omega|\phi(0)|\lambda_0\rangle$$

$$\langle\lambda_\mathbf{q}| T\left[\phi(x_2)\phi(x_3)\dots\phi(x_n)\right]|\Omega\rangle\,.$$

To make the integral meaningful, we include a convergence factor $i\epsilon$ to the exponential. With this convention,

$$I_+ = \int_{T_+}^{\infty} dx^0 e^{ix^0(p^0-E_\mathbf{q}(\lambda)+i\epsilon)} \int d^3x e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{q})} \sum_\lambda \int \frac{d^3q}{(2\pi)^3 2E_\mathbf{q}(\lambda)} \langle\Omega|\phi(0)|\lambda_0\rangle$$

$$\langle\lambda_\mathbf{q}| T\left[\phi(x_2)\phi(x_3)\dots\phi(x_n)\right]|\Omega\rangle\,.$$

We can now do the x^0 and $\mathbf{x}$ integrals,

$$\int_{T_+}^{\infty} dx^0 e^{ix^0(p^0-E_\mathbf{q}(\lambda)+i\epsilon)} = \frac{ie^{iT_+(p^0-E_\mathbf{q}(\lambda)+i\epsilon)}}{p^0-E_\mathbf{q}(\lambda)+i\epsilon} \tag{5.25}$$

$$\int d^3x e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{q})} = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\,. \tag{5.26}$$

In total,

$$I_+ = \sum_\lambda \frac{1}{2E_\mathbf{p}(\lambda)} \frac{ie^{iT_+(p^0-E_\mathbf{p}(\lambda)+i\epsilon)}}{p^0-E_\mathbf{p}(\lambda)+i\epsilon} \langle\Omega|\phi(0)|\lambda_0\rangle\langle\lambda_\mathbf{p}| T\left[\phi(x_2)\phi(x_3)\dots\phi(x_n)\right]|\Omega\rangle\,.$$

We recall that the sum over λ includes all the eigenstates of the Hamiltonian, one of which is the lowest-energy 1-particle state. The contribution of this state to I_+ is $[E_{\mathbf{p}} = E_{\mathbf{p}}(m)]$,

$$\frac{1}{2E_{\mathbf{p}}} \frac{ie^{iT_+(p^0-E_{\mathbf{p}}+i\epsilon)}}{p^0 - E_{\mathbf{p}} + i\epsilon} \langle\Omega|\phi(0)|(m,\mathbf{0})\rangle\langle(m,\mathbf{p})| \, T\left[\phi(x_2)\phi(x_3)\dots\phi(x_n)\right]|\Omega\rangle \, .$$

$$= \frac{1}{2E_{\mathbf{p}}} \frac{ie^{iT_+(p^0-E_{\mathbf{p}}+i\epsilon)}}{p^0 - E_{\mathbf{p}} + i\epsilon} \sqrt{Z}\, \langle(m,\mathbf{p})| \, T\left[\phi(x_2)\phi(x_3)\dots\phi(x_n)\right]|\Omega\rangle \, .$$

$$\xrightarrow{p^0 \to E_{\mathbf{p}}} \frac{1}{2E_{\mathbf{p}}} \frac{i\sqrt{Z}}{p^0 - E_{\mathbf{p}} + i\epsilon} \langle(m,\mathbf{p})| \, T\left[\phi(x_2)\dots\phi(x_n)\right]|\Omega\rangle \, .$$

Now, because

$$\frac{1}{p^2 - m^2 + i\epsilon} = \frac{1}{(p^0 - E_{\mathbf{p}} + i\epsilon)(p^0 + E_{\mathbf{p}} - i\epsilon)} \qquad (5.27)$$

$$\xrightarrow{p^0 \to E_{\mathbf{p}}} \frac{1}{(p^0 - E_{\mathbf{p}} + i\epsilon)2E_{\mathbf{p}}} \, ,$$

we can write

$$I_+ \underset{p^0 \to E_{\mathbf{p}}}{\sim} \frac{i\sqrt{Z}}{p^2 - m^2 + i\epsilon} \langle(m,\mathbf{p})| \, T\left[\phi(x_2)\dots\phi(x_n)\right]|\Omega\rangle \, .$$

This notation signifies that the rest of terms are not singular in the indicated limit.

Region II:

In the second region the relevant integral is of the form

$$\int_{T_-}^{T_+} dx^0 e^{ip^0x^0} \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} \langle\Omega|T\left[\phi(x)\phi(x_2)\dots\phi(x_n)\right]|\Omega\rangle \, . \qquad (5.28)$$

Since the integration domain is finite and the p^0 dependence $e^{ip^0x^0}$ of the integrand is analytic, this region does not give rise to singularities. Note that the region I was singular only because the upper limit of the integral was taken to infinity.

Region III:

In the third region T_- is the earliest moment so we have to evaluate,

$$I_- \equiv \int_{-\infty}^{T_-} dx^0 e^{ip^0 x^0} \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} \langle\Omega| T\left[\phi(x_2)\ldots\phi(x_n)\right]\phi(x)|\Omega\rangle\,. \quad (5.29)$$

The calculation goes almost exactly as in region I, and we find,

$$I_- \underset{p^0\to -E_{\mathbf{p}}}{\sim} \frac{i\sqrt{Z}}{p^2-m^2+i\epsilon}\langle\Omega|T\left[\phi(x_2)\ldots\phi(x_n)\right]|(m,-\mathbf{p})\rangle\,, \quad (5.30)$$

so the 1-particle singularity is found at $p^0 \to -E_{\mathbf{p}}$.

Up to now, we have established the following:

$$\int d^4x e^{ip\cdot x}\langle\Omega|T\left[\phi(x)\phi(x_2)\phi(x_3)\ldots\phi(x_n)\right]|\Omega\rangle \quad (5.31)$$

$$\underset{p^0\to E_{\mathbf{p}}}{\sim} \frac{i\sqrt{Z}}{p^2-m^2+i\epsilon}\langle\mathbf{p}|\,T\left[\phi(x_2)\ldots\phi(x_n)\right]|\Omega\rangle \qquad (\text{from } x^0 \gg 0 \text{ limit})$$

$$\underset{p^0\to -E_{\mathbf{p}}}{\sim} \frac{i\sqrt{Z}}{p^2-m^2+i\epsilon}\langle\Omega|T\left[\phi(x_2)\ldots\phi(x_n)\right]|-\mathbf{p}\rangle \quad (\text{from } x^0 \ll 0 \text{ limit})$$

In other words, the field operator $\phi(x)$ creates 1-particle states when $x^0 \to \pm\infty$. It can also create multi-particle states but the corresponding analytic structure will be different.

Note also that if $n=2$ we recover Eq. (5.19), which at some level justifies the introduction of the "convergence factor" $i\epsilon$.

Let's now consider Fourier transform over all variables

$$\left[\prod_{i=1}^{n}\int d^4x_i e^{ip_i\cdot x_i}\right]\langle\Omega|T\left[\phi(x_1)\phi(x_2)\ldots\phi(x_n)\right]|\Omega\rangle\,,$$

and evaluate the contribution from region

$$x_1^0, x_2^0 > T \tag{5.32}$$

$$x_3^0, \dots x_n^0 < -T \tag{5.33}$$

so that in the time-ordered product the fields $\phi(x_1)$ and $\phi(x_2)$ are the leftmost,

$$\langle\Omega|T\left[\phi(x_1)\dots\right]|\Omega\rangle = \langle\Omega|T\left[\phi(x_1)\phi(x_2)\right]T\left[\phi(x_3)\dots\phi(x_n)\right]|\Omega\rangle\,.$$

The integration with respect to the first two variables becomes now

$$\left[\prod_{i=1}^{2}\int_{T_+}^{\infty}dx_i^0\int d^3x_i e^{ip_i\cdot x_i}\right]\langle\Omega|T\left[\phi(x_1)\phi(x_2)\dots\phi(x_n)\right]|\Omega\rangle$$

$$= \sum_\lambda\int\frac{d^3q}{(2\pi)^3 2E_{\mathbf{q}}(\lambda)}\left[\prod_{i=1}^{2}\int_{T_+}^{\infty}dx_i^0\int d^3x_i e^{ip_i\cdot x_i}\right]$$

$$\times\,\langle\Omega|T\left[\phi(x_1)\phi(x_2)\right]|\lambda_{\mathbf{q}}\rangle\,\langle\lambda_{\mathbf{q}}|T\left[\phi(x_3)\dots\phi(x_n)\right]|\Omega\rangle\,.$$

The previous expression contains a term,

$$\langle\Omega|\,T\left[\phi(x_1)\phi(x_2)\right]|\lambda_{\mathbf{q}}\rangle\,.$$

A condition for this to give something nonzero is clearly that

$$T\left[\phi(x_1)\phi(x_2)\right]|\lambda_{\mathbf{q}}\rangle = C|\Omega\rangle + \dots \tag{5.34}$$

where rest of the terms will give zero since the eigenstates of the Hamiltonian operator are (or can be chosen in the degenerate case to be) orthogonal. In other words, two consecutive field operators should be able to annihilate all exitations of $|\lambda_{\mathbf{q}}\rangle$. Equivalently,

$$\langle\Omega|\,T\left[\phi(x_1)\phi(x_2)\right] = C\langle\lambda_{\mathbf{q}}| + \dots \tag{5.35}$$

i.e. two consecutive field operators should be able to create the state $|\lambda_{\mathbf{q}}\rangle$ by acting on the vacuum. Let us suppose that $|\lambda_{\mathbf{q}}\rangle$ consists of two independent 1-particle states:

$$\int\frac{d^3q}{(2\pi)^3 2E_{\mathbf{q}}(\lambda)}|\lambda_{\mathbf{q}}\rangle\,\langle\lambda_{\mathbf{q}}| = \frac{1}{2!}\int\frac{d^3q_1}{(2\pi)^3 2E_{\mathbf{q}_1}}\int\frac{d^3q_2}{(2\pi)^3 2E_{\mathbf{q}_2}}|\mathbf{q}_1\mathbf{q}_2\rangle\,\langle\mathbf{q}_1\mathbf{q}_2|\,.$$

Based on our earlier results we know that $\phi(x)$ can annihilate a 1-particle state,

$$\phi(x)|\mathbf{q}\rangle = \sqrt{Z}e^{-iq\cdot x}|\Omega\rangle + \cdots \, , \tag{5.36}$$

so it's reasonable to assume that when hitting the above 2-particle state $\phi(x)$ can annihilate either one of the independent exitations,

$$\phi(x)|\mathbf{q}_1\mathbf{q}_2\rangle = \sqrt{Z}e^{-iq_1\cdot x}|\mathbf{q}_2\rangle + \sqrt{Z}e^{-iq_2\cdot x}|\mathbf{q}_1\rangle + \cdots \, . \tag{5.37}$$

Under this assumption,

$$\phi(y)\phi(x)|\mathbf{q}_1\mathbf{q}_2\rangle = Z\left[e^{-iq_1\cdot x}e^{-iq_2\cdot y} + e^{-iq_2\cdot x}e^{-iq_1\cdot y}\right]|\Omega\rangle + \cdots \tag{5.38}$$

This is naturally independent of the order of operators $\phi(y)$ and $\phi(x)$. Thus,

$$\left[\prod_{i=1}^{2}\int_{T_+}^{\infty} dx_i^0 \int d^3x_i e^{ip_i\cdot x_i}\right]\langle\Omega|T\left[\phi(x_1)\phi(x_2)\ldots\phi(x_n)\right]|\Omega\rangle$$

$$= \frac{1}{2!}\int\frac{d^3q_1}{(2\pi)^3 2E_{\mathbf{q}_1}}\int\frac{d^3q_2}{(2\pi)^3 2E_{\mathbf{q}_2}}\left[\prod_{i=1}^{2}\int_{T_+}^{\infty} dx_i^0 \int d^3x_i e^{ip_i\cdot x_i}\right]$$

$$\times\, Z\left[e^{-iq_1\cdot x_1}e^{-iq_2\cdot x_2} + e^{-iq_2\cdot x_1}e^{-iq_1\cdot x_2}\right]\,\langle\mathbf{q}_1\mathbf{q}_2|T\left[\phi(x_3)\ldots\phi(x_n)\right]|\Omega\rangle + \cdots \, ,$$

$$\underset{p_i^0\to E_{\mathbf{p}_i}}{\sim}\left[\prod_{i=1}^{2}\frac{i\sqrt{Z}}{p_i^2 - m^2 + i\epsilon}\right]\,\langle\mathbf{p}_1\mathbf{p}_2|T\left[\phi(x_3)\ldots\phi(x_n)\right]|\Omega\rangle \, .$$

The fact that only two independent 1-particle states lead to a singularity structure like this is non-trivial and to more carefully argue this point would require a wave-packet treatment, see e.g. Peskin 7.2. The rest $n-2$ states lead to a similar singularity structure in the opposite limit $T \to -\infty$,

$$\left[\prod_{i=3}^{n}\int_{-\infty}^{-T} dx_i^0 \int d^3x_i e^{ip_i\cdot x_i}\right]\langle\mathbf{p}_1\mathbf{p}_2|T\left[\phi(x_3)\ldots\phi(x_n)\right]|\Omega\rangle$$

$$\underset{p_i^0\to -E_{\mathbf{p}_i}}{\sim}\left[\prod_{i=3}^{n}\frac{i\sqrt{Z}}{p_i^2 - m^2 + i\epsilon}\right]\,\langle\mathbf{p}_1\mathbf{p}_2|-\mathbf{p}_3\ldots-\mathbf{p}_n\rangle \, .$$

The states $\langle \mathbf{p}_1\mathbf{p}_2|$ and $|-\mathbf{p}_3\ldots-\mathbf{p}_n\rangle$ that appear in these formulae are Heisenberg-picture states – they do not depend on time. However, they were "created" by time-dependent operators at $T\to\infty$ and $T\to-\infty$ so they look like 2- and $(n-2)$-particle states only in these limits. We thus identify them as the "out" and "in" states used in deriving the cross section back in Section 3.5. The factor

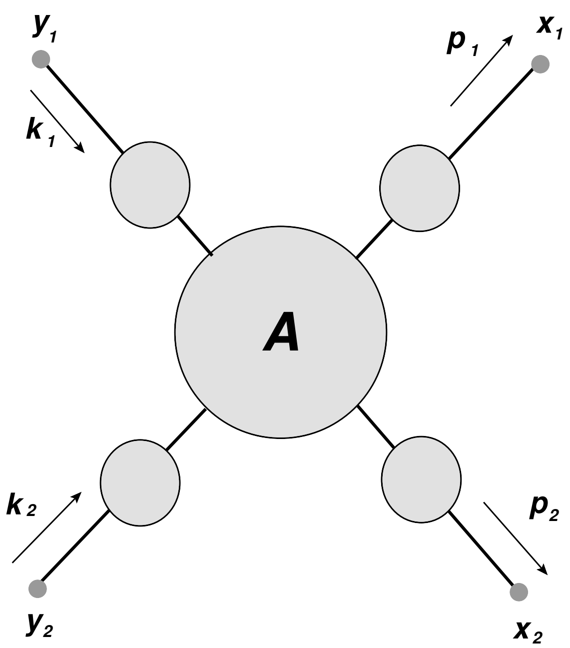
$$\langle \mathbf{p}_1\mathbf{p}_2|-\mathbf{p}_3\ldots-\mathbf{p}_n\rangle = {}_{\mathrm{out}}\langle \mathbf{p}_1\mathbf{p}_2|-\mathbf{p}_3\ldots-\mathbf{p}_n\rangle_{\mathrm{in}}\,, \tag{5.39}$$

is thus exactly the S-matrix element we defined earlier. By this construction we get the **LSZ reduction theorem**:

$$\left[\prod_{i=1}^{n}\int d^4x_i e^{ip_i\cdot x_i}\right]\left[\prod_{j=1}^{m}\int d^4y_j e^{-ik_j\cdot y_j}\right]\langle\Omega|T\left[\phi(x_1)\ldots\phi(x_n)\phi(y_1)\ldots\phi(y_m)\right]|\Omega\rangle$$

$$\underset{\substack{p_i^0\to E_{\mathbf{p}_i}\\ k_j^0\to E_{\mathbf{k}_j}}}{\sim}\quad\left[\prod_{i=1}^{n}\frac{i\sqrt{Z}}{p_i^2-m^2+i\epsilon}\right]\left[\prod_{j=1}^{m}\frac{i\sqrt{Z}}{k_j^2-m^2+i\epsilon}\right]\langle\mathbf{p}_1\ldots\mathbf{p}_n|S|\mathbf{k}_1\ldots\mathbf{k}_m\rangle\,.$$

How to use this theorem? As an example, let's consider a 4-point function $\langle\Omega|T\left[\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)\right]|\Omega\rangle$:



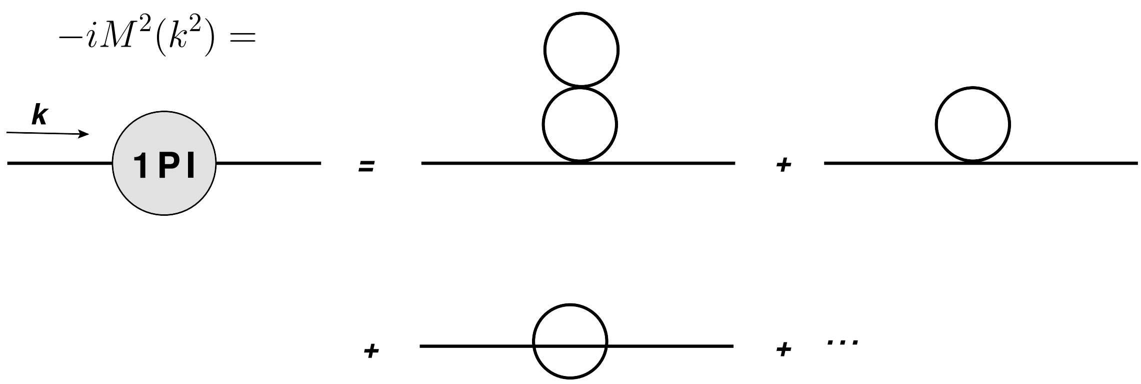
Based on the momentum-space Feynman rules, this corresponds to an expression

$$\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \times A \tag{5.40}$$

$$\times\, e^{ik_1\cdot y_1}\,[\mathbf{FP}(k_1)]\, e^{ik_2\cdot y_2}\,[\mathbf{FP}(k_2)]\, e^{-ip_1\cdot x_1}\,[\mathbf{FP}(p_1)]\, e^{-ip_2\cdot x_2}\,[\mathbf{FP}(p_2)]\ .$$

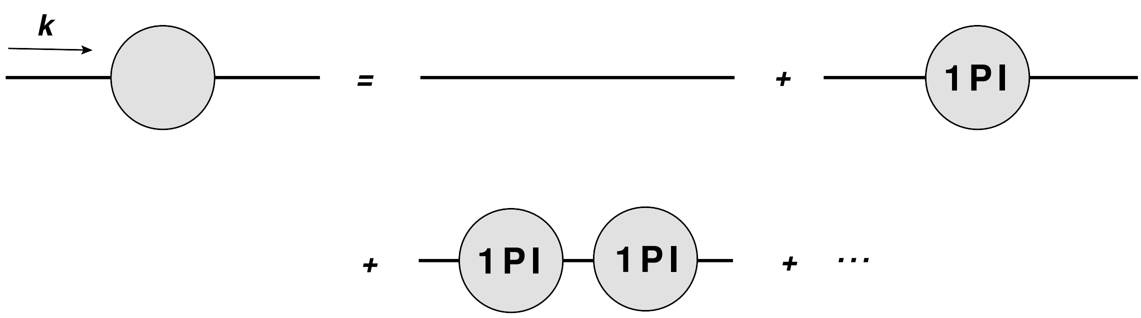
Here the factor $A = A(k_1, k_2, p_1, p_2)$ contains a sum of all amputated 4-point diagrams and the factor $\mathbf{FP}(k)$ signifies the "complete" propagator. More exactly, let us denote by $-iM^2(k^2)$ the sum of all **1-particle irreducible (1PI)** 2-point functions,

$$-iM^2(k^2) =$$



A diagram is 1PI if it does not split into two separate parts if we cut one line. The complete propagator $\mathbf{FP}(k)$ is the sum,

$$\mathbf{FP}(k) =$$



$$
\begin{aligned}
&= \frac{i}{k^2 - m_0^2 + i\epsilon} + \frac{i}{k^2 - m_0^2 + i\epsilon}[-iM^2(k^2)]\frac{i}{k^2 - m_0^2 + i\epsilon} \\
&+ \frac{i}{k^2 - m_0^2 + i\epsilon}[-iM^2(k^2)]\frac{i}{k^2 - m_0^2 + i\epsilon}[-iM^2(k^2)]\frac{i}{k^2 - m_0^2 + i\epsilon} + \cdots \\
&= \frac{i}{k^2 - m_0^2 + i\epsilon}\left[1 + \left(\frac{M^2(k^2)}{k^2 - m_0^2 + i\epsilon}\right) + \left(\frac{M^2(k^2)}{k^2 - m_0^2 + i\epsilon}\right)^2 + \cdots\right] \\
&= \frac{i}{k^2 - m_0^2 + i\epsilon}\frac{1}{1 - \frac{M^2(k^2)}{k^2 - m_0^2 + i\epsilon}} = \frac{i}{k^2 - m_0^2 + i\epsilon}\frac{k^2 - m_0^2 + i\epsilon}{k^2 - m_0^2 + i\epsilon - M^2(k^2)} \\
&= \frac{i}{k^2 - m_0^2 - M^2(k^2) + i\epsilon}\,.
\end{aligned} \tag{5.41}
$$

What we have called the physical mass m is the solution to the equation,

$$
k^2 - m_0^2 - M^2(k^2) \stackrel{k^2 \to m^2}{=} 0\,. \tag{5.42}
$$

That is, the physical mass of a particle is defined as the location of the pole of the propagator. This is sometimes called the **pole mass**. In this limit, $\mathbf{FP}(k)$ is of the form,

$$
\mathbf{FP}(k) \stackrel{k^2 \to m^2}{=} \frac{iZ}{k^2 - m^2 + i\epsilon} + \cdots \tag{5.43}
$$

$$
Z^{-1} = 1 - \frac{dM^2(k^2)}{dk^2}\bigg|_{k^2 = m^2}, \tag{5.44}
$$

where the rest of the terms are not singular in the limit $k^2 \to m^2$. Let us take this limit in Eq. (5.40), and do a Fourier transform,

$$\left[\prod_{i=1}^{2}\int d^4x_i e^{iP_i\cdot x_i}\right]\left[\prod_{j=2}^{m}\int d^4y_j e^{-iK_j\cdot y_j}\right]\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\times A$$

$$\times\, e^{ik_1\cdot y_1}\left[\mathbf{FP}(k_1)\right]e^{ik_2\cdot y_2}\left[\mathbf{FP}(k_2)\right]e^{-ip_1\cdot x_1}\left[\mathbf{FP}(p_1)\right]e^{-ip_2\cdot x_2}\left[\mathbf{FP}(p_2)\right]$$

$$=\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}$$

$$\times\left[\prod_{i=1}^{2}\int d^4x_i e^{ix_i\cdot(P_i-p_i)}\right]\left[\prod_{j=2}^{m}\int d^4y_j e^{-iy_j\cdot(K_j-k_j)}\right]\times A(p_1,p_2,k_1,k_2)$$

$$\times\frac{iZ}{p_1^2-m^2+i\epsilon}\frac{iZ}{p_2^2-m^2+i\epsilon}\frac{iZ}{k_1^2-m^2+i\epsilon}\frac{iZ}{k_2^2-m^2+i\epsilon}$$

$$=A(P_1,P_2,K_1,K_2)\times\left[\prod_{i=1}^{2}\frac{iZ}{P_i^2-m^2+i\epsilon}\right]\left[\prod_{i=1}^{2}\frac{iZ}{K_i^2-m^2+i\epsilon}\right].$$

According to the LSZ theorem this corresponds to the expression,

$$\left[\prod_{i=1}^{n}\frac{i\sqrt{Z}}{P_i^2-m^2+i\epsilon}\right]\left[\prod_{j=1}^{m}\frac{i\sqrt{Z}}{K_j^2-m^2+i\epsilon}\right]\langle\mathbf{P}_1\mathbf{P}_2|S|\mathbf{K}_1\mathbf{K}_2\rangle\,.$$

We thus find, for $2\to 2$ scattering,

$$\langle\mathbf{P}_1\mathbf{P}_2|S|\mathbf{K}_1\mathbf{K}_2\rangle=(\sqrt{Z})^4\,\times A(P_1,P_2,K_1,K_2)\,. \qquad (5.45)$$

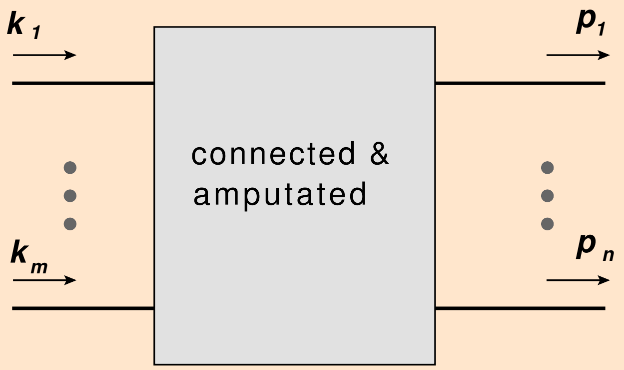
By definition, the amplitude A contains only the **amputated diagrams**. They also need to be **fully connected**, since otherwize we would not get the proper singularity structure (Ex.).

The mass parameter m_0 that appears in the Lagrangian does not correspond the physical mass m, though the two are related ny $m^2=m_0^2+M^2(m^2)$. Since $M^2=\mathcal{O}(\lambda)$, in the leading-order calculations one can set m_0 and m to be equal. Also $Z=1+\mathcal{O}(\lambda)$, we can set Z to unity in leading-order calculations. We will come back to this later in the course.

In other words, the scattering matrix is just

$\langle \mathbf{p}_1 \ldots \mathbf{p}_n | S | \mathbf{k}_1 \ldots \mathbf{k}_m \rangle =$

$(\sqrt{Z})^{n+m} \times$



with the propagators from the external lines stripped away.

It's left as an challenge (Ex.) to deduce how the LSZ theorem leads to Eq. (3.114),

$$\langle \mathbf{k}_1 \cdots \mathbf{k}_n | S | \mathbf{k}_A \mathbf{k}_B \rangle = \left[\sqrt{Z_1} \sqrt{Z_2} \prod_{i=1}^{n} \sqrt{Z_i} \right] \tag{5.46}$$

$$\times \left[{}_I\langle \mathbf{k}_1 \cdots \mathbf{k}_n | T \left\{ \exp \left[-i \int dt H_I(x) \right\} \right] | \mathbf{k}_A \mathbf{k}_B \rangle_I \right]_{\substack{\text{connected} \\ \text{amputated}}}$$

which we have used earlier to derive scattering amplitudes.

For fermion fields the LSZ formula can be derived similarly as we did here for scalar fields. The principal difference is that for fermion fields we now have (Ex.)

$$\langle \Omega | \psi(0) | p, s \rangle_{\text{particle}} = \sqrt{Z}\, u_s(p)\,, \tag{5.47}$$

$$\langle \Omega | \overline{\psi}(0) | p, s \rangle_{\text{antiparticle}} = -\sqrt{Z}\, \overline{v}_s(p)\,. \tag{5.48}$$

Otherwize the derivation of the LSZ theorem is nearly identical – result is:

$$
\underbrace{\left[\prod_{i=1}^{n_f}\int d^4x_i e^{ip_i\cdot x_i}\right]}_{\text{out fermions}}
\underbrace{\left[\prod_{j=1}^{n_{\overline{f}}}\int d^4x_j' e^{ip_j'\cdot x_j'}\right]}_{\text{out antifermions}}
\underbrace{\left[\prod_{j=1}^{m_f}\int d^4y_j e^{-ik_j\cdot y_j}\right]}_{\text{in fermions}}
\underbrace{\left[\prod_{i=1}^{m_{\overline{f}}}\int d^4y_i' e^{-ik_i'\cdot y_i'}\right]}_{\text{in antifermions}}
$$

$$
\langle\Omega|T\left[\underbrace{\prod_{i=1}^{n_f}\psi_{\ell_i}(x_i)}_{\text{out fermions}}\ \underbrace{\prod_{i=1}^{n_{\overline{f}}}\overline{\psi}_{\ell_i'}(x_i')}_{\text{out antifermions}}\ \underbrace{\prod_{i=1}^{m_f}\overline{\psi}_{h_i}(y_i)}_{\text{in fermions}}\ \underbrace{\prod_{i=1}^{m_{\overline{f}}}\psi_{h_i'}(y_i')}_{\text{in antifermions}}\right]|\Omega\rangle
$$

$$
\underset{\substack{p_i^0,p_j'^0\to E_{\mathbf{p}_i}\\ k_j^0,k_i'^0\to E_{\mathbf{k}_j}}}{\sim}
\underbrace{\left[\prod_{i=1}^{n_f}\frac{i\sqrt{Z}\sum_{r_i}[u_{r_i}(p_i)]_{\ell_i}}{p_i^2-m^2+i\epsilon}\right]}_{\text{out fermions}}
\underbrace{\left[\prod_{j=1}^{m_f}\frac{i\sqrt{Z}\sum_{s_i}[\overline{u}_{s_i}(k_i)]_{h_i}}{k_i^2-m^2+i\epsilon}\right]}_{\text{in fermions}}
$$

$$
\underbrace{\left[\prod_{j=1}^{n_{\overline{f}}}\frac{i\sqrt{Z}\sum_{r_i'}[\overline{v}_{r_i'}(p_i')]_{\ell_i'}}{(p_i')^2-m^2+i\epsilon}\right]}_{\text{out antifermions}}
\underbrace{\left[\prod_{i=1}^{m_{\overline{f}}}\frac{i\sqrt{Z}\sum_{s_i'}[v_{s_i'}(k_i')]_{h_i'}}{(k_i')^2-m^2+i\epsilon}\right]}_{\text{in antifermions}}
$$

$$
\langle(\mathbf{p}_i,r_i);(\mathbf{p'}_i,r_i')|S|(\mathbf{k}_i,s_i);(\mathbf{k'}_i,s_i')\rangle\,. \tag{5.49}
$$

where,

$$n_f = \text{number of outgoing fermions}$$

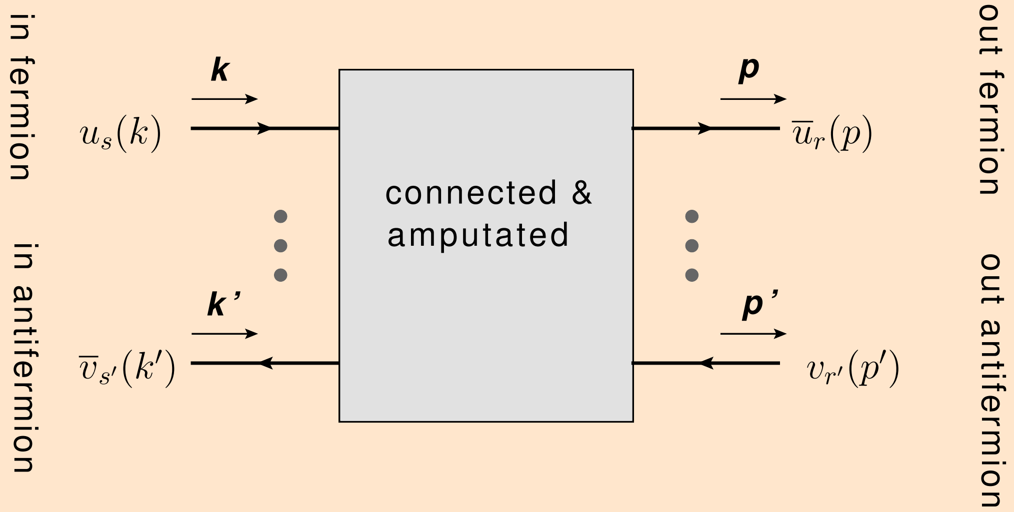
$$m_f = \text{number of incoming fermions}$$

$$n_{\overline{f}} = \text{number of outgoing antifermions}$$

$$m_{\overline{f}} = \text{number of incoming antifermions}$$

It follows from this that effectively one again cancels the fermion propagators corresponding to the incoming and outgoing particles and replaces them with the appropriate Dirac spinors:

$$\langle((\mathbf{p}_i, r_i); (\mathbf{p'}_i, r'_i)|S|(\mathbf{k}_i, s_i); (\mathbf{k'}_i, s'_i)\rangle = (\sqrt{Z})^{n_f+m_f+n_{\overline{f}}+m_{\overline{f}}} \quad \times$$



For vector particles the LSZ formula is

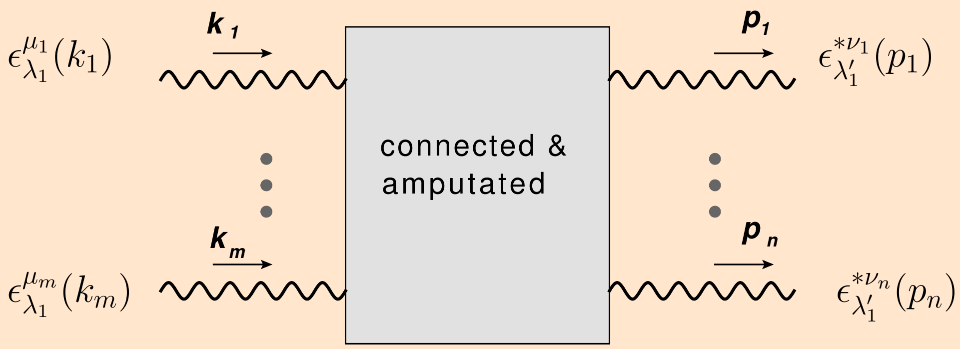
$$\left[\prod_{i=1}^{n} \int d^4x_i e^{ip_i\cdot x_i}\right] \left[\prod_{j=1}^{m} \int d^4y_j e^{-ik_j\cdot y_j}\right] \tag{5.50}$$

$$\langle\Omega|T\left[A^{\mu_1}(x_1)\ldots A^{\mu_n}(x_n)A^{\nu_1}(y_1)\ldots A^{\mu_m}(y_m)\right]|\Omega\rangle$$

$$\underset{\substack{p_i^0\to E_{\mathbf{p}_i}\\ k_j^0\to E_{\mathbf{k}_j}}}{\sim} \left[\prod_{i=1}^{n} \frac{i\sqrt{Z}\sum_{\lambda'_i}\epsilon^{\mu_i}_{\lambda'_i}(p_i)}{p_i^2-m^2+i\epsilon}\right] \left[\prod_{j=1}^{m} \frac{i\sqrt{Z}\sum_{\lambda_j}\epsilon^{*\nu_i}_{\lambda_j}(k_j)}{k_j^2-m^2+i\epsilon}\right] \langle(\mathbf{p}_i, \lambda'_i)|S|(\mathbf{k}_j, \lambda_j)\rangle\,.$$

For massless vector particles like photons there is a subtlety in the derivation of this result as the 1-particle state with zero invariant mass is not isolated in the spectral density but there are also multiparticle states with zero invariant mass (Itzykson-Zuber, 5-1-5). Nevertheless, it again follows that to obtain the S-matrix element one effectively cancels the propagators corresponding to the incoming and outgoing particles and replaces them with the appropriate polarization vectors:

$$\langle (\mathbf{p}_i, \lambda_i')|S|(\mathbf{k}_i, \lambda_i)\rangle = (\sqrt{Z})^{n+m} \quad \times$$



Usually we are interested in a mix of scalar, fermion and vector particles, and the most "general" LSZ formula can be obtained by combining the above three. Of course, all particles have their own specific field-strength renormalization factor Z and mass m.

5.3 Optical theorem

The optical theorem is a consequence of the unitarity of the scattering matrix, $S^\dagger S = 1$. Using the split $S = 1 + iT$, as defined in Eq. (3.98),

$$(1 - iT^\dagger)(1 + iT) = 1 + i(T - T^\dagger) + T^\dagger T = 1$$

$$\Longrightarrow -i(T - T^\dagger) = T^\dagger T\,. \tag{5.51}$$

Let $|\mathbf{p}_1\mathbf{p}_2\rangle$ and $|\mathbf{k}_1\mathbf{k}_2\rangle$ represent 2-particle states of the interacting theory. By using the completeness of states,

$$\langle \mathbf{p}_1\mathbf{p}_2|T^\dagger T|\mathbf{k}_1\mathbf{k}_2\rangle \tag{5.52}$$

$$= \sum_n \left(\prod_{i=1}^{n} \int \frac{d^3q}{(2\pi)^3 2E_{\mathbf{q}_i}}\right) \langle \mathbf{p}_1\mathbf{p}_2|T^\dagger|\mathbf{q}_1\ldots\mathbf{q}_n\rangle\langle\mathbf{q}_1\ldots\mathbf{q}_n|T|\mathbf{k}_1\mathbf{k}_2\rangle\,.$$

On the other hand, based on the definition (3.99) of the invariant matrix element,

$$\langle \mathbf{k}_1 \cdots \mathbf{k}_n | iT | \mathbf{k}_A \mathbf{k}_B \rangle = (2\pi)^4 \delta^{(4)} \left(k_A + k_B - \sum_i k_i \right) i\mathcal{M} \left(k_A, k_B \to k_f \right) ,$$

so

$$\begin{aligned} &\langle \mathbf{p}_1 \mathbf{p}_2 | T^\dagger T | \mathbf{k}_1 \mathbf{k}_2 \rangle \qquad (5.53) \\ =&\sum_n \left(\prod_{i=1}^n \int \frac{d^3 q}{(2\pi)^3 2E_{\mathbf{q}_i}} \right) \left[\mathcal{M}^* \left(p_1, p_2 \to \mathbf{q}_1 \ldots \mathbf{q}_n \right) \right] \left[\mathcal{M} \left(k_1, k_2 \to \mathbf{q}_1 \ldots \mathbf{q}_n \right) \right] \\ \times &(2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - q \right) (2\pi)^4 \delta^{(4)} \left(k_1 + k_2 - q \right) \end{aligned}$$

Similarly,

$$\begin{aligned} -i \langle \mathbf{p}_1 \mathbf{p}_2 | (T - T^\dagger) | \mathbf{k}_1 \mathbf{k}_2 \rangle = &-i \left[\mathcal{M} \left(k_1, k_2 \to p_1, p_2 \right) - \mathcal{M}^* \left(p_1, p_2 \to k_1, k_2 \right) \right] \\ &\times (2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - k_1 - k_2 \right) . \qquad (5.54) \end{aligned}$$

Equating the last two equations gives us an identity,

$$\begin{aligned} &- i \left[\mathcal{M} \left(k_1, k_2 \to p_1, p_2 \right) - \mathcal{M}^* \left(p_1, p_2 \to k_1, k_2 \right) \right] \qquad (5.55) \\ &\times (2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - k_1 - k_2 \right) \\ =&\sum_n \left(\prod_{i=1}^n \int \frac{d^3 q}{(2\pi)^3 2E_{\mathbf{q}_i}} \right) \left[\mathcal{M}^* \left(p_1, p_2 \to \mathbf{q}_1 \ldots \mathbf{q}_n \right) \right] \left[\mathcal{M} \left(k_1, k_2 \to \mathbf{q}_1 \ldots \mathbf{q}_n \right) \right] \\ \times &(2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - q \right) (2\pi)^4 \delta^{(4)} \left(k_1 + k_2 - p_1 - p_2 \right) , \end{aligned}$$

so when $k_1 + k_2 = p_1 + p_2$,

$$\begin{aligned} &- i \left[\mathcal{M} \left(k_1, k_2 \to p_1, p_2 \right) - \mathcal{M}^* \left(p_1, p_2 \to k_1, k_2 \right) \right] \qquad (5.56) \\ &= \sum_n \left(\prod_{i=1}^n \int \frac{d^3 q}{(2\pi)^3 2E_{\mathbf{q}_i}} \right) \left[\mathcal{M}^* \left(p_1, p_2 \to \mathbf{q}_1 \ldots \mathbf{q}_n \right) \right] \left[\mathcal{M} \left(k_1, k_2 \to \mathbf{q}_1 \ldots \mathbf{q}_n \right) \right] \\ &\times (2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - q \right) . \end{aligned}$$

We used here 2-particle states but this is not any restriction. More generally (and in a shorter form),

$$-i\left[\mathcal{M}\left(a\to b\right)-\mathcal{M}^{*}\left(b\to a\right)\right]=\sum_{f}\int d\Gamma_{f}\left[\mathcal{M}^{*}\left(b\to f\right)\right]\left[\mathcal{M}\left(a\to f\right)\right]\,, \tag{5.57}$$

where it is implicit that $p_a = p_b$. This is one of forms of optical theorem.

In a particular case in which $|a\rangle = |b\rangle = |\mathbf{k}_1, \mathbf{k}_2\rangle$,

$$\begin{aligned} & -i\left[\mathcal{M}\left(\mathbf{k}_1,\mathbf{k}_2\to\mathbf{k}_1,\mathbf{k}_2\right)-\mathcal{M}^{*}\left(\mathbf{k}_1,\mathbf{k}_2\to\mathbf{k}_1,\mathbf{k}_2\right)\right] \\ & = 2\mathrm{Im}\mathcal{M}\left(\mathbf{k}_1,\mathbf{k}_2\to\mathbf{k}_1,\mathbf{k}_2\right)\,, \end{aligned} \tag{5.58}$$

and

$$2\mathrm{Im}\mathcal{M}\left(\mathbf{k}_1,\mathbf{k}_2\to\mathbf{k}_1,\mathbf{k}_2\right)=\sum_{f}\int d\Gamma_{f}|\mathcal{M}\left(\mathbf{k}_1,\mathbf{k}_2\to f\right)|^{2}\,. \tag{5.59}$$

The right-hand side is, up to the flux factor $F = 1/\left(4E_{\mathbf{k}_1}E_{\mathbf{k}_2}|v_{\mathbf{k}_1} - v_{\mathbf{k}_2}|\right)$, the total cross section for process "$\mathbf{k}_1, \mathbf{k}_2 \to$ **anything**". In the center-of-mass frame,

$$\begin{aligned} 4E_{\mathbf{k}_1}E_{\mathbf{k}_2}|v_{\mathbf{k}_1}-v_{\mathbf{k}_2}| &= 4E_{\mathbf{k}_1}E_{\mathbf{k}_2}\left|\frac{\mathbf{k}_1}{E_{\mathbf{k}_1}}-\frac{-\mathbf{k}_1}{E_{-\mathbf{k}_2}}\right| \\ &= 4E_{\mathbf{k}_1}E_{\mathbf{k}_2}\left|\mathbf{k}_1\right|\frac{E_{\mathbf{k}_1}+E_{\mathbf{k}_2}}{E_{\mathbf{k}_1}E_{\mathbf{k}_2}} \\ &= 4\left|\mathbf{k}_1\right|\sqrt{s}\,, \end{aligned} \tag{5.60}$$

so that (still in the center-of-mass frame),

$$\mathrm{Im}\mathcal{M}\left(\mathbf{k}_1,\mathbf{k}_2\to\mathbf{k}_1,\mathbf{k}_2\right)=2\sqrt{s}\left|\mathbf{k}_1\right|\sigma_{\mathrm{total}}\left(\mathbf{k}_1,\mathbf{k}_2\to X\right) \tag{5.61}$$

In other words, the imaginary part of the matrix element for **elastic** $\mathbf{k}_1, \mathbf{k}_2 \to \mathbf{k}_1, \mathbf{k}_2$ scattering is related to the total cross section $\mathbf{k}_1, \mathbf{k}_2 \to X$.

A process $\mathbf{k}_1, \mathbf{k}_2 \to \mathbf{k}_1, \mathbf{k}_2$ is often called **forward scattering** (the colliding particles continue along their initial trajectory).

Analytical properties of matrix elements:

We have already calculated some simple matrix elements in ϕ^4, Yukawa and QED cases but these matrix elements were always real. A non-zero imaginary part is typical for more complex diagrams involving loops, so that the intermediate (virtual) particles can be on shell and the $+i\epsilon$ prescription in the propagator becomes relevant. However, if the center-of-mass energy $\sqrt{s}$ is too low for the intermediate particles to go on shell, the matrix element remains real.

Physically s is always real, but let's be more liberal and extend the matrix element $\mathcal{M}(s)$, defined by the Feynman rules, to the complex plane in s. We define s_0 as the (real) threshold energy for the intermediate virtual particles to go on shell. If $s < s_0$ and $s \in \Re$, also $\mathcal{M}(s) \in \Re$, so that

$$\mathcal{M}(s) = [\mathcal{M}(s^*)]^* \, . \tag{5.62}$$

This equation is true for all $s < s_0$. From the complex analysis we know that in this case the above equation is fulfilled also in a larger region in the complex plane, where the functions are analytic. Let ϵ to be a small real number. Then,

$$\mathcal{M}(s+i\epsilon) = \mathrm{Re}\,\mathcal{M}(s+i\epsilon) + i\mathrm{Im}\,\mathcal{M}(s+i\epsilon) \tag{5.63}$$

$$\begin{aligned} [\mathcal{M}(s+i\epsilon)^*]^* &= [\mathcal{M}(s-i\epsilon)]^* \\ &= [\mathrm{Re}\,\mathcal{M}(s-i\epsilon) + i\mathrm{Im}\,\mathcal{M}(s-i\epsilon)]^* \\ &= \mathrm{Re}\,\mathcal{M}(s-i\epsilon) - i\mathrm{Im}\,\mathcal{M}(s-i\epsilon) \, . \end{aligned} \tag{5.64}$$

According to Eq. (5.62) these two are equal, so

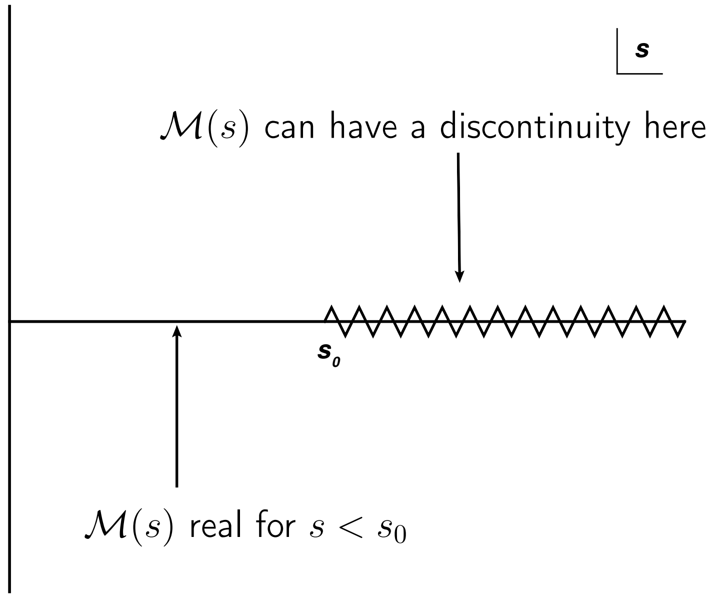
$$\mathrm{Re}\,\mathcal{M}(s+i\epsilon) = \mathrm{Re}\,\mathcal{M}(s-i\epsilon) \, , \tag{5.65}$$

$$\mathrm{Im}\,\mathcal{M}(s+i\epsilon) = -\mathrm{Im}\,\mathcal{M}(s-i\epsilon) \, . \tag{5.66}$$

We see that if the imaginary part of $\mathcal{M}(s)$ is non zero, there is a **discontinuity** across the real line,

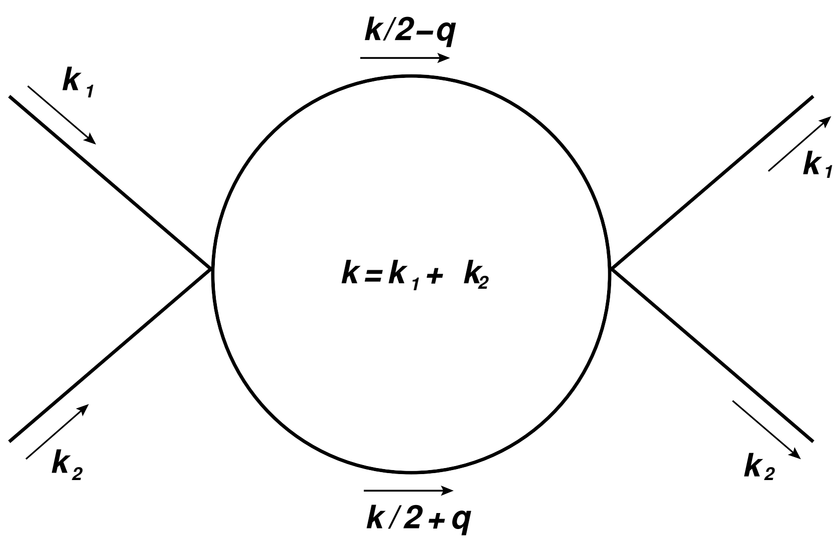
$$\mathcal{M}(s + i\epsilon) - \mathcal{M}(s - i\epsilon) = 2i\text{Im}\,\mathcal{M}(s + i\epsilon)\ . \tag{5.67}$$

When $s < s_0$, the imaginary part is zero and there's no discontiuity there, but when $s > s_0$ we can expect a discontinuity across the real line:



An example and the Cutkosky rules:

Let us now look into the matrix element (particularly its imaginary part) for process $\mathbf{k}_1, \mathbf{k}_2 \to \mathbf{k}_1, \mathbf{k}_2$ in ϕ^4 theory at order $\mathcal{O}(\lambda^2)$. Consider the diagram,



This clearly contains a closed loop. We will develop a systematic way to evaluate diagrams like this a bit later in the course. However, now we are mainly interested in the imaginary part of the diagram which we can calculate by considering the difference between $\mathcal{M}(s+i\epsilon)$ and $\mathcal{M}(s-i\epsilon)$. Our strategy is thus to compute the discontinuity across the real axis.

Using the Feynman rules we can readily write down the matrix element corresponding to the above diagram,

$$\begin{aligned} i\mathcal{M} &= \frac{1}{2}(-i\lambda)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{(k/2-q)^2 - m^2 + i\epsilon} \frac{i}{(k/2+q)^2 - m^2 + i\epsilon} \\ &= \frac{\lambda^2}{2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{(k/2-q)^2 - m^2 + i\epsilon} \frac{1}{(k/2+q)^2 - m^2 + i\epsilon}, \end{aligned} \tag{5.68}$$

where the overall $1/2$ is a combinatoric factor. We will use the residue theorem to evaluate the q^0 integral, so let us first solve for the locations of the poles. In the center-of-mass frame:

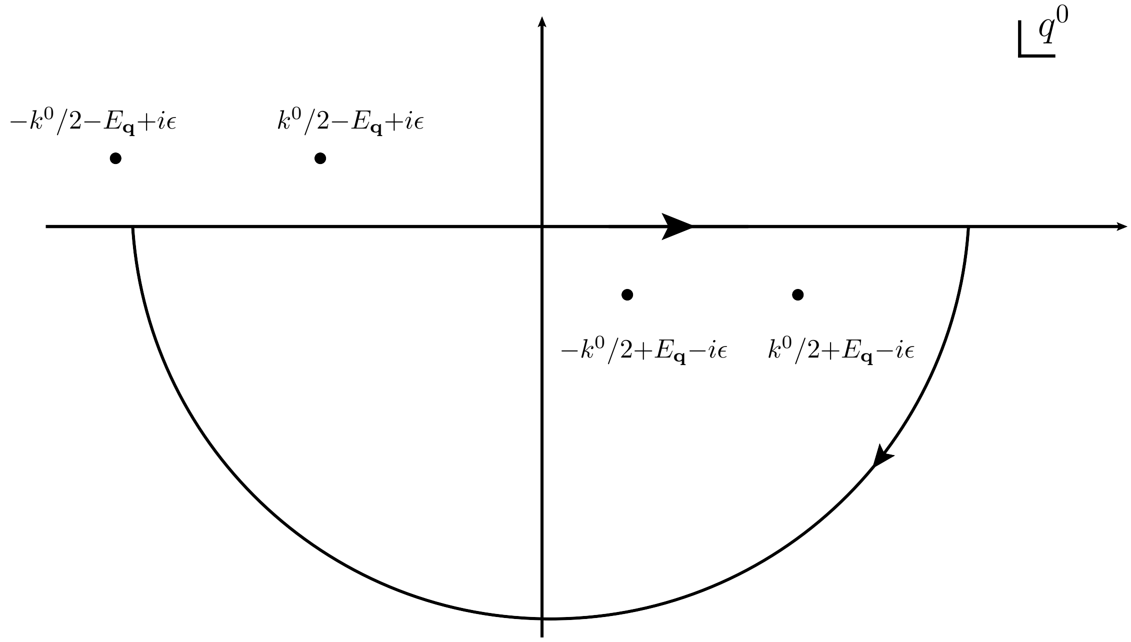
$$\begin{aligned} (k/2 \pm q)^2 - m^2 + i\epsilon &= (k^0/2 \pm q^0)^2 - \mathbf{q}^2 - m^2 + i\epsilon \\ &= (k^0/2 \pm q^0)^2 - E_{\mathbf{q}}^2 + i\epsilon = 0 \qquad \big| E_{\mathbf{q}}^2 \equiv \mathbf{q}^2 + m^2 \end{aligned} \tag{5.69}$$

$$\implies (k^0/2 \pm q^0)^2 = E_{\mathbf{q}}^2 - i\epsilon$$

$$k^0/2 \pm q^0 = \pm (E_{\mathbf{q}} - i\epsilon) \qquad \big| E_{\mathbf{q}} > 0$$

$$\underline{q^0 = \frac{k^0}{2} \pm (E_{\mathbf{q}} - i\epsilon)}, \quad \underline{q^0 = -\frac{k^0}{2} \pm (E_{\mathbf{q}} - i\epsilon)}\ .$$

We see that the integrand has in total 4 poles in the complex q^0 plane:



The integrand is strongly suppressed in the limit $|q^0| \to \infty$ so we can close the integration contour either in the upper or the lower half plane. We choose the lower one, so the integration contour encloses the poles $q^0 = -k^0/2 + E_{\mathbf{q}} - i\epsilon$ and $q^0 = k^0/2 + E_{\mathbf{q}} - i\epsilon$:

$$\int dq^0 \frac{1}{(k/2-q)^2 - m^2 + i\epsilon} \frac{1}{(k/2+q)^2 - m^2 + i\epsilon} \tag{5.70}$$

$$= -2\pi i \left[\text{Res}\left(q^0 = -k^0/2 + E_{\mathbf{q}} - i\epsilon\right) + \text{Res}\left(q^0 = k^0/2 + E_{\mathbf{q}} - i\epsilon\right) \right]$$

The residues are:

- $\text{Res}\left(q^0 = -k^0/2 + E_{\mathbf{q}} - i\epsilon\right)$ (5.71)

$$= \lim_{q^0 \to -k^0/2+E_{\mathbf{q}}-i\epsilon} \frac{q^0 - \left(-k^0/2 + E_{\mathbf{q}} - i\epsilon\right)}{\left[(k/2-q)^2 - m^2 + i\epsilon\right]\left[(k/2+q)^2 - m^2 + i\epsilon\right]}$$

$$= \lim_{q^0 \to -k^0/2+E_{\mathbf{q}}-i\epsilon} \frac{q^0 - \left(-k^0/2 + E_{\mathbf{q}} - i\epsilon\right)}{\left[q^0 - (k^0/2 - E_{\mathbf{q}} + i\epsilon)\right]\left[q^0 - (k^0/2 + E_{\mathbf{q}} - i\epsilon)\right]}$$

$$\times \frac{1}{\left[q^0 - (-k^0/2 - E_{\mathbf{q}} + i\epsilon)\right]\left[q^0 - (-k^0/2 + E_{\mathbf{q}} - i\epsilon)\right]}$$

$$= \frac{1}{\left[2E_{\mathbf{q}} - k^0\right]\left[-k^0\right]\left[2E_{\mathbf{q}}\right]}$$

- $\text{Res}\left(q^0 = k^0/2 + E_{\mathbf{q}} - i\epsilon\right)$ (5.72)

$$= \lim_{q^0 \to k^0/2+E_{\mathbf{q}}-i\epsilon} \frac{q^0 - \left(k^0/2 + E_{\mathbf{q}} - i\epsilon\right)}{\left[q^0 - (k^0/2 - E_{\mathbf{q}} + i\epsilon)\right]\left[q^0 - (k^0/2 + E_{\mathbf{q}} - i\epsilon)\right]}$$

$$\times \frac{1}{\left[q^0 - (-k^0/2 - E_{\mathbf{q}} + i\epsilon)\right]\left[q^0 - (-k^0/2 + E_{\mathbf{q}} - i\epsilon)\right]}$$

$$= \frac{1}{\left[2E_{\mathbf{q}}\right]\left[2E_{\mathbf{q}} + k^0\right]\left[k^0\right]}$$

Thus,

$$\int dq^0 \frac{1}{(k/2-q)^2 - m^2 + i\epsilon} \frac{1}{(k/2+q)^2 - m^2 + i\epsilon} \tag{5.73}$$

$$= -2\pi i \left[\frac{1}{\left[2E_{\mathbf{q}} - k^0\right]\left[-k^0\right]\left[2E_{\mathbf{q}}\right]} + \frac{1}{\left[2E_{\mathbf{q}}\right]\left[2E_{\mathbf{q}} + k^0\right]\left[k^0\right]}\right]$$

The latter term does not lead to discontinuities $\left(E_{\mathbf{q}} > 0,\ \text{Re}\, k^0 > 0\right)$ so we forget that one. By going to spherical coordinates,

$$d^3q = d\Omega |\mathbf{q}|^2 d|\mathbf{q}|\,. \tag{5.74}$$

Since $E_{\mathbf{q}}^2 = \mathbf{q}^2 + m^2$, we have $2E_{\mathbf{q}} dE_{\mathbf{q}} = 2|\mathbf{q}|d|\mathbf{q}|$, i.e.

$$d^3q = d\Omega |\mathbf{q}| E_{\mathbf{q}} dE_{\mathbf{q}}\,. \tag{5.75}$$

The relevant part of the matrix element thus becomes,

$$i\mathcal{M} = \frac{-2\pi i\frac{\lambda^2}{2}}{(2\pi)^4}\int d^3q \frac{1}{[2E_{\mathbf{q}} - k^0]\,[-k^0]\,[2E_{\mathbf{q}}]} \tag{5.76}$$

$$= \frac{-i\frac{\lambda^2}{2}}{(2\pi)^3}\int \frac{d\Omega|\mathbf{q}|E_{\mathbf{q}}dE_{\mathbf{q}}}{[k^0 - 2E_{\mathbf{q}}]\,[k^0]\,[2E_{\mathbf{q}}]}$$

$$= \frac{-i\frac{\lambda^2}{2}}{(2\pi)^3}\frac{4\pi}{2}\int_m^\infty dE_{\mathbf{q}}\frac{\sqrt{E_{\mathbf{q}}^2 - m^2}}{[k^0 - 2E_{\mathbf{q}}]\,[k^0]}\,.$$

The integrand clearly has a pole at $E_{\mathbf{q}} = k^0/2$. However, if $k^0 < 2m$,

$$k^0 - 2E_{\mathbf{q}} < 0\,, \tag{5.77}$$

and $\mathcal{M}$ is real (though infinite...), as expected. If then $k^0 > 2m$, the pole is in the contour of integration and the result will depend on which side of the real axis $k^0 = \sqrt{s}$ is. Based on Eq. (5.67) what we need is a difference between these integrals at $\sqrt{s} = k^0 + i\epsilon$ and $\sqrt{s} = k^0 - i\epsilon$. Let's therefore calculate,

$$\frac{-i\lambda^2}{8\pi^2}\int_m^\infty \frac{dE_{\mathbf{q}}}{k^0}\sqrt{E_{\mathbf{q}}^2 - m^2}\left[\frac{1}{k^0 - 2E_{\mathbf{q}} + i\epsilon} - \frac{1}{k^0 - 2E_{\mathbf{q}} - i\epsilon}\right]\,.$$

By combining the denominators,

$$\left[\frac{1}{k^0 - 2E_{\mathbf{q}} + i\epsilon} - \frac{1}{k^0 - 2E_{\mathbf{q}} - i\epsilon}\right] = \frac{-2i\epsilon}{\left(k^0 - 2E_{\mathbf{q}}\right)^2 + \epsilon^2} \tag{5.78}$$

$$\xrightarrow{\epsilon\to 0} -2i\pi\delta\left(k^0 - 2E_{\mathbf{q}}\right)\,, \tag{5.79}$$

where we used a δ-function identity

$$\delta(x) = \lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon}{x^2 + \epsilon^2}\,. \tag{5.80}$$

Thus the discontinuity of $i\mathcal{M}$ is

$$
\begin{aligned}
i\delta\mathcal{M} &= \frac{-i\lambda^2}{8\pi^2}\int_m^\infty \frac{dE_{\mathbf{q}}}{k^0}\sqrt{E_{\mathbf{q}}^2 - m^2}\left[\frac{1}{k^0 - 2E_{\mathbf{q}} + i\epsilon} - \frac{1}{k^0 - 2E_{\mathbf{q}} - i\epsilon}\right] \\
&= \frac{-\lambda^2}{4\pi}\int_m^\infty \frac{dE_{\mathbf{q}}}{k^0}\sqrt{E_{\mathbf{q}}^2 - m^2}\,\delta\left(k^0 - 2E_{\mathbf{q}}\right) \\
&= \frac{-\lambda^2}{8\pi}\frac{1}{k^0}\sqrt{\left(\frac{k^0}{2}\right)^2 - m^2}\,. \qquad (5.81)
\end{aligned}
$$

According to Eq. (5.67) we have,

$$
\delta\mathcal{M} = \frac{i\lambda^2}{8\pi}\frac{1}{k^0}\sqrt{\left(\frac{k^0}{2}\right)^2 - m^2} = 2i\mathrm{Im}\,\mathcal{M}(s) \qquad (5.82)
$$

$$
\Longrightarrow \mathrm{Im}\ \mathcal{M}(s) = \frac{\lambda^2}{16\pi}\frac{1}{\sqrt{s}}\sqrt{\frac{s}{4} - m^2}\,. \qquad (5.83)
$$

In the center-of-mass frame $k_1^2 = m^2 = E_1^2 - \mathbf{p}_{\mathrm{cm}}^2 = s/4 - \mathbf{p}_{\mathrm{cm}}^2$, so $|\mathbf{p}_{\mathrm{cm}}| = \sqrt{s/4 - m^2}$, and our final form for the imaginary part is

$$
\mathrm{Im}\ \mathcal{M}(s) = \frac{\lambda^2}{16\pi}\frac{|\mathbf{p}_{\mathrm{cm}}|}{\sqrt{s}}\,. \qquad (5.84)
$$

The optical theorem (5.61) says that this should correspond to the total cross section,

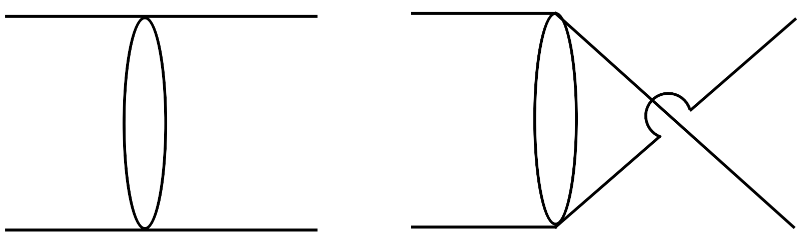
$$
\mathrm{Im}\ \mathcal{M}(s) = \frac{\lambda^2}{16\pi}\frac{|\mathbf{p}_{\mathrm{cm}}|}{\sqrt{s}} = 2\sqrt{s}|\mathbf{p}_{\mathrm{cm}}|\sigma_{\mathrm{total}}(\mathbf{k}_1, \mathbf{k}_2 \to X)\,. \qquad (5.85)
$$

or

$$
\sigma_{\mathrm{total}}(\mathbf{k}_1, \mathbf{k}_2 \to X) = \frac{\lambda^2}{32\pi s}\,. \qquad (5.86)
$$

This agrees exactly with the result (3.132) we computed earlier! Thus, we have explicitly verified the optical theorem in this particular case.

The following loop diagrams are of the same order in coupling constant:



These, however, do not produce a discontinuity like the diagram considered above.

Cutkosky rules:

The calculation above was straightforward, yet a bit messy. We can reach the same result through an easier method. We note that the contribution of the imaginary part came from a kinematical point,

$$q^0 = -k^0/2 + E_{\mathbf{q}} - i\epsilon \tag{5.87}$$

$$E_{\mathbf{q}} = \sqrt{\mathbf{q}^2 + m^2} = k^0/2\,, \tag{5.88}$$

at which both virtual particles are on shell,

$$\left[(k/2 \pm q)^2 - m^2\right] = 0\,. \tag{5.89}$$

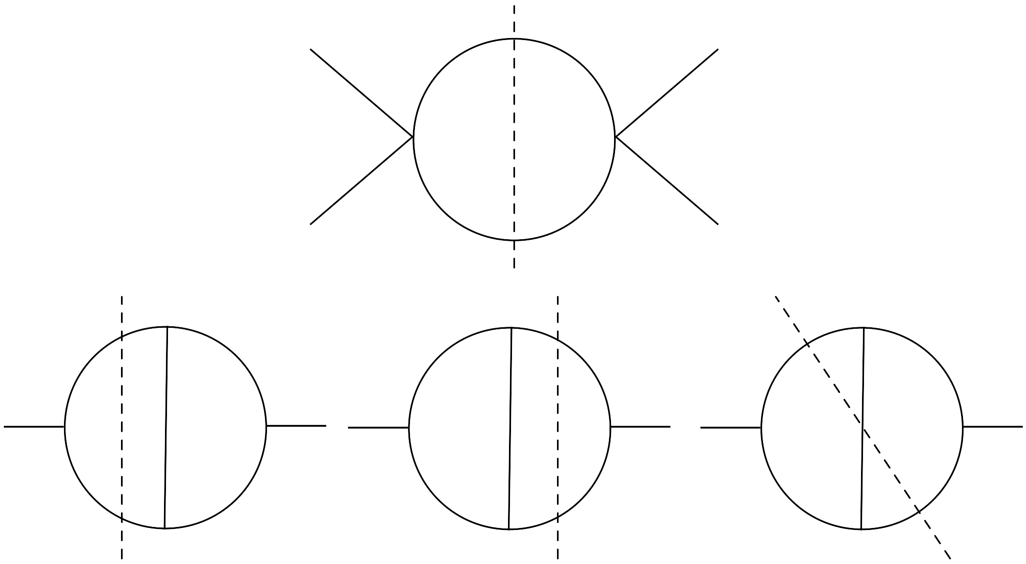
The contribution of a specific kinematic point is naturally obtained by setting appropriate δ functions. Indeed, if we replace in the original matrix element (5.68) the propagators by the following δ functions,

$$\frac{i}{(k/2+q)^2 - m^2 + i\epsilon} \longrightarrow -2\pi i\delta\left[(k/2+q)^2 - m^2\right] \tag{5.90}$$

$$\frac{i}{(k/2-q)^2 - m^2 + i\epsilon} \longrightarrow -2\pi i\delta\left[(k/2-q)^2 - m^2\right]\,, \tag{5.91}$$

it is straightforward to check (Ex.) that the result is nothing but the discontinuity $i\delta\mathcal{M}$ in Eq. (5.81). This is, of course, not by accident but an example of the so-called **Cutkosky rules**.

The Cutkosky rulse use the notion of a **cut diagram**. We call a cut such a line that splits the diagram into two connected pieces:



The first one is just the diagram considered above, and the latter ones a examples of cut diagrams in ϕ^3 theory.

Cutkosky rules for calculating the discontinuity of a diagram:

1. Draw all the possible cuts. Retain only those in which the virtual particles crossed by the cut can be kinematically on shell at the same time.

2. In cut propagators replace

$$\frac{1}{p^2 - m^2 + i\epsilon} \to -2\pi i \delta\left(p^2 - m^2\right) ,$$

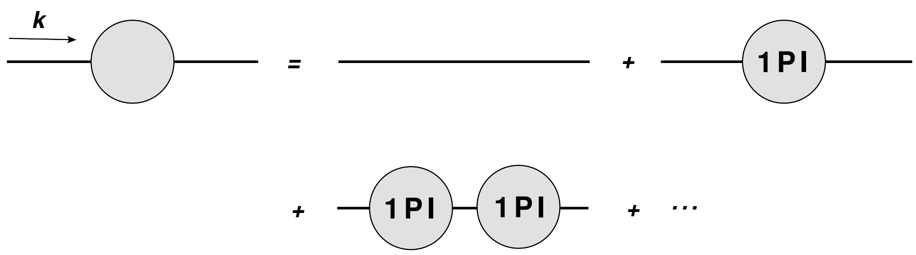
and integrate over undetermined momenta.

3. Sum the results of all possible cuts.

5.4 Unstable particles

We will now derive a general expression for the decay widths of unstable particles. We denoted eariler by $\mathbf{FP}(k)$ the propagator including all virtual corrections,

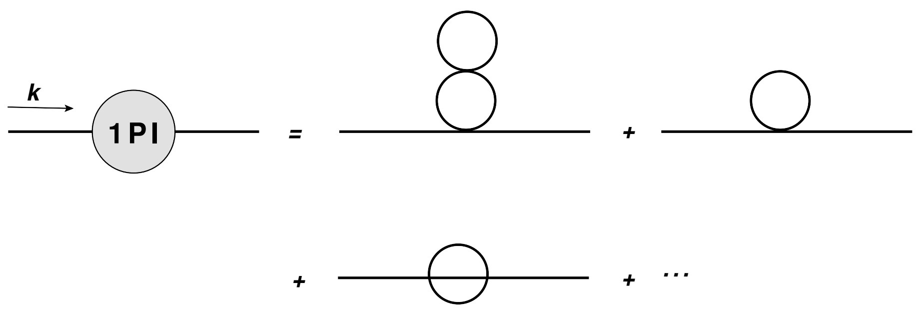
$$\mathbf{FP}(k) =$$



$$= \frac{i}{k^2 - m_0^2 - M^2(k^2) + i\epsilon},$$

where each 1PI blob corresponds to $-iM^2(k^2)$,

$$-iM^2(k^2) =$$



Let's now apply the LSZ theorem in the case that the initial and final state contain the same single particle. Clearly, all the amputated diagrams are exactly those that comprise $-iM^2(k^2)$. Both sides of the diagram contribute one $\sqrt{Z}$, so

$$i\mathcal{M}(k \to k) = -iZM^2(k^2)\,. \tag{5.92}$$

On the other hand, the optical theorem (5.59) gives,

$$2\mathrm{Im}\mathcal{M}\left(k \to k\right) = \sum_f \int d\Gamma_f |\mathcal{M}\left(k \to f\right)|^2\,. \tag{5.93}$$

If we consider e.g. the ϕ^4 theory, a on-shell ϕ particle cannot decay. Thus the right-hand side of Eq. (5.93) is zero, from which it follows that also $\mathrm{Im}\mathcal{M}\left(k \to k\right) = 0$, which means that $\mathcal{M}\left(k \to k\right)$ is real. Thus also $M^2(k^2)$ is real and the equation

$$k^2 - m_0^2 - M^2(k^2) \overset{k^2 \to m^2}{=} 0\,, \tag{5.94}$$

has a real-valued solution m. Near $k^2 \sim m^2$ the propagator $\mathbf{FP}(k)$ is of the form,

$$\mathbf{FP}(k) \overset{k^2 \to m^2}{=} \frac{iZ}{k^2 - m^2 + i\epsilon} + \cdots \tag{5.95}$$

$$Z^{-1} = 1 - \frac{dM^2(k^2)}{dk^2}\bigg|_{k^2=m^2}\,. \tag{5.96}$$

Let us the suppose that we have several interacting particles (e.g. Yukawa teory) arranged so that ϕ can decay. In this case the right-hand side of Eq. (5.93) is $\neq 0$, and thus $\mathcal{M}\left(k \to k\right)$ and also $M^2(k^2)$ have an imaginary part. We will generalize the definition of the physical mass m to be the solution of the equation,

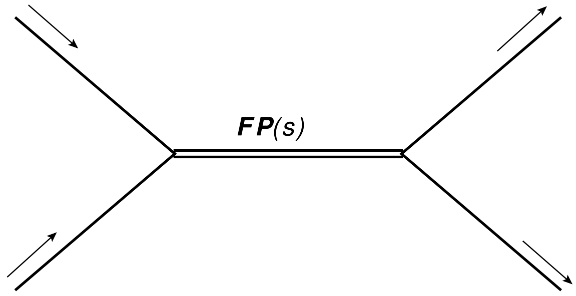
$$k^2 - m_0^2 - \mathrm{Re}\,M^2(k^2) \overset{k^2 \to m^2}{=} 0\,. \tag{5.97}$$

In the neighbourhood of $k^2 \sim m^2$ the propagator looks like,

$$\mathbf{FP}(k) \overset{k^2 \to m^2}{=} \frac{iZ}{k^2 - m^2 - iZ\mathrm{Im}\,M^2(k^2) + i\epsilon} + \cdots \tag{5.98}$$

$$Z^{-1} = 1 - \frac{d\mathrm{Re}\,M^2(k^2)}{dk^2}\bigg|_{k^2=m^2}\,. \tag{5.99}$$

If this propagator occurs e.g. in diagram,



the cross section will be of the form,

$$\sigma \propto \left| \frac{1}{s - m^2 - iZ\mathrm{Im}\, M^2(s^2)} \right|^2 = \frac{1}{\left[s - m^2\right]^2 + \left[Z\mathrm{Im}\, M^2(s^2)\right]^2} \quad (5.100)$$

If the imaginary part $\mathrm{Im}\, M^2(s)$ is "small", the above function is peaked around $s = m^2$, and we can approximate $\mathrm{Im}\, M^2(s) \approx \mathrm{Im}\, M^2(m^2)$, if $s \sim m^2$. Then,

$$\sigma \propto \frac{1}{\left(s - m^2\right)^2 + \left(m\Gamma\right)^2} \quad (5.101)$$

$$\Gamma = -\frac{Z}{m}\mathrm{Im}\, M^2(m^2)\,. \quad (5.102)$$

The form, (5.101) is generally known as the **Breit-Wigner resonance**, and the constant Γ is the **decay width**. From Eqs. (5.92) and (5.93) we now get,

$$\Gamma = -\frac{Z}{m}\mathrm{Im}\, M^2(k^2) = \frac{1}{m}\mathrm{Im}\mathcal{M}\left(k \to k\right) \;= \frac{1}{2m}\sum_f \int d\Gamma_f |\mathcal{M}\left(k \to f\right)|^2\,.$$

so,

$$\Gamma = \frac{1}{2m}\sum_f \int d\Gamma_f |\mathcal{M}\left(k \to f\right)|^2\,. \quad (5.103)$$

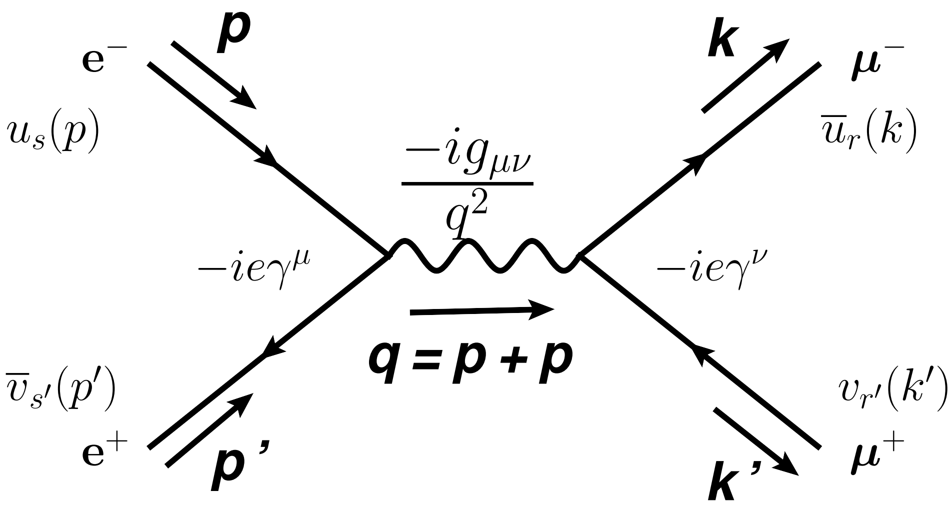
We can interpret the decay width Γ also as the **decay rate**, and its inverse $1/\Gamma$ as the particle's **lifetime**.

6 Basic QED processes

In this section we will go through some QED results and related calculational techniques. The content of this section is probably familiar to most from the Particle Physics course.

6.1 $e^+e^- \to \mu^+\mu^-$

Probably the most simple QED process is the electron-positron annihilation into muon-antimuon pair:



We work in the Feynman gauge in which this diagram corresponds to the matrix element,

$$i\mathcal{M} = [\overline{u}_r(k)\,(-ie\gamma^\nu)\,v_{r'}(k')]\ \frac{-ig_{\mu\nu}}{q^2}\ [\overline{v}_{s'}(p')\,(-ie\gamma^\mu)\,u_s(p)] \tag{6.1}$$

$$= \frac{ie^2}{q^2}\,[\overline{u}_r(k)\gamma^\mu v_{r'}(k')]\,[\overline{v}_{s'}(p')\gamma_\mu u_s(p)]\ . \tag{6.2}$$

To compute the cross section we need $|\mathcal{M}|^2$ so we complex conjugate this:

$$(i\mathcal{M})^* = \frac{-ie^2}{q^2}\,[\overline{u}_r(k)\gamma^\nu v_{r'}(k')]^*\,[\overline{v}_{s'}(p')\gamma_\nu u_s(p)]^*\ . \tag{6.3}$$

The complex conjugation of each of the terms in square brackets proceeds with the aid of a couple of γ-matrix identities,

$$
\begin{aligned}
[\overline{u}_r(k)\gamma^\nu v_{r'}(k')]^* &= [\overline{u}_r(k)\gamma^\nu v_{r'}(k')]^\dagger = v^\dagger_{r'}(k')\gamma^{\nu\dagger}\overline{u}^\dagger_r(k) \\
&= v^\dagger_{r'}(k')\gamma^{\nu\dagger}\left[u^\dagger_r(k)\gamma^0\right]^\dagger \\
&= v^\dagger_{r'}(k')\gamma^{\nu\dagger}\gamma^{0\dagger}u_r(k) \\
&= v^\dagger_{r'}(k')\gamma^0\gamma^0\gamma^{\nu\dagger}\gamma^0 u_r(k) \qquad \Big|\gamma^{0\dagger} = \gamma^0\,,\ \gamma^0\gamma^0 = 1 \\
&= \overline{v}_{r'}(k')\left[\gamma^0\gamma^{\nu\dagger}\gamma^0\right]u_r(k) \\
&= \overline{v}_{r'}(k')\gamma^\nu u_r(k) \qquad \Big|\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu\,.
\end{aligned}
\tag{6.4}
$$

The second one goes analogously, so we have our $|\mathcal{M}|^2$,

$$
\begin{aligned}
|\mathcal{M}|^2 = \frac{e^4}{q^4}&\left[\overline{u}_r(k)\gamma^\mu v_{r'}(k')\right]\left[\overline{v}_{r'}(k')\gamma^\nu u_r(k)\right] \\
&\left[\overline{v}_{s'}(p')\gamma_\mu u_s(p)\right]\left[\overline{u}_s(p)\gamma_\nu v_{s'}(p')\right]\,.
\end{aligned}
\tag{6.5}
$$

By writing out the indices of the matrix products we see that the terms in the upper and lower lines can be turned into a trace:

$$
\begin{aligned}
&\left[\overline{u}_r(k)\gamma^\mu v_{r'}(k')\right]\left[\overline{v}_{r'}(k')\gamma^\nu u_r(k)\right] \\
&= [\overline{u}_r(k)]_a[\gamma^\mu]_{ab}[v_{r'}(k')]_b\ [\overline{v}_{r'}(k')]_c[\gamma^\nu]_{cd}[u_r(k)]_d \\
&= [u_r(k)]_d[\overline{u}_r(k)]_a[\gamma^\mu]_{ab}[v_{r'}(k')]_b\ [\overline{v}_{r'}(k')]_c[\gamma^\nu]_{cd} \\
&= [u_r(k)\overline{u}_r(k)]_{da}[\gamma^\mu]_{ab}[v_{r'}(k')\overline{v}_{r'}(k')]_{bc}[\gamma^\nu]_{cd} \\
&= [u_r(k)\overline{u}_r(k)\gamma^\mu]_{db}[v_{r'}(k')\overline{v}_{r'}(k')\gamma^\nu]_{bd} \\
&= [u_r(k)\overline{u}_r(k)\gamma^\mu v_{r'}(k')\overline{v}_{r'}(k')\gamma^\nu]_{dd} \\
&= \mathrm{Tr}\left[u_r(k)\overline{u}_r(k)\gamma^\mu v_{r'}(k')\overline{v}_{r'}(k')\gamma^\nu\right]\,,
\end{aligned}
\tag{6.6}
$$

and we have,

$$|\mathcal{M}|^2 = \frac{e^4}{q^4} \mathrm{Tr}\left[u_r(k)\overline{u}_r(k)\gamma^\mu v_{r'}(k')\overline{v}_{r'}(k')\gamma^\nu\right] \qquad (6.7)$$

$$\mathrm{Tr}\left[v_{s'}(p')\overline{v}_{s'}(p')\gamma_\mu u_s(p)\overline{u}_s(p)\gamma_\nu\right] .$$

In the most simple case the electron and positron beams are **unpolarized** so that they contain both spin states in same proportion. In this case we **average** over the initial-state spins,

$$\frac{1}{2}\sum_s \frac{1}{2}\sum_{s'} |\mathcal{M}|^2 . \qquad (6.8)$$

The particle detectors seldomly resolve the spin states of the muons and in this case we also sum over the final-state spins,

$$\overline{|\mathcal{M}|^2} \equiv \frac{1}{2}\sum_s \frac{1}{2}\sum_{s'} \sum_r \sum_{r'} |\mathcal{M}|^2 . \qquad (6.9)$$

This is handy, as we can directly use the relations (2.22),

$$\sum_{s=1}^{2} u_s(p)\overline{u}_s(p) = \not{p} + m , \qquad (6.10)$$

$$\sum_{s=1}^{2} v_s(p)\overline{v}_s(p) = \not{p} - m . \qquad (6.11)$$

Thus, computing $|\mathcal{M}|^2$ reduces to traces of γ matrics,

$$|\mathcal{M}|^2 = \frac{e^4}{4q^4} \mathrm{Tr}\left[(\not{k} + m_\mu)\,\gamma^\mu\,(\not{k}' - m_\mu)\,\gamma^\nu\right] \mathrm{Tr}\left[(\not{p}' - m_e)\,\gamma_\mu\,(\not{p} + m_e)\,\gamma_\nu\right] . \qquad (6.12)$$

To simplify expressions like this there are tons of formulae, of which the most important are collected in the next page.

$$\mathrm{Tr}\left[\mathbb{1}\right] = 4 \tag{6.13}$$

$$\mathrm{Tr}\left[\text{odd n.o. of } \gamma \text{ matrices}\right] = 0 \tag{6.14}$$

$$\mathrm{Tr}\left[\gamma^\mu \gamma^\nu\right] = 4 g^{\mu\nu} \tag{6.15}$$

$$\mathrm{Tr}\left[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma\right] = 4\left[g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}\right] \tag{6.16}$$

$$\mathrm{Tr}\left[\gamma^5\right] = 0 \tag{6.17}$$

$$\mathrm{Tr}\left[\gamma^\mu \gamma^\nu \gamma^5\right] = 0 \tag{6.18}$$

$$\mathrm{Tr}\left[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5\right] = -4 i \epsilon^{\mu\nu\rho\sigma} \tag{6.19}$$

$$\epsilon^{\alpha\beta\mu\nu} \epsilon_{\alpha\beta\rho\sigma} = -2\left(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho\right) \tag{6.20}$$

$$\epsilon^{\alpha\beta\mu\nu} \epsilon_{\alpha\beta\mu\sigma} = -6 \delta^\nu_\sigma \tag{6.21}$$

$$\epsilon^{\alpha\beta\mu\nu} \epsilon_{\alpha\beta\mu\nu} = -24 \tag{6.22}$$

$$\gamma^\mu \gamma_\mu = 4 \tag{6.23}$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -2 \gamma^\nu \tag{6.24}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4 g^{\nu\rho} \tag{6.25}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2 \gamma^\sigma \gamma^\rho \gamma^\nu \tag{6.26}$$

$$\gamma^5 \gamma^\mu \gamma^\nu \gamma^\eta = g^{\nu\eta} \gamma^5 \gamma^\mu - g^{\mu\eta} \gamma^5 \gamma^\nu + g^{\mu\nu} \gamma^5 \gamma^\eta - i \epsilon^{\mu\nu\eta\rho} \gamma_\rho$$

$$\mathrm{Tr}\left[\gamma^{\mu_1} \gamma^{\mu_2} \cdots \gamma^{\mu_n}\right] = \mathrm{Tr}\left[\gamma^{\mu_n} \cdots \gamma^{\mu_2} \gamma^{\mu_1}\right] \tag{6.27}$$

By using these identities, computing the traces is easy:

- $$\mathrm{Tr}\left[\left(\not{p}' - m_e\right)\gamma_\mu\left(\not{p} + m_e\right)\gamma_\nu\right] \tag{6.28}$$

$$= \mathrm{Tr}\left[\not{p}'\gamma_\mu\not{p}\gamma_\nu\right] - m_e\underbrace{\mathrm{Tr}\left[\gamma_\mu\not{p}\gamma_\nu\right]}_{=0} + m_e\underbrace{\mathrm{Tr}\left[\not{p}'\gamma_\mu\gamma_\nu\right]}_{=0} - m_e^2\mathrm{Tr}\left[\gamma_\mu\gamma_\nu\right]$$

$$= p'^\rho p^\sigma \mathrm{Tr}\left[\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu\right] - 4m_e^2 g_{\mu\nu}$$

$$= p'^\rho p^\sigma 4\left[g_{\rho\mu}g_{\sigma\nu} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\nu}g_{\mu\sigma}\right] - 4m_e^2 g_{\mu\nu}$$

$$= 4\left[p'_\mu p_\nu - (p'\cdot p)g_{\mu\nu} + p'_\nu p_\mu\right] - 4m_e^2 g_{\mu\nu}$$

$$= 4\left[p'_\mu p_\nu + p'_\nu p_\mu - g_{\mu\nu}\left(m_e^2 + p'\cdot p\right)\right] \; .$$

- $$\mathrm{Tr}\left[\left(\not{k} + m_\mu\right)\gamma^\mu\left(\not{k}' - m_\mu\right)\gamma^\nu\right] \tag{6.29}$$

$$= 4\left[k^\mu k'^\nu + k^\nu k'^\mu - g_{\mu\nu}\left(m_\mu^2 + k'\cdot k\right)\right] \; .$$

Since the mass of the electron $m_e \approx 0.5\,\mathrm{MeV}$ is much smaller than the muon mass $m_\mu \approx 105\,\mathrm{MeV}$, we can safely set $m_e = 0$. Our matrix-element squared becomes then,

$$|\mathcal{M}|^2 = \frac{4e^4}{q^4}\left[k^\mu k'^\nu + k^\nu k'^\mu - g_{\mu\nu}\left(m_\mu^2 + k'\cdot k\right)\right]\left[p'_\mu p_\nu + p'_\nu p_\mu - g_{\mu\nu}\left(p'\cdot p\right)\right]$$

$$= \frac{4e^4}{q^4}\Big[(k\cdot p')(k'\cdot p) + (k\cdot p)(k'\cdot p') - (p\cdot p')(k\cdot k')$$

$$+ (k\cdot p)(k'\cdot p') + (k\cdot p')(k'\cdot p) - (p\cdot p')(k\cdot k')$$

$$- 2(p\cdot p')\left(m_\mu^2 + k'\cdot k\right) + 4\left(m_\mu^2 + k'\cdot k\right)(p\cdot p')\Big]$$

$$= \frac{8e^4}{q^4}\Big[(k\cdot p')(k'\cdot p) + (k\cdot p)(k'\cdot p') - (p\cdot p')(k\cdot k')$$

$$+ \left(m_\mu^2 + k'\cdot k\right)(p\cdot p')\Big]$$

$$= \frac{8e^4}{q^4}\Big[(k\cdot p')(k'\cdot p) + (k\cdot p)(k'\cdot p') + m_\mu^2(p\cdot p')\Big] \tag{6.30}$$

This is explicitly Lorentz invariant. More concrete expression is obtained by choosing some frame of reference. Typical choices include **the rest frame** of some massive particle, **center-of-mass frame** of some selected particles, or **brick-wall/Breit frame** in which the momentum of a projectile particle gets reversed. Here the easiest choice is the center-of-mass frame of the e^+e^- pair (which is the center-of-mass frame of the $\mu^+\mu^-$ pair as well). Let's choose the momenta as follows:

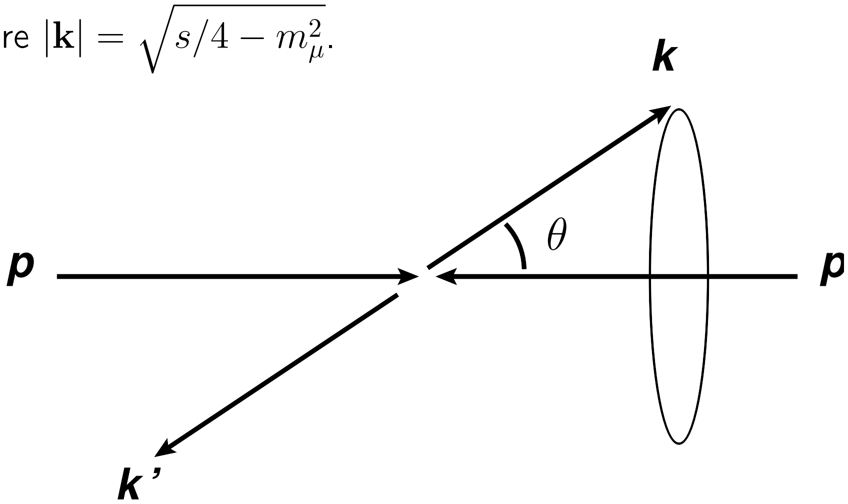
$$p = \left(\frac{\sqrt{s}}{2}, 0, 0, \frac{\sqrt{s}}{2}\right) \tag{6.31}$$

$$p' = \left(\frac{\sqrt{s}}{2}, 0, 0, -\frac{\sqrt{s}}{2}\right) \tag{6.32}$$

$$k = \left(\frac{\sqrt{s}}{2}, |\mathbf{k}|\sin\theta\sin\phi, |\mathbf{k}|\sin\theta\cos\phi, |\mathbf{k}|\cos\theta\right) \tag{6.33}$$

$$k' = \left(\frac{\sqrt{s}}{2}, -|\mathbf{k}|\sin\theta\sin\phi, -|\mathbf{k}|\sin\theta\cos\phi, -|\mathbf{k}|\cos\theta\right), \tag{6.34}$$

where $|\mathbf{k}| = \sqrt{s/4 - m_\mu^2}$.



The required dot products are easy to evaluate,

$$k \cdot p = k' \cdot p' = \frac{1}{2}\left[\frac{s}{2} - \sqrt{s}|\mathbf{k}|\cos\theta\right] \tag{6.35}$$

$$k \cdot p' = k' \cdot p = \frac{1}{2}\left[\frac{s}{2} + \sqrt{s}|\mathbf{k}|\cos\theta\right] \tag{6.36}$$

$$p \cdot p' = \frac{s}{2} \tag{6.37}$$

$$q^2 = (k + k')^2 = s \tag{6.38}$$

We substitute these into the expression of the squared matrix element:

$$
\begin{aligned}
|\mathcal{M}|^2 &= \frac{8e^4}{q^4}\Big[(k\cdot p')(k'\cdot p) + (k\cdot p)(k'\cdot p') + m_\mu^2(p\cdot p')\Big] \qquad (6.39)\\
&= \frac{8e^4}{q^4}\left\{\frac{1}{4}\left[\frac{s}{2} + \sqrt{s}|\mathbf{k}|\cos\theta\right]^2 + \frac{1}{4}\left[\frac{s}{2} - \sqrt{s}|\mathbf{k}|\cos\theta\right]^2 + m_\mu^2\frac{s}{2}\right\}\\
&= \frac{8e^4}{q^4}\left\{\frac{1}{4}\left[\frac{s^2}{2} + 2s|\mathbf{k}|^2\cos^2\theta\right] + m_\mu^2\frac{s}{2}\right\}\\
&= \frac{s^2e^4}{q^4}\left\{\left[1 + \frac{4|\mathbf{k}|^2}{s}\cos^2\theta\right] + \frac{4m_\mu^2}{s}\right\}\\
&= \frac{s^2e^4}{q^4}\left\{\left[\frac{4\left(\frac{s}{4} - m_\mu^2\right)}{s}\cos^2\theta\right] + \left(1 + \frac{4m_\mu^2}{s}\right)\right\}\\
&= e^4\left[\left(1 - \frac{4m_\mu^2}{s}\right)\cos^2\theta + \left(1 + \frac{4m_\mu^2}{s}\right)\right]
\end{aligned}
$$

We then use the general formula for the cross section (3.109). The flux factor F is now

$$
F = 4\sqrt{(p\cdot p')^2 - m_e^2m_e^2} = 4\sqrt{(s/2)^2} = 2s\,, \qquad (6.40)
$$

and the two-particle phase-space element (3.111) in the center-of-mass frame,

$$
\Gamma_2 = \int d\Omega\frac{|\mathbf{k}|}{16\pi^2\sqrt{s}} = \int d\Omega\frac{1}{16\pi^2\sqrt{s}}\sqrt{\frac{s}{4} - m_\mu^2}\,. \qquad (6.41)
$$

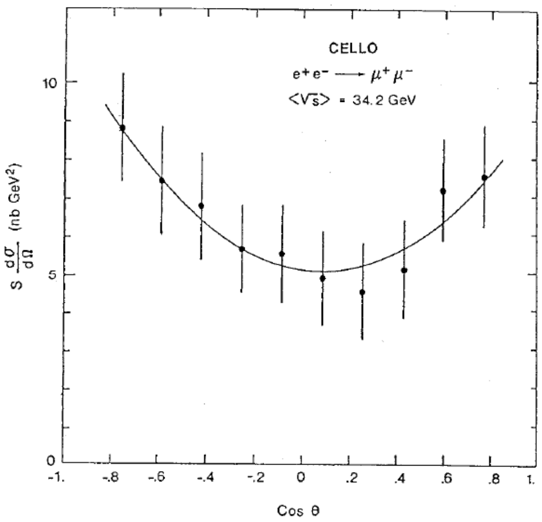
Putting all together gives us the leading-order differential cross section,

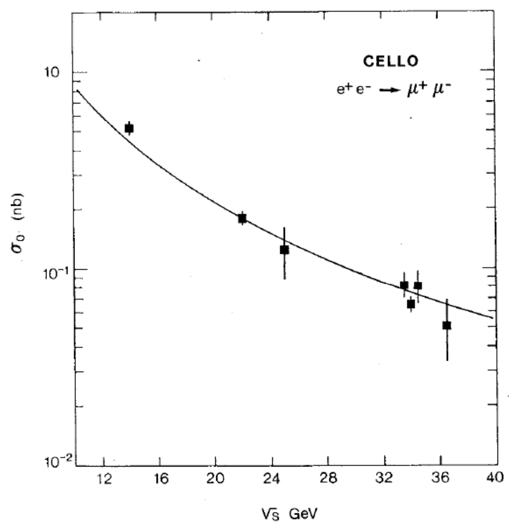
$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{d\Gamma_2}{F}|\mathcal{M}|^2 \qquad (6.42)\\
&= \frac{1}{2s}\frac{1}{16\pi^2\sqrt{s}}\sqrt{\frac{s}{4} - m_\mu^2}\,e^4\left[\left(1 - \frac{4m_\mu^2}{s}\right)\cos^2\theta + \left(1 + \frac{4m_\mu^2}{s}\right)\right]\\
&= \frac{e^4}{(4\pi)^2}\frac{1}{4s}\sqrt{1 - \frac{4m_\mu^2}{s}}\left[\left(1 - \frac{4m_\mu^2}{s}\right)\cos^2\theta + \left(1 + \frac{4m_\mu^2}{s}\right)\right],
\end{aligned}
$$

or, in terms of the fine-structure constant $\alpha = e^2/4\pi$,

$$\frac{d\sigma^{e^+e^-\to\mu^+\mu^-}}{d\Omega} = \frac{\alpha^2}{4s}\sqrt{1-\frac{4m_\mu^2}{s}}\left[\left(1-\frac{4m_\mu^2}{s}\right)\cos^2\theta + \left(1+\frac{4m_\mu^2}{s}\right)\right]. \tag{6.43}$$

The picture below shows the measured angular distribution [Z.Phys. C14 (1982) 283]. The data clearly exhibits a minimum near $\cos\theta = 0$ which corresponds to 90° scattering in the center-of-mass frame.





Our leading-order result (6.43) predicts a completely symmetric angular distribution around $\cos\theta = 0$. More accurate measurements have revealed that the angular distribution is not completely symmetric which can be explained by weak interactions (Z-boson interchange).

The total cross section is obtained by integrating over the angular variables:

$$\sigma_{\text{total}}^{e^+e^-\to\mu^+\mu^-} = \int d\Omega \frac{d\sigma^{e^+e^-\to\mu^+\mu^-}}{d\Omega} = \int_0^{2\pi} d\phi \int_{-1}^{1} (d\cos\theta) \frac{d\sigma^{e^+e^-\to\mu^+\mu^-}}{d\Omega}. \tag{6.44}$$

The differential cross section does not depend on the azimuthal angle ϕ so we only have two types of integrals,

$$\int_0^{2\pi} d\phi \int_{-1}^{1} (d\cos\theta) = 4\pi \tag{6.45}$$

$$\int_0^{2\pi} d\phi \int_{-1}^{1} (d\cos\theta)\cos^2\theta = \frac{4\pi}{3}\,. \tag{6.46}$$

By using these,

$$\begin{aligned}\sigma_{\text{total}}^{e^+e^-\to\mu^+\mu^-} &= \frac{4\pi\alpha^2}{4s}\sqrt{1-\frac{4m_\mu^2}{s}}\left[\left(1-\frac{4m_\mu^2}{s}\right)\frac{1}{3}+\left(1+\frac{4m_\mu^2}{s}\right)\right] \\ &= \frac{\pi\alpha^2}{3s}\sqrt{1-\frac{4m_\mu^2}{s}}\left[4+2\frac{4m_\mu^2}{s}\right],\end{aligned} \tag{6.47}$$

so all in all,

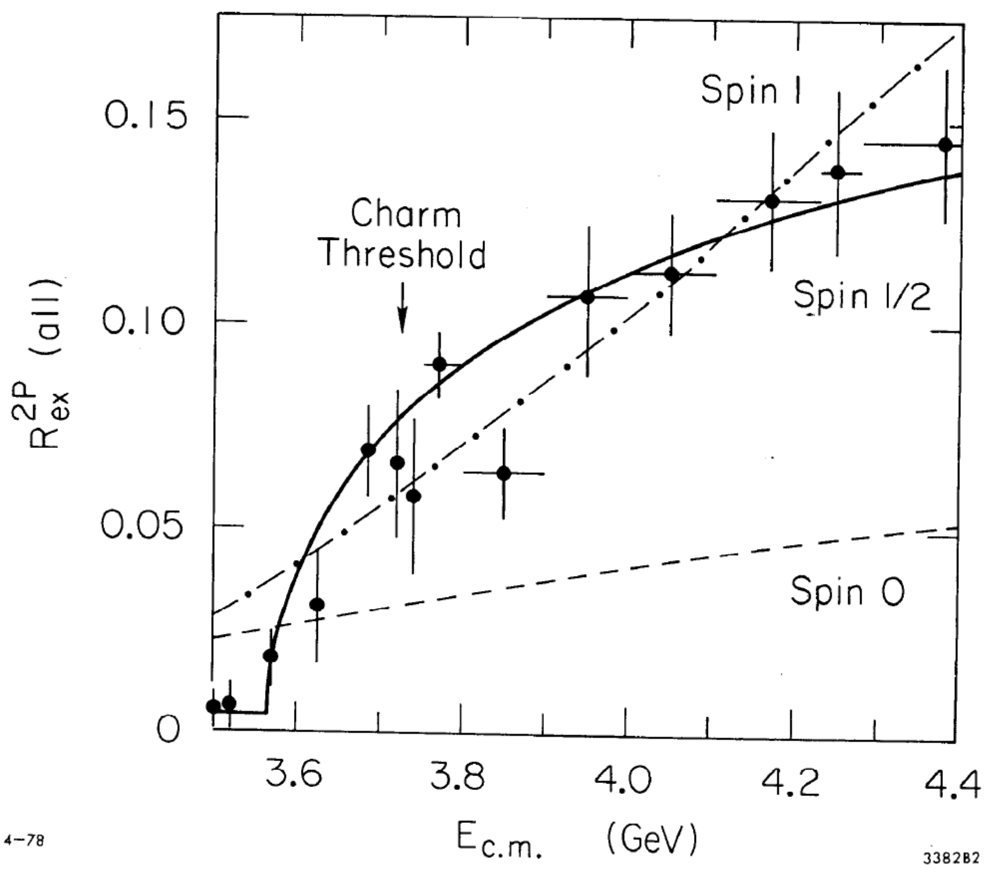
$$\sigma_{\text{total}}^{e^+e^-\to\mu^+\mu^-} = \frac{4\pi\alpha^2}{3s}\sqrt{1-\frac{4m_\mu^2}{s}}\left[1+\frac{2m_\mu^2}{s}\right]. \tag{6.48}$$

In the previous picture some experimental results for the total cross sections were shown as well and the curve is the QED prediction $\sim 1/s$.

Historically, the total cross section has been used to measure properties of the τ lepton by investigating the ratio,

$$R = \frac{\sigma_{\text{total}}^{e^+e^-\to\tau^+\tau^-}}{\sigma_{\text{total}}^{e^+e^-\to\mu^+\mu^-}}\,. \tag{6.49}$$

First, to produce a $\tau^+\tau^-$ pair a threshold energy $\sqrt{s} > 2m_\tau$ is needed, but at large-enough energies the ratio should tend to unity. In the picture below we show some results for this ratio [Phys. Rev. Lett. 41, 13]:



In the picture we clearly see a threshold near $3.5\,\text{GeV} < \sqrt{s} < 3.6\,\text{GeV}$ which is consistent with the τ mass $m_\tau \approx 1.78\,\text{GeV}$. In addition, the masses affect how quickly the ratio grows. By using Eq. (6.48) as a template, one can deduce the mass of the τ particle. The τ measurement involves only part of the decay channels so the ratio does not tend to unity. If one assumes that the τ particle is not a fermion but a spin-0 or spin-1 particle, the ratio would behave rather differently.

6.2 Helicity breakdown of $\mathrm{e}^+\mathrm{e}^- \to \mu^+\mu^-$ process

In the previous section we summed/averaged over all the spin states. By doing so we naturally lose information. We will now learn how to carry out the $e^+e^- \to \mu^+\mu^-$ in specific spin configurations. In principle, this allows to more thoroughly test the theory and leads to a more complete understanding of the QED interaction.

To begin with we the high-energy limit by forgetting the masses. So we set $m_e = m_\mu = 0$. In this limit,

$$\frac{d\sigma^{e^+e^-\to\mu^+\mu^-}}{d\Omega} \to \frac{\alpha^2}{4s}\left[1+\cos^2\theta\right] \tag{6.50}$$

$$\sigma_{\text{total}}^{e^+e^-\to\mu^+\mu^-} \to \frac{4\pi\alpha^2}{3s}\,. \tag{6.51}$$

A practical choice for the spin parts $(\xi_s,\, \eta_s)$ of the spinors at the zero-mass limit are the eigenstates of the helicity operator $\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}$. see Sect. 2.1. It is easy to check (Ex.) that in the zero-mass case we can replace the helicity operator by the γ^5 matrix,

$$\gamma^5 u_s(p) \overset{m=0}{=} (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})\, u_s(p)\,, \tag{6.52}$$

$$\gamma^5 v_s(p) \overset{m=0}{=} (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})\, v_s(p)\,. \tag{6.53}$$

Let us now denote by $u^\pm(p)$ and $v^\pm(p)$ the spinors in the helicity basis,

$$\gamma^5 u^\pm(p) \overset{m=0}{=} (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})\, u^\pm(p) = \pm u^\pm(p)\,, \tag{6.54}$$

$$\gamma^5 v^\pm(p) \overset{m=0}{=} (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})\, v^\pm(p) = \pm v^\pm(p)\,. \tag{6.55}$$

Define

$$P_R = \frac{1}{2}\left(1+\gamma^5\right)\,, \qquad P_L = \frac{1}{2}\left(1-\gamma^5\right)\,, \tag{6.56}$$

which fulfill

$$P_i^2 = P_i\,, \qquad P_R + P_L = 1\,, \qquad P_R\, P_L = 0\,. \tag{6.57}$$

From these relations we see that P_R and P_L are **projection operators**. By using the above relations we have,

$$\begin{aligned}
P_R\, u^+(p) &\overset{m=0}{=} u^+(p) & \qquad P_R\, v^+(p) &\overset{m=0}{=} v^+(p) \\
P_R\, u^-(p) &\overset{m=0}{=} 0 & P_R\, v^-(p) &\overset{m=0}{=} 0 \\
P_L\, u^+(p) &\overset{m=0}{=} 0 & P_L\, v^+(p) &\overset{m=0}{=} 0 \\
P_L\, u^-(p) &\overset{m=0}{=} u^-(p) & P_L\, v^-(p) &\overset{m=0}{=} v^-(p)
\end{aligned}\;. \tag{6.58}$$

The projections $P_R u_s(p)$ and $P_L u_s(p)$ (and the same for v spinors) are called the **right- and left-handed components of the spinor**.

Let's now return tot the squared matrix element (6.5),

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\left[\overline{u}_r(k)\gamma^\mu v_{r'}(k')\right]\left[\overline{v}_{r'}(k')\gamma^\nu u_r(k)\right] \qquad (6.59)$$
$$\left[\overline{v}_{s'}(p')\gamma_\mu u_s(p)\right]\left[\overline{u}_s(p)\gamma_\nu v_{s'}(p')\right],$$

and inspect the factor $\left[\overline{v}_{s'}(p')\gamma_\mu u_s(p)\right]$ in the helicity basis. If, for example, $u_s(p) = u_+(p)$, then

$$\overline{v}_{s'}(p')\gamma_\mu u_+(p) = \overline{v}_{s'}(p')\gamma_\mu P_R\, u_+(p)\,. \qquad (6.60)$$

On the other hand, since γ^5 is Hermitean and anticommutes with all other γ matrices,

$$\begin{aligned}\overline{v}_{s'}(p')\gamma_\mu u_+(p) &= v^\dagger_{s'}(p')\gamma^0\gamma_\mu P_R\, u_+(p) \qquad (6.61)\\ &= v^\dagger_{s'}(p')P_R\gamma^0\gamma_\mu\, u_+(p)\\ &= \left[P_R\, v_{s'}(p')\right]^\dagger \gamma^0\gamma_\mu\, u_+(p)\\ &= \begin{cases}\left[v_{s'}(p')\right]^\dagger \gamma^0\gamma_\mu\, u_+(p) & \text{if} \quad s' = +\\ 0 & \text{if} \quad s' = -\end{cases}.\end{aligned}$$

In other words, the both spinors have to be either right- or left-handed – the mixed configurations vanish. From Sect. 2.4 we recall that **a right-handed v spinor corresponds to a left-handed positron, and vice versa**. We can deduce that the helicity states of colliding electron and positron have to be opposite to give a non-zero cross section.

Let's now focus on a specific initial- and final-state helicity configuration. As an example we look at the case $e^-_R e^+_L \to \mu^-_R \mu^+_L$,

$$|\mathcal{M}(e^-_R e^+_L \to \mu^-_R\mu^+_L)|^2 = \frac{e^4}{q^4}\left[\overline{u}_+(k)\gamma^\mu v_+(k')\right]\left[\overline{v}_+(k')\gamma^\nu u_+(k)\right] \qquad (6.62)$$
$$\left[\overline{v}_+(p')\gamma_\mu u_+(p)\right]\left[\overline{u}_+(p)\gamma_\nu v_+(p')\right].$$

By using the projection operators, we can formally sum over the helicity states,

$$|\mathcal{M}(e_R^- e_L^+ \to \mu_R^- \mu_L^+)|^2 = \frac{e^4}{q^4} \sum_{r,r'} \left[\overline{u}_r(k)\gamma^\mu P_R\, v_{r'}(k')\right] \left[\overline{v}_{r'}(k')\gamma^\nu u_r(k)\right]$$

$$\sum_{s,s'} \left[\overline{v}_{s'}(p')\gamma_\mu P_R\, u_s(p)\right] \left[\overline{u}_s(p)\gamma_\nu v_{s'}(p')\right] ,$$

where P_Rs pick the case in which all spinors are right-handed. As earlier, we can turn this into a trace,

$$|\mathcal{M}(e_R^- e_L^+ \to \mu_R^- \mu_L^+)|^2 = \frac{e^4}{q^4} \tag{6.63}$$

$$\sum_{r,r'} \mathrm{Tr}\left[u_r(k)\overline{u}_r(k)\gamma^\mu P_R\, v_{r'}(k')\overline{v}_{r'}(k')\gamma^\nu\right]$$

$$\sum_{s,s'} \mathrm{Tr}\left[v_{s'}(p')v_{s'}(p')\gamma_\mu P_R\, u_s(p)\overline{u}_s(p)\gamma_\nu\right]$$

$$= \frac{e^4}{4q^4} \mathrm{Tr}\left[\not{k}\gamma^\mu(1+\gamma^5)\not{k}'\gamma^\nu\right] \mathrm{Tr}\left[\not{p}'\gamma_\mu(1+\gamma^5)\not{p}\gamma_\nu\right] .$$

Let's open the traces:

- $$\mathrm{Tr}\left[\not{p}'\gamma_\mu(1+\gamma^5)\not{p}\gamma_\nu\right] = \mathrm{Tr}\left[\not{p}'\gamma_\mu\not{p}\gamma_\nu\right] + \mathrm{Tr}\left[\not{p}'\gamma_\mu\gamma^5\not{p}\gamma_\nu\right] \tag{6.64}$$

$$= 4\left[p'_\mu p_\nu + p'_\nu p_\mu - g_{\mu\nu}\left(p'\cdot p\right)\right] - 4i\epsilon_{\alpha\nu\beta\mu}p^\alpha p'^\beta$$

- $$\mathrm{Tr}\left[\not{k}\gamma^\mu(1+\gamma^5)\not{k}'\gamma^\nu\right] = \mathrm{Tr}\left[\not{k}\gamma^\mu\not{k}'\gamma^\nu\right] + \mathrm{Tr}\left[\not{k}\gamma^\mu\gamma^5\not{k}'\gamma^\nu\right] \tag{6.65}$$

$$= 4\left[k^\mu k'^\nu + k^\nu k'^\mu - g_{\mu\nu}\left(k'\cdot k\right)\right] - 4i\epsilon^{\eta\nu\sigma\mu}k^\sigma k'^\eta ,$$

and substitute back to the expression for the matrix element squared,

$$|\mathcal{M}(e_R^- e_L^+ \to \mu_R^- \mu_L^+)|^2 = \frac{4e^4}{q^4} \tag{6.66}$$

$$\left[p'_\mu p_\nu + p'_\nu p_\mu - g_{\mu\nu}(p' \cdot p) - i\epsilon_{\alpha\nu\beta\mu} p^\alpha p'^\beta\right]$$

$$[k^\mu k'^\nu + k^\nu k'^\mu - g_{\mu\nu}(k' \cdot k) - i\epsilon^{\eta\nu\sigma\mu} k^\sigma k'^\eta]$$

$$= \frac{4e^4}{q^4}\left\{2\Big[(k \cdot p')(k' \cdot p) + (k \cdot p)(k' \cdot p')\Big] - \underbrace{\epsilon_{\alpha\nu\beta\mu}\epsilon^{\eta\nu\sigma\mu}}_{-2\left[\delta^\sigma_\beta \delta^\eta_\alpha - \delta^\sigma_\alpha \delta^\eta_\beta\right]} k^\sigma k'^\eta p^\alpha p'^\beta\right\}$$

$$= \frac{8e^4}{q^4}\Big[(k \cdot p')(k' \cdot p) + (k \cdot p)(k' \cdot p') + (k \cdot p')(k' \cdot p) - (k \cdot p)(k' \cdot p')\Big]$$

$$= \frac{16e^4}{q^4}(k \cdot p')(k' \cdot p) = \frac{16e^4}{s^2} \times \frac{1}{4}\left[\frac{s}{2} + \sqrt{s}|\mathbf{k}|\cos\theta\right]^2$$

$$= \frac{4e^4}{s^2} \times \left[\frac{s}{2} + \sqrt{s}\frac{\sqrt{s}}{2}\cos\theta\right]^2 = e^4\left(1 + \cos\theta\right)^2 .$$

The differential cross section is then,

$$\begin{aligned} d\sigma(e_R^- e_L^+ \to \mu_R^- \mu_L^+) &= \frac{d\Gamma_2}{F}|\mathcal{M}(e_R^- e_L^+ \to \mu_R^- \mu_L^+)|^2 \\ &= \frac{d\Omega}{2s}\frac{1}{16\pi^2\sqrt{s}}\frac{\sqrt{s}}{2}e^4\left(1 + \cos\theta\right)^2 \\ &= d\Omega\frac{\alpha^2}{4s}\left(1 + \cos\theta\right)^2 , \end{aligned} \tag{6.67}$$

so,

$$\frac{d\sigma(e_R^- e_L^+ \to \mu_R^- \mu_L^+)}{d\Omega} = \frac{\alpha^2}{4s}\left(1 + \cos\theta\right)^2 .$$

Othe helicity combinations are computed in the same way. The results are:

$$\frac{d\sigma(e_R^- e_L^+ \to \mu_R^- \mu_L^+)}{d\Omega} = \frac{d\sigma(e_L^- e_R^+ \to \mu_L^- \mu_R^+)}{d\Omega} = \frac{\alpha^2}{4s}\left(1 + \cos\theta\right)^2 ,$$

$$\frac{d\sigma(e_R^- e_L^+ \to \mu_L^- \mu_R^+)}{d\Omega} = \frac{d\sigma(e_L^- e_R^+ \to \mu_R^- \mu_L^+)}{d\Omega} = \frac{\alpha^2}{4s}\left(1 - \cos\theta\right)^2 .$$

In total there are $2^4 = 16$ different possible helicity configurations but 12 of these yield zero. By summing the 4 non-zero terms and dividing by a factor of 4 (for the spin averaging), we reproduce the unpolarized cross section.

6.3 Non-relativistic limit of $e^+e^- \to \mu^+\mu^-$

Let us now examine the limit in which $\sqrt{s} \sim 2m_\mu$, so that $\mu^-\mu^+$ pair can barely be produced. The cross section (6.43) will then become,

$$\frac{d\sigma^{e^+e^-\to\mu^+\mu^-}}{d\Omega} = \frac{\alpha^2}{4s}\underbrace{\sqrt{1-\frac{4m_\mu^2}{s}}}_{2|\mathbf{k}|/\sqrt{\mathbf{s}}}\left[\underbrace{\left(1-\frac{4m_\mu^2}{s}\right)}_{\approx 0}\cos^2\theta + \underbrace{\left(1+\frac{4m_\mu^2}{s}\right)}_{\approx 2}\right]$$

$$\xrightarrow{\sqrt{s}\sim 2m_\mu} = \frac{\alpha^2}{s}\frac{|\mathbf{k}|}{\sqrt{s}}, \tag{6.68}$$

with no angular dependence. In what follows we will compute this with explicit spinors. Our starting point will be the matrix element (6.2),

$$i\mathcal{M} = \frac{ie^2}{q^2}\left[\overline{u}_r(k)\gamma^\mu v_{r'}(k')\right]\left[\overline{v}_{s'}(p')\gamma_\mu u_s(p)\right]. \tag{6.69}$$

Since $m_\mu \approx 210 m_e$, the incoming electron and positron are very much relativistic so for them we will not use any non-relativistic approximations. If the electron travels in to $+z$ direction and the positron to the $-z$ direction, we write

$$u_s(p) = \sqrt{E_\mathbf{p}+m}\begin{pmatrix} I \\ \frac{\sigma^3|\mathbf{p}|}{E+m}\end{pmatrix}\xi_s \approx \sqrt{|\mathbf{p}|}\begin{pmatrix} I \\ \sigma^3\end{pmatrix}\xi_s \tag{6.70}$$

$$v_{s'}(p) = \sqrt{E_\mathbf{p}+m}\begin{pmatrix} \frac{-\sigma^3|\mathbf{p}|}{E+m} \\ I\end{pmatrix}\eta_{s'} \approx \sqrt{|\mathbf{p}|}\begin{pmatrix} -\sigma^3 \\ I\end{pmatrix}\eta_{s'}. \tag{6.71}$$

Using these,

$$\overline{v}_{s'}(p')\gamma^\mu u_s(p) = v^\dagger_{s'}(p')\gamma^0\gamma^\mu u_s(p) \tag{6.72}$$

$$= |\mathbf{p}|\eta^\dagger_{s'}\begin{pmatrix}-\sigma^3 & I\end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\gamma^\mu\begin{pmatrix} I \\ \sigma^3 \end{pmatrix}\xi_s$$

$$= |\mathbf{p}|\eta^\dagger_{s'}\begin{pmatrix}-\sigma^3 & -I\end{pmatrix}\gamma^\mu\begin{pmatrix} I \\ \sigma^3 \end{pmatrix}\xi_s$$

$$= \begin{cases} -\sqrt{s}\eta^\dagger_{s'}\sigma^i\xi_s & \text{if} \quad \mu = 1,2 \\ 0 & \text{if} \quad \mu = 0,3 \end{cases} .$$

Let's first look at the case that e^- is right-handed and e^+ left-handed (so both spinors are right-handed),

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})\xi_s = \sigma^3\xi_s = \xi_s \tag{6.73}$$

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}'})\eta_{s'} = -\sigma^3\eta_{s'} = \eta_{s'} , \tag{6.74}$$

i.e.

$$\xi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \eta_\uparrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{6.75}$$

Then,

$$\overline{v}_\uparrow(p')\gamma^\mu u_\uparrow(p) = v^\dagger_\uparrow(p')\gamma^0\gamma^\mu u_\uparrow(p) = -\sqrt{s}\begin{cases} 0 & \text{if} \quad \mu = 0 \\ 1 & \text{if} \quad \mu = 1 \\ i & \text{if} \quad \mu = 2 \\ 0 & \text{if} \quad \mu = 3 \end{cases} . \tag{6.76}$$

Correspondigly, if e^- is left-handed ja e^+ is right-handed (both spinors left-handed), the spin parts are

$$\xi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad \eta_\downarrow = \begin{pmatrix} -1 \\ 0 \end{pmatrix} . \tag{6.77}$$

In our convention in which $\eta_s = \left(-i\sigma^2\right)\xi_s^*$. In this case,

$$\overline{v}_\downarrow(p')\gamma^\mu u_\downarrow(p) = v_\downarrow^\dagger(p')\gamma^0\gamma^\mu u_\downarrow(p) = \sqrt{s}\begin{cases} 0 & \text{if} \quad \mu = 0 \\ 1 & \text{if} \quad \mu = 1 \\ -i & \text{if} \quad \mu = 2 \\ 0 & \text{if} \quad \mu = 3 \end{cases}. \tag{6.78}$$

The produced muons are non relativistic so,

$$u_r(k) = \sqrt{E_\mathbf{k} + m_\mu}\begin{pmatrix} I \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{k}}{E_\mathbf{k}+m_\mu} \end{pmatrix}\xi_r \approx \sqrt{2m_\mu}\begin{pmatrix} I \\ 0 \end{pmatrix}\xi_r \tag{6.79}$$

$$v_{r'}(k') = \sqrt{E_{\mathbf{k}'} + m_\mu}\begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\mathbf{k}'}{E_{\mathbf{k}'}+m_\mu} \\ I \end{pmatrix}\eta_{r'} \approx \sqrt{2m_\mu}\begin{pmatrix} 0 \\ I \end{pmatrix}\eta_{r'}\,, \tag{6.80}$$

with these

$$\overline{u}_r(k)\gamma^\mu v_{r'}(k') = u_r^\dagger(k)\gamma^0\gamma^\mu v_{r'}(k') \tag{6.81}$$

$$= 2m_\mu\xi_r^\dagger\,(I \quad 0)\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\gamma^\mu\begin{pmatrix} 0 \\ I \end{pmatrix}\eta_{r'}$$

$$= 2m_\mu\xi_r^\dagger\,(I \quad 0)\,\gamma^\mu\begin{pmatrix} 0 \\ I \end{pmatrix}\eta_{r'}$$

$$= \begin{cases} 0 & \text{if} \quad \mu = 0 \\ 2m_\mu\xi_r^\dagger\sigma^i\eta_{r'} & \text{if} \quad \mu = i \end{cases}.$$

Thus, our matrix element becomes,

$$
\begin{aligned}
i\mathcal{M} &= \frac{ie^2}{q^2} g_{\mu\nu} \left[\overline{u}_r(k)\gamma^\mu v_{r'}(k')\right]\left[\overline{v}_{s'}(p')\gamma^\nu u_s(p)\right] \qquad (6.82)\\
&= \pm\frac{ie^2}{q^2} \times \Big[g_{00}\times 0 + g_{11}\left(-\sqrt{s}\right)\times 2m_\mu \xi_r^\dagger \sigma^1 \eta_{r'} \\
&\qquad + g_{22}(\pm i)\left(-\sqrt{s}\right)\times 2m_\mu \xi_r^\dagger \sigma^2 \eta_{r'} + g_{33}\times 0\Big] \\
&= \pm\frac{-\sqrt{s}2m_\mu ie^2}{q^2}\Big[+ g_{11}\xi_r^\dagger\sigma^1\eta_{r'} + g_{22}(\pm i)\xi_r^\dagger\sigma^2\eta_{r'}\Big] \\
&= \pm\frac{\sqrt{s}2m_\mu ie^2}{q^2}\Big[+ \xi_r^\dagger\sigma^1\eta_{r'} + (\pm i)\xi_r^\dagger\sigma^2\eta_{r'}\Big] \\
&= \begin{cases} \frac{4\sqrt{s}m_\mu ie^2}{q^2}\,\xi_r^\dagger \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \eta_{r'} \approx 2ie^2\,\xi_r^\dagger \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\eta_{r'}\,, & e_R^- e_L^+ \\ -\frac{4\sqrt{s}m_\mu ie^2}{q^2}\,\xi_r^\dagger \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \eta_{r'} \approx 2ie^2\,\xi_r^\dagger \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\eta_{r'}\,, & e_L^- e_R^+ \end{cases} .
\end{aligned}
$$

We easily see that $\mathcal{M} \neq 0$ in the $e_R^- e_L^+$ case only when

$$
\xi_r = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \eta_{r'} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{6.83}
$$

and $\mathcal{M} \neq 0$ in $e_L^- e_R^+$ case only when

$$
\xi_r = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \eta_{r'} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \tag{6.84}
$$

so both muons have their spin projection into z direction either $+1/2$ or $-1/2$ (again, recall that the spin projection of an atiparticle is opposite to the projection of the v spinor). The complex conjugation gives,

$$
\left[i\mathcal{M}(e_R^- e_L^-)\right]^* \approx -2ie^2\,\eta_{r'}^\dagger \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \xi_r\,,
$$

so

$$|\mathcal{M}(e_R^- e_L^-)|^2 \approx 4e^4\, \eta_{r'}^\dagger \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \xi_r \xi_r^\dagger \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \eta_{r'} \qquad (6.85)$$

$$= 4e^4\, \mathrm{Tr}\left[\eta_{r'}\eta_{r'}^\dagger \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \xi_r \xi_r^\dagger \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right].$$

The spin sums are fairly simple, e.g.

$$\sum_{r'} \eta_{r'}\eta_{r'}^\dagger = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (6.86)$$

such that

$$\sum_{r,r'} |\mathcal{M}(e_R^- e_L^-)|^2 \approx 4e^4\, \mathrm{Tr}\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = 4e^4\, \mathrm{Tr}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 4e^4 .$$

Dividing by the flux factor $2s$ and multiplying by the phase-space element $\frac{|\mathbf{k}|}{16\pi^2\sqrt{s}}$ turns this a cross section,

$$\frac{d\sigma\left(e_R^- e_L^+ \to \mu^-\mu^+\right)}{d\Omega} = \frac{1}{2s}\frac{|\mathbf{k}|}{16\pi^2\sqrt{s}}4e^4 = \frac{\alpha^2}{s}\frac{2|\mathbf{k}|}{\sqrt{s}} . \qquad (6.87)$$

The opposite spin configuration gives the same result,

$$\frac{d\sigma\left(e_L^- e_R^+ \to \mu^-\mu^+\right)}{d\Omega} = \frac{d\sigma\left(e_R^- e_L^+ \to \mu^-\mu^+\right)}{d\Omega} , \qquad (6.88)$$

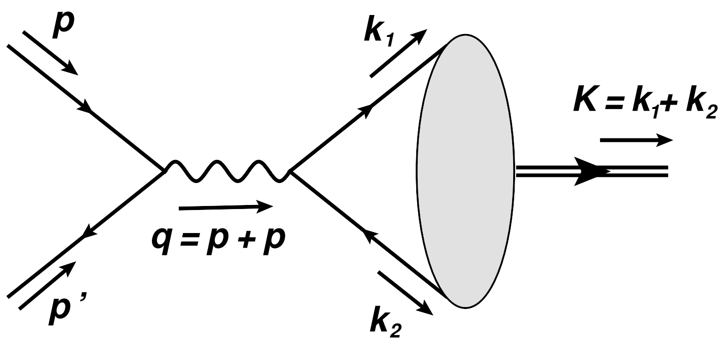
and the unpolarized cross section becomes,

$$\frac{1}{4}\left[\frac{d\sigma\left(e_L^- e_R^+ \to \mu^-\mu^+\right)}{d\Omega} + \frac{d\sigma\left(e_R^- e_L^+ \to \mu^-\mu^+\right)}{d\Omega}\right] = \frac{\alpha^2}{s}\frac{|\mathbf{k}|}{\sqrt{s}} , \qquad (6.89)$$

which agrees with Eq. (6.68).

6.3.1 Bound states

Although the preceding calculation was formally ok it may be in doubt: if the muon pair is produced to almost rest the Coulomb interaction can in principle tie them together to a bound state. The bound $\mu^+\mu^-$ state has not yet been observed but e.g. e^+e^- bound state (positronium) is well known. A state like this can hardly be decribed with plane-wave spinors as we have done in the previous section.



In the previous section we saw that if the projections of the electron and positron spins to the z axis are both either $+1/2$ or $-1/2$, also both outgoing muons have their spin projection to the z axis either $+1/2$ or $-1/2$. All the particles are thus either in "spin up" or "spin down" state. Let us denote the corresponding matrix element (6.82) as

$$\mathcal{M}\left(\uparrow\uparrow\rightarrow \mathbf{k}_1\uparrow\mathbf{k}_2\uparrow\right) = \mathcal{M}\left(\downarrow\downarrow\rightarrow \mathbf{k}_1\downarrow\mathbf{k}_2\downarrow\right) = 2e^2\,. \tag{6.90}$$

This is independent of the momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ (at least in our non-relativistic limit).

The system of two non-relativistic muons can be described by the methods of Quantum Mechanics I course — the treatment is nearly identical with that of the hydrogen atom. The Hamiltonian for a two-muon system is

$$\hat{H} = \frac{\hat{p}_1^2}{2m_\mu} + \frac{\hat{p}_2^2}{2m_\mu} + V(|\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2|)\,, \tag{6.91}$$

where $V(\mathbf{x}_1, \mathbf{x}_2)$ is just the Coulomb potential and $\hat{p}_i = -i\nabla_{\mathbf{x_i}}$. The position of the center-of-mass of this system, and the relative position of the muons are given by the operators,

$$\hat{\mathbf{R}} \equiv \frac{m_\mu\hat{\mathbf{x}}_1 + m_\mu\hat{\mathbf{x}}_2}{m_\mu + m_\mu} = \frac{1}{2}\left(\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2\right) \tag{6.92}$$

$$\hat{\mathbf{r}} \equiv \left(\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2\right)\,, \tag{6.93}$$

and the corresponding conjugate momentum operators are,

$$\hat{\mathbf{P}} = \hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2 = -i\nabla_{\mathbf{R}}\,, \tag{6.94}$$

$$\hat{\mathbf{p}} = \frac{1}{2}\left(\hat{\mathbf{p}}_1 - \hat{\mathbf{p}}_2\right) = -i\nabla_{\mathbf{r}}\,, \tag{6.95}$$

fulfilling the usual canonical commutation relations,

$$\left[\hat{R}_k, \hat{P}_\ell\right] = \left[\hat{r}_k, \hat{p}_\ell\right] = i\hbar\delta_{k\ell}\,. \tag{6.96}$$

With these the Hamiltonian can be written as,

$$\hat{H} = \frac{\hat{\mathbf{P}}^2}{2(2m_\mu)} + \frac{\hat{\mathbf{p}}^2}{2(m_\mu/2)} + V(|\hat{\mathbf{r}}|)\,. \tag{6.97}$$

The solution of the Schrödinger equation,

$$\hat{H}\,\psi\left(\mathbf{r},\mathbf{R}\right) = E\,\psi\left(\mathbf{r},\mathbf{R}\right) \tag{6.98}$$

can be written as

$$\psi\left(\mathbf{r},\mathbf{R}\right) = \psi\left(\mathbf{r}\right)e^{-i\mathbf{P}\cdot\mathbf{R}/\hbar}\,, \tag{6.99}$$

and the total energy of the two-muon system is

$$E = \frac{\mathbf{P}^2}{2M} + \epsilon\,, \tag{6.100}$$

where ϵ is the energy related to the mutual motion of the muons, given by

$$\left[\frac{\hat{\mathbf{p}}^2}{2(m_\mu/2)} + V(|\hat{\mathbf{r}}|)\right]\psi\left(\mathbf{r}\right) = \epsilon\psi\left(\mathbf{r}\right)\,. \tag{6.101}$$

In spherical coordinates the general solution to this equation is of the form

$$\psi_{n\ell m}\left(\mathbf{r}\right) = R_{n\ell}(r)\,Y_{\ell m}(\theta,\phi)\,, \tag{6.102}$$

where $R_{n\ell}(r)$ are radial wave functions and $Y_{\ell m}(\theta,\varphi)$ spherical harmonics:

$$Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{1,\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\varphi}$$

$$Y_{20}(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

$$Y_{2,\pm1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\varphi}$$

$$Y_{2,\pm2}(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\varphi}.$$

n	l	R_{nl}
1	0	$2\left(\frac{Z}{a}\right)^{\frac{3}{2}} e^{-\frac{Zr}{a}}$
2	0	$\frac{1}{\sqrt{2}}\left(\frac{Z}{a}\right)^{\frac{3}{2}}\left(1-\frac{Zr}{2a}\right) e^{-\frac{Zr}{2a}}$
2	1	$\frac{1}{2\sqrt{6}}\left(\frac{Z}{a}\right)^{\frac{5}{2}} r e^{-\frac{Zr}{2a}}$
3	0	$\frac{2}{3\sqrt{3}}\left(\frac{Z}{a}\right)^{\frac{3}{2}}\left(1-\frac{2Zr}{3a}+\frac{2}{27}(\frac{Zr}{a})^2\right) e^{-\frac{Zr}{3a}}$
3	1	$\frac{8}{27\sqrt{6}}\left(\frac{Z}{a}\right)^{\frac{5}{2}} r\left(1-\frac{Zr}{6a}\right) e^{-\frac{Zr}{3a}}$
3	2	$\frac{4}{81\sqrt{30}}\left(\frac{Z}{a}\right)^{\frac{7}{2}} r^2 e^{-\frac{Zr}{3a}}$

In our case $Z = 1$ and $a = \frac{8\pi\epsilon_0\hbar^2}{m_\mu e^2}$.

The wave function in momentum space is obtained by a Fourier transform,

$$\tilde{\psi}(\mathbf{k}) = \int d^3x e^{i\mathbf{k}\cdot\mathbf{x}}\psi(\mathbf{x})\,, \tag{6.103}$$

which we assume to be normalized such that

$$\int \frac{d^3k}{(2\pi)^3}|\tilde{\psi}(\mathbf{k})|^2 = 1\,. \tag{6.104}$$

We will now form a wave function that corresponds to a bound $\mu^-\mu^+$ pair that moves with an overall momentum $\mathbf{K}$. We do this by weighting each state $|\mathbf{k}_1\uparrow\mathbf{k}_2\uparrow\rangle$ by an appropriate probability density $\tilde{\psi}(\mathbf{k})$:

$$|B(\mathbf{K})\uparrow\uparrow\rangle = \sqrt{2E_\mathbf{K}}\int\frac{d^3k}{(2\pi)^3}\tilde{\psi}(\mathbf{k})\frac{1}{\sqrt{2E_{\mathbf{k}_1}}}\frac{1}{\sqrt{2E_{\mathbf{k}_2}}}|\mathbf{k}_1\uparrow\mathbf{k}_2\uparrow\rangle\Big|_{\substack{\mathbf{K}=\mathbf{k}_1+\mathbf{k}_2\\ \mathbf{k}=(\mathbf{k}_1-\mathbf{k}_2)/2}}\,,$$

in which $E_\mathbf{K} = \sqrt{\mathbf{K}^2+M^2}$ and $M = 2m_\mu$. The various normalization factors have been chosen such that the normalization agrees with usual 1-particle states (Ex.),

$$\langle B(\mathbf{K}')\uparrow\uparrow|B(\mathbf{K})\uparrow\uparrow\rangle = 2E_\mathbf{K}(2\pi)^3\delta^{(3)}\left(\mathbf{K}'-\mathbf{K}\right)\,. \tag{6.105}$$

Other possible spin-0 and spin-1 states are obtained by replacing $|\mathbf{k}_1 \uparrow \mathbf{k}_2 \uparrow\rangle$ by the configuration of interest:

$$\begin{aligned}
&S=0 && \tfrac{1}{\sqrt{2}}\Big[|\mathbf{k}_1 \uparrow \mathbf{k}_2 \downarrow\rangle - |\mathbf{k}_1 \downarrow \mathbf{k}_2 \uparrow\rangle\Big] \\
&S=1 \quad m_s=1: && |\mathbf{k}_1 \uparrow \mathbf{k}_2 \uparrow\rangle \\
&S=1 \quad m_s=0: && \tfrac{1}{\sqrt{2}}\Big[|\mathbf{k}_1 \uparrow \mathbf{k}_2 \downarrow\rangle + |\mathbf{k}_1 \downarrow \mathbf{k}_2 \uparrow\rangle\Big] \\
&S=1 \ \ m_s=-1: && |\mathbf{k}_1 \downarrow \mathbf{k}_2 \downarrow\rangle
\end{aligned} \tag{6.106}$$

We already agreed that $\sqrt{s} \sim 2m_\mu$, so $\mathbf{K} \approx 0$, and the relevant state vector is thus,

$$|B \uparrow\uparrow\rangle \equiv |B(\mathbf{0}) \uparrow\uparrow\rangle = \sqrt{2M} \int \frac{d^3k}{(2\pi)^3} \tilde{\psi}(\mathbf{k}) \frac{1}{\sqrt{2m_\mu}} \frac{1}{\sqrt{2m_\mu}} |\mathbf{k} \uparrow -\mathbf{k} \uparrow\rangle \,. \tag{6.107}$$

The invariant matrix element for producing a state like this in e^+e^- collision is thus,

$$\begin{aligned}
\mathcal{M}\left(\uparrow\uparrow \to B \uparrow\uparrow\right) &= \sqrt{2M} \int \frac{d^3k}{(2\pi)^3} \tilde{\psi}^*(\mathbf{k}) \frac{1}{\sqrt{2m_\mu}} \frac{1}{\sqrt{2m_\mu}} \mathcal{M}\left(\uparrow\uparrow \to \mathbf{k}_1 \uparrow \mathbf{k}_2 \uparrow\right) \\
&= \sqrt{2M} \int \frac{d^3k}{(2\pi)^3} \tilde{\psi}^*(\mathbf{k}) \frac{1}{\sqrt{2m_\mu}} \frac{1}{\sqrt{2m_\mu}} 2e^2 \qquad (6.108) \\
&= \sqrt{2M} \int \frac{d^3k}{(2\pi)^3} \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \psi^*(\mathbf{x}) \frac{1}{\sqrt{2m_\mu}} \frac{1}{\sqrt{2m_\mu}} 2e^2 \\
&= \sqrt{2M} \int d^3x \delta^{(3)}(\mathbf{x}) \psi^*(\mathbf{x}) \frac{1}{\sqrt{2m_\mu}} \frac{1}{\sqrt{2m_\mu}} 2e^2 \\
&= \sqrt{\frac{2}{M}} \big[2e^2\big] \psi^*(\mathbf{0}) \,.
\end{aligned}$$

The "spin down" case give the same result so

$$\mathcal{M}\left(\uparrow\uparrow \to B \uparrow\uparrow\right) = \mathcal{M}\left(\downarrow\downarrow \to B \downarrow\downarrow\right) = \sqrt{\frac{2}{M}} \big[2e^2\big] \psi^*(\mathbf{0}) \,. \tag{6.109}$$

Because the radial wave function $R_{n\ell}(r) = 0$ if $\ell > 0$, the produced state always has $\ell = 0$. The muon pair is thus produced to a state $^3S_1$.

We will now form the cross section for producing the bound state. Let us assume an unpolarized case so we average over the initial-state spins and sum over final-state spins (though we know that only 2 combinations are $\neq 0$),

$$\begin{aligned}\overline{|\mathcal{M}\left(e^+e^- \to B\right)|^2} &= \frac{1}{4} \times \left[|\mathcal{M}\left(\uparrow\uparrow \to B \uparrow\uparrow\right)|^2 + |\mathcal{M}\left(\downarrow\downarrow \to B \downarrow\downarrow\right)|^2\right] \\ &= \frac{2}{4} \times \frac{2}{M}\left[4e^4\right]|\psi^*(\mathbf{0})|^2 = \frac{4e^4}{M}|\psi^*(\mathbf{0})|^2 . \end{aligned} \tag{6.110}$$

The flux factor is again just $2s$ but we now effectively have only one particle in the final state,

$$\begin{aligned}\sigma\left(e^+e^- \to B\right) &= \frac{1}{2s}\int \frac{d^3K}{2E_{\mathbf{K}}(2\pi)^3}(2\pi)^4\delta^{(4)}\left(p + p' - K\right) \times \frac{4e^4}{M}|\psi^*(\mathbf{0})|^2 \\ &= \frac{1}{2s}\frac{1}{2E_{\mathbf{K}}}(2\pi)\delta\left(p^0 + p'^0 - K^0\right)\frac{4e^4}{M}|\psi^*(\mathbf{0})|^2 \\ &= \frac{1}{s}\frac{e^4}{M^2}(2\pi)|\psi^*(\mathbf{0})|^2\delta\left(\sqrt{s} - M\right) \\ &= \frac{32\pi^3\alpha^2}{M^4}|\psi^*(\mathbf{0})|^2\delta\left(\sqrt{s} - M\right) . \end{aligned} \tag{6.111}$$

Since $\delta\left(\sqrt{s} - M\right) = 2M\delta\left(s - M^2\right)$, we have our final form,

$$\sigma\left(e^+e^- \to B\right) = \frac{64\pi^3\alpha^2}{M^4}|\psi^*(\mathbf{0})|^2\delta\left(s - M^2\right) . \tag{6.112}$$

Of course, the bound $\mu^+\mu^-$ pair is not a stable state and annihilates back to e^+e^- pair ($^3S_1$) or into two photons ($^1S_0$), so the δ-function peak brodens into a some kind of resonance. In practice, some broadening is also caused by the spread of the e^+e^- beam energies which is easily large enough such that all energy levels with different n quantum numbers, are produced.

Following the same steps as above, we can also compute the decay width,

$$\Gamma\left(B \to e^+e^-\right) = \frac{16\pi\alpha^2}{3}\frac{|\psi^*(\mathbf{0})|^2}{M^2}\,. \tag{6.113}$$

In more complicated calculations the use of explicit spinors can get tedious. However, it is possible to reduce the calculation to the usual γ-matrix algebra when the mutual momentum is almost zero, $\mathbf{k} \approx 0$ (as it has to be to produce the bound state). The momenta of the individual muons are then the same, $\mathbf{k}_1 = \mathbf{k}_2 \approx \mathbf{K}/2$. In this case the matrix element is always of the form,

$$\begin{aligned}&\mathcal{M}\left[(p,s);(p',s') \to B(K,S)\right] \qquad (6.114)\\ =&\frac{\sqrt{2E_{\mathbf{K}}}}{2E_{\mathbf{K}/2}}\psi^*(\mathbf{0})\mathcal{M}\left[(p,s);(p',s') \to \left(\frac{\mathbf{K}}{2},\frac{\mathbf{K}}{2},S\right)\right]\,,\end{aligned}$$

where the matrix element in the left-hand side corresponds to a free muon pair in a specific spin state S. In this form, everything related to the bound state is just a multiplicative factor and the summation/averaging over spins can be done for the matrix element only,

$$\overline{\mathcal{M}\left[p,p' \to B(K)\right]^2} = \frac{2E_{\mathbf{K}}}{4E^2_{\mathbf{K}/2}}|\psi^*(\mathbf{0})|^2\overline{\mathcal{M}\left[p,p' \to \left(\frac{\mathbf{K}}{2},\frac{\mathbf{K}}{2}\right)\right]^2}\,.$$

When computing the righ-hand side, the following identities hold for spin-1 triplet (Ex.),

$\mathbf{S=1}$

$$v_\uparrow(k)\overline{u}_\uparrow(k) = \frac{-1}{\sqrt{2}}\not{\epsilon}^*_1(K)\left(\frac{\not{K}+M}{2}\right) \tag{6.115}$$

$$v_\downarrow(k)\overline{u}_\downarrow(k) = \frac{-1}{\sqrt{2}}\not{\epsilon}^*_{-1}(K)\left(\frac{\not{K}+M}{2}\right) \tag{6.116}$$

$$\frac{1}{\sqrt{2}}\left[v_\uparrow(k)\overline{u}_\downarrow(k) + v_\downarrow(k)\overline{u}_\uparrow(k)\right] = \frac{-1}{\sqrt{2}}\not{\epsilon}^*_0(K)\left(\frac{\not{K}+M}{2}\right)\,, \tag{6.117}$$

where the polarization vectors have been defined for $\mathbf{K} = (E_{\mathbf{K}},0,0,\mathbf{K})$ in the form,

$$\epsilon_1^\mu(K) = \frac{-1}{\sqrt{2}}(0, 1, i, 0) \tag{6.118}$$

$$\epsilon_{-1}^\mu(K) = \frac{1}{\sqrt{2}}(0, 1, -i, 0) \tag{6.119}$$

$$\epsilon_0^\mu(K) = \frac{1}{M}(|\mathbf{K}|, 0, 0, E_{\mathbf{K}}) \ , \tag{6.120}$$

and in other cases they are obtained by an appropriate Lorentz-transformation. The first two correspond to the positive and negative helicities of a spin-1 particle, encountered also in Eqs. (6.76) and (6.78). The last one corresponds to a "helicity zero" state for a massive spin-1 particle. These polarization vectors fulfill the completeness relation,

$$\sum_{\lambda=-1,0,1} \epsilon_\lambda^\mu(K)\epsilon_\lambda^{*\nu}(K) = -g^{\mu\nu} + \frac{K_\mu K_\nu}{M^2} \, . \tag{6.121}$$

For the Spin-0 singlet we have,

$\mathbf{S = 0}$

$$\frac{1}{\sqrt{2}}\left[v_\uparrow(k)\overline{u}_\downarrow(k) - v_\downarrow(k)\overline{u}_\uparrow(k)\right] = \frac{1}{\sqrt{2}}\not{\gamma}^5\left(\frac{\not{K} + M}{2}\right) \ , \tag{6.122}$$

though it cannot be produced in e^+e^- collisions (at least to first approximation).

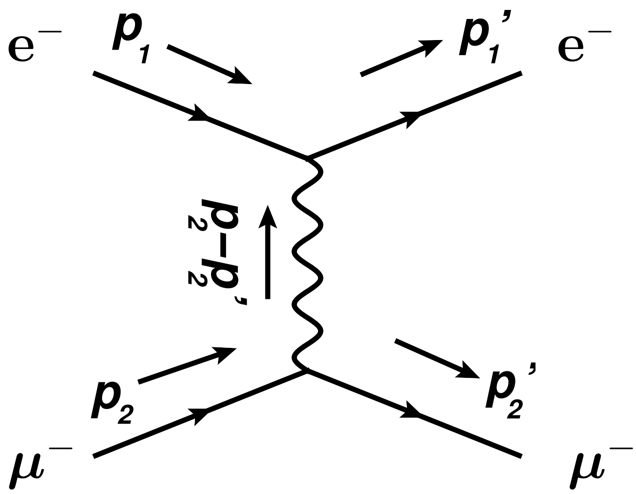
Starting from the matrix element (6.2) for $e^-e^+ \to \mu^-\mu^+$ scattering,

$$i\mathcal{M} = \frac{ie^2}{q^2}\left[\overline{u}_r(k)\gamma^\mu v_{r'}(k')\right]\left[\overline{v}_{s'}(p')\gamma_\mu u_s(p)\right] \ , \tag{6.123}$$

it is now straightforward to reproduce (Ex.) e.g. the total cross section (6.112) without explicit spinor representations. It is also easy to check that for spin-0 state, the cross section is zero.

6.4 $e^-\mu^- \to e^-\mu^-$ and crossing symmetry

Let us now inspect the elastic $e^-\mu^- \to e^-\mu^-$ scattering:



In comparison to the $e^-e^+ \to \mu^-\mu^+$ scattering the diagram is essentially the same but $90°$ rotated. As we will shortly see, this structural similarity can be taken advantage of in calculating the matrix-element squared. We start by writing down the matrix element:

$$i\mathcal{M} = \overline{u}_{s'}(p_1')(-ie\gamma^\mu)u_s(p_1)\frac{-ig_{\mu\nu}}{(p_2-p_2')^2}\overline{u}_{r'}(p_2')(-ie\gamma^\nu)u_r(p_2) \tag{6.124}$$

$$= \frac{ie^2}{(p_2-p_2')^2}\left[\overline{u}_{s'}(p_1')\gamma^\mu u_s(p_1)\right]\left[\overline{u}_{r'}(p_2')\gamma_\mu u_r(p_2)\right] .$$

We square this,

$$|\mathcal{M}|^2 = \frac{e^4}{(p_2-p_2')^4}\left[\overline{u}_{s'}(p_1')\gamma^\mu u_s(p_1)\right]\left[\overline{u}_s(p_1)\gamma^\nu u_{s'}(p_1')\right] \tag{6.125}$$

$$\times\left[\overline{u}_{r'}(p_2')\gamma_\mu u_r(p_2)\right]\left[\overline{u}_r(p_2)\gamma_\nu u_{r'}(p_2')\right]$$

$$= \frac{e^4}{(p_2-p_2')^4}\mathrm{Tr}\left[u_{s'}(p_1')\overline{u}_{s'}(p_1')\gamma^\mu u_s(p_1)\overline{u}_s(p_1)\gamma^\nu\right]$$

$$\times\,\mathrm{Tr}\left[u_{r'}(p_2')\overline{u}_{r'}(p_2')\gamma_\mu u_r(p_2)\overline{u}_r(p_2)\gamma_\nu\right] ,$$

and average over the inital-state spins and sum over final-state spins:

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\sum_{s,s',r,r'} |\mathcal{M}|^2 \tag{6.126}$$

$$= \frac{1}{4}\frac{e^4}{(p_2-p_2')^4}\mathrm{Tr}\left[(\not{p}_1' + m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu\right]$$

$$\times\ \mathrm{Tr}\left[(\not{p}_2' + m_\mu)\gamma_\mu(\not{p}_2 + m_\mu)\gamma_\nu\right] .$$

Let's now compare this to the squared matrix element (6.12) for the process $\mathbf{e}^-(p)\mathbf{e}^+(p') \to \boldsymbol{\mu}^-(k)\boldsymbol{\mu}^+(k')$,

$$\frac{e^4}{4(k+k')^4}\,\mathrm{Tr}\left[(\not{k} + m_\mu)\,\gamma^\mu\,(\not{k}' - m_\mu)\,\gamma^\nu\right]\mathrm{Tr}\left[(\not{p}' - m_e)\,\gamma_\mu\,(\not{p} + m_e)\,\gamma_\nu\right] .$$

If we make the following replacements,

$$p \to p_1\,, \quad p' \to -p_1'\,, \quad k \to p_2'\,, \quad k' \to -p_2\,, \tag{6.127}$$

the result is formally the same as in Eq. (6.126). We can thus obtain the result of opening the trace directly from Eq. (6.30) just by making the replacements indicated above:

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{(p_2-p_2')^4}\Big[(p_2'\cdot p_1')(p_2\cdot p_1) + (p_2'\cdot p_1)(p_2\cdot p_1') - m_\mu^2(p_1\cdot p_1')\Big] \tag{6.128}$$

What we have here is a particular example of a more general principle which we call the **crossing symmetry**:

initial-state fermion $\Longleftrightarrow$ final-state antifermion

$$\left|\overline{\mathcal{M}}\left[f(k),\ldots\to\ldots\right]\right|^2 = -\left|\overline{\mathcal{M}}\left[\ldots\to\ldots,\overline{f}(-k)\right]\right|^2$$

initial-state antifermion $\Longleftrightarrow$ final-state fermion

$$\left|\overline{\mathcal{M}}\left[\overline{f}(k),\ldots\to\ldots\right]\right|^2 = -\left|\overline{\mathcal{M}}\left[\ldots\to\ldots,f(-k)\right]\right|^2$$

initial-state boson $\Longleftrightarrow$ final-state boson

$$\left|\overline{\mathcal{M}}\left[\phi(k)\,,\ldots\rightarrow\ldots\right]\right|^2 = \left|\overline{\mathcal{M}}\left[\ldots\rightarrow\ldots,\overline{\phi}(-k)\right]\right|^2$$

The extra minus signs in front of spin-summed matrix elements $|\overline{\mathcal{M}}|^2$ for fermion-antifermion interchange originate from the spin sums,

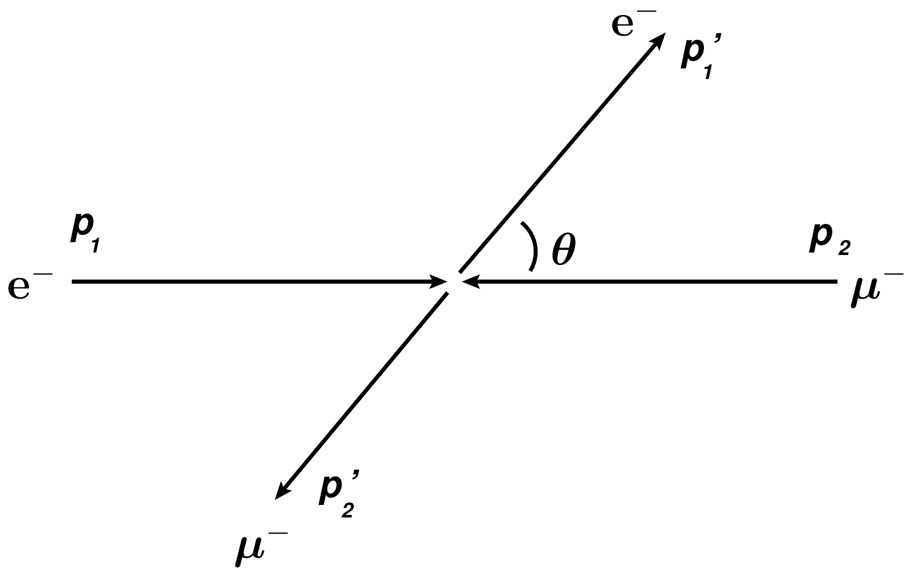
$$\sum_s u_s(p)\overline{u}_s(p) = \not{p} + m \xrightarrow{p\rightarrow -k} -\not{k} + m$$
$$= -(\not{k} - m) = -\sum_s v_s(p)\overline{v}_s(p)$$

In fact, the crossing symmetry is still a much stronger theorem and, with proper phase conventions for the spinors (Peskin 5.4),

$$\mathcal{M}\left[\overline{\phi}(k)\,,\ldots\rightarrow\ldots\right] = \mathcal{M}\left[\ldots\rightarrow\ldots,\phi(-k)\right]$$

where ϕ denotes any particle type.

The kinematics is quite different than in the $\mathbf{e}^-\mathbf{e}^+ \to \boldsymbol{\mu}^-\boldsymbol{\mu}^+$ case. We choose here the center-of-mass frame of the incoming particles:



and choose the momenta as follows:

$$p_1 = (|\mathbf{k}|, 0, 0, |\mathbf{k}|) \tag{6.129}$$

$$p_2 = (E, 0, 0, -|\mathbf{k}|) \tag{6.130}$$

$$p_1' = (|\mathbf{k}|, |\mathbf{k}|\sin\theta\cos\phi, |\mathbf{k}|\sin\theta\sin\phi, |\mathbf{k}|\cos\theta) \tag{6.131}$$

$$p_2' = (E, -|\mathbf{k}|\sin\theta\cos\phi, -|\mathbf{k}|\sin\theta\sin\phi, -|\mathbf{k}|\cos\theta)\ . \tag{6.132}$$

The needed dot products are easy to calculate:

$$p_1 \cdot p_2 = p_1' \cdot p_2' = |\mathbf{k}|E + |\mathbf{k}|^2 = \frac{1}{2}\left(s - m_\mu^2\right) \tag{6.133}$$

$$p_1 \cdot p_1' = |\mathbf{k}|^2 (1 - \cos\theta) \tag{6.134}$$

$$p_1 \cdot p_2' = p_2 \cdot p_1' = |\mathbf{k}|E + |\mathbf{k}|^2 \cos\theta\,. \tag{6.135}$$

Substitute these into (6.128),

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{4|\mathbf{k}|^4 (1-\cos\theta)^2}\Big[\left(|\mathbf{k}|E + |\mathbf{k}|^2\right)^2 + \left(|\mathbf{k}|E + |\mathbf{k}|^2\cos\theta\right)^2 - m_\mu^2 |\mathbf{k}|^2 (1-\cos\theta)\Big] \tag{6.136}$$

$$= \frac{2e^4}{|\mathbf{k}|^2 (1-\cos\theta)^2} \Big[(E + |\mathbf{k}|)^2 + (E + |\mathbf{k}|\cos\theta)^2 - m_\mu^2 (1-\cos\theta) \Big]$$

The flux facor is now

$$F = 4\sqrt{(p_1 \cdot p_2)^2 - m_e^2 m_\mu^2} \approx 4\sqrt{(p_1 \cdot p_2)^2} = 2(s - m_\mu^2)\,, \qquad (6.137)$$

and the two-particle phase-space element in center-of-mass frame we get from Eq. (3.111),

$$\Gamma_2 = \int d\Omega \frac{|\mathbf{k}|}{16\pi^2\sqrt{s}}\,. \qquad (6.138)$$

Then we just pack these together to form a cross section,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{1}{F}\frac{d\Gamma_2}{d\Omega}\overline{|\mathcal{M}|^2} = \frac{1}{2(s-m_\mu^2)}\frac{|\mathbf{k}|}{16\pi^2\sqrt{s}} \qquad (6.139)$$

$$\times \frac{2e^4}{|\mathbf{k}|^2 (1-\cos\theta)^2} \Big[(E + |\mathbf{k}|)^2 + (E + |\mathbf{k}|\cos\theta)^2 - m_\mu^2 (1-\cos\theta) \Big]$$

$$= \frac{\alpha^2}{2s|\mathbf{k}|^2 (1-\cos\theta)^2} \Big[s + (E + |\mathbf{k}|\cos\theta)^2 - m_\mu^2 (1-\cos\theta) \Big]$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{\alpha^2}{2s|\mathbf{k}|^2 (1-\cos\theta)^2} \Big[s + (E + |\mathbf{k}|\cos\theta)^2 - m_\mu^2 (1-\cos\theta) \Big]$$

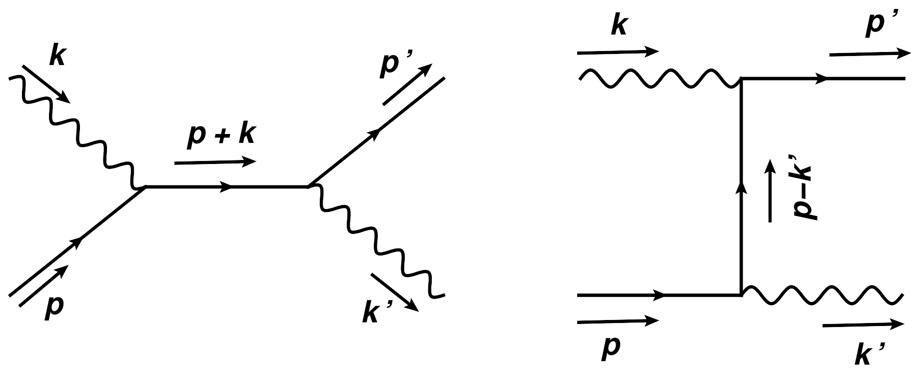
The factor $(1-\cos\theta)^2$ in the denominator makes the cross section strongly singular in the limit $\theta \to 0$,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} \sim \frac{1}{(1-\cos\theta)^2} \sim \frac{1}{\left(\frac{1}{2!}\theta^2 + \cdots\right)^2} \sim \frac{1}{\theta^4}\,, \quad \text{as } \theta \to 0\,. \qquad (6.140)$$

The total cross section is thus infinite. The singularity originates from the fact that in the limit $\theta \to 0$ the exchanged virtual photon goes on shell, $q^2 \to 0$. The infinite total cross section can be seen as an analogy to the infinite range of the Coulomb force.

6.5 Compton scattering

A bit harder process with external photons is the elastic $e^-\gamma \to e^-\gamma \to$ scattering:



At least the matrix element is easy to write down:

$$i\mathcal{M} = \Bigg[\overline{u}_{s'}(p')\left[-ie\gamma^\mu\right]\frac{i\left(\not{p}+\not{k}+m\right)}{(p+k)^2-m^2}\left[-ie\gamma^\nu\right]u_s(p) \tag{6.141}$$

$$+\,\overline{u}_{s'}(p')\left[-ie\gamma^\nu\right]\frac{i\left(\not{p}-\not{k}'+m\right)}{(p-k')^2-m^2}\left[-ie\gamma^\mu\right]u_s(p)\Bigg]\,\epsilon_{\lambda,\nu}(k)\epsilon^*_{\lambda',\mu}(k')$$

$$= -ie^2\Bigg[\overline{u}_{s'}(p')\not{\epsilon}^*_{\lambda'}(k')\frac{\left(\not{p}+\not{k}+m\right)}{(p+k)^2-m^2}\not{\epsilon}_\lambda(k)u_s(p)$$

$$+\,\overline{u}_{s'}(p')\not{\epsilon}_\lambda(k)\frac{\left(\not{p}-\not{k}'+m\right)}{(p-k')^2-m^2}\not{\epsilon}^*_{\lambda'}(k')u_s(p)\Bigg]$$

The denominators can be expressed in a shorter form as,

- $$(p+k)^2-m^2 = \underbrace{p^2}_{=m^2}+\underbrace{k^2}_{=0}+2p\cdot k-m^2 = 2p\cdot k \tag{6.142}$$
- $$(p-k')^2-m^2 = -2p\cdot k'\,, \tag{6.143}$$

and using the anticommutator $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$ and the Dirac equation

$(\not{p}-m)u_s(p)=0,$

$$\begin{aligned}
\left(\not{p}+m\right)\not{\epsilon}_\lambda(k)u_s(p) &= \left[-\not{\epsilon}_\lambda(k)\not{p}+2\epsilon_\lambda(k)\cdot p+m\not{\epsilon}_\lambda(k)\right]u_s(p)\\
&= 2\left[\epsilon_\lambda(k)\cdot p\right]u_s(p)+\not{\epsilon}_\lambda(k)\underbrace{\left[-\not{p}+m\right]u_s(p)}_{=0}\\
&= 2\left[\epsilon_\lambda(k)\cdot p\right]u_s(p)\,.
\end{aligned}$$

A significant simplification is obtained if we now choose specific gauge for the polarization vectors. In the Coulomb gauge the polarization vectors are explicitly transverse and in the rest frame of the initial-state electron $p=(m,\mathbf{0})$ we can choose the polarization vectors such that $\epsilon^0=\epsilon'^0=0$. Then,

$$p\cdot\epsilon=p\cdot\epsilon'=0\,, \tag{6.144}$$

in any other frame as well. Thus,

$$\left(\not{p}+m\right)\not{\epsilon}_\lambda(k)u_s(p)=2\left[\epsilon_\lambda(k)\cdot p\right]u_s(p)=0\,.$$

Thanks to this, the matrix element simplifies quite significantly:

$$i\mathcal{M}=-ie^2\left[\overline{u}_{s'}(p')\not{\epsilon}^*_{\lambda'}(k')\frac{\not{k}\not{\epsilon}_\lambda(k)}{2p\cdot k}u_s(p)+\overline{u}_{s'}(p')\not{\epsilon}_\lambda(k)\frac{-\not{k}'\not{\epsilon}^*_{\lambda'}(k')}{-2p\cdot k'}u_s(p)\right].$$

We still speed up the notation by setting $u_s(p)=u$, $\overline{u}_{s'}(p')=\overline{u}'$, $\epsilon_\lambda(k)=\epsilon$, $\epsilon^*_{\lambda'}(k')=\epsilon^{*'}$, so

$$i\mathcal{M}=-ie^2\overline{u}'\left[\frac{\not{\epsilon}^{'*}\not{k}\not{\epsilon}}{2p\cdot k}+\frac{-\not{\epsilon}\not{k}'\not{\epsilon}^{'*}}{-2p\cdot k'}\right]u\,.$$

Doesn't look so bad. Now we square this to form $|\mathcal{M}|^2$:

$$\begin{aligned}
|\mathcal{M}|^2 &= \frac{e^4}{4(p\cdot k)^2}\overline{u}'\left[\not{\epsilon}^{'*}\not{k}\not{\epsilon}\right]u\,\overline{u}\left[\not{\epsilon}^*\not{k}\not{\epsilon}'\right]u'+\frac{e^4}{4(p\cdot k')^2}\overline{u}'\left[\not{\epsilon}\not{k}'\not{\epsilon}^{'*}\right]u\,\overline{u}\left[\not{\epsilon}'\not{k}'\not{\epsilon}^*\right]u'\\
&\quad+\frac{e^4}{4(p\cdot k)(p\cdot k')}\left[\overline{u}'\left[\not{\epsilon}^{'*}\not{k}\not{\epsilon}\right]u\,\overline{u}\left[\not{\epsilon}'\not{k}'\not{\epsilon}^*\right]u'+\overline{u}'\left[\not{\epsilon}\not{k}'\not{\epsilon}^{'*}\right]u\overline{u}\left[\not{\epsilon}^*\not{k}\not{\epsilon}'\right]u'\right]\\
&= \frac{e^4}{4(p\cdot k)^2}\mathrm{Tr}\,\left\{u'\overline{u}'\left[\not{\epsilon}^{'*}\not{k}\not{\epsilon}\right]u\,\overline{u}\left[\not{\epsilon}^*\not{k}\not{\epsilon}'\right]\right\}
\end{aligned}$$

$$+\frac{e^4}{4(p\cdot k')^2}\mathrm{Tr}\left\{u'\overline{u}'\left[\not\epsilon\not k'\not\epsilon'^*\right]u\,\overline{u}\left[\not\epsilon'\not k'\not\epsilon^*\right]\right\}$$

$$+\frac{e^4}{4(p\cdot k)(p\cdot k')}\mathrm{Tr}\left\{u'\overline{u}'\left[\not\epsilon'^*\not k\not\epsilon\right]u\,\overline{u}\left[\not\epsilon'\not k'\not\epsilon^*\right]\right\}$$

$$+\frac{e^4}{4(p\cdot k)(p\cdot k')}\mathrm{Tr}\left\{u'\overline{u}'\left[\not\epsilon\not k'\not\epsilon'^*\right]u\overline{u}\left[\not\epsilon^*\not k\not\epsilon'\right]\right\}$$

We suppose that the e^- beams is unpolarized and sum over the spins of the final-state electron, but leave the photon polarizations open:

$$\overline{|\mathcal{M}|^2}=\frac{1}{2}\sum_{s,s'}|\mathcal{M}|^2=\frac{e^4}{8}\left[\frac{\mathbf{I}}{(p\cdot k)^2}+\frac{\mathbf{II}+\mathbf{III}}{(p\cdot k)(p\cdot k')}+\frac{\mathbf{IV}}{(p\cdot k')^2}\right] \tag{6.145}$$

$$\mathbf{I}=\mathrm{Tr}\left\{(\not p'+m)\left[\not\epsilon'^*\not k\not\epsilon\right](\not p+m)\left[\not\epsilon^*\not k\not\epsilon'\right]\right\}$$

$$\mathbf{II}=\mathrm{Tr}\left\{(\not p'+m)\left[\not\epsilon'^*\not k\not\epsilon\right](\not p+m)\left[\not\epsilon'\not k'\not\epsilon^*\right]\right\}$$

$$\mathbf{III}=\mathrm{Tr}\left\{(\not p'+m)\left[\not\epsilon\not k'\not\epsilon'^*\right](\not p+m)\left[\not\epsilon^*\not k\not\epsilon'\right]\right\}$$

$$\mathbf{IV}=\mathrm{Tr}\left\{(\not p'+m)\left[\not\epsilon\not k'\not\epsilon'^*\right](\not p+m)\left[\not\epsilon'\not k'\not\epsilon^*\right]\right\}$$

There are traces of 8 γ matrices to evaluate, but they are not so bad at the end. We will still simplify the treatment by assuming the polarization vectors to be real (linear polarization). Here we compute explicitly the case $\mathbf{I}$:

$$\mathbf{I}=\mathrm{Tr}\left\{\not p'\not\epsilon'\not k\not\epsilon\not p\not\epsilon\not k\not\epsilon'\right\}+m^2\mathrm{Tr}\left\{\not\epsilon'\not k\not\epsilon\not\epsilon\not k\not\epsilon'\right\} \tag{6.146}$$

We open the traces:

$$\bullet\,\mathrm{Tr}\left[\not p'\not\epsilon'\not k\underbrace{\not\epsilon\not p\not\epsilon}_{2p\cdot\epsilon-\not\epsilon\not p}\not k\not\epsilon'\right]=2\underbrace{(p\cdot\epsilon)}_{=0}\mathrm{Tr}\left[\not p'\not\epsilon'\not k\not\epsilon\not k\not\epsilon'\right]-\mathrm{Tr}\left[\not p'\not\epsilon'\not k\underbrace{\not\epsilon\not\epsilon}_{-1}\not p\not k\not\epsilon'\right]$$

$$=+\mathrm{Tr}\left[\not p'\not\epsilon'\not k\not p\not k\not\epsilon'\right]$$

$$=2(k\cdot p)\mathrm{Tr}\left[\not p'\not\epsilon'\not k\not\epsilon'\right]-\mathrm{Tr}\left[\not p'\not\epsilon'\underbrace{\not k\not k}_{0}\not p\not\epsilon'\right]$$

$$= 2(k\cdot p)2(k\cdot\epsilon')\mathrm{Tr}\left[\not p'\not\epsilon'\right] - 2(k\cdot p)\mathrm{Tr}\left[\not p'\not k\underbrace{\not\epsilon'\not\epsilon'}_{-1}\right]$$

$$= 2(k\cdot p)2(k\cdot\epsilon')4(p'\cdot\epsilon') + 2(k\cdot p)4(p'\cdot k)$$

$$= 8(k\cdot p)\left[2(k\cdot\epsilon')(p'\cdot\epsilon') + (p'\cdot k)\right]$$

- $\mathrm{Tr}\left[\not\epsilon'\not k\not\epsilon\not\epsilon\not k\not\epsilon'\right] \overset{\not\epsilon\not\epsilon=-1}{=} -\mathrm{Tr}\left[\not\epsilon'\not k\not k\not\epsilon'\right] \overset{\not k\not k=k^2=0}{=} 0\,.$

So the trace $\mathbf{I}$ is pretty simple,

$$\mathbf{I} = 8(k\cdot p)\left[2(k\cdot\epsilon')(p'\cdot\epsilon') + (p'\cdot k)\right]\,. \tag{6.147}$$

The case $\mathbf{IV}$ is no more difficult and in addition $\mathbf{II} = \mathbf{III}$. Summing all the contributions gives (Ex.),

$$\overline{|\mathcal{M}(\lambda,\lambda')|^2} = e^4\left[\frac{(k\cdot k')^2}{(k\cdot p)(k'\cdot p)} + 4\left(\epsilon\cdot\epsilon'\right)^2\right]\,. \tag{6.148}$$

Rather simple result. This is Lorentz invariant but we still have to keep in mind that the polarization vectors have been chosen such that in the rest frame of the initial electron $\epsilon^0 = \epsilon'^0 = 0$. If we wish, we can at this moment also sum over the final-state photon polarizations and average over the initial-state polarizations. To accomplish this, we can use (Ex.)

$$\frac{1}{2}\sum_{\lambda,\lambda'}\left(\epsilon\cdot\epsilon'\right)^2 = \frac{1}{2}\left(1+\cos^2\theta\right)\,, \tag{6.149}$$

where the angle θ refers to the scattering angle in the electron rest frame (see the figure below), $\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}' = \cos\theta$. With this result (Ex.),

$$\frac{1}{2}\sum_{\lambda,\lambda'}\overline{|\mathcal{M}(\lambda,\lambda')|^2} = 2e^4\left[\frac{(k\cdot k')^2}{(k\cdot p)(k'\cdot p)} + \left(1+\cos^2\theta\right)\right] \tag{6.150}$$

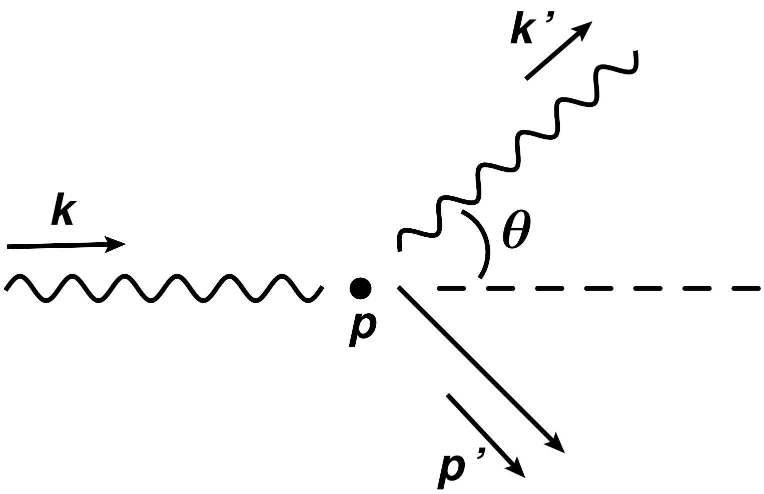
$$= 2e^4\left[\frac{(k\cdot k')^2}{(k\cdot p)(k'\cdot p)} + 2 - 2m^2\frac{k\cdot k'}{(k\cdot p)(k'\cdot p)} + m^4\left[\frac{k\cdot k'}{(k\cdot p)(k'\cdot p)}\right]^2\right].$$

An alternative way to obtain this result is to use the polarization sum (4.20),

$$\sum_{\lambda=1,2} \epsilon^{\mu}_{\mathbf{k},\lambda}\epsilon^{*\nu}_{\mathbf{k},\lambda} = -g^{\mu\nu} + \frac{k^\mu \overline{k}^\nu + k^\nu \overline{k}^\mu}{k\cdot\overline{k}}\,, \qquad \overline{k} = (k^0, -\mathbf{k})$$

already before opening the traces.

Let us then check the kinematics in the rest frame of the incoming electron. This is often called as the **laboratory frame** in this context which probably is a historical relic as in the original Compton-scattering experiments the photons (X rays) scattered off fixed target – electrons bound to an atom.



We may choose the momenta as,

$$p = (m, 0, 0, 0) \tag{6.151}$$

$$k = (\omega, 0, 0, \omega) \tag{6.152}$$

$$k' = (\omega', \omega'\sin\theta\cos\phi, \omega'\sin\theta\sin\phi, \omega'\cos\theta) \tag{6.153}$$

$$p' = p + k - k' \tag{6.154}$$

The dot products that occur in $\overline{|\mathcal{M}|^2}$ are,

$$k \cdot k' = \omega\omega' \left(1 - \cos\theta\right) \tag{6.155}$$

$$k \cdot p = m\omega \tag{6.156}$$

$$k' \cdot p = m\omega' \tag{6.157}$$

so

$$\overline{|\mathcal{M}|^2} = e^4 \left[\frac{\omega^2\omega'^2 \left(1 - \cos\theta\right)^2}{m^2\omega\omega'} + 4\left(\epsilon \cdot \epsilon'\right)^2\right] . \tag{6.158}$$

On the other hand,

$$m^2 = p'^2 = (p + k - k')^2 = m^2 + 2p \cdot (k - k') - 2k \cdot k' \tag{6.159}$$

$$= m^2 + 2m(\omega - \omega') - 2\omega\omega'(1 - \cos\theta)$$

$$\Longrightarrow (1 - \cos\theta) = \frac{m(\omega - \omega')}{\omega\omega'} = m\left[\frac{1}{\omega'} - \frac{1}{\omega}\right] . \tag{6.160}$$

Substituting this into Eq. (6.158), we get our final matrix element squared:

$$\overline{|\mathcal{M}|^2} = e^4 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2 + 4\left(\epsilon \cdot \epsilon'\right)^2\right] . \tag{6.161}$$

Then we form the cross section. The flux factor F is here,

$$4\sqrt{(p \cdot k)^2 - m^2 m_\gamma^2} = 4\sqrt{(m\omega)^2} = 4m\omega\,. \tag{6.162}$$

Then the phase space:

$$\int d\Gamma_2^{\mathrm{LAB}} = \int \left[\frac{d^3p'}{(2\pi)^3 2E_{\mathbf{p}'}}\right]\left[\frac{d^3k'}{(2\pi)^3 2E_{\mathbf{k}'}}\right](2\pi)^4\,\delta^{(4)}\left(p + k - p' - k'\right)$$

$$= \int \left[\frac{d^3k'}{(2\pi)^3 2\omega'}\right]\left[\frac{1}{2E_{\mathbf{p}'}}\right](2\pi)\,\delta\left(m + \omega - \omega' - E_{\mathbf{p}'}\right) . \tag{6.163}$$

To get rid of the remaining δ function we express $E_{\mathbf{p}'}$ in terms of other variables,

$$E_{\mathbf{p}'} = \sqrt{|\mathbf{p}'|^2 + m^2} = \sqrt{|\mathbf{p} + \mathbf{k} - \mathbf{k}'|^2 + m^2} \tag{6.164}$$
$$= \sqrt{|\mathbf{k} - \mathbf{k}'|^2 + m^2} = \sqrt{\omega^2 + \omega'^2 - 2\mathbf{k}\cdot\mathbf{k}' + m^2}$$
$$= \sqrt{\omega^2 + \omega'^2 - 2\omega\omega'\cos\theta + m^2}\,.$$

The relevant δ function is thus

$$\delta\left(m + \omega - \omega' - \sqrt{\omega^2 + \omega'^2 - 2\omega\omega'\cos\theta + m^2}\right) \tag{6.165}$$

$$= \frac{E_{\mathbf{p}'}}{m + \omega(1-\cos\theta)}\delta\left(\omega' - \frac{\omega}{1 + \frac{\omega}{m}(1-\cos\theta)}\right)\,. \tag{6.166}$$

In spherical coordinates $d^3k = d\phi d\cos\theta\omega'^2 d\omega'$, so integrating over ω' we have,

$$\int d\Gamma_2^{\mathrm{LAB}} = \int \left[\frac{d\phi d\cos\theta\omega'^2}{(2\pi)^2 2\omega'}\right]\left[\frac{1}{2E_{\mathbf{p}'}}\right]\frac{E_{\mathbf{p}'}}{m + \omega(1-\cos\theta)} \tag{6.167}$$
$$= \int \left[\frac{d\phi d\cos\theta\omega'^2}{(2\pi)^2 2\omega'}\right]\frac{1}{2}\frac{\omega'}{m\omega} = \frac{1}{16\pi^2}\int d\Omega\left[\frac{\omega'^2}{m\omega}\right]\,,$$

Then we just pack everything together:

$$\frac{d\sigma(e^-\gamma \to e^-\gamma)}{d\phi d\cos\theta} = \frac{1}{F}\frac{d\Gamma_n}{d\Omega}\overline{|\mathcal{M}(e^-\gamma \to e^-\gamma)|^2} \tag{6.168}$$
$$= \frac{1}{4m\omega}\frac{1}{16\pi^2}\left[\frac{\omega'^2}{m\omega}\right]e^4\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2 + 4\left(\epsilon\cdot\epsilon'\right)^2\right]$$
$$= \frac{1}{4m\omega}\frac{1}{16\pi^2}\left[\frac{\omega'^2}{m\omega}\right]16\pi^2\alpha^2\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2 + 4\left(\epsilon\cdot\epsilon'\right)^2\right]$$
$$= \frac{\alpha^2}{4m}\left[\frac{\omega'^2}{m\omega^2}\right]\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2 + 4\left(\epsilon\cdot\epsilon'\right)^2\right]$$

Our final result is the famous **Klein-Nishina formula** in the laboratory frame:

$$\frac{d\sigma\left[e^-\gamma(\lambda)\to e^-\gamma(\lambda')\right]}{d\phi d\cos\theta}=\frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-2+4\left(\epsilon\cdot\epsilon'\right)^2\right]$$

If the initial-state photons are unpolarized, we may use the summation (4.20)

$$\sum_{\lambda=1,2}\epsilon^{\mu}_{\mathbf{k},\lambda}\epsilon^{*\nu}_{\mathbf{k},\lambda}=-g^{\mu\nu}+\frac{k^\mu\overline{k}^\nu+k^\nu\overline{k}^\mu}{k\cdot\overline{k}}\,,\qquad \overline{k}=(k^0,-\mathbf{k})\,.$$

such that the spin-averaged cross section becomes,

$$\frac{d\sigma\left[e^-\gamma\to e^-\gamma(\lambda')\right]}{d\phi d\cos\theta}=\frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-2\left(\hat{\mathbf{k}}\cdot\boldsymbol{\epsilon}'\right)^2\right]\,.$$

We can deduce that the final-state photon's polarization is preferably perpendicular to the plane set by $\mathbf{k}$ and $\mathbf{k}'$. Indeed, the Compton scattering can be used to prepare photon beams of definite polarization by e.g. shooting high-energy electron beams with a laser.

The fully unpolarized cross section is recovered if we still sum over the final-state polarization. The result is, in the lab frame,

$$\frac{d\sigma\left[e^-\gamma\to e^-\gamma\right]}{d\phi d\cos\theta}=\frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\theta\right]\,. \qquad (6.169)$$

Since ω' is a function of $\cos\theta$ the calculation of the total cross section is not completely trivial, though doable. In the low- and high-energy limits this is gets easier.

Low-energy limit:

In the limit of low energies $\omega\ll m_e$. From Eq. (6.160) we see that

$$\frac{\omega'}{\omega}=\frac{1}{1+\frac{\omega}{m}(1-\cos\theta)}\xrightarrow{\omega\ll m_e}1\,, \qquad (6.170)$$

so the cross section simplifies to

$$\frac{d\sigma\left[e^-\gamma \to e^-\gamma\right]}{d\phi d\cos\theta} \xrightarrow{\omega \ll m_e} \frac{\alpha^2}{2m^2}\left[2-\sin^2\theta\right] = \frac{\alpha^2}{2m^2}\left[1+\cos^2\theta\right] . \tag{6.171}$$

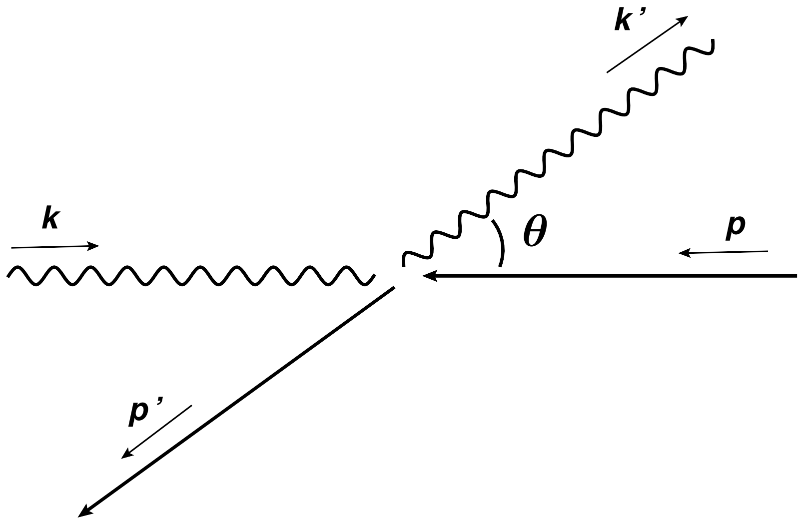
This is easy to integrate,

$$\sigma\left(e^-\gamma \to e^-\gamma\right) \xrightarrow{\omega \ll m_e} = \frac{8\pi\alpha^2}{3m^2} \tag{6.172}$$

This result is known as the **Thomson cross section**, and it can also be derived from classical electrodynamics. Although $\omega \ll m_e$, still (easily) $\omega \gg E_{\text{bonding}}$ for outer electrons of an atom ($m_e \sim 510\,\text{keV}$, $E_{\text{bonding}} \sim 10\ldots1000\,\text{eV}$ for outer electrons).

High-energy limit:

When $\omega \gg m_e$ it is useful to work in center-of-mass coordinates.



$$p = (E, 0, 0, -|\mathbf{k}|) \ , \quad E = \sqrt{\mathbf{k}^2 + m^2} \approx |\mathbf{k}| + \frac{m^2}{2|\mathbf{k}|} \tag{6.173}$$

$$k = (|\mathbf{k}|, 0, 0, |\mathbf{k}|) \tag{6.174}$$

$$k' = (|\mathbf{k}|, |\mathbf{k}| \sin\theta \cos\phi, |\mathbf{k}| \sin\theta \sin\phi, |\mathbf{k}| \cos\theta) \tag{6.175}$$

$$p' = p + k - k' \tag{6.176}$$

The required dot products are,

$$k \cdot k' = |\mathbf{k}|^2 (1 - \cos\theta) \tag{6.177}$$

$$k \cdot p = E|\mathbf{k}| + |\mathbf{k}|^2 \approx 2|\mathbf{k}|^2 + m^2/2 \tag{6.178}$$

$$k' \cdot p = E|\mathbf{k}| + |\mathbf{k}|^2 \cos\theta \approx |\mathbf{k}|^2 (1 + \cos\theta) + m^2/2 \tag{6.179}$$

In addition $\sqrt{s} = E + |\mathbf{k}| \approx 2|\mathbf{k}| + m^2/2|\mathbf{k}|$. The matrix-element squared is obtained from Eq. (6.150),

$$\overline{|\mathcal{M}|^2} \approx 2e^4 \left[\frac{(k \cdot k')^2}{(k \cdot p)(k' \cdot p)} + 2 \right]$$

$$\approx 2e^4 \left[\frac{|\mathbf{k}|^4 (1 - \cos\theta)^2}{(2|\mathbf{k}|^2 + m^2/2)(|\mathbf{k}|^2 (1 + \cos\theta) + m^2/2)} + 2 \right]$$

$$\approx 2e^4 \left[\frac{\frac{s^2}{16} (1 - \cos\theta)^2}{\frac{s}{2} \left[\frac{s}{4} (1 + \cos\theta) + m^2/2\right]} + 2 \right]$$

$$\approx 2e^4 \left[\frac{\frac{1}{2} (1 - \cos\theta)^2}{1 + \cos\theta + \frac{2m^2}{s}} + 2 \right]$$

In the hig-energy limit we can directly use the cross-section formula (3.112),

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{cm}} = \frac{|\mathcal{M}|^2}{64\pi^2 s} = \frac{1}{64\pi^2 s} 2e^4 \left[\frac{\frac{1}{2} (1 - \cos\theta)^2}{1 + \cos\theta + \frac{2m^2}{s}} + 2 \right] \tag{6.180}$$

$$= \frac{\alpha^2}{4s} \left[\frac{(1 - \cos\theta)^2}{1 + \cos\theta + \frac{2m^2}{s}} + 4 \right],$$

so that

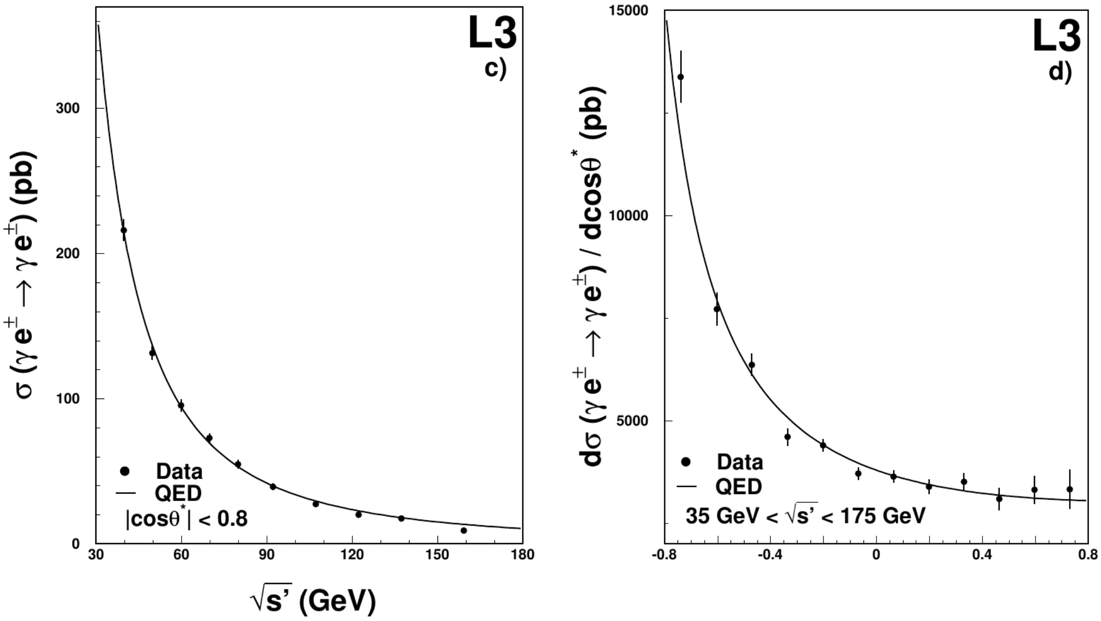
$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{cm}} \xrightarrow{\sqrt{s} \gg m} \frac{\alpha^2}{4s} \left[\frac{(1 - \cos\theta)^2}{1 + \cos\theta + \frac{2m^2}{s}} + 4 \right]$$

The cross section is again peaks when $\theta \sim \pi$. However, the mass of the electron keeps it finite unlike in the $e^-\mu^- \to e^-\mu^-$ case.

In the limit $m^2 \ll s$ the integral accumulates mainly from the region $\theta \sim \pi$, and the leading terms is,

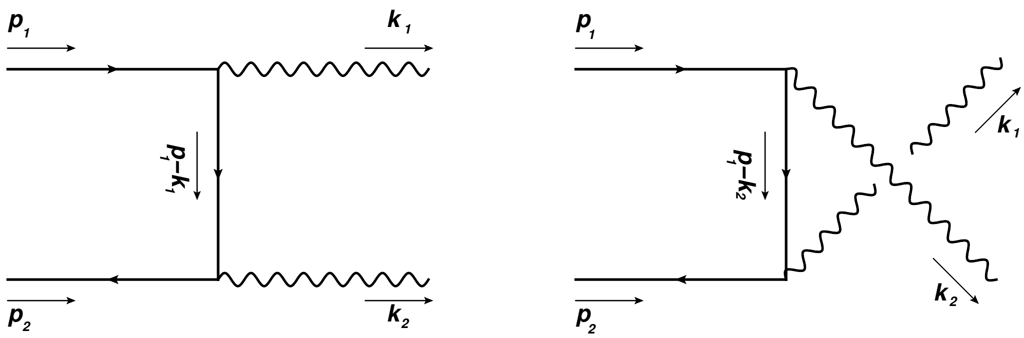
$$\sigma_{\text{tot}} = \int d\cos\theta d\phi \left(\frac{d\sigma}{d\Omega}\right)_{\text{cm}} \xrightarrow{\sqrt{s} \gg m} \frac{2\pi\alpha^2}{s} \log\left[\frac{s}{m^2}\right] \tag{6.181}$$

Both the angular and $\sqrt{s}$ dependencies are well in line with the measurements, as the following pictures show [Phys.Lett. B616 (2005) 145-158].

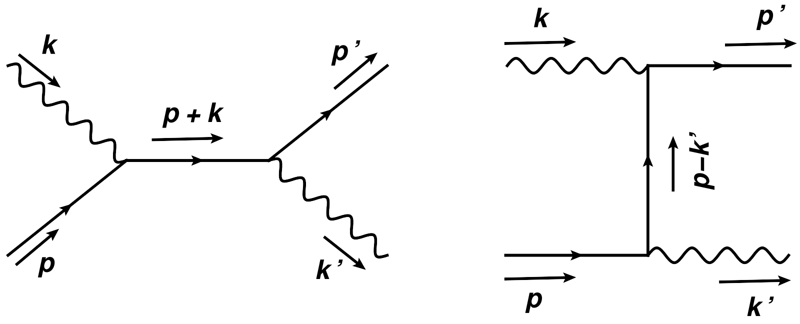


6.6 The e^+e^- pair annihilation into two photons

As our last QED example we look at $e^+e^- \to \gamma\gamma$ reaction:



If we compare these to the diagrams corresponding to the Compton scattering,



we notice that that we can use the crossing symmetry by making the replacements

$$k \to -k_1\,, \quad k' \to k_2\,, \quad p' \to -p_2\,, \quad p \to p_1\,, \tag{6.182}$$

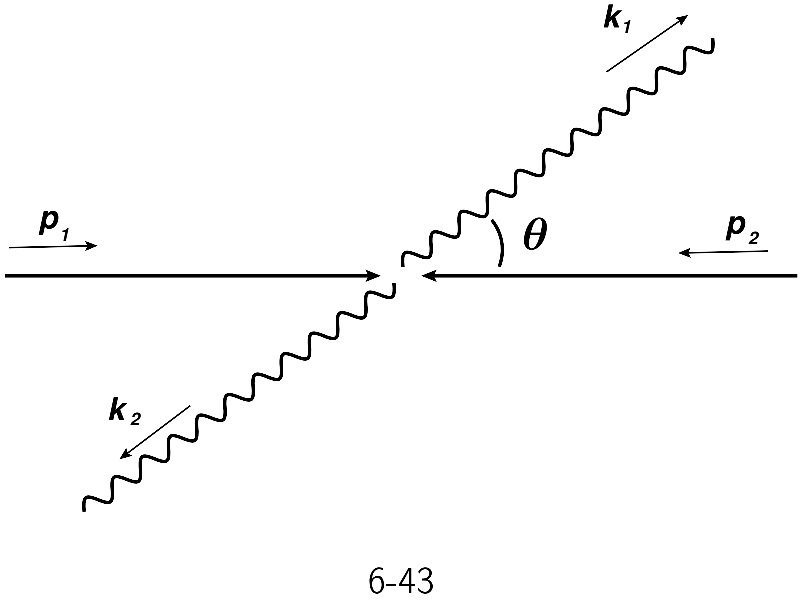
in the Compton case and multiplying the squared matrix element by -1 (only one fermionic interchange). By making these actions in Eq. (6.150), we have

$$\overline{|\mathcal{M}|^2} = \tag{6.183}$$

$$-\,2e^4\left[\frac{(-k_1\cdot k_2)^2}{(-k_1\cdot p_1)(k_2\cdot p_1)} + 2 - 2m^2\frac{-k_1\cdot k_2}{(-k_1\cdot p)(k_2\cdot p_1)} + m^4\left[\frac{-k_1\cdot k_2}{(k\cdot p_1)(k_2\cdot p_1)}\right]^2\right]$$

$$= 2e^4\left[\frac{(k_1\cdot k_2)^2}{(k_1\cdot p_1)(k_2\cdot p_1)} - 2 + 2m^2\frac{k_1\cdot k_2}{(k_1\cdot p_1)(k_2\cdot p_1)} - m^4\left[\frac{k_1\cdot k_2}{(k\cdot p_1)(k_2\cdot p_1)}\right]^2\right].$$

We do the kinematics in the ceter-of-mass frame:



Parametrize the momenta as

$$p_1 = (E, 0, 0, |\mathbf{p}|) \ , \quad E = \frac{\sqrt{s}}{2} = \sqrt{|\mathbf{p}|^2 + m^2} \tag{6.184}$$

$$p_2 = (E, 0, 0, -|\mathbf{p}|) \tag{6.185}$$

$$k_1 = (E, E\sin\theta\cos\phi, E\sin\theta\sin\phi, E\cos\theta) \tag{6.186}$$

$$k_2 = p_1 + p_2 - k_1 \tag{6.187}$$

so the dot products are easy to compute,

$$k_1 \cdot k_2 = 2E^2 \tag{6.188}$$

$$k_1 \cdot p_1 = E^2 - E|\mathbf{p}|\cos\theta \tag{6.189}$$

$$k_1 \cdot p_2 = k_2 \cdot p_1 = E^2 + E|\mathbf{p}|\cos\theta \tag{6.190}$$

$$p_1 \cdot p_2 = E^2 + |\mathbf{p}|^2 = 2|\mathbf{p}|^2 + m^2 \tag{6.191}$$

To build the cross section we need the flux factor,

$$F = 4\sqrt{(p_1 \cdot p_2)^2 - m^4} = 4\sqrt{4|\mathbf{p}|^4 + m^4 + 4m^2|\mathbf{p}|^2 - m^4} \tag{6.192}$$

$$= 4\sqrt{(p_1 \cdot p_2)^2 - m^4} = 4|\mathbf{p}|\sqrt{s}\,. \tag{6.193}$$

In the case of only two final-state particles we can use the result (3.111) for the phase space,

$$\Gamma_2 = \int d\Omega \frac{|\mathbf{k}_1|}{16\pi^2\sqrt{s}} = \int d\Omega \frac{E}{16\pi^2\sqrt{s}}\,, \tag{6.194}$$

so

$$\frac{1}{F}\Gamma_2 = \int d\Omega \frac{E}{16\pi^2\sqrt{s}} \frac{1}{4|\mathbf{p}|\sqrt{s}} = \int d\Omega \frac{1}{64\pi^2 s}\left(\frac{E}{|\mathbf{p}|}\right)\,. \tag{6.195}$$

The cross section is obtained by multiplying this with the squared matrix element (counting both photons) and dividing by 2! since we have two identical particles in the final state,

$$\frac{d\sigma}{d\phi d\cos\theta} \tag{6.196}$$

$$= \frac{1}{2!}\frac{1}{64\pi^2 s}\left(\frac{E}{|\mathbf{p}|}\right)\left[\overline{|\mathcal{M}(\theta,\phi)|^2} + \overline{|\mathcal{M}(\pi-\theta,\pi+\phi)|^2}\right]$$

$$= \frac{\alpha^2}{s}\left(\frac{E}{|\mathbf{p}|}\right)\left[\frac{E^2+|\mathbf{p}|^2\cos^2\theta}{m^2+|\mathbf{p}|^2\sin^2\theta} + \frac{2m^2}{m^2+|\mathbf{p}|^2\sin^2\theta} - \frac{2m^4}{\left(m^2+|\mathbf{p}|^2\sin^2\theta\right)^2}\right].$$

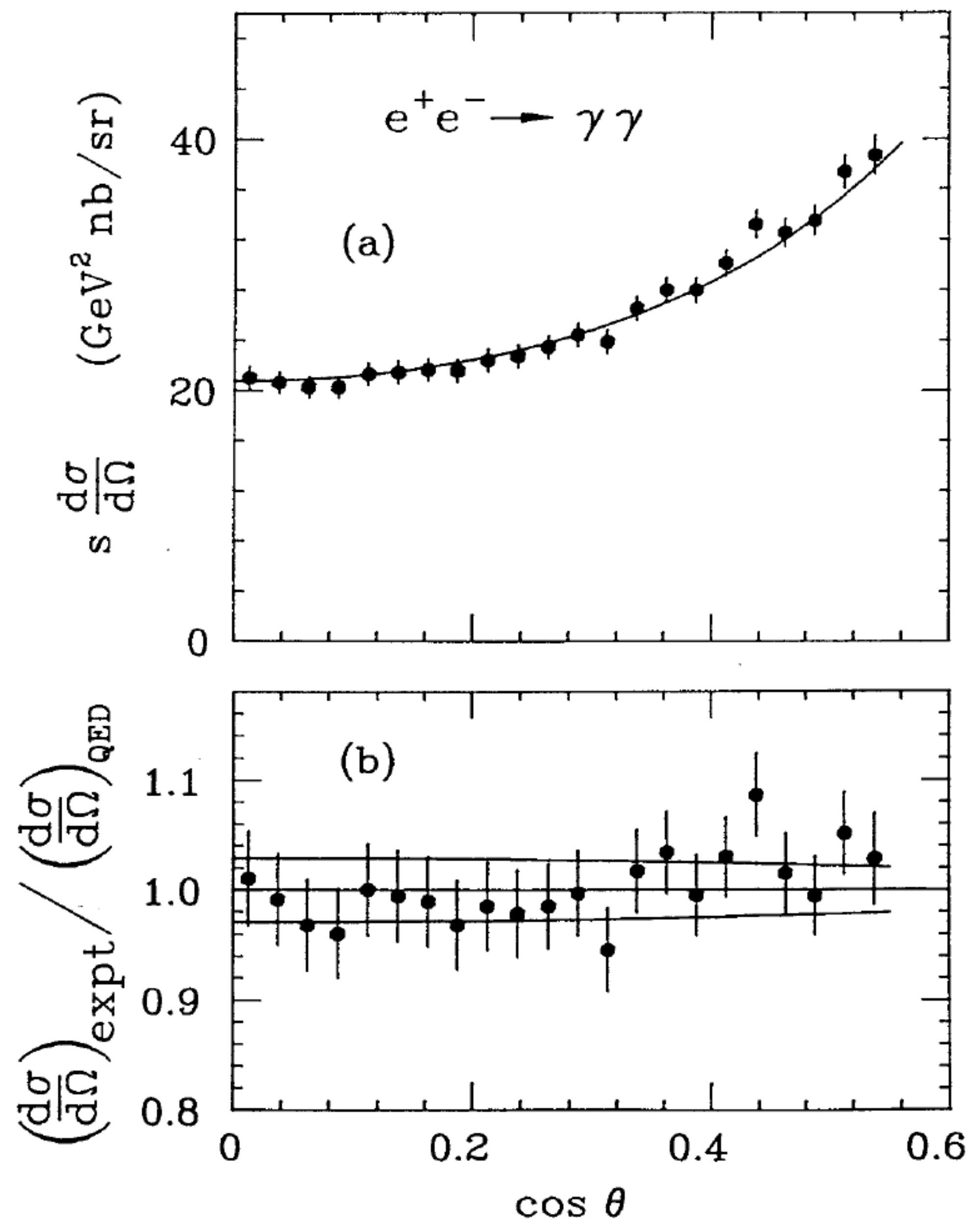
The cross section is peaked near $\theta \sim 0$ and $\theta \sim \pi$. Outside these regions and at the high-energy limit $\sqrt{s} \gg m$, this simplifies to

$$\frac{d\sigma}{d\phi d\cos\theta} \xrightarrow[|\cos\theta|\ll 1]{\sqrt{s}\gg m} \frac{\alpha^2}{s}\frac{1+\cos^2\theta}{\sin^2\theta}. \tag{6.197}$$

The leading term for the total cross section is,

$$\sigma_{\text{tot}} \xrightarrow{\sqrt{s}\gg m} \approx \frac{2\pi\alpha^2}{s}\log\left(\frac{s}{4m^2}\right). \tag{6.198}$$

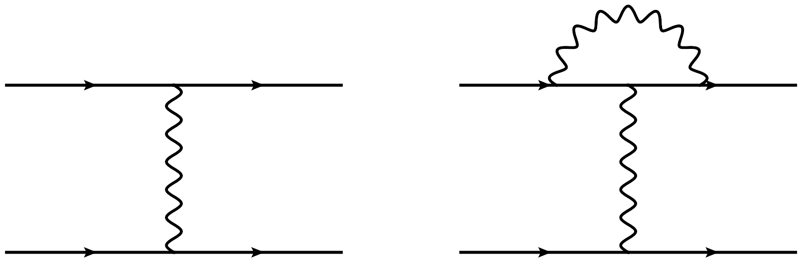
The next picture plots the QED prediction for the angular dependence using (6.197). Clearly, already the leading-order calculation reproduces nicely the measurements [Phys.Rev. D34 (1986) 3286].



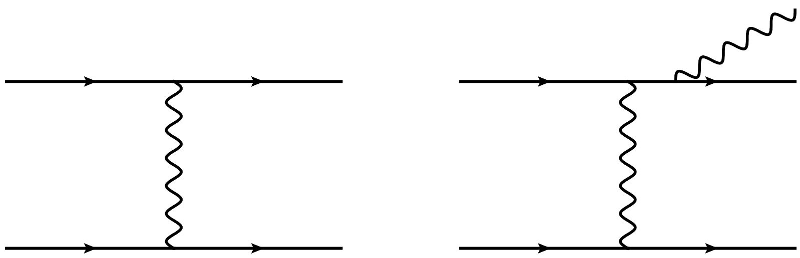
$e^+e^- \to \gamma\gamma$
(a)
$s\frac{d\sigma}{d\Omega}$ (GeV² nb/sr)
0
20
40
(b)
$\left(\frac{d\sigma}{d\Omega}\right)_{expt} / \left(\frac{d\sigma}{d\Omega}\right)_{QED}$
0.8
0.9
1.0
1.1
0
0.2
0.4
0.6
cos θ

7 Introduction to radiative corrections and renormalization

In the previous section we calculated QED cross section at leading order by so-called **tree-level diagarms**. The matrix elements are in this case just algebraic expressions. The higher-order corrections include closed **loops**, so that the matrix element contains d^4p-type integrals.



Also the renormalization constant $\sqrt{Z}$ which the LSZ-theorem entails should be accounted for, corresponding in essence to closed loops in external legs. In addition, almost all leading-order calculations receive corrections from higher-order tree-level diagrams. For example, in $e^+e^- \to e^+e^-$ process the final state can contain very low-energy photons which no particle detector can observe. The extra photons can also carry a higher energy if the are emitted into places where there are no detectors or where one cannot even place one (e.g. the beam pipe). For these resons all measurements are to some extent **inclusive** meaning e.g. in the case of $e^+e^- \to e^+e^-$ process that the final state can contain also other particles than only the e^+e^- pair.



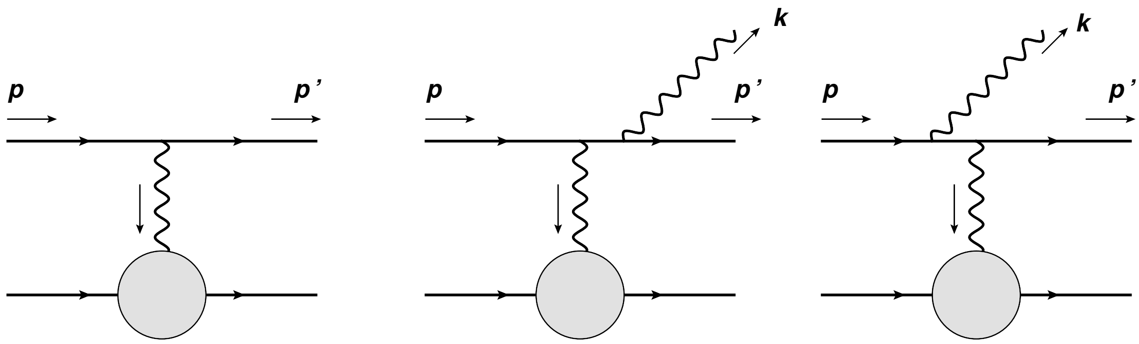
The difficulty is that nearly all higher-order calculations yield infinities. Broadly, there are three types of divergences: If the energy of the emitted or virtual photon momentum k in the above diagrams go to zero, all particles go on shell. In this case the denominators of the propagators vanish, e.g.

$$\frac{1}{(p-k)^2 - m^2} = \frac{1}{-2k\cdot p + k^2} \xrightarrow{k\to 0} \infty\,.$$

These are so-called **infrared divergences**. Another class of inifinities is met in the limit when the momentum of the virtual photon gets large. These are known as **ultraviolet divergences**. The third class is formed by **collinear divergences**. In this section we explore the structure of these divergences and how they either disappear or are removed consistently.

7.1 Braking radiation

Let us start by looking at the photon emission from the initial- and final-state electron:



The lower part of the diagrams is not of our interest here – the electron could scatter off from whatever target. We thus mark the lower part with a grey blob. We write the leading-order matrix element (the left-most diagram) in the form

$$i\mathcal{M}^0(p,p') = [\overline{u}_{s'}(p')(-ie\gamma^\mu)u_s(p)] \times \frac{-ig_{\mu\nu}}{(p-p')^2 + i\epsilon}\Phi^\nu(p-p')\,, \quad (7.1)$$

where Φ^ν contains everything that is there in the lower part of the diagram.

The matrix element for the radiation diagrams are written correspondigly:

$$i\mathcal{M}^{\text{rad}} = \Bigg[\overline{u}_{s'}(p')\big(-ie\not{\epsilon}^*_\lambda(k)\big)\frac{i(\not{p}'+\not{k}+m)}{(p'+k)^2-m^2}(-ie\gamma^\mu)u_s(p) \tag{7.2}$$

$$+\overline{u}_{s'}(p')(-ie\gamma^\mu)\frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2}\big(-ie\not{\epsilon}^*_\lambda(k)\big)u_s(p)\Bigg]$$

$$\times\frac{-ig_{\mu\nu}}{(p-p'-k)^2+i\epsilon}\Phi^\nu(p-p'-k)\,.$$

As noted already, the electron propagators become singular in the limit $k\to 0$. Taking this limit in the numerator, our expression simplifies to

$$i\mathcal{M}^{\text{rad}} \xrightarrow{k\to 0} e\Bigg[\overline{u}_{s'}(p')\not{\epsilon}^*_\lambda(k)\frac{(\not{p}'+m)}{(p'+k)^2-m^2}(-ie\gamma^\mu)u_s(p) \tag{7.3}$$

$$+\overline{u}_{s'}(p')(-ie\gamma^\mu)\frac{(\not{p}+m)}{(p-k)^2-m^2}\not{\epsilon}^*_\lambda(k)u_s(p)\Bigg]$$

$$\times\frac{-ig_{\mu\nu}}{(p-p')^2+i\epsilon}\Phi^\nu(p-p')\,.$$

As in the calculation of the Compton scattering, the denominators of the propagators can be written as,

$$\bullet\,(p'+k)^2-m^2 = 2p'\cdot k \tag{7.4}$$

$$\bullet\,(p-k)^2-m^2 = -2p\cdot k\,, \tag{7.5}$$

and by using the anticommutator $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$ and Dirac equation $(\not{p}-m)u_s(p)=0$,

$$\begin{aligned}\big(\not{p}+m\big)\,\not{\epsilon}^*_\lambda(k)u_s(p) &= \left[-\not{\epsilon}^*_\lambda(k)\not{p}+2\epsilon^*_\lambda(k)\cdot p+m\not{\epsilon}^*_\lambda(k)\right]u_s(p)\\ &= 2\left[\epsilon^*_\lambda(k)\cdot p\right]u_s(p)+\not{\epsilon}^*_\lambda(k)\underbrace{\left[-\not{p}+m\right]u_s(p)}_{=0}\\ &= 2\left[\epsilon^*_\lambda(k)\cdot p\right]u_s(p)\,.\end{aligned}$$

Similarly,

$$\overline{u}_{s'}(p')\not{\epsilon}^*_\lambda(k)\left(\not{p}'+m\right) = 2\left[\epsilon^*_\lambda(k)\cdot p'\right]\overline{u}_{s'}(p')\,.$$

Thus,

$$i\mathcal{M}^{\text{rad}} \xrightarrow{k\to 0} e\Bigg[\overline{u}_{s'}(p')2\left[\epsilon^*_\lambda(k)\cdot p'\right]\frac{1}{2p'\cdot k}(-ie\gamma^\mu)u_s(p) \tag{7.6}$$

$$+\,\overline{u}_{s'}(p')(-ie\gamma^\mu)\frac{1}{-2p\cdot k}2\left[\epsilon^*_\lambda(k)\cdot p\right]u_s(p)\Bigg] \times \frac{-ig_{\mu\nu}}{(p-p')^2+i\epsilon}\Phi^\nu(p-p')$$

$$= i\mathcal{M}^0(p,p')\times e\left[\frac{\epsilon^*_\lambda(k)\cdot p'}{p'\cdot k} - \frac{\epsilon^*_\lambda(k)\cdot p}{p\cdot k}\right].$$

We see that in the limit $k \to 0$ the leading-order matrix element $\mathcal{M}^0$ and the part associated with the soft radiation factorize. Due to this property, the contribution of the soft radiation to the cross section is

$$d\sigma^{\text{rad}}(p,p') = d\sigma^0(p,p') \tag{7.7}$$

$$\times \int \frac{d^3k}{2|\mathbf{k}|(2\pi)^3}e^2\left[\frac{\epsilon^*_\lambda(k)\cdot p'}{p'\cdot k} - \frac{\epsilon^*_\lambda(k)\cdot p}{p\cdot k}\right]\left[\frac{\epsilon_\lambda(k)\cdot p'}{p'\cdot k} - \frac{\epsilon_\lambda(k)\cdot p}{p\cdot k}\right],$$

where $d\sigma^0(p,p')$ is the leading-order result (whatever it is). We are not interested in the photon polarizations here, so we sum over them using (4.20),

$$\sum_{\lambda=1,2}\epsilon^\mu_{\mathbf{k},\lambda}\epsilon^{*\nu}_{\mathbf{k},\lambda} = -g^{\mu\nu} + \frac{k^\mu\overline{k}^\nu + k^\nu\overline{k}^\mu}{k\cdot\overline{k}}, \qquad \overline{k} = (k^0,-\mathbf{k}).$$

It's easy to see that the second term in the sum gives zero, as it must due to the Ward identity. Thus it is enough to account for the $-g^{\mu\nu}$ part:

$$\sum_\lambda\left[\frac{\epsilon^*_\lambda(k)\cdot p'}{p'\cdot k} - \frac{\epsilon^*_\lambda(k)\cdot p}{p\cdot k}\right]\left[\frac{\epsilon_\lambda(k)\cdot p'}{p'\cdot k} - \frac{\epsilon_\lambda(k)\cdot p}{p\cdot k}\right] \tag{7.8}$$

$$= \frac{2p\cdot p'}{(p\cdot k)(p'\cdot k)} - \frac{m^2}{(p\cdot k)^2} - \frac{m^2}{(p'\cdot k)^2}.$$

$$d\sigma^{\text{rad}}(p,p') = d\sigma^0(p,p') \tag{7.9}$$

$$\times \int \frac{d^3k}{2|\mathbf{k}|(2\pi)^3}e^2\left[\frac{2p\cdot p'}{(p\cdot k)(p'\cdot k)} - \frac{m^2}{(p\cdot k)^2} - \frac{m^2}{(p'\cdot k)^2}\right]$$

The denominators contain terms,

$$p \cdot k = |\mathbf{k}| \left(E_{\mathbf{p}} - |\mathbf{p}| \cos\theta_{p,k}\right)$$

$$p' \cdot k = |\mathbf{k}| \left(E_{\mathbf{p}'} - |\mathbf{p}'| \cos\theta_{p',k}\right) ,$$

which can go to zero when $|\mathbf{k}| \to 0$, or close to zero when $\cos\theta_{p,k} \sim 1$. Thus, **most of the photon radiation is soft and is in the direction of the initial- or final-state electron**. Due to the momentum conservation the speed of the electrons gets reduced but the direction does not change on the average. Thus, the word **braking radiation**. The same result can be derived from classical electromagnetism.

Since there is no lower bound for the photon energy, $d\sigma^{\mathrm{rad}}(p, p') = \infty$. However, we provisionally **regularize** the integral by giving the photon a small mass μ which translates into a lower bound for the radiated photon energy. Going to spherical coordinates, we have the following integral to be done:

$$\frac{1}{2(2\pi)^3} \int_\mu d|\mathbf{k}||\mathbf{k}| \int d\Omega_{\mathbf{k}} \left[\frac{2p \cdot p'}{(p \cdot k)(p' \cdot k)} - \frac{m^2}{(p \cdot k)^2} - \frac{m^2}{(p' \cdot k)^2}\right] .$$

The angular part gives (Ex.),

$$\int d\Omega_{\mathbf{k}} \left[\frac{2p \cdot p'}{(p \cdot k)(p' \cdot k)} - \frac{m^2}{(p \cdot k)^2} - \frac{m^2}{(p' \cdot k)^2}\right] \tag{7.10}$$

$$= \frac{8\pi}{|\mathbf{k}|^2} \left[\frac{\left(1 + \frac{2m^2}{-q^2}\right)}{\sqrt{1 + \frac{4m^2}{-q^2}}} \log\left(\frac{\sqrt{1 + \frac{4m^2}{-q^2}} + 1}{\sqrt{1 + \frac{4m^2}{-q^2}} - 1}\right) - 1\right] ,$$

where $q^2 \equiv (p - p')^2$. If the momentum exchange $-q^2 \gg m^2$, this simplifies to

$$\xrightarrow{q^2 \gg m^2} \frac{8\pi}{|\mathbf{k}|^2} \left[\log\left(\frac{-q^2}{m^2}\right) - 1\right] .$$

The upper bound for the remaining $d|\mathbf{k}|$ integral is not important (as long as it remains within the physical phase space). For dimensional reasons we can take the upper bound to be proportional to $\sqrt{-q^2}$:

$$\int_{\mu}^{c\sqrt{-q^2}} \frac{d|\mathbf{k}|}{|\mathbf{k}|} = \log\left(\frac{\sqrt{-q^2}}{\mu}\right) + \log(c)\,, \tag{7.11}$$

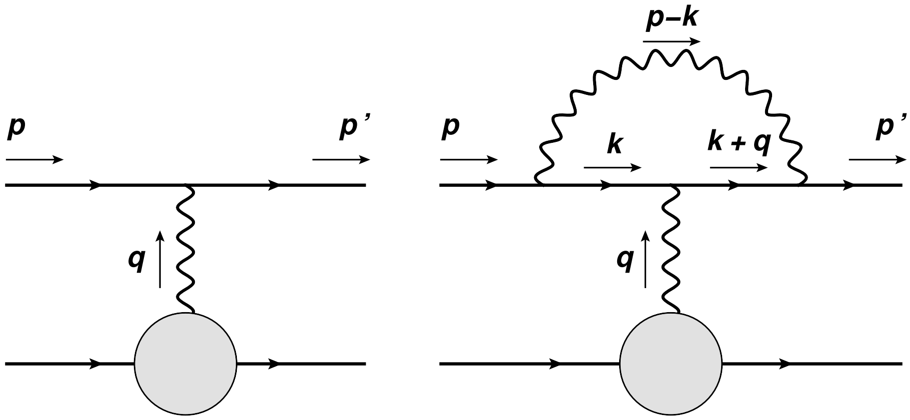
where $\log(c)$ is some finite constant. Thus, the IR-divegent part acquires the form,

$$\begin{aligned} d\sigma^{\mathrm{rad,IR}}(p,p') &= d\sigma^0(p,p') \\ &\quad \times \frac{\alpha}{\pi}\log\left(\frac{-q^2}{\mu^2}\right)\left[\frac{\left(1+\frac{2m^2}{-q^2}\right)}{\sqrt{1+\frac{4m^2}{-q^2}}}\log\left(\frac{\sqrt{1+\frac{4m^2}{-q^2}}+1}{\sqrt{1+\frac{4m^2}{-q^2}}-1}\right)-1\right] \\ &\xrightarrow{q^2\gg m^2} d\sigma^0(p,p')\times\frac{\alpha}{\pi}\log\left(\frac{-q^2}{\mu^2}\right)\left[\log\left(\frac{-q^2}{m^2}\right)-1\right] \end{aligned} \tag{7.12}$$

In the limit $q^2 \gg m^2$ a product of two logarithms emerges. This is known as the **Sudakov double logarithm**. The approximations we have made in the limit $|\mathbf{k}| \to 0$ are all valid, so the found IR divergence is not an artefact but a true property of the theory – and its difficulty.

The regularizing parameter μ is of course unphysical and our final result cannot depend on it if the theory is to have any predictive power. However, as noted in the beginning, also the virtual corrections entail IR divergences and it turns out, as we will see, that when all relevant contributions are summed, the IR divergences disappear. Note that the our result for the braking radiation is also divergent in the $m \to 0$ limit (collinear divergence).

7.2 Virtual vertex correction



We wrote out leading-order matrix element in the form,

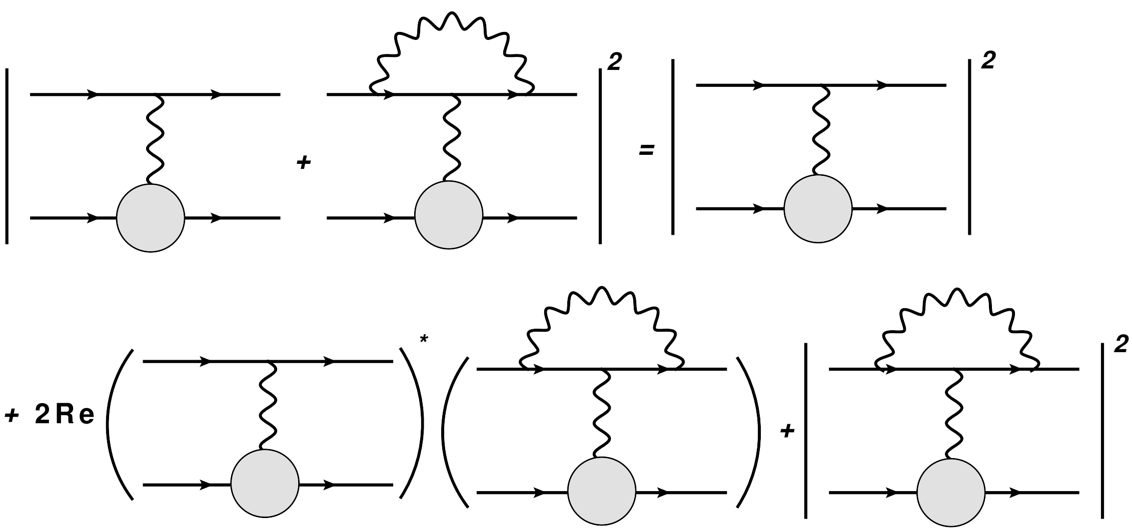
$$i\mathcal{M}^0(p,p') = ie\,[\overline{u}_{s'}(p')\gamma^\mu u_s(p)] \times \frac{ig_{\mu\nu}}{(p-p')^2 + i\epsilon}\Phi^\nu(p-p')\,. \quad (7.13)$$

In general, since the factors $\overline{u}_{s'}(p')$ and $u_s(p)$ are always the outermost in any amputated diagram, we can write this type of diagram as

$$i\mathcal{M}(p,p') = ie\,[\overline{u}_{s'}(p')\Gamma^\mu u_s(p)] \times \frac{ig_{\mu\nu}}{(p-p')^2 + i\epsilon}\Phi^\nu(p-p')\,, \quad (7.14)$$

where, to leading order, $\Gamma^\mu = \gamma^\mu$.

The inital and final states in the above leading- and loop-corrected diagrams are the same. Thus they must be summed before squaring. Thus, we get three terms:



The "cross term" coming with the factor $2\mathrm{Re}$ is of the same order in coupling as the real radiation contribution so this one is what we will need.

We now write the matrix element corresponding to the loop-corrected diagram (in Feynman gauge):

$$i\mathcal{M}^{\rm loop} = \int \frac{d^4k}{(2\pi)^4} \overline{u}_{s'}(p')\left[-ie\gamma^\nu\right] \frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2+i\epsilon}\left[-ie\gamma^\mu\right]\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}$$

$$\times\left[-ie\gamma^\rho\right]u_s(p) \times \frac{-ig_{\nu\rho}}{(p-k)^2+i\epsilon} \times \frac{-ig_{\mu\nu}}{(p-p')^2+i\epsilon}\Phi^\nu(p-p') \qquad (7.15)$$

$$= ie\left[-ie^2\int\frac{d^4k}{(2\pi)^4}\frac{\overline{u}_{s'}(p')\gamma^\nu(\not{k}+\not{q}+m)\gamma^\mu(\not{k}+m)\gamma_\nu u_s(p)}{\left[(k+q)^2-m^2+i\epsilon\right]\left[k^2-m^2+i\epsilon\right]\left[(p-k)^2+i\epsilon\right]}\right]$$

$$\times\frac{ig_{\mu\nu}}{(p-p')^2+i\epsilon}\Phi^\nu(p-p')\,.$$

Comparing to (7.14) we can identify,

$$\overline{u}_{s'}(p')\Gamma^\mu_{\rm loop}u_s(p) = \qquad (7.16)$$

$$-ie^2\int\frac{d^4k}{(2\pi)^4}\frac{\overline{u}_{s'}(p')\gamma^\nu(\not{k}+\not{q}+m)\gamma^\mu(\not{k}+m)\gamma_\nu u_s(p)}{\left[(k+q)^2-m^2+i\epsilon\right]\left[k^2-m^2+i\epsilon\right]\left[(p-k)^2+i\epsilon\right]}\,.$$

By using the γ-matrix identities (6.24)-(6.26), we can sum over the free index ν,

$$\overline{u}_{s'}(p')\Gamma^\mu_{\rm loop}u_s(p) = \qquad (7.17)$$

$$2ie^2\int\frac{d^4k}{(2\pi)^4}\frac{\overline{u}_{s'}(p')\left[\not{k}\gamma^\mu(\not{k}+\not{q})+m^2\gamma^\mu-2m(2k+q)^\mu\right]u_s(p)}{\left[(k+q)^2-m^2+i\epsilon\right]\left[k^2-m^2+i\epsilon\right]\left[(p-k)^2+i\epsilon\right]}\,.$$

This type of integrals are most conveniently computed by the **Feynman parametrizations**:

$$\frac{1}{AB} = \int_0^1 dxdy\, \delta(1-x-y) \frac{1}{[xA+yB]^2} \tag{7.18}$$

$$\frac{1}{AB^n} = \int_0^1 dxdy\, \delta(1-x-y) \frac{ny^{n-1}}{[xA+yB]^{n+1}} \tag{7.19}$$

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 dx_1 \ldots dx_n\, \delta(1-\sum x_i) \frac{(n-1)!}{[x_1 A_1 + \ldots + x_n A_n]^n} \tag{7.20}$$

We use the last one of these to process Eq. (7.17):

$$\begin{aligned} &\frac{1}{[(k+q)^2 - m^2 + i\epsilon]\, [k^2 - m^2 + i\epsilon]\, [(p-k)^2 + i\epsilon]} \\ &= \int dxdydz \delta(1-x-y-z) \frac{2}{D^3}\,, \end{aligned} \tag{7.21}$$

where the denominator D reads:

$$\begin{aligned} D &= x\left[k^2 - m^2 + i\epsilon\right] + y\left[(k+q)^2 - m^2 + i\epsilon\right] + z\left[(p-k)^2 + i\epsilon\right] \\ &= x\left[k^2 - m^2\right] + y\left[k^2 + q^2 + 2k\cdot q - m^2\right] + z\left[p^2 + k^2 - 2p\cdot k\right] \\ &+ \underbrace{(x+y+z)}_{=1} i\epsilon \\ &= x\left[-m^2\right] + y\left[q^2 + 2k\cdot q - m^2\right] + z\left[p^2 - 2p\cdot k\right] + k^2 + i\epsilon \\ &= \underline{k^2 + 2k\cdot(yq - zp)} + yq^2 + zp^2 - (x+y)m^2 + i\epsilon\,. \end{aligned} \tag{7.22}$$

The underlined terms are the only ones that depend on the integration variable k. By completing the square,

$$\begin{aligned} &k^2 + 2k\cdot(yq - zp) \\ =&(k + yq - zp)^2 - y^2q^2 - z^2p^2 + 2yzp\cdot q\,. \end{aligned} \tag{7.23}$$

Denoting $\ell \equiv k + yq - zp$, we have

$$
\begin{aligned}
D &= (k + yq - zp)^2 - y^2q^2 - z^2p^2 + 2yzp\cdot q + yq^2 + zp^2 - (x+y)m^2 + i\epsilon \\
&= \ell^2 - y^2q^2 + 2yz\underbrace{(p\cdot q)}_{-q^2/2} + yq^2 - (\underbrace{x+y}_{=1-z} + z^2 - z)m^2 + i\epsilon \\
&= \ell^2 + \underbrace{(-y^2 - yz + y)}_{xy}q^2 - \underbrace{(1 - z + z^2 - z)}_{=(1-z)^2}m^2 + i\epsilon \\
&= \ell^2 + xyq^2 - (1-z)^2m^2 + i\epsilon\,.
\end{aligned}
$$

So finally,

$$
D = \ell^2 - \Delta + i\epsilon\,, \tag{7.24}
$$

$$
\Delta = -xyq^2 + (1-z)^2m^2\,. \tag{7.25}
$$

We want to do a change of variables $d^4k \to d^4\ell$, so also in the numerator all the momenta k should be written in terms of ℓ,

$$
\begin{aligned}
N =& \not{k}\gamma^\mu(\not{k} + \not{q}) + m^2\gamma^\mu - 2m(2k+q)^\mu \\
&= (\not{\ell} - y\not{q} + z\not{p})\gamma^\mu(\not{\ell} - y\not{q} + z\not{p} + \not{q}) + m^2\gamma^\mu \\
&- 2m\left[2\ell^\mu + (1-2y)q^\mu + 2zp^\mu\right]
\end{aligned} \tag{7.26}
$$

We can simplify this using the identities,

$$
\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^\mu}{D^n} = 0\,, \tag{7.27}
$$

$$
\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^\mu\ell^\nu}{D^n} = \frac{1}{4}g^{\mu\nu}\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^2}{D^n}\,. \tag{7.28}
$$

The nullity of the first one follows from the fact that the integral is odd (D depends only on ℓ^2). The second one is a symmetric tensor so the right-hand side should be of the form $Cg^{\mu\nu}$, and the factor C can be found by contracting each side of the equation by $g_{\mu\nu}$. One can also prove it more directly. Anyway, we can use these to simplify our numerator:

$$N = \not{\ell}\gamma^\mu\not{\ell} + \Big[-y\not{q} + z\not{p}\Big]\gamma^\mu\Big[(1-y)\not{q} + z\not{p}\Big] + m^2\gamma^\mu - 2m\left[(1-2y)q^\mu + 2zp^\mu\right]$$

$$= -\frac{1}{2}\gamma^\mu\ell^2 + \Big[-y\not{q} + z\not{p}\Big]\gamma^\mu\Big[(1-y)\not{q} + z\not{p}\Big] + m^2\gamma^\mu - 2m\left[(1-2y)q^\mu + 2zp^\mu\right] .$$

Thus, at this stage we have massaged the vertex correction into a form,

$$\overline{u}_{s'}(p')\Gamma^\mu_{\text{loop}}u_s(p) = \int dxdydz\delta(1-x-y-z) \times 4ie^2\int\frac{d^4\ell}{(2\pi)^4}\frac{\overline{u}_{s'}(p')\left[-\frac{1}{2}\gamma^\mu\ell^2 + \cdots\right]u_s(p)}{\left(\ell^2 - \Delta + i\epsilon\right)^3} . \tag{7.29}$$

Wick's rotation

The advantage of the Feynman parametrization is that the 4-D loop integral becomes relatively simple as the integrand depends only on the scalar ℓ^2. Indeed, integrals like,

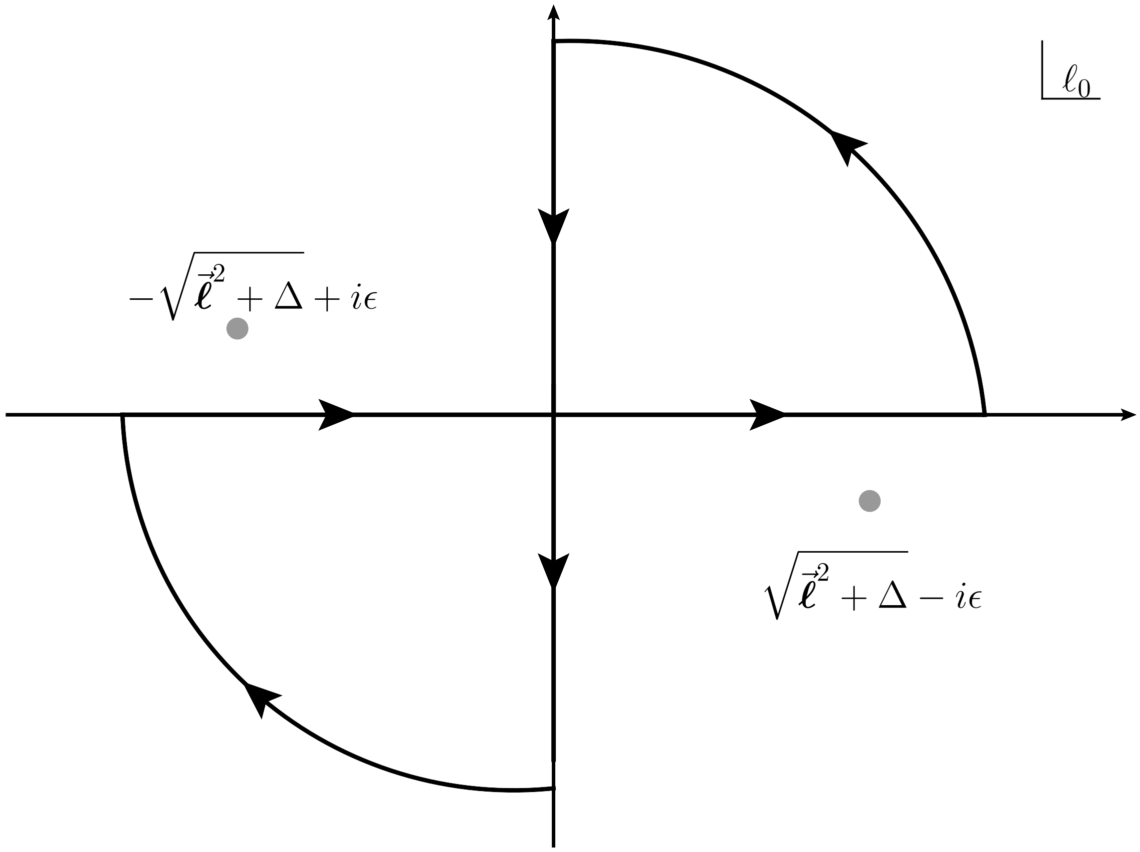
$$\int\frac{d^4\ell}{(2\pi)^4}\frac{F(\ell^2;\ldots)}{\left(\ell^2 - \Delta + i\epsilon\right)^m} \tag{7.30}$$

can be evaluated by so-callled **Wick's rotation**. The idea is to reduce the integral in the Minkowksi space time to a usual Euklidian integral. The factor in the denominator is

$$\ell^2 - \Delta + i\epsilon = \ell_0^2 - \left(\vec{\boldsymbol{\ell}}^2 + \Delta\right) + i\epsilon \tag{7.31}$$

For a scattering process $q^2 < 0$, so Δ is positive, $\Delta = -xyq^2 + (1-z)^2m^2 > 0$. The integrand has poles at

$$\ell_0 = \pm\sqrt{\vec{\boldsymbol{\ell}}^2 + \Delta} \mp i\epsilon\,. \tag{7.32}$$



By integrating along the indicated closed contour, there are no poles inside so the integral vanishes,

$$\left[\int_{-\infty}^{\infty}+\int_{\mathrm{arc}\,1}+\int_{i\infty}^{-i\infty}+\int_{\mathrm{arc}\,2}\right]\frac{F(\ell^2;\ldots)}{(\ell^2-\Delta+i\epsilon)^m}d\ell_0=0\,. \tag{7.33}$$

Assuming that

$$|\ell_0|\left|\frac{F(\ell^2;\ldots)}{(\ell^2-\Delta+i\epsilon)^m}\right|\xrightarrow{|\ell_0|\to\infty}0\,, \tag{7.34}$$

the arc integrals yield zero and thus

$$\int_{-\infty}^{\infty}\frac{F(\ell^2;\ldots)}{(\ell^2-\Delta+i\epsilon)^m}d\ell_0=-\int_{i\infty}^{-i\infty}\frac{F(\ell^2;\ldots)}{(\ell^2-\Delta)^m}d\ell_0\,. \tag{7.35}$$

The latter integral countour is along the imaginary axis so we can parametrize it as $\ell_0=-it$, $d\ell_0=-idt$. In addition,

$$\ell^2=\ell_0^2-\vec{\boldsymbol{\ell}}^2=-t^2-\vec{\boldsymbol{\ell}}^2\,, \tag{7.36}$$

so our integral becomes,

$$-\int_{i\infty}^{-i\infty} \frac{F(\ell^2;\ldots)}{(\ell^2-\Delta)^m} d\ell_0 = i\int_{-\infty}^{\infty} \frac{F(-t^2-\vec{\boldsymbol{\ell}}^2;\ldots)}{\left(-t^2-\vec{\boldsymbol{\ell}}^2-\Delta\right)^m} dt \tag{7.37}$$

$$= i(-1)^m \int_{-\infty}^{\infty} \frac{F(-t^2-\vec{\boldsymbol{\ell}}^2;\ldots)}{\left(t^2+\vec{\boldsymbol{\ell}}^2+\Delta\right)^m} dt$$

By defining an Euklidian 4-D vector $\ell_{\mathrm{E}}^0 = t$ and $\boldsymbol{\ell}_{\mathrm{E}} = \boldsymbol{\ell}$, we finally have

$$\int_{-\infty}^{\infty} \frac{d^4\ell}{(2\pi)^4} \frac{F(\ell^2;\ldots)}{(\ell^2-\Delta+i\epsilon)^m} = i(-1)^m \int_{-\infty}^{\infty} \frac{d^4\ell_{\mathrm{E}}}{(2\pi)^4} \frac{F(-\ell_{\mathrm{E}}^2;\ldots)}{(\ell_{\mathrm{E}}^2+\Delta)^m}\,. \tag{7.38}$$

Since the integrand depends only on ℓ_{E}^2, we can use the spherical coordinates,

$$\int d^4\ell_{\mathrm{E}} = \int d\Omega_4 \int |\ell_{\mathrm{E}}|^3 d|\ell_{\mathrm{E}}|\,. \tag{7.39}$$

We can parametrize this by using, in addition to the usual 3-D coordinates, an additional angle $0 < \omega < \pi$,

$$\ell_{\mathrm{E}} = \left(|\ell_{\mathrm{E}}|\cos\omega, \hat{\boldsymbol{\ell}}|\ell_{\mathrm{E}}|\sin\omega\right)\,, \tag{7.40}$$

where $\hat{\boldsymbol{\ell}}$ is a 3-D unit vector. As a result, the angular part is just (Ex.),

$$\int d\Omega_4 = 2\pi^2\,. \tag{7.41}$$

Computing the radial parts is also straightforward, and we finally have (Ex.),

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2-\Delta+i\epsilon)^m} = \frac{i(-1)^m}{(4\pi)^2} \frac{1}{(m-1)(m-2)\Delta^{m-2}} \tag{7.42}$$

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^m} = \frac{i(-1)^{m+1}}{(4\pi)^2} \frac{2}{(m-1)(m-2)(m-3)\Delta^{m-3}} \tag{7.43}$$

In the first one we must have $m > 2$, and in the latter $m > 3$.

Unfortunately we cannot directly apply the above identities to evaluate the ℓ^2 term in Eq. (7.29) since the the exponent there is exactly 3 and the integral does not converge at large momenta $\longleftrightarrow$ UV divergence. We provisionally regularize the integral by the **Pauli-Villars method**, in which the original photon propagator is replaced by the difference,

$$\frac{1}{(p-k)^2+i\epsilon} \xrightarrow{\textbf{Pauli-Villars}} \frac{1}{(p-k)^2+i\epsilon} - \frac{1}{(p-k)^2-\Lambda^2+i\epsilon}\,.$$

For large values of k, this difference behaves as,

$$\frac{1}{(p-k)^2+i\epsilon} - \frac{1}{(p-k)^2-\Lambda^2+i\epsilon} \sim \frac{\Lambda^2}{(k^2)^2}\,,$$

so there is one power of k^2 more in the downstairs which improves the convergence. At the end of the calculations we would like to take $\Lambda \to \infty$ such that the added term in the propagator effectively drops out. By doing this change in Eq. (7.16), we obtain an extra term which is otherwize identical with the original ℓ^2 term, but in which the Δ factor becomes,

$$\Delta_\Lambda = -xyq^2 + (1-z)^2m^2 + z\Lambda^2\,. \tag{7.44}$$

In effect, the ℓ^2 term in Eq. (7.29) is replaced by,

$$\int \frac{d^4\ell}{(2\pi)^4}\left[\frac{\ell^2}{\left(\ell^2-\Delta+i\epsilon\right)^3} - \frac{\ell^2}{\left(\ell^2-\Delta_\Lambda+i\epsilon\right)^3}\right] \tag{7.45}$$

$$= i\int \frac{d^4\ell_{\mathrm{E}}}{(2\pi)^4}\left[\frac{\ell_{\mathrm{E}}^2}{\left(\ell_{\mathrm{E}}^2+\Delta\right)^3} - \frac{\ell_{\mathrm{E}}^2}{\left(\ell_{\mathrm{E}}^2+\Delta_\Lambda\right)^3}\right] = \frac{i}{(4\pi)^2}\log\left(\frac{\Delta_\Lambda}{\Delta}\right)$$

The rest of what is in Eq. (7.29) can be directly integrated by identities (7.42), and we find,

$$\overline{u}_{s'}(p')\Gamma^\mu_{\text{loop}}u_s(p) = \frac{\alpha}{2\pi}\int dx dy dz \delta(1-x-y-z) \tag{7.46}$$

$$\times\, \overline{u}_{s'}(p')\Bigg\{\gamma^\mu \log\left(\frac{\Delta_\Lambda}{\Delta}\right) + \frac{1}{\Delta}\Bigg[\Big[-y\not{q} + z\not{p}\Big]\gamma^\mu\Big[(1-y)\not{q} + z\not{p}\Big]$$

$$+ m^2\gamma^\mu - 2m\Big[(1-2y)q^\mu + 2zp^\mu\Big]\Bigg]\Bigg\}u_s(p)\,.$$

For the moment we have managed to regularize the UV infinity but the above expression is also IR divergent. For example, the $m^2\gamma^\mu$ term is of the form,

$$\int_0^1 dxdydz\delta(1-x-y-z)\frac{1}{\Delta} \tag{7.47}$$

$$= \int_0^1 dz \int_0^{1-z} dy \frac{1}{-(1-y-z)yq^2 + (1-z)^2 m^2} \qquad \Big| y = (1-z)\xi$$

$$= \int_0^1 dz \int_0^1 d\xi \frac{1}{-(1-z)(1-\xi)\xi q^2 + (1-z)m^2}$$

$$= \int_0^1 \frac{dz}{1-z} \int_0^1 d\xi \frac{1}{-(1-\xi)\xi q^2 + m^2},$$

where the z integral is logarithmically divergent. We now regularize this in the same manner as in the case of braking radiation, giving the photon provisionally a small mass μ. Thus, we make the following replacement in the photon propagator,

$$\frac{1}{(p-k)^2 + i\epsilon} \to \frac{1}{(p-k)^2 - \mu^2 + i\epsilon}.$$

By doing this in (7.16), we just effectively add an extra term $z\mu^2$ into Δ:

$$\Delta = -xyq^2 + (1-z)^2 m^2 \longrightarrow -xyq^2 + (1-z)^2 m^2 + z\mu^2. \tag{7.48}$$

This regularizes the IR divergence.

Let's now isolate the IR-divergent part. As we saw above, the divergence comes from the corner of the parameter space where $z \to 1$, so $x, y \to 0$. The relevant integral is thus of the form,

$$\int_0^1 dxdydz\delta(1-x-y-z)\frac{F(x,y,z)}{\Delta} \tag{7.49}$$

$$= \int_0^1 dz \int_0^{1-z} dy \frac{F(1-y-z, y, z)}{-(1-y-z)yq^2 + (1-z)^2 m^2 + z\mu^2} \qquad \Big| y = (1-z)\xi$$

$$= \int_0^1 dz \int_0^1 d\xi \frac{(1-z)F\left[(1-z)(1-\xi), (1-z)\xi, z\right]}{-(1-z)^2(1-\xi)\xi q^2 + (1-z)^2 m^2 + z\mu^2}.$$

We expand the numerator around $z = 1$,

$$F[(1-z)(1-\xi), (1-z)\xi, z] = F[0,0,1] + \mathcal{O}(1-z)\,, \tag{7.50}$$

where the $\mathcal{O}(1-z)$ terms will cancel the IR divergence. In addition,

$$\frac{1}{-(1-z)^2(1-\xi)\xi q^2 + (1-z)^2 m^2 + z\mu^2} \tag{7.51}$$

$$= \frac{1 + \mathcal{O}(1-z)}{-(1-z)^2(1-\xi)\xi q^2 + (1-z)^2 m^2 + \mu^2}, \tag{7.52}$$

so from the view point of IR divergence we can concentrate on the integral,

$$\int_0^1 dz \int_0^1 d\xi \frac{(1-z)F[0,0,1]}{-(1-z)^2(1-\xi)\xi q^2 + (1-z)^2 m^2 + \mu^2}\,.$$

We can thus set $z = 1$ and $x, y = 0$ in the numerator of the latter half of Eq. (7.46). Then,

$$\overline{u}_{s'}(p')\Gamma^\mu_{\rm IR} u_s(p) \to \frac{\alpha}{2\pi}\int dxdydz\delta(1-x-y-z) \tag{7.53}$$

$$\times\, \overline{u}_{s'}(p')\frac{1}{\Delta}\Big[\not{p}\gamma^\mu\big(\not{q}+\not{p}\big) + m^2\gamma^\mu - 2m\big(q^\mu + 2p^\mu\big)\Big]u_s(p) = \ldots$$

$$= \frac{\alpha}{2\pi}\int dxdydz\delta(1-x-y-z) \times \overline{u}_{s'}(p')\frac{1}{\Delta}\Big[\gamma^\mu\big(q^2 - 2m^2\big)\Big]u_s(p)\,,$$

where reaching the latter form requires some Dirac algebra. The numerator does not contain anything that would depend on the integration variables, so we only need to integrate Δ^{-1}. The integral is (Ex.),

$$\int dxdydz\delta(1-x-y-z) \times \frac{1}{-xyq^2 + (1-z)^2 m^2 + z\mu^2} \tag{7.54}$$

$$= \frac{1}{2}\frac{1}{q^2\sqrt{1+4\beta}}\log\left(\frac{-q^2}{\mu^2}\right)\left[-2\log\frac{\sqrt{1+4\beta}+1}{\sqrt{1+4\beta}-1}\right] + \text{finite terms}\,,$$

where $\beta = \frac{m^2}{-q^2}$. All in all, the structure of the vertex correction (7.46) is

$$\overline{u}_{s'}(p')\Gamma^{\mu}_{\text{loop}}u_s(p) = \frac{\alpha}{2\pi}\left[\overline{u}_{s'}(p')\gamma^{\mu}u_s(p)\right] \times \frac{1}{2}\left\{ \log\left(\frac{\Lambda^2}{-q^2}\right) \right. \tag{7.55}$$

$$\left. + \frac{\left(1+\frac{2m^2}{-q^2}\right)}{\sqrt{1+4\beta}} \log\left(\frac{-q^2}{\mu^2}\right)\left[-2\log\frac{\sqrt{1+4\beta}+1}{\sqrt{1+4\beta}-1}\right] \right\}.$$

$$+ \text{finite terms}$$

The spinor structure is identical with the leading-order calculation so the contribution to the cross section can be directly obtained from the above expression recalling the factor of 2 from 2Re:

$$d\sigma^{\text{vertex}}(p,p') = d\sigma^0(p,p') \times \frac{\alpha}{2\pi}\left\{ \log\left(\frac{\Lambda^2}{-q^2}\right) \right. \tag{7.56}$$

$$\left. + \frac{1+2\beta}{\sqrt{1+4\beta}} \log\left(\frac{-q^2}{\mu^2}\right)\left[-2\log\frac{\sqrt{1+4\beta}+1}{\sqrt{1+4\beta}-1}\right] \right\}.$$

$$+ \text{finite terms}$$

Comparing this to the result we had from the braking radiation (7.12),

$$d\sigma^{\text{rad,IR}}(p,p') = d\sigma^0(p,p') \tag{7.57}$$

$$\times \frac{\alpha}{\pi}\log\left(\frac{-q^2}{\mu^2}\right)\left[\frac{1+2\beta}{\sqrt{1+4\beta}}\log\left(\frac{\sqrt{1+4\beta}+1}{\sqrt{1+4\beta}-1}\right) - 1\right],$$

we notice that if the two contributions are added, the vertex correction almost miraculously cancels a big chunk of the IR divergence that come from the braking radiation,

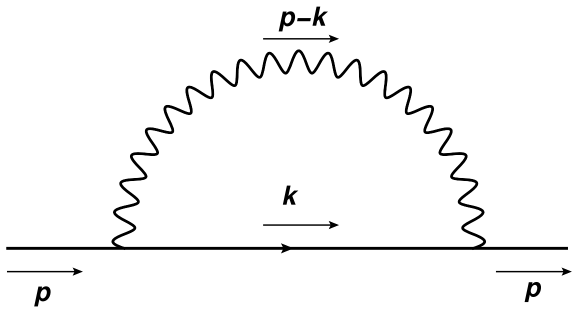
$$d\sigma^{\text{rad}}(p,p') + d\sigma^{\text{vertex}}(p,p') = \text{finite terms} \tag{7.58}$$

$$+ d\sigma^0(p,p') \times \frac{\alpha}{2\pi}\left\{ \log\left(\frac{\Lambda^2}{-q^2}\right) - 2\log\left(\frac{-q^2}{\mu^2}\right) \right\}.$$

This still contains part of the IR divergence and the UV infinity from the vertex loop. In the next section we sort out their destiny.

7.3 The electron self energy

Based on the LSZ theorem the external electron legs are to be multiplied by the renormalization constant $\sqrt{Z}$ defined as the pole of the full propagator. The first QED contribution is given by the following diagram:



As part of a larger diagram, this piece will correspond to an expression,

$$
\begin{aligned}
&\int \frac{d^4k}{(2\pi)^4}\frac{i(\not{p}+m)}{p^2-m^2}(-ie\gamma_\mu)\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}(-ie\gamma_\nu)\frac{i(\not{p}+m)}{p^2-m^2}\frac{-ig^{\mu\nu}}{(p-k)^2+i\epsilon} \\
&= \frac{i(\not{p}+m)}{p^2-m^2}\left[-e^2\int\frac{d^4k}{(2\pi)^4}\gamma_\mu\frac{(\not{k}+m)}{k^2-m^2+i\epsilon}\gamma^\mu\frac{1}{(p-k)^2+i\epsilon}\right]\frac{i(\not{p}+m)}{p^2-m^2} \\
&= \frac{i(\not{p}+m)}{p^2-m^2}\left[-i\Sigma_2(p)\right]\frac{i(\not{p}+m)}{p^2-m^2}\,, \qquad (7.59)
\end{aligned}
$$

when we define

$$
-i\Sigma_2(p) \equiv -e^2\int\frac{d^4k}{(2\pi)^4}\gamma_\mu\frac{\not{k}+m}{k^2-m^2+i\epsilon}\gamma^\mu\frac{1}{(p-k)^2+i\epsilon}\,. \qquad (7.60)
$$

We can process the loop integral with the methods of the previous section. By using the Feynman parameters,

$$
\begin{aligned}
&\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2+i\epsilon}\frac{1}{(p-k)^2+i\epsilon} \qquad (7.61) \\
&= \int_0^1 dx\int\frac{d^4k}{(2\pi)^4}\frac{1}{\left[k^2-2x(p\cdot k)+xp^2-(1-x)m^2+i\epsilon\right]^2}\,.
\end{aligned}
$$

Completing the square, $k^2 - 2x(p\cdot k) = (k-xp)^2 - x^2p^2$, and defining a new integration variable $\ell \equiv k - xp$,

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2+i\epsilon}\frac{1}{(p-k)^2+i\epsilon} = \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\left[\ell^2-\Delta+i\epsilon\right]^2},$$

where $\Delta = -x(1-x)p^2 + (1-x)m^2$. In the numerator,

$$\gamma_\mu(\not{k}+m)\gamma^\mu = -2\not{k} + 4m \to -2(\not{\ell}+x\not{p}) + 4m\,. \tag{7.62}$$

Dropping the term linear in ℓ (integrates to zero), we get

$$\int \frac{d^4k}{(2\pi)^4}\frac{\gamma_\mu(\not{k}+m)\gamma^\mu}{k^2-m^2+i\epsilon}\frac{1}{(p-k)^2+i\epsilon} = \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\frac{-2x\not{p}+4m}{\left[\ell^2-\Delta+i\epsilon\right]^2}.$$

When k is large, the integral behaves as $\int d^4k/k^4$ which gives a logarithmic UV divergence. In addition, in the limit $x \to 1$ we see that $\Delta \to 0$ which yields an IR divergence. We regulate these using the same technique as in the case of vertex correction. The infrared divergence gets regulated by giving the photon a small mass μ^2,

$$\frac{1}{(p-k)^2+i\epsilon} \to \frac{1}{(p-k)^2-\mu^2+i\epsilon}, \tag{7.63}$$

and the Pauli-Villars prescription removes the UV divergence when we include the subtraction term,

$$\frac{1}{(p-k)^2+i\epsilon} \to \frac{1}{(p-k)^2-\mu^2+i\epsilon} - \frac{1}{(p-k)^2-\Lambda^2+i\epsilon}. \tag{7.64}$$

Doing this,

$$\int \frac{d^4k}{(2\pi)^4}\frac{\gamma_\mu(\not{k}+m)\gamma^\mu}{k^2-m^2+i\epsilon}\frac{1}{(p-k)^2+i\epsilon} \longrightarrow \tag{7.65}$$

$$\int_0^1 dx\,\left(-2x\not{p}+4m\right)\int \frac{d^4\ell}{(2\pi)^4}\left[\frac{1}{\left[\ell^2-\Delta+i\epsilon\right]^2} - \frac{1}{\left[\ell^2-\Delta_\Lambda+i\epsilon\right]^2}\right],$$

where now

$$\Delta = -x(1-x)p^2 + (1-x)m^2 + x\mu^2\,, \tag{7.66}$$

$$\Delta_\Lambda = -x(1-x)p^2 + (1-x)m^2 + x\Lambda^2\,. \tag{7.67}$$

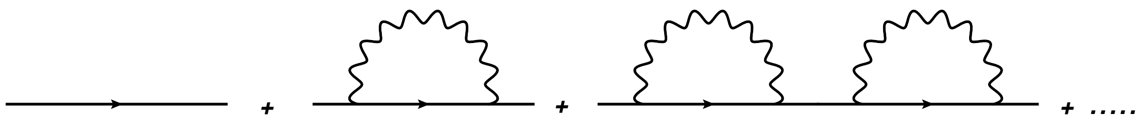
By doing the Wick rotation,

$$\int \frac{d^4\ell}{(2\pi)^4}\left[\frac{1}{\left[\ell^2-\Delta+i\epsilon\right]^2}-\frac{1}{\left[\ell^2-\Delta_\Lambda+i\epsilon\right]^2}\right]=\frac{i}{(4\pi)^2}\log\left(\frac{\Delta_\Lambda}{\Delta}\right),$$

so overall, when $\Lambda^2\to\infty$, we find

$$\Sigma_2(p)=\frac{\alpha}{2\pi}\int_0^1 dx\,(-x\not{p}+2m)\log\left(\frac{x\Lambda^2}{-x(1-x)p^2+(1-x)m^2+x\mu^2}\right) \tag{7.68}$$

We now proceed as in Section 5, and sum the contribution of the just-computed diagram to all orders. At this point we should also remember that the mass m in the above expression should actually be the unphysical bare mass m_0.



This diagrammatic sum corresponds to an expression,

$$\begin{aligned}&\frac{i(\not{p}+m_0)}{p^2-m_0^2}+\frac{i(\not{p}+m_0)}{p^2-m_0^2}\left[-i\Sigma_2(p)\right]\frac{i(\not{p}+m_0)}{p^2-m_0^2} \\ &\quad+\frac{i(\not{p}+m_0)}{p^2-m_0^2}\left[-i\Sigma_2(p)\right]\frac{i(\not{p}+m_0)}{p^2-m_0^2}\left[-i\Sigma_2(p)\right]\frac{i(\not{p}+m_0)}{p^2-m_0^2}+\cdots\end{aligned} \tag{7.69}$$

By using a shorter form,

$$\frac{i(\not{p}+m_0)}{p^2-m_0^2}=\frac{i}{\not{p}-m_0}, \tag{7.70}$$

where $(\not{p}-m_0)^{-1}$ refers to the inverse of $(\not{p}-m_0)$ we can write the above sum as,

$$\frac{i}{\not p - m_0} + \frac{i}{\not p - m_0}\left[-i\Sigma_2(p)\right]\frac{i}{\not p - m_0} \tag{7.71}$$

$$+\frac{i}{\not p - m_0}\left[-i\Sigma_2(p)\right]\frac{i}{\not p - m_0}\left[-i\Sigma_2(p)\right]\frac{i}{\not p - m_0} + \cdots$$

$$=\frac{i}{\not p - m_0} + \frac{i}{\not p - m_0}\frac{\Sigma_2(p)}{\not p - m_0} + \frac{i}{\not p - m_0}\left(\frac{\Sigma_2(p)}{\not p - m_0}\right)^2 + \cdots$$

$$=\frac{i}{\not p - m_0}\left[1 + \frac{\Sigma_2(p)}{\not p - m_0} + \left(\frac{\Sigma_2(p)}{\not p - m_0}\right)^2 + \cdots\right].$$

where we used the fact that $(\not p - m_0)$ and its inverse commute with $\Sigma_2(p)$. Formally this is a geometric series which we can sum:

$$\frac{i}{\not p - m_0}\left[1 + \frac{\Sigma_2(p)}{\not p - m_0} + \left(\frac{\Sigma_2(p)}{\not p - m_0}\right)^2 + \cdots\right]$$

$$= \frac{i}{\not p - m_0}\frac{1}{1 - \frac{\Sigma_2(p)}{\not p - m_0}} = \frac{i}{\not p - m_0 - \Sigma_2(p)}. \tag{7.72}$$

More explicitly,

$$\frac{1}{\not p - m_0 - \Sigma_2(p)} = \frac{\not p\left[1 - \Sigma'(p^2)\right] + m_0\left[1 + \Sigma''(p^2)\right]}{p^2\left[1 - \Sigma'(p^2)\right]^2 - m_0^2\left[1 + \Sigma''(p^2)\right]^2}, \tag{7.73}$$

where

$$\Sigma'(p^2) \equiv -\frac{\alpha}{2\pi}\int_0^1 dx x \log\left(\frac{x\Lambda^2}{-x(1-x)p^2 + (1-x)m^2 + x\mu^2}\right),$$

$$\Sigma''(p^2) \equiv 2\frac{\alpha}{2\pi}\int_0^1 dx \log\left(\frac{x\Lambda^2}{-x(1-x)p^2 + (1-x)m^2 + x\mu^2}\right).$$

Based on the general discussion of Section 5 the summed propagator (7.73) should have a pole at the physical mass, $p^2 = m^2$. We find this as a solution of the equation

$$\left[p^2\left[1 - \Sigma'(p^2)\right]^2 - m_0^2\left[1 + \Sigma''(p^2)\right]^2\right]\Big|_{p^2=m^2} = 0. \tag{7.74}$$

Near this pole, the summed propagator is of the form,

$$Z_2 \frac{i(\not p + m)}{p^2 - m^2}\,, \tag{7.75}$$

where Z_2 is the renormalization factor related to the electron field (the one that appears in the LSZ theorem). After a bit of tinkering, we find (Ex.),

$$\begin{aligned} m^2 &= m_0^2 \times \left[\!\!\left[1 + \frac{\alpha}{2\pi} \int_0^1 dx\,(4-2x) \log\left[\frac{x\Lambda^2}{(1-x)^2 m_0^2 + x\mu^2}\right] \right]\!\!\right] \\ Z_2 &= 1 + \frac{\alpha}{2\pi} \int_0^1 dx \left[\!\!\left[-x \log\left[\frac{x\Lambda^2}{(1-x)^2 m^2 + x\mu^2}\right] \right.\right. \\ &\qquad\qquad \left.\left. + (2-x)\,\frac{2m^2 x(1-x)}{(1-x)^2 m^2 + x\mu^2} \right]\!\!\right] \end{aligned} \tag{7.76}$$

To this order in coupling constant α, we can equally well use m^2 or m_0^2 in what is inside the double square brackets.

The results above indicate that the mass parameter m_0 that appears in the Lagrangian and what we call a physical mass m are different by a divergent factor. The physical mass m is of course finite which indicates that m_0 has to be divergent as well. The equation above implies that we should define,

$$m_0^2 = m^2 \times \left[\!\!\left[1 - \frac{\alpha}{2\pi} \int_0^1 dx\,(4-2x) \log\left[\frac{x\Lambda^2}{(1-x)^2 m^2 + x\mu^2}\right] \right]\!\!\right],$$

and use this definition in calculations – we recall that the quantity that appears in the Feynman rules is m_0. When doing so, part of the UV infinities coming from the loop diagrams cancel. It is important that the above definition is made only once – the same definition removes UV infinities from all kinds of processes, not just the one we have considered here. From the view point of the process considered now this does not really show up as the leading-order diagram does not contain an electron propagator. The procedure outlined here is called the **mass renormalization**. The definition of m_0 is, however, not unique and from the viewpoint of removing divergences we could add or subtract whatever finite terms. Different choices are called **schemes**. The definition above is a common one and known as the **on-shell scheme** or **pole-mass scheme**. It is also worth pointing out that in the massless case the mass renormalization is not need but $m_0 = m = 0$.

The renormalization constant Z_2 contains both UV- and IR divergences. They are easily isolated from the complete expression (7.76),

UV part:

$$\frac{\alpha}{2\pi}\int_0^1 dx \left[\!\left[-x\log\left[\frac{x\Lambda^2}{(1-x)^2m^2+x\mu^2}\right]\right]\!\right] \tag{7.77}$$

$$=\frac{\alpha}{2\pi}\int_0^1 dx \left[\!\left[-x\log\left(\frac{\Lambda^2}{m^2}\right)\right]\!\right] + \cdots$$

$$=-\frac{1}{2}\frac{\alpha}{2\pi}\log\left(\frac{\Lambda^2}{m^2}\right) + \cdots$$

IR part:

$$\frac{\alpha}{2\pi}\int_0^1 dx \left[\!\left[(2-x)\,\frac{2m^2x(1-x)}{(1-x)^2m^2+x\mu^2}\right]\!\right] \tag{7.78}$$

$$=2m^2\frac{\alpha}{2\pi}\int_0^1 dx \left[\!\left[[1+(1-x)]\,\frac{[-(1-x)+1]\,(1-x)}{(1-x)^2m^2+x\mu^2}\right]\!\right]$$

$$=2m^2\frac{\alpha}{2\pi}\int_0^1 dx \frac{(1-x)}{(1-x)^2m^2+\mu^2}\left[1+\mathcal{O}(1-x)\right]$$

$$=\frac{\alpha}{2\pi}\log\left(\frac{m^2}{\mu^2}\right)$$

The renormalization factor Z_2 is thus,

$$Z_2 = 1 + \frac{\alpha}{2\pi}\left[-\frac{1}{2}\log\left(\frac{\Lambda^2}{m^2}\right) + \log\left(\frac{m^2}{\mu^2}\right)\right] + \text{finite terms}$$

According to the LSZ theorem each external electron enters the scattering amplitude with a factor of $\sqrt{Z_2}$, so in total we have a factor Z_2^2 multiplying the cross section. To order α,

$$Z_2^2 = 1 + \frac{\alpha}{2\pi}\left[-\log\left(\frac{\Lambda^2}{m^2}\right) + 2\log\left(\frac{m^2}{\mu^2}\right)\right] + \text{finite terms}\,.$$

The contribution of the external-leg corrections to the cross section is thus,

$$\begin{aligned} d\sigma^{\text{external leg}}(p,p') &= d\sigma^0(p,p') \times \left(Z_2^2 - 1\right) \\ &= d\sigma^0(p,p') \times \frac{\alpha}{2\pi}\left[-\log\left(\frac{\Lambda^2}{m^2}\right) + 2\log\left(\frac{m^2}{\mu^2}\right)\right] \\ &\quad + \text{finite terms} \end{aligned} \tag{7.79}$$

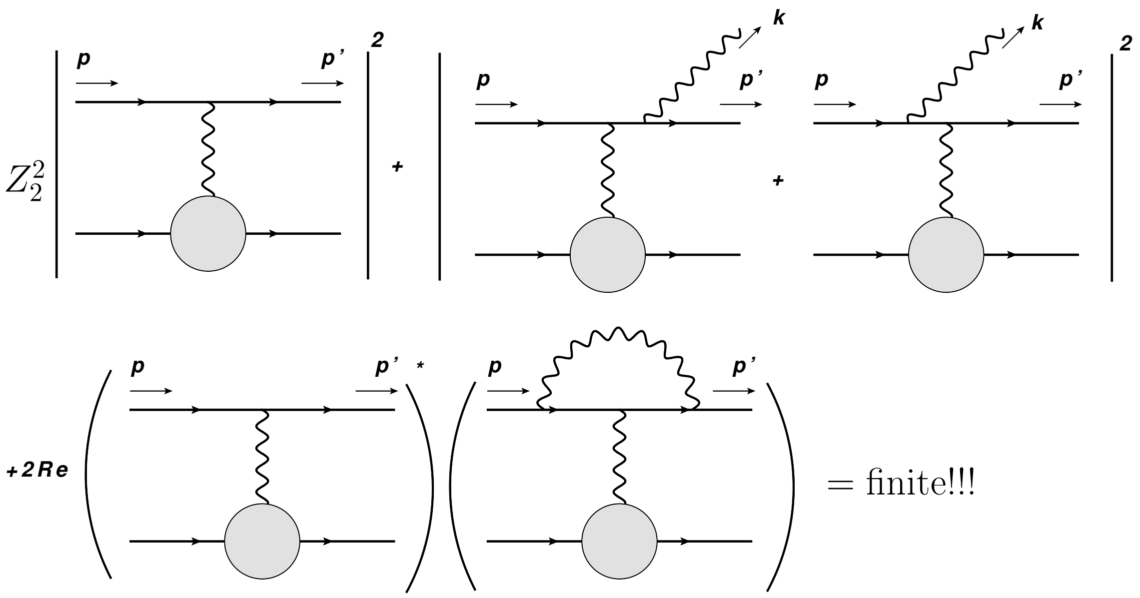
To close this section, we compare the obtained result with Eq. (7.58), the sum of braking radiation and vertex correction,

$$\begin{aligned} d\sigma^{\text{rad}}(p,p') + d\sigma^{\text{vertex}}(p,p') &= \text{finite terms} \\ + d\sigma^0(p,p') \times \frac{\alpha}{2\pi}&\left\{\log\left(\frac{\Lambda^2}{-q^2}\right) - 2\log\left(\frac{-q^2}{\mu^2}\right)\right\}. \end{aligned} \tag{7.80}$$

Remarkably, the divergence structure is exactly the same but the signs are the opposite! Thus the sum of all three contributions is finite

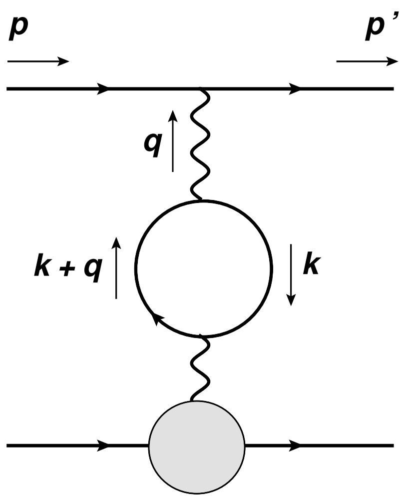
$$\begin{aligned} & d\sigma^{\text{rad}}(p,p') + d\sigma^{\text{vertex}}(p,p') + d\sigma^{\text{external leg}}(p,p') \\ &= d\sigma^0(p,p') \times \frac{\alpha}{2\pi}\left\{\log\left(\frac{m^2}{-q^2}\right) - 2\log\left(\frac{-q^2}{m^2}\right)\right\} + \cdots \\ &= \text{a finite number}\;\;. \end{aligned} \tag{7.81}$$

We have now seen how different radiation/loop diagrams can yield infinities but when appropriately combined, it is possible to find a finite result. The cancellation of infrared divegences is known as the **Kinoshita-Lee-Nauenberg theorem**, and in the case of UV divergences what we have seen is part of the **renormalization** which we will discuss more in the following section.



7.4 Photon self energy

A diagram which yields a contribution of the same order in the QED coupling as the previous diagrams is the one in which we draw an electron loop on the photon line:



This is also a virtual correction so it does not change the kinematics. In the Feynman gauge this corresponds to a matrix element,

$$i\mathcal{M}^{\gamma}(p,p') = -ie\left[\overline{u}_{s'}(p')\gamma^{\mu}u_s(p)\right] \times \frac{-ig_{\mu\alpha}}{q^2+i\epsilon}\left[i\Pi^{\alpha\beta}(q)\right]\frac{-ig_{\beta\nu}}{q^2+i\epsilon}\Phi^{\nu}(q)\,,$$

where

$$\begin{aligned} i\Pi^{\alpha\beta}(q) &= -(-ie)^2\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^{\beta}\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}\gamma^{\alpha}\frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2+i\epsilon}\right] \\ &= -e^2\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^{\beta}\frac{(\not{k}+m)}{k^2-m^2+i\epsilon}\gamma^{\alpha}\frac{(\not{k}+\not{q}+m)}{(k+q)^2-m^2+i\epsilon}\right] \qquad (7.82) \\ &= -4e^2\int\frac{d^4k}{(2\pi)^4}\frac{k^{\alpha}(k+q)^{\beta}+k^{\beta}(k+q)^{\alpha}-g^{\alpha\beta}\left(k^2+k\cdot q-m^2\right)}{\left[k^2-m^2+i\epsilon\right]\left[(k+q)^2-m^2+i\epsilon\right]}\,. \end{aligned}$$

The overall minus sign comes from the factor (-1) in the context of closed fermion loop. We proceed as in the previous loop calculations and use the

Feynman parametrization:

$$\frac{1}{[k^2 - m^2 + i\epsilon]\,[(k+q)^2 - m^2 + i\epsilon]} \tag{7.83}$$

$$= \int_0^1 dxdy\delta\,(1-x-y)\,\frac{1}{\Big[y\,[k^2 - m^2 + i\epsilon] + x\,[(k+q)^2 - m^2 + i\epsilon]\Big]^2}$$

$$= \int_0^1 dxdy\delta\,(1-x-y)\,\frac{1}{\Big[(k^2 - m^2 + i\epsilon)(x+y) + x\,[2k\cdot q + q^2]\Big]^2}$$

$$= \int_0^1 dx\frac{1}{\Big[k^2 + 2xk\cdot q - m^2 + xq^2 + i\epsilon\Big]^2}\,.$$

We complete the square, $k^2 + 2xk\cdot q = (k+xq)^2 - x^2q^2$, so that

$$\frac{1}{[k^2 - m^2 + i\epsilon]\,[(k+q)^2 - m^2 + i\epsilon]} = \int_0^1 dx\frac{1}{\big[\ell^2 - \Delta + i\epsilon\big]^2}\,, \tag{7.84}$$

with

$$\ell = k + xq\,, \tag{7.85}$$

$$\Delta = m^2 - x(1-x)q^2 > 0\,. \tag{7.86}$$

In the numerator of (7.82),

$$k^\alpha(k+q)^\beta + k^\beta(k+q)^\alpha - g^{\alpha\beta}\left(k^2 + k\cdot q - m^2\right) \tag{7.87}$$

$$= (\ell - xq)^\alpha((\ell - xq) + q)^\beta + (\ell - xq)^\beta((\ell - xq) + q)^\alpha$$

$$- g^{\alpha\beta}\left((\ell - xq)^2 + (\ell - xq)\cdot q - m^2\right)$$

$$\hat{=} 2\ell^\alpha\ell^\beta - g^{\alpha\beta}\ell^2 - 2x(1-x)q^\alpha q^\beta + g^{\alpha\beta}\left(m^2 + x(1-x)q^2\right)\,,$$

where we discarded the terms linear in ℓ. Thus, at this point,

$$i\Pi^{\alpha\beta}(q) = -4e^2\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{\big[\ell^2 - \Delta + i\epsilon\big]^2} \tag{7.88}$$

$$\Big[2\ell^\alpha\ell^\beta - g^{\alpha\beta}\ell^2 - 2x(1-x)q^\alpha q^\beta + g^{\alpha\beta}\left(m^2 + x(1-x)q^2\right)\Big]\,.$$

This is again UV divergent but this time there's no IR divergence since $\Delta > 0$ due to the electron mass. We could use the Pauli-Villars regularization but for fermion loops it's not as convenient as with photon loops. At this point we well shift to the modern **dimensional regularization**.

Dimensional regularization

The idea is super simple: A typical loop integral is of the form,

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\left[\ell^2 - \Delta + i\epsilon\right]^2} . \tag{7.89}$$

by Wick's rotation,

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\left[\ell^2 - \Delta + i\epsilon\right]^2} = i \int \frac{d^4\ell_E}{(2\pi)^4} \frac{1}{\left[\ell_E^2 + \Delta\right]^2} . \tag{7.90}$$

This is clearly infinite,

$$\int \frac{d^4\ell_\mathrm{E}}{(2\pi)^4} \frac{1}{\left(\ell_\mathrm{E}^2 + \Delta\right)^2} = \int \frac{d\Omega_4}{(2\pi)^4} \int_0^\infty \frac{d|\ell_\mathrm{E}||\ell_\mathrm{E}|^3}{\left(\ell_\mathrm{E}^2 + \Delta\right)^2} \sim \log(\infty) . \tag{7.91}$$

If, instead of 4 space-time dimensions, we have d dimensions,

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\left[\ell^2 - \Delta + i\epsilon\right]^2} , \tag{7.92}$$

performing the Wick rotation,

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\left[\ell^2 - \Delta + i\epsilon\right]^2} = i \int \frac{d^d\ell_E}{(2\pi)^d} \frac{1}{\left[\ell_E^2 + \Delta\right]^2} , \tag{7.93}$$

we find a finite result:

$$\int \frac{d^d\ell_\mathrm{E}}{(2\pi)^d} \frac{1}{\left(\ell_\mathrm{E}^2 + \Delta\right)^2} = \int \frac{d\Omega_d}{(2\pi)^d} \int_0^\infty \frac{d|\ell_\mathrm{E}||\ell_\mathrm{E}|^{d-1}}{\left(\ell_\mathrm{E}^2 + \Delta\right)^2} < \infty , \text{ if } d < 4. \tag{7.94}$$

Thus, we can regularize the UV divergence by **reducing** the number of space-time dimensions. Also the IR divergence can be regularized by this method but in this case we need to **increase** the number of dimensions.

Sometimes – or actually very often – both are regularized at once by dim.reg. which is bit of a tricky business.

The angular integral in d dimensions goes with a Gaussian integral,

$$(\sqrt{\pi})^d = \left(\int dx e^{-x^2}\right)^d = \int d^d x \exp\left[-\sum_{i=1}^{d} x_i^2\right] \tag{7.95}$$
$$= \int d\Omega_d \int_0^\infty dx x^{d-1} e^{-x^2},$$

and making a change of variables $y = x^2$, $dy = 2xdx$,

$$(\sqrt{\pi})^d = \left(\int d\Omega_d\right) \frac{1}{2} \int_0^\infty dy y^{(d/2-1)} e^{-y}.$$

We can identify here the integral representation of the Γ function,

$$\Gamma(z) = \int_0^\infty dx x^{z-1} e^{-x}, \quad \mathrm{Re}\,(z) > 0 \tag{7.96}$$

so

$$(\sqrt{\pi})^d = \left(\int d\Omega_d\right) \frac{1}{2} \Gamma\left(\frac{d}{2}\right).$$

Thus,

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{7.97}$$

Also the radial part of (7.94) can be turned into Γ functions:

$$\int_0^\infty \frac{d|\ell_{\mathrm{E}}||\ell_{\mathrm{E}}|^{d-1}}{(|\ell_{\mathrm{E}}|^2 + \Delta)^2} = \frac{1}{2} \int_0^\infty \frac{d|\ell_{\mathrm{E}}|^2 (|\ell_{\mathrm{E}}|^2)^{d/2-1}}{(|\ell_{\mathrm{E}}|^2 + \Delta)^2} \tag{7.98}$$

We do a change of variables, $x = \Delta/(|\ell_{\mathrm{E}}|^2 + \Delta)$, $dx = -d|\ell_{\mathrm{E}}|^2 \Delta/(|\ell_{\mathrm{E}}|^2 +$

$\Delta)^2$, $x : 1 \to 0$,

$$\int_0^\infty \frac{d|\ell_E||\ell_E|^{d-1}}{(|\ell_E|^2+\Delta)^2} = \frac{1}{2}\int_0^1 dx \frac{(|\ell_E|^2+\Delta)^2}{\Delta}\frac{(\frac{\Delta}{x}-\Delta)^{d/2-1}}{(|\ell_E|^2+\Delta)^2} \qquad (7.99)$$

$$= \frac{1}{2}\Delta^{d/2-2}\int_0^1 dx x^{1-d/2}(1-x)^{d/2-1}.$$

The remaining x integral matches with the definition of the so-called β function which, in turn, is related to Γ function,

$$B(\alpha,\beta) \equiv \int_0^1 dx x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad (7.100)$$

when $\mathrm{Re}\,(\alpha) > 0$ and $\mathrm{Re}\,(\beta) > 0$. This follows directly from the definition of the Γ function (7.96). By using the above identity,

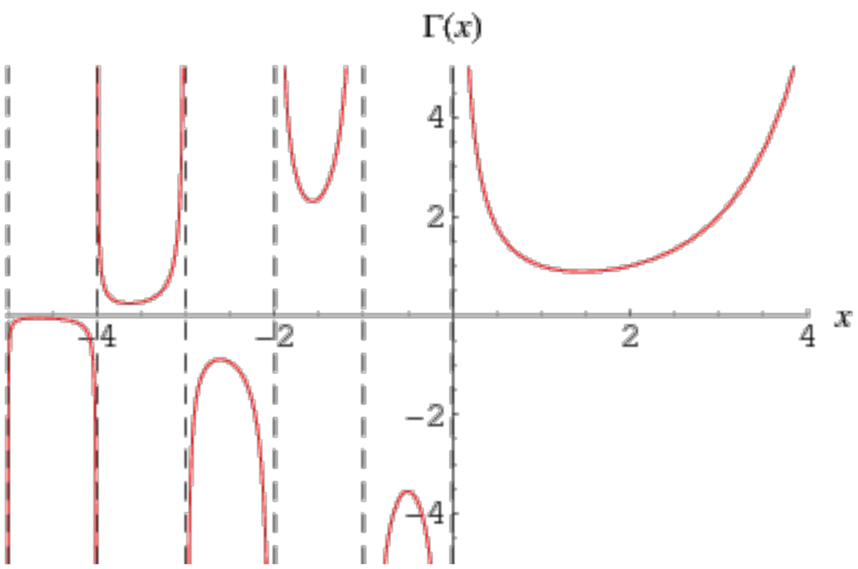
$$\int_0^\infty \frac{d|\ell_E||\ell_E|^{d-1}}{(|\ell_E|^2+\Delta)^2} = \frac{1}{2}\Delta^{\frac{d}{2}-2}\frac{\Gamma(2-d/2)\Gamma(d/2)}{\Gamma(2)}.$$

In total,

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^2} = \frac{2\pi^{d/2}}{\Gamma(d/2)}\frac{1}{(2\pi)^d}\frac{1}{2}\Delta^{\frac{d}{2}-2}\Gamma(2-d/2)\Gamma(d/2) \qquad (7.101)$$

$$= \frac{1}{(4\pi)^{d/2}}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\Gamma(2-d/2), \quad d<4.$$

The $\Gamma(z)$ function is singular at $z = 0$,



and thus $\Gamma(2-d/2)$ is singular when $d=4$. It is customary to write $d=4-\epsilon$, where $\epsilon>0$, and by using the definition of the Γ function,

$$\Gamma\left(2-\frac{d}{2}\right)=\Gamma\left(\frac{\epsilon}{2}\right)=\frac{2}{\epsilon}-\gamma_E+\mathcal{O}(\epsilon), \tag{7.102}$$

where γ_E is the Euler-Mascheroni constant,

$$\gamma_E\equiv-\int_0^\infty e^{-x}\log x\approx 0.5772\,. \tag{7.103}$$

By using this expansion, we can finally write the singularity structure of the integral (7.101) explicitly,

$$\begin{aligned}\int\frac{d^d\ell_{\rm E}}{(2\pi)^d}\frac{1}{(\ell_{\rm E}^2+\Delta)^2}&=\frac{1}{(4\pi)^{d/2}}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\Gamma(2-d/2) \\ &\overset{\epsilon\to 0}{=}\frac{1}{(4\pi)^2}\left[\frac{2}{\epsilon}-\gamma_E-\log\Delta+\log(4\pi)\right]\,.\end{aligned} \tag{7.104}$$

We see that the logarithmic UV divergence corresponds in dimensional regulariztion to $1/\epsilon$ pole. It should be born in mind that **the parameter ϵ appearing here has nothing to do with the ϵ that appears in the propagators!**

With a bit of tinkering, one can verify the following general identities,

$$\int\frac{d^N\ell}{(2\pi)^N}\frac{1}{\left[\ell^2-\Delta+i\epsilon\right]^m}=\frac{i(-1)^m}{(4\pi)^{N/2}}\frac{\Gamma\left(m-N/2\right)}{\Gamma\left(m\right)}\left(\frac{1}{\Delta}\right)^{m-N/2} \tag{7.105}$$

$$\int\frac{d^N\ell}{(2\pi)^N}\frac{\ell^2}{\left[\ell^2-\Delta+i\epsilon\right]^m}=\frac{-i(-1)^m}{(4\pi)^{N/2}}\frac{N}{2}\frac{\Gamma\left(m-N/2-1\right)}{\Gamma\left(m\right)}\left(\frac{1}{\Delta}\right)^{m-N/2-1}$$

When the dimension of the space time is N, the energy-momentum vectors are of the form,

$$p^\mu=(p^0,p^1,p^2,\ldots,p^{N-1}), \tag{7.106}$$

and thus also the indices of the metric tensor $g^{\mu\nu}$ run from 0 to $N-1$,

$$g^{\mu\nu}g_{\mu\nu}=N\,. \tag{7.107}$$

For this reason also the γ-matrix algebra slightly changes. This is not unique, but usually the following identities are retained intact,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,, \quad \mathrm{Tr}\,(I) = 4\,, \tag{7.108}$$

and it follows that

$$\gamma^\mu\gamma^\nu\gamma_\mu = -(N-2)\gamma^\nu \tag{7.109}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\rho\nu} + (N-4)\gamma^\nu\gamma^\rho \tag{7.110}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-N)\gamma^\nu\gamma^\rho\gamma^\sigma \tag{7.111}$$

$$\int \frac{d^N\ell}{(2\pi)^N}\frac{\ell^\mu\ell^\nu}{D(\ell^2)} = \frac{1}{N}g^{\mu\nu}\int \frac{d^N\ell}{(2\pi)^N}\frac{\ell^2}{D(\ell^2)}\,. \tag{7.112}$$

Lastly, the QED coupling becomes dimensionful quantity. Since the action,

$$S = \int d^4x\mathcal{L}_{\mathrm{QED}} \tag{7.113}$$

is dimensionless, in 4 dimensions we have $\dim[\mathcal{L}_{\mathrm{QED}}] = 4$ (in dimensions of mass). The QED Lagrangian density was,

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi - e\overline{\psi}\gamma^\mu\psi A_\mu\,, \tag{7.114}$$

so we can infer,

$$\dim[\psi] = 3/2\,, \tag{7.115}$$

$$\dim[A] = 1\,, \tag{7.116}$$

$$\dim[e] = 0\,. \tag{7.117}$$

When the space-time dimension is N, we have $\dim[\mathcal{L}^N_{\mathrm{QED}}] = N$, and

$$\dim[\psi] = (N-1)/2\,, \tag{7.118}$$

$$\dim[A] = N/2 - 1\,, \tag{7.119}$$

$$\dim[e] = 2 - N/2\,. \tag{7.120}$$

When $N = 4-\epsilon$, then $\dim[e] = \epsilon/2$. Often, the dimension of the spacetime is written explicitly using an arbitrary mass scale μ_D as,

$$e \to e\mu_D^{2-N/2}\,. \tag{7.121}$$

Let's now continue with the photon self-energy diagram from Eq. (7.88), but now in N dimensions,

$$i\Pi^{\alpha\beta}(q) = -4e^2\mu_D^{4-N}\int_0^1 dx \int \frac{d^N\ell}{(2\pi)^N}\frac{1}{\left[\ell^2-\Delta+i\epsilon\right]^2} \tag{7.122}$$

$$\left[2\ell^\alpha\ell^\beta - g^{\alpha\beta}\ell^2 - 2x(1-x)q^\alpha q^\beta + g^{\alpha\beta}\left(m^2+x(1-x)q^2\right)\right]$$

$$= -4e^2\mu_D^{4-N}\int_0^1 dx \int \frac{d^N\ell}{(2\pi)^N}\frac{1}{\left[\ell^2-\Delta+i\epsilon\right]^2}$$

$$\left[(2/N-1)g^{\alpha\beta}\ell^2 - 2x(1-x)q^\alpha q^\beta + g^{\alpha\beta}\left(m^2+x(1-x)q^2\right)\right].$$

The required ℓ integrals are,

- $$\int \frac{d^N\ell}{(2\pi)^N}\frac{1}{\left[\ell^2-\Delta+i\epsilon\right]^2} = \frac{i}{(4\pi)^{N/2}}\Gamma\left(2-N/2\right)\left(\frac{1}{\Delta}\right)^{2-N/2} \tag{7.123}$$
- $$\int \frac{d^N\ell}{(2\pi)^N}\frac{(2/N-1)\ell^2}{\left[\ell^2-\Delta+i\epsilon\right]^2} = \frac{-i}{(4\pi)^{N/2}}\frac{N}{2}(2/N-1)\Gamma\left(1-N/2\right)\left(\frac{1}{\Delta}\right)^{2-N/2-1}$$

$$= \frac{-i}{(4\pi)^{N/2}}(1-\frac{N}{2})\Gamma\left(1-N/2\right)\left(\frac{1}{\Delta}\right)^{2-N/2-1}$$

$$= \frac{-i}{(4\pi)^{N/2}}\Gamma\left(2-N/2\right)\left(\frac{1}{\Delta}\right)^{2-N/2-1}\,.$$

Using these, we get,

$$i\Pi^{\alpha\beta}(q) = \tag{7.124}$$

$$= -4e^2\mu_D^{4-N}\int_0^1 dx\left[\frac{i}{(4\pi)^{N/2}}\left(\frac{1}{\Delta}\right)^{2-N/2}\Gamma\left(2-N/2\right)\right]$$

$$\left[(-m^2+x(1-x)q^2)g^{\alpha\beta}-2x(1-x)q^\alpha q^\beta+g^{\alpha\beta}\left(m^2+x(1-x)q^2\right)\right].$$

The lowest line simplifies to

$$2x(1-x)\left[q^2g^{\alpha\beta}-q^\alpha q^\beta\right], \tag{7.125}$$

so finally,

$$i\Pi^{\alpha\beta}(q) = \left[q^2g^{\alpha\beta}-q^\alpha q^\beta\right]\times i\Pi(q^2) \tag{7.126}$$

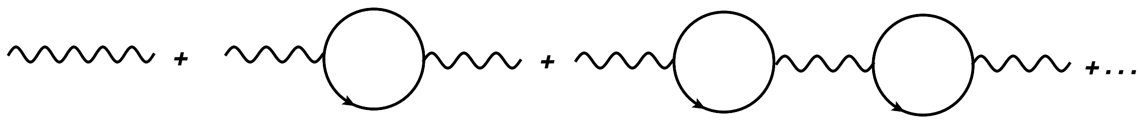
$$i\Pi(q^2) = \frac{-8ie^2\mu_D^{4-N}}{(4\pi)^{N/2}}\int_0^1 dx x(1-x)\left[\left(\frac{1}{\Delta}\right)^{2-N/2}\Gamma\left(2-N/2\right)\right]$$

$$\overset{\epsilon\to 0}{=} \frac{-2i\alpha}{\pi}\int_0^1 dx x(1-x)\left[\frac{2}{\epsilon}-\gamma_E+\log\frac{\mu_D^2}{\Delta}+\log(4\pi)\right]$$

We note that $\Pi^{\alpha\beta}(q)$ fulfills the Ward identity,

$$q_\alpha\Pi^{\alpha\beta}(q) = q_\beta\Pi^{\alpha\beta}(q) = 0\,, \tag{7.127}$$

as we might have expected. We proceed as in the electron self-energy calculation and sum the obtained result to all orders,



This corresponds to,

$$\frac{-ig_{\mu\nu}}{q^2}+\frac{-ig_{\mu\alpha}}{q^2}i\Pi^{\alpha\beta}(q)\frac{-ig_{\beta\nu}}{q^2}+\frac{-ig_{\mu\alpha}}{q^2}i\Pi^{\alpha\beta}(q)\frac{-ig_{\beta\rho}}{q^2}i\Pi^{\rho\sigma}(q)\frac{-ig_{\sigma\nu}}{q^2}+\cdots$$

and after some small tinkering,

$$\frac{-i}{q^2\left[1-\Pi(q)\right]}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)-i\left(\frac{q_\mu q_\nu}{q^4}\right). \tag{7.128}$$

Those terms which are proportional to $q_\mu q_\nu$ will, according to the Ward identity, yield zero in scattering amplitudes so only the $g_{\mu\nu}$ term is relevant. The full propagator thus reads,

$$\frac{-ig_{\mu\nu}}{q^2\left[1-\Pi(q)\right]}. \tag{7.129}$$

The summed propagator clearly has a pole at $q^2 = 0$ so the **photon remains massless**. Close to the pole the propagator behaves, obviously, as

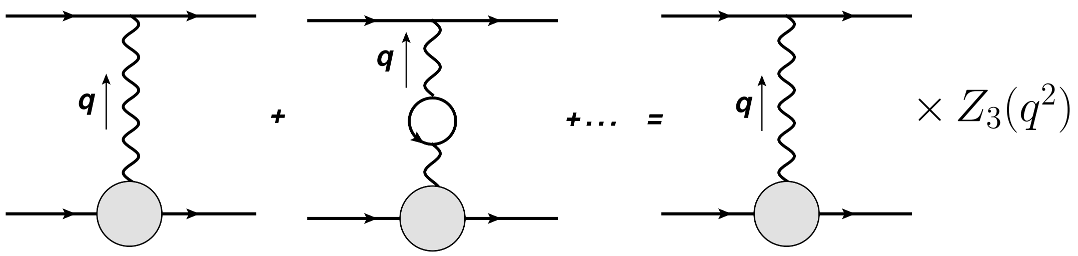
$$\frac{-ig_{\mu\nu}Z_3}{q^2}, \tag{7.130}$$

where Z_3 is the renormalization constant related to the photon field,

$$Z_3 = \frac{1}{\left[1-\Pi(0)\right]} = 1-\frac{\alpha}{3\pi}\left[\frac{2}{\epsilon}-\gamma_E+\log\frac{\mu_D^2}{m^2}+\log(4\pi)\right]. \tag{7.131}$$

This is what we would use (according to the LSZ theorem) if our scattering amplitude contains external photons.

Now we don't have external photons in the game, but the virtual electron loop yields a multiplicative factor $Z_3(q^2) \equiv 1/\big[1-\Pi(q^2)\big]$:



So where should we stuff the UV-divergence that $Z_3(q^2)$ entails? In analogy to the mass renormalization, this infinity is absorbed into a redefinition of the electric charge — **charge renormalization**. We now denote the charge that appears in the original Lagrangian by e_0 and call it the **bare charge**. Since an internal photon propagator always starts and ends to a vertex factor $-ie_0\gamma^\mu$, it is natural to share the contribution of $Z_3(q^2)$ evenly with both. In addition, as $Z_3(q^2)$ depends on the scale q^2, we define an **effective charge/coupling** or **running charge/coupling**,

$$e_{\text{eff}}(q^2) \equiv e_0\sqrt{Z_3(q^2)}\,, \tag{7.132}$$

or in terms of the fine-structure constant $\alpha = e^2/4\pi$,

$$\alpha_{\text{eff}}(q^2) \equiv \alpha_0\, Z_3(q^2)\,. \tag{7.133}$$

This would indicate that the measured charge will depend on a scale (momentum transfer). The charge that an experimentalist will measure is definitely a finite number, so because $Z_3(q^2)$ is infinite, also the bare charge α_0 has to be infinite as well.

The effective coupling $\alpha_{\text{eff}}(q^2)$ thus depends on the scale. How? According to the definition,

$$\alpha_{\text{eff}}(q^2) = \frac{\alpha_0}{1-\Pi(q^2)}\,, \tag{7.134}$$

so

$$\frac{1}{\alpha_{\text{eff}}(q^2)} = \frac{1}{\alpha_0} - \frac{\Pi(q^2)}{\alpha_0}\,. \tag{7.135}$$

The low-energy measurement give $\alpha \equiv \alpha_{\text{eff}}(0) \approx 1/137$, so we use this as a reference value,

$$\begin{aligned}\frac{1}{\alpha_{\text{eff}}(q^2)} &= \frac{1}{\alpha_0} - \frac{\Pi(0)}{\alpha_0} + \frac{\Pi(0)}{\alpha_0} - \frac{\Pi(q^2)}{\alpha_0} \\ &= \frac{1}{\alpha} - \frac{1}{\alpha_0}\left[\Pi(q^2) - \Pi(0)\right]\end{aligned} \tag{7.136}$$

According to Eq. (7.126),

$$\begin{aligned}\Pi(q^2) - \Pi(0) &= \frac{-2\alpha_0}{\pi}\int_0^1 dx x(1-x)\log\frac{m^2}{m^2 - x(1-x)q^2} \\ &\xrightarrow{-q^2\gg m^2} \frac{\alpha_0}{3\pi}\left[\log\left(\frac{-q^2}{m^2}\right) - \frac{5}{3}\right],\end{aligned} \tag{7.137}$$

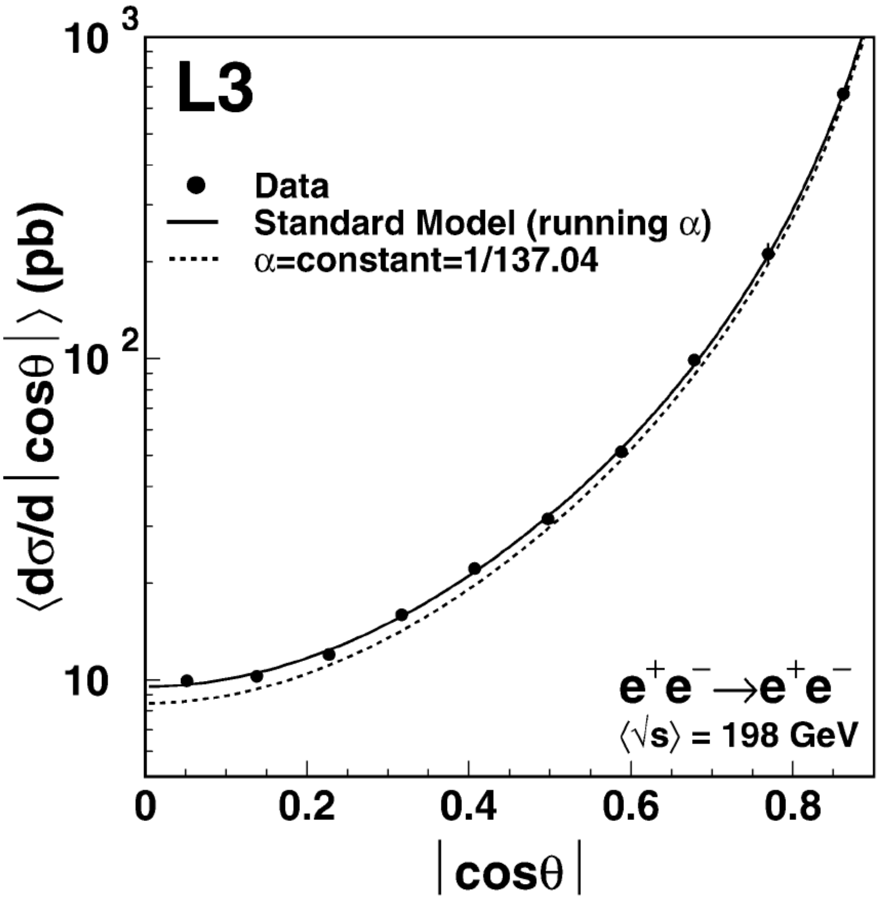
so

$$\frac{1}{\alpha_{\text{eff}}(q^2)} = \frac{1}{\alpha} - \frac{1}{3\pi}\left[\log\left(\frac{-q^2}{m^2}\right) - \frac{5}{3}\right]. \tag{7.138}$$

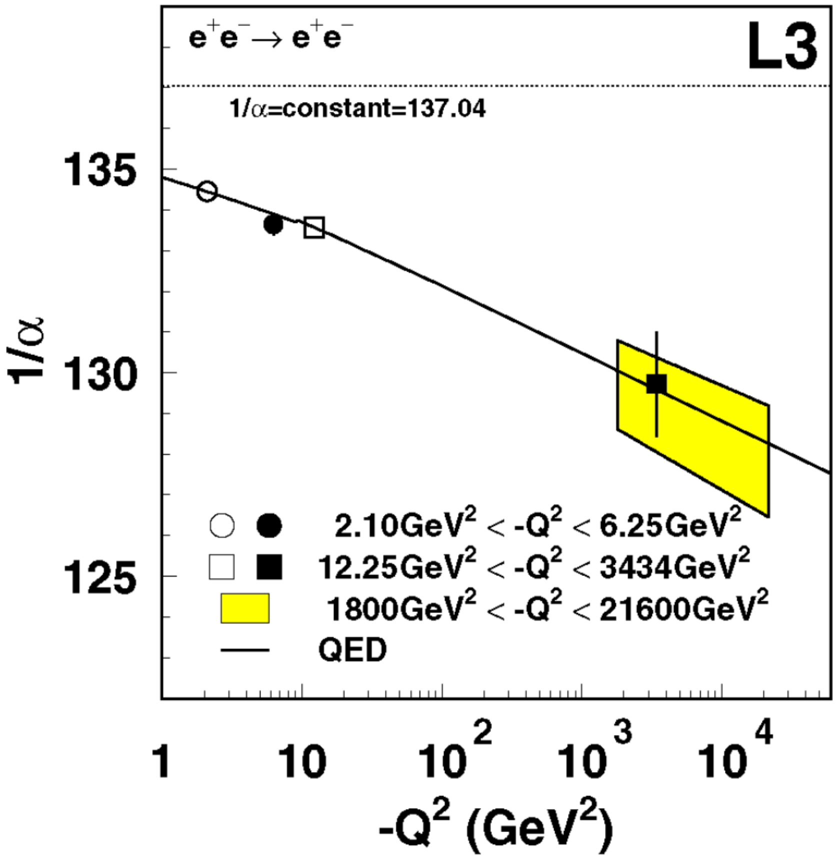
This gives the final form of the scale-dependent coupling (to first order),

$$\alpha_{\text{eff}}(q^2) = \frac{\alpha}{1 - \frac{\alpha}{3\pi}\log\left(\frac{-q^2}{m^2}\right)}, \quad -q^2 \gg m^2. \tag{7.139}$$

When $-q^2$ grows, the denominator of the equation above diminishes, so the coupling becomes stronger. The change is relatively slow (logarithmic) but it has been verified experimentally. Below we show some result from the LEP collider for the angular dependence in $e^+e^- \to e^+e^-$ process [Phys.Lett. B623 (2005) 26-36].



Without a scale-dependent coupling the shape of the theoretical curve deviates from the measurements. Accounting for the scale dependence in coupling even visually improves the correspondence. Below still the extracted $\alpha_{\text{eff}}(q^2)$.



The measurements thus clearly prefer the scale dependence of the coupling constant.

One can, of course, always express the physical cross sections also in terms of scale-independent coupling e.g. $\alpha = \alpha_{\text{eff}}(q^2 = 0) \approx 1/137$ which also removes the $1/\epsilon$ poles and dependence of the unphysical parameter μ_D^2 perfectly fine. However, in this case our expression for the cross section would explicitly involve powers of logarithms of the form $\alpha \log(-q^2/m^2)$ which can be large if $-q^2 \gg m^2$ and thereby worsen the convergence of the perturbative series. By expressing the cross sections in terms of running coupling $\alpha_{\text{eff}}(q^2)$ effectively resums these logarithms into the definition of the coupling stabilizing the perturbative series. The fact that $\alpha_{\text{eff}}(q^2)$ resums such logarithms to all orders can be seen also by expanding Eq. (7.139) in powers of α.

The scale dependence or **running** of the coupling is often expressed in terms of the so-called **β function**,

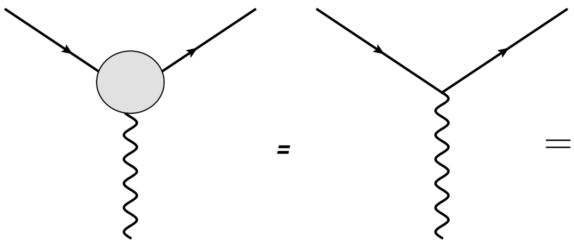
$$\beta(Q^2) \equiv Q^2 \frac{d\alpha_{\text{eff}}(Q^2)}{dQ^2}, \qquad Q^2 \equiv -q^2 . \tag{7.140}$$

From Eq. (7.139) we can easily check that for QED (to lowest order),

$$\beta(Q^2) = \frac{\alpha^2_{\text{eff}}(Q^2)}{3\pi}, \quad Q^2 \gg m^2 . \tag{7.141}$$

This also clearly shows that the coupling constant monotonically increases as the scale Q^2 grows.

The fact that the behaviour of QED coupling $\alpha_{\text{eff}}(Q^2)$ is completely dictated by the photons self energy diagram is not general but is specific to QED. Let us denote the UV-divergent part of the loop-corrected vertex by $1/\tilde{Z}_1$,



$$= -ie\gamma^\mu \times \tilde{Z}_1^{-1}(q^2)$$

According to Eq. (7.56), with Pauli-Villars regularization,

$$\tilde{Z}_1^{-1}(q^2) = 1 + \frac{\alpha_0}{2\pi}\left[\frac{1}{2}\log\left(\frac{\Lambda^2}{-q^2}\right) + \cdots\right] .$$

or the same in dimensional regularization (Ex.),

$$\tilde{Z}_1^{-1}(q^2) = 1 + \frac{\alpha_0}{2\pi}\frac{1}{2}\left[\frac{2}{\epsilon} - \gamma_E + \log(4\pi) + \log\left(\frac{\mu_D^2}{-q^2}\right) + \cdots\right] . \tag{7.142}$$

We then denote the UV-divergent part of the electron self-energy (after mass renormalization) by $\tilde{Z}_2$. According to Eq. (7.76), with Pauli-Villars regularization,

$$\tilde{Z}_2(q^2) = 1 - \frac{\alpha}{2\pi}\left[\frac{1}{2}\log\left(\frac{\Lambda^2}{-q^2}\right) + \cdots\right]$$

which in dimensional regularization corresponds to (Ex.),

$$\tilde{Z}_2(q^2) = 1 - \frac{\alpha_0}{2\pi}\frac{1}{2}\left[\frac{2}{\epsilon} - \gamma_E + \log(4\pi) + \log\left(\frac{\mu_D^2}{-q^2}\right) + \cdots\right]. \quad (7.143)$$

Both external electrons contribute by $\sqrt{\tilde{Z}_2}$. Finally, we denote by $\tilde{Z}_3(q^2)$ the UV-divergent part of the photon self-energy correction,

$$\tilde{Z}_3(q^2) = 1 - \frac{\alpha_0}{3\pi}\left[\frac{2}{\epsilon} + \log\left(\frac{\mu_D^2}{-q^2}\right) - \gamma_E + \log(4\pi)\cdots\right]$$

In general we should define the scale-dependent coupling by

$$e_{\text{eff}}(q^2) \equiv e_0\frac{\tilde{Z}_2(q^2)\sqrt{\tilde{Z}_3(q^2)}}{\tilde{Z}_1(q^2)}, \quad (7.144)$$

but in QED it so happens that $\tilde{Z}_2(q^2)/\tilde{Z}_1(q^2)$ is not UV divergent so only the photon self-energy correction is enough to renormalize the QED coupling. In other theories (e.g. QCD), this may not be the case and all the legs i connecting to a given vertex will give one $\sqrt{\tilde{Z}_i}$ and the vertex-correction itself one $\tilde{Z}_1^{-1}$.

Schemes and scales

What terms to include into the renormalizaton factors $\tilde{Z}_i(q^2)$ when defining the running coupling constant by Eq. (7.144) is not unique. Different choices are called **renormalization schemes**. In dimensional regularization by far the most common is the so-called **modified minimal subtraction**

scheme or just $\overline{\text{MS}}$ **scheme** in short. In this scheme one defines,

$$\tilde{Z}_3(q^2) \overset{\overline{\text{MS}}}{=} 1 - \frac{\alpha_0}{3\pi}\left[\frac{2}{\epsilon} - \gamma_E + \log(4\pi) + \log\left(\frac{\mu_D^2}{-q^2}\right)\right], \tag{7.145}$$

$$\tilde{Z}_2(q^2) \overset{\overline{\text{MS}}}{=} 1 - \frac{\alpha_0}{2\pi}\frac{1}{2}\left[\frac{2}{\epsilon} - \gamma_E + \log(4\pi) + \log\left(\frac{\mu_D^2}{-q^2}\right)\right], \tag{7.146}$$

$$\tilde{Z}_1^{-1}(q^2) \overset{\overline{\text{MS}}}{=} 1 + \frac{\alpha_0}{2\pi}\frac{1}{2}\left[\frac{2}{\epsilon} - \gamma_E + \log(4\pi) + \log\left(\frac{\mu_D^2}{-q^2}\right)\right]. \tag{7.147}$$

so the definition absorbs not only the $1/\epsilon$ pole but also factors γ_E ja $\log(4\pi)$ typical to the dimensional regularization. In the so-called **minimal subtraction scheme** or **MS scheme** in short, these terms are left out from the definition.

To some extent, the choice of scheme affects e.g. what kind of β function we get. At least the first five terms of the QED β function have been calculated. In the $\overline{\text{MS}}$ scheme the first three terms are,

$$\beta(Q^2) = \frac{\alpha_{\text{eff}}^2(Q^2)}{3\pi} + \frac{\alpha_{\text{eff}}^3(Q^2)}{4\pi^2} - \frac{31\alpha_{\text{eff}}^4(Q^2)}{288\pi^3}. \tag{7.148}$$

Another ambiguity is related to the scale q^2. As we see from the definition (7.144), we can express e_0 in terms of whatever scale q^2. It is natural to tie this scale to some invariant scale that appears in the process but there is no single correct way to choose this. The chosen scale is called the **renormalization scale**.

In a physical observable, two different renormalization schemes or scale choices formally differ by a factor that is higher order in coupling than the precision of the calculation. In this sense all schemes and scales are equally good. Numerically they are not exactly equal, though. By performing the calculation in more than one scheme and with several scale choices serves as a tool to test the perturbative reliability of the result.