6. Renormalization

Renormaization, at first encounter, may appear to be but a technical – and dubious – method to remove the mathematical divergences plaguing the QFT pertubation expansion. In reality it is much more, and the divergences are just a minor issue whose solution calls for a regularization of the theory.

Appearance of divergences is actually almost inevitable result from causality of the theory:

$$\text{causality} \longrightarrow \text{locality} \longrightarrow \text{divergences}$$

$$\uparrow\downarrow$$

only singular interactions possible : $\sim \phi(x)^4$, $\bar{\psi}(x) \not{A}(x) \psi(x)$ etc.

Cf. Coulomb interaction : $V(|x_1 - x_2|) \propto \dfrac{e^4}{|x_1 - x_2|}$.

Bare and physical variables

Let us again use the $\lambda\phi^4$-theory as an example. We write

$$\mathcal{L} \equiv \frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{1}{2} m_0^2 \phi_0^2 - \frac{\lambda_0}{4!} \phi_0^4 \qquad (6.1)$$

where ϕ_0 is the unrenormalized (bare) field and m_0 and λ_0 are the bare mass and coupling parameters. The first observation to make is that none of these objects is a directly observable quantity! For example a scattering event measurement say, for process $\phi\phi \to \phi\phi$ actually tells nothing about λ_0. Instead

$$\sigma_{bare} \propto \int d\Omega \, \frac{|M|^2}{\#} \propto a_0 \lambda_0^2 \Big(1 + \underbrace{\overset{=\infty}{\tilde{b}_1} \lambda_0 + \overset{=\infty}{\tilde{b}_2} \lambda_0^2 + \dots}_{\text{infinite series of infinite terms}}\Big) = \infty . \qquad (6.2)$$

Technically the renormalization program consists of absorbing the infinities in the perturbative expansions such as (6,2) into the bare variables ϕ_0, λ_0 and m_0, by defining new, renormalized parameters

$$\begin{aligned} \phi_0 &\equiv Z_\phi^{1/2} \phi_R \\ \lambda_0 &\equiv Z_\phi^{-2} Z_\lambda \lambda_R \\ m_0^2 &\equiv m_R^2 + \delta m^2 \end{aligned} \qquad (6,3)$$

where ϕ_R, λ_R and m_R are finite physical parameters, and all infinities present in bare parameters are included in factors $Z_\phi^{1/2}$, Z_λ and δm^2. (We shall do this shortly to 1-loop level.)

Physically renormaligability means that the unknown bare parameters of the theory do not define its long distance behaviour. Instead, the theory can be defined based only on the measurable parameters. That is all measurable predictions of the theory are set once parameters m_R, λ_R and ϕ_R are determined.

Equations (6.3) merely connect bare and renormalized parameters to each others. These relations are far from being meaningless however. For example from the fact that λ_0, m_0^2 and ϕ_0 are just pure numbers, whereas the functions δm^2, Z_ϕ and Z_λ in general depend on external momenta q entering the graphs (there are some scheme-dependence to this statement), we infer that renormalized parameters depend on scale E.g.

$$\lambda_R \longrightarrow \lambda_R(Q^2) \qquad (6.4)$$

is a scale-dependent, running coupling. [Renormalization group]

Renormalization program can be set up in many different ways.

- In canonical renormalization one takes (6.1) as a starting point and compute first the divergent greens functions of the model. These are functions of m_0^2, λ_0 and ϕ_0, from which the measurable quantities can be computed. (One can concentrate only on 1PI-functions, because all the rest can be expressed in terms of them.) The divergences in 1PI-graphs are absorbed into bare parameters through redefinitions (6.3) after which all renormalized greens functions are finite.

(identical) Eventually the method of choice in these lectures.

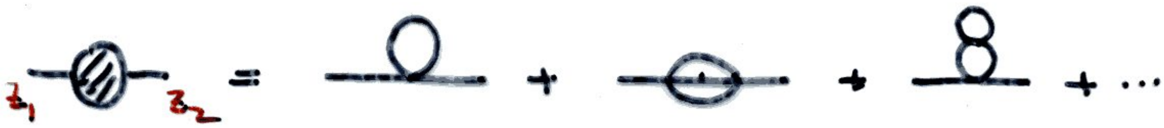
- Another possibility in the BPHZ-method, where one rewrites the lagrangian from the outset in terms of the renormalized parameters using (6.3). Then $\mathcal{L}$ is divided into a renormalized lagrangian and a counter-term lagrangian. In this approach greens functions are immediately written in terms of physical quantities and infinities are eliminated by adjusting the counter-term lagrangian. (This method is philosophically closer to the fundamental idea behind renormalizability.)

6.1 Renormalization of $\lambda\phi^4$-theory (canonical)

Let us start from the Lagrange function (6.1), and consider different Greens functions. The simplest is of course the propagator (2-point function). We have already shown that the diagrammatic series for the propagator can be arranged into a form:

(6.5)

Where



(6.6)

contains all irreducible 1PI-graphs. Let us now denote this sum by

$$-i\Pi(z_1, z_2) = -i\Pi(z_1 - z_2) \tag{6.7}$$

$\uparrow$ translational invariance

Then the series (6.5) corresponds to an equation

$$G(x-y) = \Delta_F(x-y) - \int_{z_1 z_2} \Delta_F(x-z_1)\, i\Pi(z_1-z_2)\, \Delta_F(z_2-y) + \dots \tag{6.8}$$

Move to momentum space by F-transforming w.r.t. $x-y$:

$$\underset{\substack{\uparrow \\ \text{by transl.} \\ \text{invariance.}}}{G(p^2)} = \Delta_F(p^2) - \int_{x-y} e^{-ip\cdot(x-y)} \int_{z_1 z_2} \underbrace{\Delta_F(x-z_1)}_{=\int_{q_1} \Delta_F(q_1^2)\, e^{iq_1\cdot(x-z_1)}}\, i\Pi(z_1-z_2)\, \Delta_F(z_2-y) + \dots$$

$$= \Delta_F(p^2) - \int_{q_1,q_2,x-y} e^{-ip\cdot(x-y)+iq_1\cdot x - iq_2\cdot y}\,\Delta_F(q_1^2)\times$$

$$\times \underbrace{\int_{z_1 z_2} i\Pi(z_1-z_2)\, e^{-iq_1\cdot z_1 + iq_2\cdot z_2}}\,\Delta_F(q_2^2) + \dots$$

$$= \underbrace{\int_{\bar{z}} e^{-i\bar{z}(q_1-q_2)}}_{=(2\pi)^4\delta^4(q_1-q_2)}\ \underbrace{\int_{\Delta z} i\Pi(\Delta z)\, e^{i\Delta z\cdot\frac{q_1+q_2}{2}}}_{i\Pi\left(\left(\frac{q_1+q_2}{2}\right)^2\right)}$$

$$= \Delta_F(p^2) - \int_{q,x-y} e^{i(p-q)\cdot(x-y)}\,\Delta_F(q^2)\, i\Pi(q^2)\,\Delta_F(q^2) + \dots$$

$$= \Delta_F(p^2) - \Delta_F(p^2)\, i\Pi(p^2)\,\Delta_F(p^2) + \dots$$

$$= \Delta_F(p^2)\left(1 - i\Pi\Delta_F + (-i\Pi\Delta_F)^2 + \dots\right) = \frac{\Delta_F(p^2)}{1 + i\Pi(p^2)\Delta_F(p^2)} \qquad (6.9)$$

Using the expression $\Delta_F(p^2) = i/(p^2 - m_0^2 + i\epsilon)$ (in terms of the bare quantities), we get

$$G(p^2) = \frac{i}{p^2 - m_0^2 - \Pi(p^2)} \qquad (6.10)$$

This is the full propagator of the interacting theory. The result is formally exact although divergence problems with $\Pi(p^2)$ need to be addressed :)

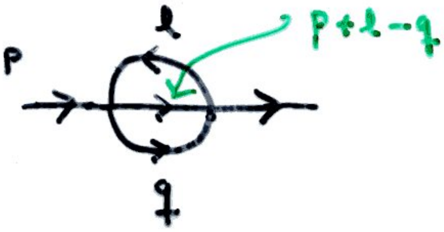
Function $\Pi(p^2)$ is thus the sum of all 1PI-graphs for 2-point function. To lowest order:

does not depend on p.

$$-i\Pi^{(1)}(p^2) = -\frac{i\lambda_0}{2}\int \frac{d^4\ell}{(2\pi)^4}\,\frac{i}{\ell^2 - m_0^2 + i\epsilon} = -i\Pi^{(1)}(0) \qquad (6.11)$$

This is clearly divergent. If we set an upper limit for the integral quadratic div.

$l_E^2 < \Lambda^2$, we find $i\Pi^{(1)} \propto \Lambda^2$. Also the second term in the expansion is divergent:

$-i\Pi^{(2a)}(p^2) =$ 

$$= \frac{1}{6}(-i\lambda_0)^2 \int \frac{d^4q}{(2\pi)^4}\frac{d^4l}{(2\pi)^4} \frac{i^3}{(q^2-m_0^2)(l^2-m_0^2)((p+l-q)^2-m_0^2)}$$

$$\sim \lambda_0^2 \int^{\Lambda} dq\, dl \, \frac{q^3 l^3}{q^2 l^2 q l} \sim \Lambda^2 \qquad (6.12)$$

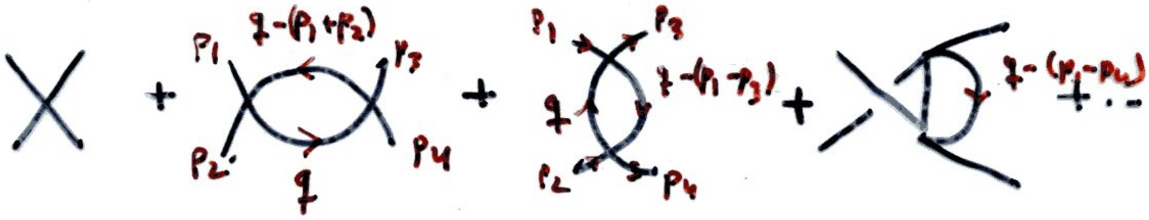
quadratic div.

This time the finite part depends also on p^2. So, it appears that the manipulations made to get (6.10) from (6.8) are meaningless. The idea is that this expansion actually is finite, when bare parameters are taken to be infinite such that they cancel the infinities from integrations. Carrying this through in practice needs regularization (eg. by cutoff) such that

$$\Pi_{reg}(p^2) < \infty \qquad (6.13)$$

after a suitable choice of m_0, λ_0 and ϕ_0, the divergent parts in Π_{reg} cancel and one can remove the regulator, resulting in a small correction controlled by renormalized coupling.

Now consider vertex function. In momentum space $\Gamma^{(n)}_{1PI}$ becomes

$\Gamma^{(4)}_{1PI} =$ 

$$= -i\lambda_0 + \Gamma(s) + \Gamma(t) + \Gamma(u) \qquad (6.14)$$

Functions $\Gamma(s)$, $\Gamma(t)$ and $\Gamma(u)$ are all of the functional form

$$\Gamma(p^2) = \frac{(-i\lambda_0)^2}{2}\int\frac{d^4q}{(2\pi)^4}\frac{i^2}{(q^2-m_0^2+i\epsilon)((q-p)^2-m_0^2+i\epsilon)} \qquad (6.15)$$

These functions turn out to be logarithmically divergent:

$$\Gamma(p^2) \sim \int^\Lambda dq\,\frac{1}{q} \sim \log\Lambda \qquad (6.16)$$

Now consider:

$$\frac{\partial\Gamma(p^2)}{\partial p^2} = \frac{1}{2p^2}\,p^\mu\frac{\partial}{\partial p^\mu}\Gamma(p^2)$$

$$= \frac{(-i\lambda_0)^2}{2p^2}\int\frac{d^4q}{(2\pi)^4}\frac{i^2\,(q-p)\cdot p}{(q^2-m_0^2+i\epsilon)((q-p)^2-m_0^2+i\epsilon)^2} \qquad (6.17)$$

This expression is finite, since at large momentum limit

$$\frac{\partial\Gamma}{\partial p^2} \sim \int^\Lambda\frac{dq}{q^2} \quad < \infty \qquad (6.18)$$

Similarly, the first derivative of the quadratically divergent function (6.12) is only logarithmically divergent and its second derivative is finite. Moreover the derivative of the function (6.11) vanishes. What these considerations teach us is that the divergences of 1PI-functions $\Gamma^{(x)}_{1PI}(p^2)$, reside in the first few terms of their Taylor expansions:

$$\Gamma^{(x)}_{1PI}(p^2) = a_0 + a_1 p^2 + \ldots + \frac{1}{n!}a_n(p^2)^n \qquad (6.19)$$

where

$$a_n \equiv \frac{\partial^n}{\partial p^{2\,n}} \Gamma^{(x)}_{1PI}(p^2)\Big|_{p^2=0} \qquad (6.20)$$

For example in the case of the function (6.15) only a_0 is divergent. So, if we write

$$\Gamma^1(p^2) \equiv \Gamma^1(0) + \tilde{\Gamma}^1(p^2) \qquad (6.21)$$

the function $\tilde{\Gamma}^1(p^2)$ is finite.

Apparent order of divergence

We can compute the apparent order of divergence of each $\Gamma^{(n)}_{1PI}$-function to arbitrary order in perturbation theory as a function of n only! Assume that an arbitrary diagram contributing to $\Gamma^{(n)}_{1PI}$ has

- n external legs (the definition)
- p internal propagators
- v vertices

Because each leg couples into one vertex and each propagator to two vertices, these numbers are bound by a relation

$$\underline{4v = 2p + n} \qquad (6.22)$$

Moreover, from Feynman rules we can deduce that the number of unconstrained loop-integrals is

each vertex induces momentum conservation

$$\underline{L = p - v + 1} \qquad (6.23)$$

momentum integral over each propagator

one overall momentum conservation

Because each integral adds four momenta to the nominator of (6.23) ($dq\,q^3$) and every propagator two momenta to the denominator we have

$$D = 4L - 2p = 4 + 2p - 4v$$
$$= \underline{4 - n}\ !\qquad (6.24)$$

Thus in $\lambda\phi^4$-theory, only 2- and 4-point functions are divergent. (Quadratically and logarithmically) All the remaining $\Gamma^{(n)}_{1PI}$ are finite.

Mass- and wave function renormalization

Based on (6.19) and (6.24) we know that only two first terms in the Taylor expansion of the function $\Pi(p^2)$ appearing in $G(p^2)$ are divergent. We can therefore write an expansion around a point $p^2 = \mu^2$:

$$\Pi(p^2) = \Pi(\mu^2) + (p^2-\mu^2)\,\Pi'(\mu^2) + \tilde{\Pi}(p^2) \qquad (6.25)$$

an arbitrary expansion point   the finite part

In the regularized theory all terms are of course finite, so we can insert (6.25) into (6.10) and manipulate it as if $\Pi \propto \lambda$ was a small number. So the expansion can be seen to make sense a'posteriori

---

- $D = 0$: $\int^{\Lambda} \frac{dq}{q} \sim \log \Lambda$; $D = 2$: $\int dq\, q \sim \Lambda^2$

  $D = -2$: $\int^{\Lambda} \frac{dq}{q^3} \sim \frac{1}{\Lambda^2}$ and so on

$$G(p^2) = \frac{i}{p^2 - m_0^2 - \Pi(\mu^2) - (p^2-\mu^2)\Pi'(\mu^2) - \tilde{\Pi}(p^2) + i\epsilon} \tag{6.26}$$

Let us now define the physical mass as the pole of the full propagator. Because μ^2 was until now arbitrary, we can set eg. the condition:

$$\boxed{m_0^2 + \Pi(\mu^2) \equiv \mu^2} \tag{6.27}$$

mass renormalization

Then

$$G(p^2) = \frac{i}{(p^2-\mu^2)\left[1-\Pi'(\mu^2)\right] - \tilde{\Pi}(p^2)} \tag{6.28}$$

Because by definition

$$\tilde{\Pi}(\mu^2) = \tilde{\Pi}'(\mu^2) = 0 \tag{6.29}$$

The μ^2 satisfying (6.27) is just the physical mass. Now denote $\mu^2 = m_R^2$ (with (6.27)); whereby $\left(\tilde{\Pi}(p^2)/(1-\Pi'(\mu^2)) \to 0 \right)$

[removing regulator.

$$G(p^2) \approx \frac{i Z_\phi}{p^2 - m_R^2 + i\epsilon} \tag{6.30}$$

where

$$Z_\phi \equiv \frac{1}{1-\Pi'(\mu^2)} \tag{6.31}$$

Now remember that

$$G(p^2) = \int d^4x \, e^{ip\cdot x} \langle 0 | T(\phi_0(x)\phi_0(0)) | 0 \rangle \tag{6.32}$$

Now define a new field

$$\boxed{\phi_R \equiv Z_\phi^{-1/2}\phi_0} \qquad (6.33)$$

wave function renormalization

Then the renormalized propagator

$$G_R(p^2) \equiv \int d^4x\, e^{-ip\cdot x} \langle 0|T(\phi_R(x)\phi_R(0))|0\rangle$$

$$= Z_\phi^{-1} G(p^2) = \frac{i}{p^2 - m_R^2 + i\epsilon} \qquad (6.34)$$

This is of course just our old Feynman propagator with m_0 replaced by the renormalized physical mass. The trick was to set

$$m_R^2 = m_0^2 + \Pi(m_R^2) \qquad (6.35)$$

infinities between these two cancel.

The quadratic divergence in $\Pi(m_R^2)$ is thus absorbed to unknown m_0. Similarly the logarithmic divergence in $\Pi'(m_R^2)$ was removed by redefinition of the field itself: $\phi_0 \to \phi_R$.

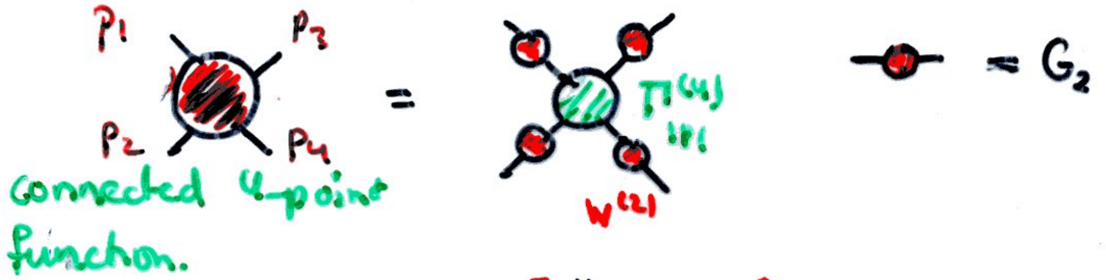
Coupling constant renormalization

We know that this is necessary because for $\Gamma^{(4)}_{1PI}$ $D = 4-n = 0$, ie $\Gamma^{(4)}$ is logarithmically divergent. Let us define the physical coupling as

freely choosable renormalization point.

$$-i\lambda_R \equiv \Gamma_R^{(4)}(s_0, t_0, u_0) \qquad (6.36)$$

On the other hand we remember the connection



$$\underline{G_0^{(4)}(s,t,u) = \left[\prod_{i=1}^{4} G(p_i^2)\right] \times \Gamma_0^{(4)}(s,t,u)} \qquad (6.37)$$

Because $G_0^{(2)} = Z_\phi G_R^{(2)}$ and of course $G_0^{(4)} = Z_\phi^2 G_R^{(4)}$, we get

$$\underline{\Gamma_R^{(4)} = Z_\phi^2 \Gamma_0^{(4)}} \qquad (6.38)$$

(More generally: $G_0^{(n)} = Z_\phi^{n/2} G_R^{(n)}$ and $\Gamma_R^{(n)} = Z_\phi^{n/2} \Gamma_0^{(n)}$.)

We thus have:

$$-i\lambda_R \equiv \Gamma_R^{(4)}(s_0,t_0,u_0) = Z_\phi^2 \Gamma_0^{(4)}(s_0,t_0,u_0)$$

and

$$= -i\lambda_0 \Big(1 + \underbrace{\frac{i}{\lambda_0}\left(\Gamma_0^{(4)}(s_0,t_0,u_0) - i\lambda_0\right)}_{= \Delta\Gamma_0^{(4)}(s_0,t_0,u_0) \;\to\; \infty \text{ as regulator is remov}}\Big) Z_\phi^2 \qquad (6.39)$$

$$\equiv -i\lambda_0 Z_\lambda^{-1} Z_\phi^2$$

i.e

$$\underline{\lambda_R = Z_\lambda^{-1} Z_\phi^2 \cdot \lambda_0} \quad \equiv \text{finite.} \qquad (6.40)$$

this defines

a) what infinities λ_0 must eat

b) scale dependence of λ_R.

We have now shown that the infinities appearing in 1PI-functions can all be absorbed to bare parameters, to all orders in PT. The actual procedure of extracting infinities and computing Z_λ, Z_ϕ and δm^2 and the finite parts of 1PI-functions is quite complicated. It turns out, at least this is my opinion, that this procedure is best organized in the so-called BPHZ-formulation.

BPHZ-method

Now that the connections between the bare and physical parameters have been established, we can use them to rewrite $\mathcal{L}$ in terms of renormalized parameters from the outset:

$\times Z_\lambda Z_\phi^2$

$$\mathcal{L} = \frac{Z_\phi}{2}\partial^\mu\phi\,\partial_\mu\phi - \frac{Z_\phi}{2}(m^2+\delta m^2)\,\phi^2 - \frac{\tilde{Z}_\lambda}{4!}\lambda\,\phi^4 \qquad (6.41)$$

where ϕ, m^2 and λ now are finite, renormalized (possibly physical) parameters, whose values need to be extracted from observations. There is no reference whatsoever to bare parameters! The idea is to divide $\mathcal{L}$ into two pieces: $\mathcal{L} = \mathcal{L}_R + \Delta\mathcal{L}$, where

$$\mathcal{L}_R \equiv \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \qquad (6.42)$$

and

$$\Delta\mathcal{L} = \frac{\overbrace{Z_\phi-1}^{\delta Z_\phi}}{2}\left[(\partial\phi)^2 - \frac{m^2}{2}\phi^2\right] - \frac{Z_\phi\overbrace{\delta m^2}^{\delta m^2}}{2}\phi^2 - \frac{\overbrace{\tilde{Z}_\lambda-1}^{\delta\tilde{Z}_\lambda}}{4!}\lambda\phi^4 \qquad (6.43)$$

Note that at tree level (that is, ignoring all loop-corrections) $Z_\phi = Z_\lambda = 1$ and $\delta m^2 = 0$, so that $\Delta\mathcal{L}^{(0)} = 0$.

The BPHZ-program goes as follows

1) Compute the propagators and vertices up to 1-loop order from the Feynman rules defined by the Lagrangian (6.42).

2) Divergent parts of the Γ_{1PI}-diagrams are extracted from the Taylor expansions (in regularized theory), and define $\Delta\mathcal{L}^{(1)}$ such that it, taken as new tree level interactions, cancels the divergences.

3) Define the 1-loop corrected Lagrange function $\mathcal{L} = \mathcal{L}_R + \Delta\mathcal{L}^{(1)}$ (which now gives finite results to 1-loop order) and compute 2-loop corrections to 1PI-functions using this $\mathcal{L}$. From the new divergences construct $\Delta\mathcal{L}^{(2)}$ ....

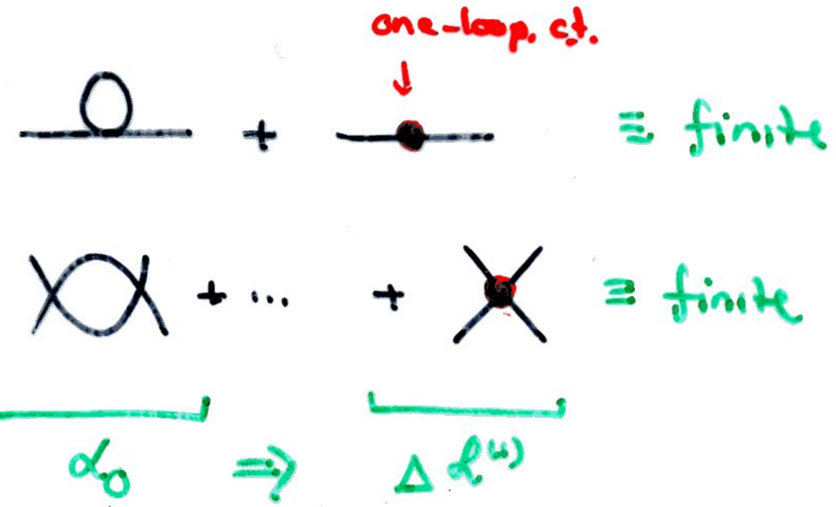
After iteration, the $\mathcal{L}$-function for the theory gets tuned to form

$$\mathcal{L} = \mathcal{L}_R + \underbrace{\Delta\mathcal{L}^{(1)} + \Delta\mathcal{L}^{(2)} + \dots + \Delta\mathcal{L}^{(n)} + \dots}_{\Delta\mathcal{L}} \qquad (6.44)$$

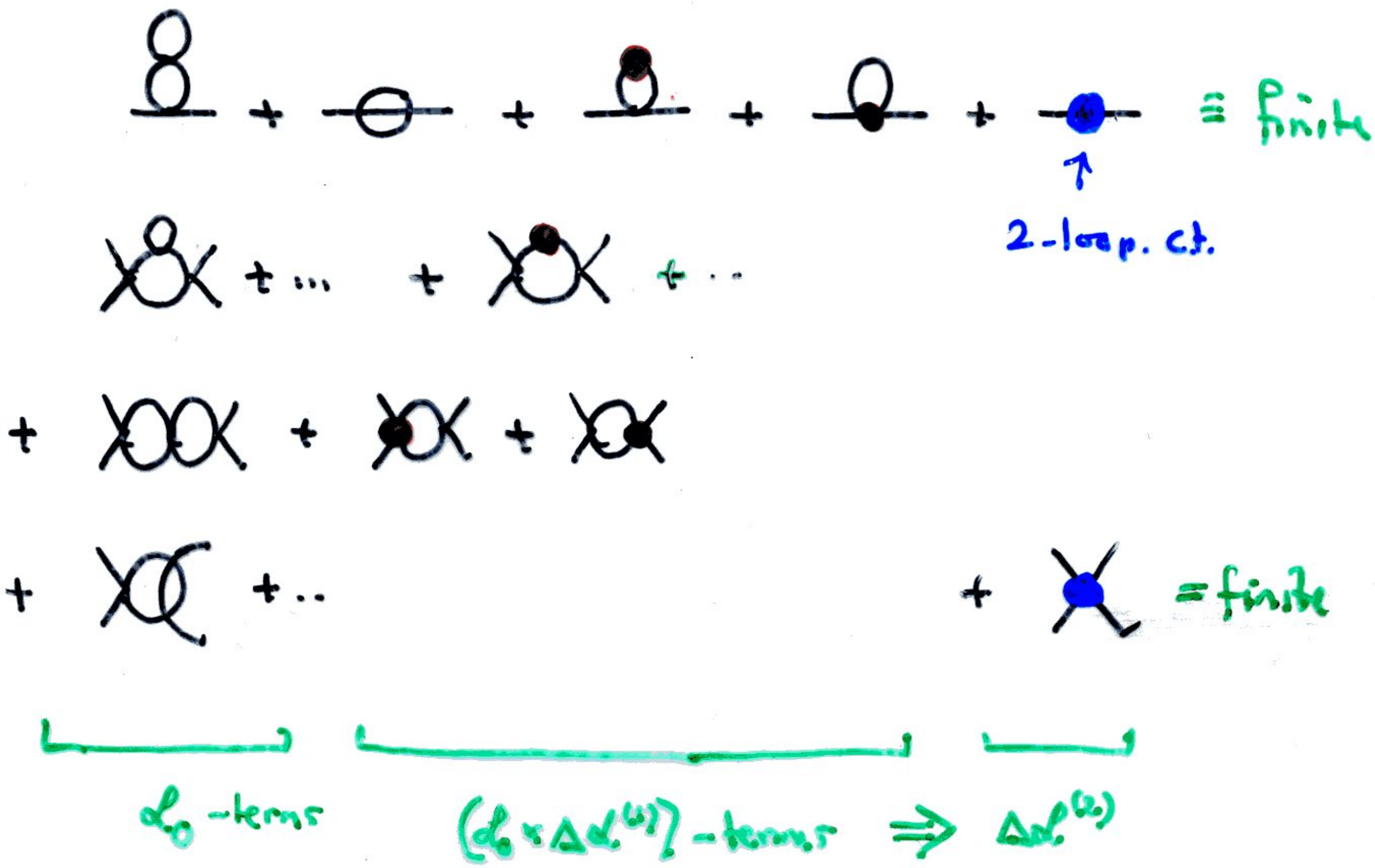
(Because we showed that in $\lambda\phi^4$-theory only 2- and 4-point functions are divergent, we know that the entire $\Delta\mathcal{L} = \sum_{n=0}^{\infty}\Delta\mathcal{L}^{(n)}$ is of the form (6.43).)

Diagrammatically:

1-loop order:



2-loops:



Now $\Delta\mathcal{L}^{(2)}_{BT} = \Delta\mathcal{L}^{(1)} + \Delta\mathcal{L}^{(2)}$ and we can continue to 3-loop graphs. In the end we can define a completely finite P.T. in terms of the renormalized parameters.

Note that while in canonical approach $\delta Z_i = \delta Z_i(\lambda_0, m_0, \phi_0)$ here $\delta Z_i = \delta Z_i(\lambda, m, \phi)$.

6.2. Regularization

An essential part of the renormalization process involves isolating singularities of divergent graphs and reorganizing PT such that predictions are finite. There are several regularization methods to achieve this goal:

1) (Euclidean) momentum cut-off $|p_E| < \Lambda$.
   (breaks gauge-invariance)
2) Discretization of the position space (lattice) $x \geq a$
   (lattice field theories)
3) Pauli-Villairs regularization
   (A covariant way to do 1.)
4) Dimensional regularization
   ($d^4x \to d^Dx$)

Methods 1 and 2 break Lorenz symmetry, so regaining the continuum limit is not always trivial. (When $\Lambda \to \infty$ and $a \to 0$.)

Pauli-Villains regularization

Add new (also negative norm) states to the Lagrange density, so as to modify the propagator

$$\frac{i}{p^2-m^2+i\epsilon} \longrightarrow \frac{i}{p^2-m^2+i\epsilon} - \sum_i \frac{i\,a_i}{p^2-M_i^2+i\epsilon} \qquad (6.45)$$

where a_i:s are chosen such that propagator introduces faster convergence than normally. For example with only one extra field with $a_1 = 1$ we get

$$\frac{i}{p^2-m^2+i\epsilon} \longrightarrow \frac{i(m^2-M^2)}{(p^2-m^2+i\epsilon)(p^2-M^2+i\epsilon)} \overset{p^2 \gg M^2, m^2}{\propto} \frac{1}{p^4}$$

This suffices to regulate all but the tadpole (Ω)-divergence in $\lambda\phi^4$ theory. (For this one needs two new fields.) We shall not perform explicit calculations in PV-method here. However, the results for Π and $\Gamma^{(4)}$ would be (cf eg. Cheng & Li p.):

$$\Pi(0) = \frac{\lambda^2}{32\pi^2} M^2$$

$$\Pi'(0) = 0$$

$$\Gamma(0) = \frac{\lambda^2}{32\pi^2} \log \frac{M^2}{m^2} \qquad (6.47)$$

to 1-loop order. Here Π and Γ were expanded around $p_i^2 = 0$. These divergences can be eliminated by choosing the counter-term lagrangian

$$\Delta\mathcal{L}^{(1)} = \frac{3i\Gamma(0)}{4!}\phi^4 + \frac{1}{2}\Pi(0)\phi^2 + \frac{1}{2}\overbrace{\Pi'(0)}^{Z_\phi - 1}(\partial_\mu\phi)^2 \qquad (6.48)$$

with this choice Greens functions become finite, and

- $Z_\phi = 1 + \Pi'(0) \simeq \frac{1}{1-\Pi'(0)} \;(= 1)$ To 1-loop order.
- $\Pi(0) = (1-Z_\phi)m^2 + Z_\phi \delta m^2 = \delta m^2 \qquad (6.49)$

$$\Rightarrow \boxed{m^2 = m_0^2 + \Pi(0)} \qquad (6.50)$$

- $\lambda Z_\lambda^{-1} = \lambda + 3i\Gamma(0) \qquad (6.51)$

Definitions (6.49) and (6.50) are not exactly the same that we encountered with the mass and wave-function renormalization in section 6.1. There we defined the theory in terms of physical on-shell mass. Equally well one can use the finite parameter (6.50), which is the mass of the theory "measured" at $p^2=0$. More precisely (6.49) and (6.50) correspond to renormalization conditions

renormalization point

$$\begin{aligned} \tilde{\Pi}(p^2=0) &\equiv 0 \\ \frac{\partial\tilde{\Pi}}{\partial p^2}(p^2=0) &\equiv 0 \end{aligned} \tag{6.52}$$

or, equivalently, in terms of the renormalized propagator:

$$\Delta_R^{-1}(p^2)\Big|_{p^2=0} \equiv -m^2 \quad ; \quad \frac{\partial\Delta_R^{-1}}{\partial p^2}\Big|_{p^2=0} \equiv +1 \tag{6.53}$$

Clearly m_R^2 and m^2 are related by

$$m_R^2 = m^2 + \Pi(m_R^2) - \Pi(0) \tag{6.54}$$

Finally, we see that in this renormalization scheme at 1-loop:

$$\begin{aligned} Z_\phi &= 1 \\ Z_\lambda &= 1 + \frac{3\lambda}{32\pi^2}\ln\frac{M^2}{m^2} \\ \delta m^2 &= \frac{\lambda}{32\pi^2}M^2 \end{aligned} \tag{6.55}$$

- The on-shell renormalizations corresponded to:

$$\Delta_R^{-1}(p^2)\Big|_{p^2=m_R^2} \equiv 0 \quad ; \quad \frac{\partial\Delta_R^{-1}}{\partial p^2}\Big|_{p^2=m_R^2} \equiv 1.$$

Dimensional regularization

The basic idea in all regularizations is that a singular, ill-defined theory (or, perturbation theory) is extended to a family of theories, labeled by some parameter (or parameters). The original theory is then (or should be-) obtained as a particular limit of this or these parameters. In the dimensional regularization by 't Hooft and Veltman ('72) this parameter is the (Euclidian) dimension of the space-time. Indeed, for example the integral

$$\int d^D q \, \frac{1}{q^2 - m^2} \sim \int^\Lambda dq \, q^{D-3} \qquad (6.56)$$

is finite for $D=1$. We can actually define the integral (6.56) for arbitrary $D \in \mathbb{C}$ (!) by continuing its representation in terms of Beta function. Divergences then become simple poles in $D-4$ as $D \to 4$. Formally

$$I_D = \# \frac{1}{4-D} + \#_2 + \#_3 \cdot (4-D) + \cdots \qquad (6.57)$$

We will need a few mathematical results:

- D-dimensional angular integral (Euclidian)

$$\int d^D q = \int_0^\infty dq \, q^{D-1} \int d^D \Omega = \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})} \int_0^\infty dq \, q^{D-1}$$

- Beta-function-representation

$$\int_0^\infty dt \, \frac{t^{n-1}}{(t+a^2)^\alpha} = (a^2)^{n-\alpha} \, \frac{\Gamma(n)\Gamma(\alpha-n)}{\Gamma(\alpha)}$$

Moreover we shall need

- Feynman parametrization

$$\frac{1}{a_1 a_2 \cdots a_n} = \int_0^1 dz_1 dz_2 \cdots dz_n \frac{\delta\left(\sum_{i=1}^n z_i - 1\right) (n-1)!}{(a_1 z_1 + a_2 z_2 + \dots + a_n z_n)^n} \tag{6.60}$$

and eventually the limit

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) \tag{6.61}$$

where $\gamma_E \approx 0{,}5772$ is the Euler-Mascheroni constant. Given these formulae, we can reconsider e.g. the 1-loop contribution to $\Pi(p^2)$:

Example 1:

$$\Gamma^{(2)}_{1PI}(1\text{-loop}) = -i\Pi^{(1)}(p^2) \mathrel{\hat{=}} -\frac{i\lambda}{2}\int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2-m^2+i\epsilon}$$

$$\longrightarrow -\frac{i\lambda}{2} \mu^{4-D} \int \frac{d^Dq}{(2\pi)^D} \frac{i}{q^2-m^2+i\epsilon} \tag{6.62}$$

arbitrary parameter with a dimension of mass, such that $\mathrm{Dim}[\Pi] = \mathrm{Dim}[m^2]$ is conserved

optional, but a common definition

IMPORTANT!

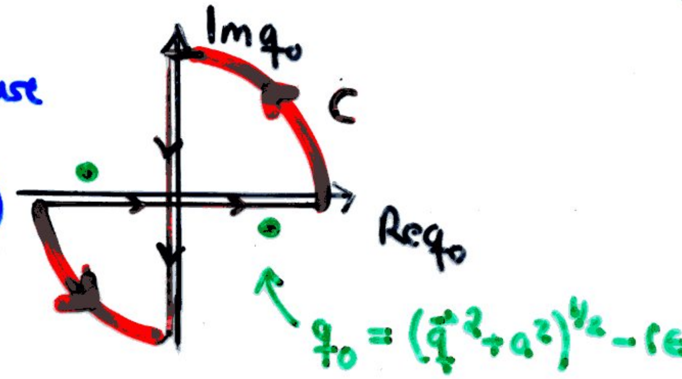
In order to be able to use (6.58) we must continue (the 4-D-version of) the integral to Euclidian time by a Wick-rotation:

$$k_{0M} \to ik_{0E} \;; \quad \begin{cases} d^4k_M \to i d^4k_E \\ k_M^2 \to -k_E^2 \end{cases} \tag{6.63}$$

- The continuation of $(2\pi)^D$ could be included in the definition of μ: by $\mu \to \frac{\mu}{4\pi}$ $\quad \mu^{4-D}\int\frac{d^Dq}{(2\pi)^D} \to \mu^{4-D}\int\frac{d^Dq}{(2\pi)^4}$, so this one is but an arbitrary def.

Wick-rotation

(Possible because of Feyman bndry cond.)



Cauchy's theorem

$$\oint f(q_0)\,dq_0 = 0 \quad (+\text{ arc-integrals} \to 0)$$

$$\Rightarrow \int_{-\infty}^{\infty} dq_0\, f(q_0) = \int_{-i\infty}^{i\infty} dq_0\, f(q_0) = i\int_{-\infty}^{\infty} dq_{0E}\, f(iq_{0E})$$

We thus get

$$-i\Pi^{(1)}(p^2) = -\frac{i\lambda}{16\pi^2}\left(4\pi\mu^2\right)^{\frac{4-D}{2}} \frac{1}{\cancel{\Gamma(\frac{D}{2})}} \underbrace{\int_0^{\infty} dq_E^2\, \frac{\frac{1}{2}(q_E^2)^{\frac{D}{2}-1}}{(q_E^2+m^2)}}_{=(m^2)^{\frac{D}{2}-1}\frac{\cancel{\Gamma(\frac{D}{2})}\,\Gamma(1-\frac{D}{2})}{\Gamma(1)}}$$

$$= -\frac{i\lambda m^2}{32\pi^2}\left(4\pi\frac{\mu^2}{m^2}\right)^{\frac{4-D}{2}} \Gamma\left(1-\frac{D}{2}\right) \qquad (6.64)$$

Now $z\Gamma(z) = \Gamma(z+1)$, so that $(4-D \equiv \epsilon)$

- $$\Gamma\left(1-\frac{D}{2}\right) = \frac{2}{2-D}\Gamma\left(2-\frac{D}{2}\right) = \frac{2}{\epsilon-2}\Gamma\left(\frac{\epsilon}{2}\right)$$

$$= -\left(1+\frac{\epsilon}{2}\right)\Gamma\left(\frac{\epsilon}{2}\right) + O(\epsilon) \qquad (6.65)$$

Moreover, with small ϵ

$$a^{\frac{\epsilon}{2}} = e^{\ln a^{\frac{\epsilon}{2}}} = e^{\frac{\epsilon}{2}\ln a} \simeq 1 + \frac{\epsilon}{2}\ln a \qquad (6.66)$$

Thus, in the limit $\epsilon \to 0$ (ie at $D \to 4$):

$$\Pi(p^2) = -\frac{\lambda m^2}{32\pi^2}\left(1+\frac{\epsilon}{2}\right)\underbrace{\Gamma\left(\frac{\epsilon}{2}\right)}_{\frac{2}{\epsilon}-\gamma_E}\cdot\left(1+\frac{\epsilon}{2}\left(\ln 4\pi + \ln\frac{\mu^2}{m^2}\right)\right) + \mathcal{O}(\epsilon)$$

$$\approx -\frac{\lambda m^2}{32\pi^2}\left(\frac{2}{\epsilon} + \underbrace{1-\gamma_E+\ln 4\pi+\ln\frac{\mu^2}{m^2}}\right) \qquad (6.67)$$

scale

Divergence

finite part - depends on arbitrary μ^2.

Example 2:

Let us now compute $\Gamma(p^2)$ appearing in the 4-point function.

$$\Gamma(p^2) = -i\frac{\lambda^2}{2}\int\frac{d^4q}{(2\pi)^4}\frac{1}{(q^2-m^2)((q-p)^2-m^2)}$$

$$\longrightarrow \frac{\lambda^2}{2}\int\frac{d^Dq_E}{(2\pi)^D}\int_0^1 dz_1\,dz_2\,\frac{\delta(1-z_1-z_2)}{\left((q_E^2+m^2)\underbrace{z_1}_{\to 1-z}+((q-p)_E^2+m^2)\underbrace{z_2}_{\to z}\right)^2} \qquad (6.68)$$

$$= \int_0^1 dz\,\frac{1}{\left(q_E^2+m^2+(p^2-2q\cdot p)z\right)^2}$$

$$= \int_0^1 dz\,\frac{1}{\left[(q-zp)_E^2+m^2+z(1-z)p_E^2\right]^2}$$

Now make a change of variables $q_E \to q' \equiv q_E - zp_E$, and change the order of integration (this is ok since regulated integrals converge)

$$\Rightarrow \Gamma(p^2) = \frac{\lambda}{32\pi^2}(4\pi\mu^2)^{\frac{4-D}{2}}\frac{1}{\Gamma\left(\frac{D}{2}\right)}\int_0^1 dz\int_0^\infty dq_E^2\,\frac{(q_E^2)^{\frac{D}{2}-1}}{(q_E^2+a^2)^2} \qquad (6.69)$$

where $a^2 \equiv m^2+z(1-z)p_E^2$. Now on the basis of eq. (6.59):

$$\int_0^\infty dq_E^2 \ldots = \frac{\Gamma(\frac{D}{2})\overbrace{\Gamma(2-\frac{D}{2})}^{\frac{\epsilon}{2}}}{\Gamma(2)}(a^2)^{\frac{D}{2}-2}$$

$$\simeq \Gamma(\tfrac{D}{2})\left(\frac{2}{\epsilon}-\gamma_E - \ln\left(m^2+z(1-z)p_E^2\right)\right)$$

Euclidian. → Minkowski (− sign)

$$\Rightarrow \Gamma(p^2) = \frac{\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon}-\gamma_E+\ln 4\pi - \int_0^1 dz \ln\left(\frac{m^2 - z(1-z)p_M^2}{\mu^2}\right)\right) \quad (6.70)$$

μ-dependence

So, the divergent part is contained in $\Gamma(0)$:

$$\Gamma(0) = \frac{\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon}-\gamma_E+\ln 4\pi - \ln\frac{\mu^2}{m^2}\right) \quad (6.71)$$

and the finite part (in $p^2=0$-scheme) (μ-independent)

$$\tilde{\Gamma}(p^2) \equiv \Gamma(p^2)-\Gamma(0) = -\frac{\lambda^2}{32\pi^2}\int_0^1 dz \ln\left(1-z(1-z)\frac{p^2}{m^2}\right). \quad (6.72)$$

$-i\epsilon$!

This can in fact be computed in terms of elementary functions:

$$\tilde{\Gamma}(p^2) = \begin{cases} \frac{\lambda^2}{32\pi^2}\left(2+\left(\frac{4m^2-p^2}{|p^2|}\right)^{1/2}\ln\left(\frac{\sqrt{4m^2-p^2}-\sqrt{|p^2|}}{\sqrt{4m^2-p^2}+\sqrt{|p^2|}}\right)\right) & ;\ p^2<0 \\ \frac{\lambda^2}{32\pi^2}\left(2+2\left(\frac{4m^2-p^2}{p^2}\right)^{1/2}\arctan\left(\sqrt{\frac{p^2}{4m^2-p^2}}\right)\right) & ;\ 0<p^2<4m^2 \\ \frac{\lambda^2}{32\pi^2}\left(2+\left(\frac{p^2-4m^2}{p^2}\right)^{1/2}\left[\ln\left(\frac{\sqrt{p^2}-\sqrt{p^2-4m^2}}{\sqrt{p^2}+\sqrt{p^2-4m^2}}\right)+i\pi\right]\right) & ;\ p^2>4m^2 \end{cases} \quad (6.73)$$

!

Complex part for $p^2-4m^2>0$. ???

6.3 S-matrix and renormalization

How to find a finite, well defined S-matrix out of this mess of definitions and divergences? To see this we will have to rethink a little about the LSZ-reduction, and talk about field strength-renormalization. To this end we shall first rewrite our LSZ-vacuum-to-vacuum amplitudes somewhat differently, using the interacting theory vacuum $|\Omega\rangle$.

Consider for example the two point function. The amplitude

$$_{out}\langle 0|\, T(\phi(x_1)\phi(x_2))\,|0\rangle_{in} \qquad (6.74)$$

can be equally well written in terms of an interacting th. vacuum as:

$$\langle \Omega|\, T(\phi(x_1)\phi(x_2))\,|\Omega\rangle \qquad (6.75)$$

where

an arbitrary reference time

$$|\Omega\rangle \equiv N \lim_{T\to(1-i\epsilon)\infty} U(t_0,-T)|0\rangle \qquad \text{(Indep. of } t_0\text{)}$$

$$\langle\Omega| \equiv \bar{N} \lim_{T\to(1-i\epsilon)\infty} \left(U(t_0,T)|0\rangle\right)^\dagger = \bar{N} \lim_{T\to(1-i\epsilon)\infty} \langle 0|\, U(T,t_0) \qquad (6.76)$$

To be defined at all $|\Omega\rangle$ and $\langle\Omega|$ must be independent of t_0. Now

$$1 \equiv \langle\Omega|\Omega\rangle = N\bar{N}\, \langle 0|\, U(T,t_0)\, U(t_0,-T)|0\rangle =$$

$$= N\bar{N}\, \langle 0|\, U(T,-T)|0\rangle \Rightarrow N\bar{N} = \frac{1}{\langle 0|\, U(T,-T)|0\rangle\Big|_{T\to(1-i\epsilon)\infty}} \qquad (6.77)$$

Then (f.ex. $x_{20} > x_{10}$)

$$\langle\Omega|\phi(x_2)\phi(x_1)|\Omega\rangle = N\bar{N}\langle 0|U(T,t_0)\overbrace{\phi(x_2)}^{=U(t_2,t_0)^\dagger\phi_0(x_2)U(t_2,t_0)}\phi(x_1)U(t_0,-T)|0\rangle$$

$$= N\bar{N}\langle 0|U(T,t_2)\phi_0(x_2)U(t_2,t_1)\phi_0(x_1)U(t_1,-T)|0\rangle$$

$$= N\bar{N}\langle 0|T\left(\phi_0(x_2)\phi_0(x_2)U(T,-T)\right)|0\rangle\Big|_{T\to(1-i\epsilon)\infty}$$

$$= \frac{\langle 0|T(\phi_0(x_1)\phi_0(x_2)\,U(T,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle} \qquad \blacksquare \qquad (6.78)$$

The point is that here one makes only reference to an interecting theory vacuum, instead of $|0\rangle$ in the definition of the Greens function

By symmetry one would expect that $(|\Omega\rangle)^* = \langle\Omega|$. This is not immediately clear from the definition, but it can be proven. First use the normalizations:

$$(|\Omega\rangle)^*|\Omega\rangle = N^*N\langle 0|U(-T,t_0)U(t_0,-T)|0\rangle = |N|^2\langle 0|0\rangle$$

$$= |N|^2 \equiv 1 \Rightarrow N \text{ is a phase.}$$

Similarly

$$\langle\Omega|(\langle\Omega|)^* \equiv 1 \Rightarrow \bar{N} \text{ is a phase} \Rightarrow \underline{N\bar{N} \text{ is a phase.}}$$

only vacuum survives.

Finally

$$\underline{(|\Omega\rangle)^*} = N^*\langle 0|U(-T,t_0) = N^*\langle 0|\overbrace{U(-T,T)}^{|0\rangle\langle 0| + \cdots}U(T,t_0)$$

$$= N^*\langle 0|U(-T,T)|0\rangle\langle 0|U(T,t_0)$$

$$= \underbrace{N^*N}_{=1}\bar{N}\langle 0|U(T,t_0) = \underline{\langle\Omega|}. \qquad (6.79)$$

So, we can write:

$$G(p^2) = \int d^4x\, e^{-ip\cdot x} \langle \Omega | T(\phi(x)\phi(0)) | \Omega \rangle \qquad (6.80)$$

Let us now consider the spectrum of this operator from a slightly different point of view. Take $|\lambda_0\rangle$ to be an eigenstate of the full H of the theory with zero momentum. Because $[H, P] = 0$, an arbitrary momentum $\vec{p}$ state can be obtained from some $|\lambda_0\rangle$ by a Lorenz boost. So, we can write a generalization of 1-particle unit operator as

Lorenz-boosted state.

$$1 = |\Omega\rangle\langle\Omega| + \sum_{\{|\lambda_0\rangle\}} \int \frac{d^3p}{(2\pi)^3 2E_p(\lambda)} |\lambda_{\vec{p}}\rangle\langle\lambda_{\vec{p}}| \qquad (6.81)$$

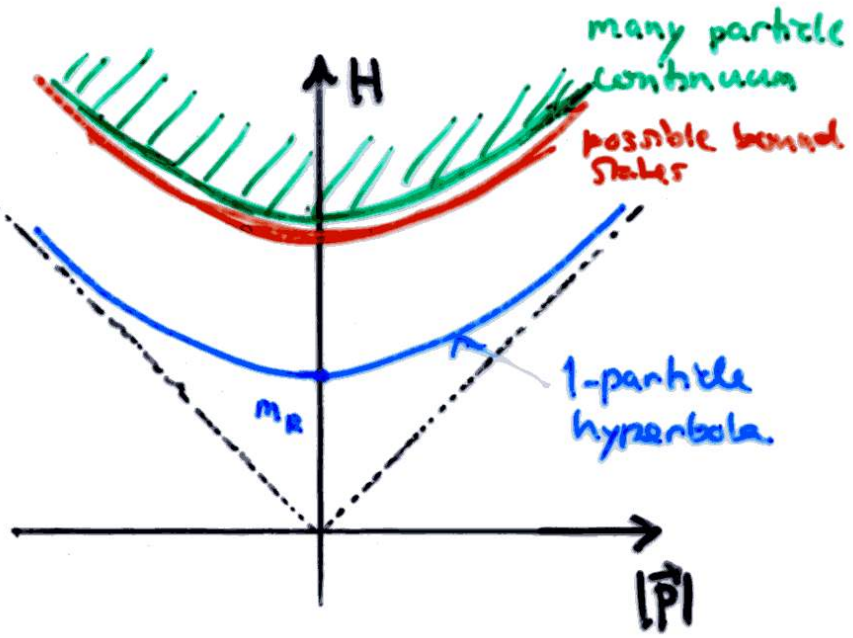
Lorenz-boosted energy

where

$$E_{\vec{p}}(\lambda) \equiv \sqrt{p^2 + m_\lambda^2} \qquad (6.82)$$

energy eigenvalue, the "mass" of the state $|\lambda_0\rangle$.

By Lorentz-covariance all excitations organize onto hyperbolas on $|\vec{p}|$, E-plane



Inserting (6.81) into (6.80), and noting that eg. for $\lambda\phi^4$-theory $\langle\Omega|\phi|\Omega\rangle = 0$ by symmetry, we get

$$\langle\Omega|\phi(x)\phi(0)|\Omega\rangle = \sum_{\{|\lambda_0\rangle\}} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} \underbrace{\langle\Omega|\phi(x)|\lambda_p\rangle}\langle\lambda_p|\phi(0)|\Omega\rangle$$

$= \langle\lambda_0|\phi(0)|\Omega\rangle$ *

$= \langle 0|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|\lambda_p\rangle$

$U^{-1}U \quad U^{-1}U$ *

$= \langle 0|\phi(0)|\lambda_p\rangle e^{-ip\cdot x}\Big|_{p_0=E_p(\lambda)}$

$= \langle 0|\phi(0)|\lambda_0\rangle e^{-ip\cdot x}\Big|_{p_0=E_p(\lambda)}$

$$= \sum_{\{\lambda_0\}} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m_\lambda^2+i\epsilon} e^{-ip\cdot x} |\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \qquad (6.83)$$

This starts to resemble the 1-particle propagator, except that there is an infinite sum of states $|\lambda_0\rangle$, while free theory contained only one excitation. Moreover, there is the extra weight factor $|\langle\Omega|\phi(0)|\lambda_0\rangle|^2$. Thus

$$G(p^2) = \sum_{\{|\lambda_0\rangle\}} \frac{i}{p^2-m_\lambda^2+i\epsilon} |\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \qquad ;\ m_\lambda^2 = m_\lambda^2(p^2)$$

$$= \int \frac{dM^2}{2\pi} \rho(M^2) \frac{i}{p^2-M^2+i\epsilon} \qquad (6.84)$$

where the spectral function ρ is given by

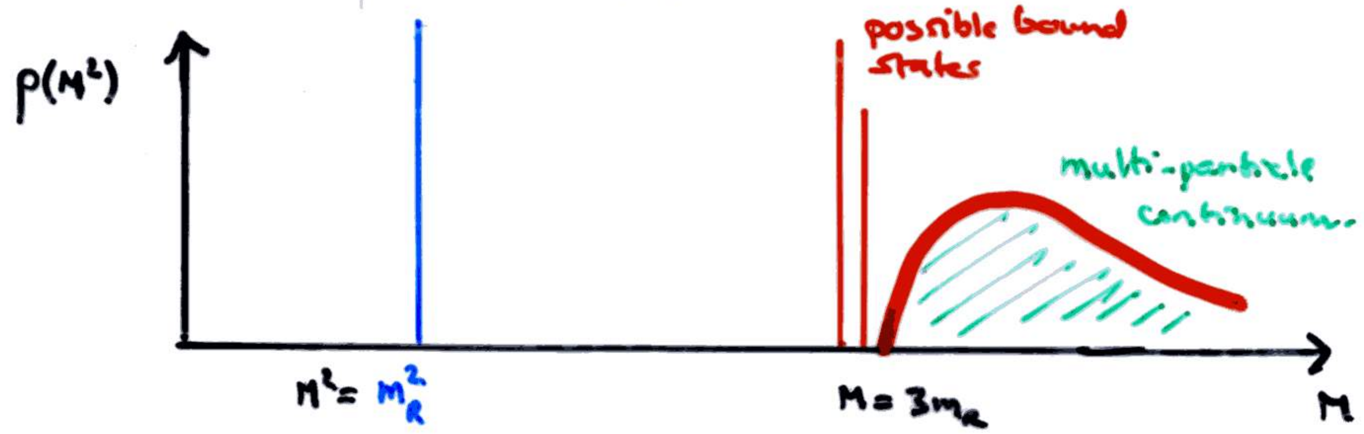
$$\rho(M^2) = \sum_{\{|\lambda_0\rangle\}} 2\pi\,\delta(M^2-m_\lambda^2)\,|\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \qquad (6.85)$$

Based on the calculations in section 6.2 we know that $\rho(M^2)$ contains

---

* $U|\lambda_p\rangle = |\lambda_0\rangle$. $\phi(0)$ and $|\Omega\rangle$ are $\mathcal{L}$-invariant!
↑ Lorentz-transformation

an isolated δ-function singularity at $M^2 = m_R^2$. In addition, the spectral function for interacting theories can contain additional singularities corresponding to bound states, and eventually multiparticle continuum (in $\lambda\phi^4$ that arises for $M^2 > (3m)^2$)



Thus

$$\rho(M^2) \simeq 2\pi\, R \cdot \delta(M^2 - m_R^2) + \left(\| + \text{▲} \right)$$

$$\Rightarrow \quad G(p^2) = \frac{iR}{p^2 - m_R^2 + i\epsilon} + \dots \qquad (6.86)$$

where, based on (6.85), the wave-function renormalization factor Z is

$$R \equiv |\langle \Omega | \phi(0) | \lambda_0^{(1)} \rangle|^2$$

int. th. vacuum

int. th. field operator.

exact 1-particle state

Clearly $R \to 1$ in the non-interacting theory. Thus R describes the projection of $\phi|\Omega\rangle$ onto 1-particle states only, out from the entire multiparticle-Fock-space that ϕ creates out from vacuum. Now, in the LSZ-reduction formalism, we started from the assumption of existence of asymptotic 1-particle states. What was meant by this was a little ambiguous however. Remember that we assumed the relation

$$\langle f|\hat{\phi}|i\rangle \longrightarrow R_{LSZ}^{1/2}\langle f|\hat{\phi}_{in}|i\rangle$$

between the interacting ($\hat{\phi}$) and noninteracting ($\hat{\phi}_{in}$) field-operators in the in-states. One can define $(R_{LSZ}^{1/2})_{fi}$ for any states fi, but the actual $Z^{1/2}$ used in reduction formalism corresponded to single particle states: $|i\rangle = |\lambda_0^{(1)}\rangle$ and $\langle f| = \langle 0|$. So we can now establish:

$$R^{1/2} \equiv \langle\Omega|\hat{\phi}(0)|\lambda_0^{(1)}\rangle \longrightarrow R_{LSZ}^{1/2}\langle 0|\hat{\phi}_{in}|\lambda_{0,free}^{(1)}\rangle = R_{LSZ}^{1/2} \tag{6.87}$$

That is, for the on-shell physical state prepared at $\pm\infty$ the field strength normalization factor is just the LSZ-correction factor.

However, when we renormalize the theory with, eg

$$\left.\frac{d\Delta^{-1}}{dp^2}\right|_{p^2=m_R^2} \equiv 1 \qquad \left(\text{and } \Delta^{-1}(m_R^2)=0\right)$$

we are setting $R=1$ as in free theory.

Another definition made in the context of LSZ-reduction concerned the definition of particle mass. It was actually defined through condition

$$m^2 \equiv p^2_{in/out}$$

where $p_{in/out}$ was the "in-" or "out-" state 4-momentum. These on the other hand were by definition physical on-shell states, so we actually took

$$m^2 \equiv m_R^2 \; ! \qquad (6.88)$$

We can now further develop the LSZ-formula (3.54): Define $\tilde{p}_i$ such that $\tilde{p}_i = p_i$ for "in-"states and $\tilde{p}_i = -q_i$ for the "out-" states. Then !?

$$_{out}\langle q_1, \dots, q_{N_f} | p_1, \dots p_{N_i} \rangle_{in} = \left[\prod_{i=1}^{N} \int d^4x_i \, e^{-i\tilde{p}_i \cdot x_i} \left(\partial^2_{x_i} + m_R^2\right) i \overset{\to 1}{R^{-1/2}} \right] G(x_1, \dots, x_N)$$

$$N = N_f + N_i$$

Z_ϕ-factors are removed when we write G using renormalized operators:

$$\langle \; | \; \rangle = \prod_{i=1}^{N} \int d^4x_i \, e^{-i\tilde{p}_i \cdot x_i} \prod_{i=1}^{N} \int \frac{d^4k_i}{(2\pi)^4} \lim_{k_i^2 \to m_R^2} -i(k_i^2 - m_R^2) \times e^{ik_i \cdot x_i} G_R(k_1, \dots, k_N)$$

$$= \prod_{i=1}^{N} \left[\lim_{\tilde{p}_i^2 \to m_R^2} -i(\tilde{p}_i^2 - m_R^2) \, G_R(\tilde{p}_i^2) \right] \cdot G_R^A(\tilde{p}_1, \dots, \tilde{p}_N)$$

In the last step the integrals over d^4k_i were first performed, each giving a factor $(2\pi)^4\,\delta^4(\tilde{p}_i - k_i)$. These then killed all d^4k_i-integrals setting $k_i \to \tilde{p}_i$ in the Greens functions. We also defined the amputated Greens function G_A through

$$G_R(k_1,\ldots,k_N) \equiv \prod_{i=1}^{N} G_R^{(2)}(k_i^2) \times G_R^A(k_1,\ldots,k_N) \qquad (6.89)$$

Finally note that only contributions from the poles of $G_R^{(2)}$-functions contribute to physical on-shell S-matrix:

$$\lim_{\tilde{p}^2 \to m_R^2} -i(\tilde{p}_i^2 - m_R^2)\, G_R(p^2) = 1 \qquad (6.90)$$

So that eventually

$$\boxed{{}_{out}\langle q_1,\ldots,q_{N_f} | p_1,\ldots,p_{N_i}\rangle_{in} = G_R^A(p_1,\ldots p_{N_i}; -q_1,\ldots -q_{N_f})} \qquad (6.91)$$

On the other hand (for $f \neq i$)

$${}_{out}\langle q_1,\ldots,q_{N_f} | p_1,\ldots,p_{N_i}\rangle_{in} = S_{fi}$$

$$\equiv -(2\pi)^4\,\delta^4\Big(\sum_i p_i - \sum_j q_j\Big)\, i\,T_{fi}$$

And furthermore, because of translational invariance

$$G_R^{A}(k_1,\dots,k_N) = (2\pi)^4\delta^4(k_1+\dots+k_N)\,\underbrace{\tilde{G}_R^{A}(k_1,\dots,k_N)}$$

a function of N-1 invariants, eg. $s_{1j} = (k_1-k_j)^2$.

So, the T-matrix element eventually is

$$\boxed{T_{fi}(\tilde{p}_1,\dots,\tilde{p}_N) = i\tilde{G}_R^{A}(\tilde{s}_{12},\dots\tilde{s}_{1N})} \qquad (6.92)$$

This is our main result for the scattering problem:

The T-matrix-element is the sum of all connected, renormalized and amputated Greens functions.*

For example $\phi\phi \to \phi\phi$:

$$\tilde{G}_R^{A}(s_{12}, s_{13}, s_{14}) = \Gamma^{(4)}_{1PI}(s,t,u) \qquad (6.93)$$

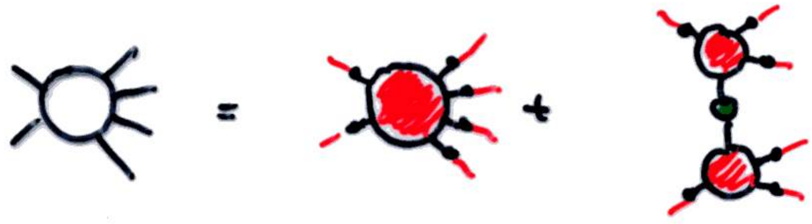
amputating

---

* Strictly speaking this assumes that the pole $p^2 = m_\pi^2$ is separated. When pole is at the end of a cut (QED) special care is needed.

Example 2: $\phi\phi \to \phi\phi\phi\phi$

$$\tilde{G}^{A}_{R} = \Gamma^{(6)}_{1PI} + \sum \Gamma^{(4)}_{1PI}\, G^{(2)}(q_i)\, \Gamma^{(4)}_{1PI} \qquad (6.94)$$



+ cross-terms

That is, we only need to compute the 1PI-functions. When these are renormalized a finite T-matrix results.

6.4. Optical theorem

We shall soon compute $\phi\phi \to \phi\phi$ scattering x-section to order λ^2. Before that let us consider an important theorem arising from the S-matrix unitarity. If we denote $S \equiv 1 - i\tilde{T}$, then

$$S^{+}S = 1 \qquad (6.95)$$

$$\Leftrightarrow \quad i(\tilde{T} - \tilde{T}^{+}) = \tilde{T}^{+}\tilde{T} \qquad (6.96)$$

Now consider the identity (6.96) in the forward scattering case: $\phi(k_1)\phi(k_2) \to \phi(k_1)\phi(k_2)$ (and return to normalization $\tilde{T} \to (2\pi)^4 \delta(\ldots)\, T$):

$$\delta^4(k_1+k_2-(k_1+k_2)) = \delta^4(0) = VT$$

$$\langle k_1 k_2 | -2\,\mathrm{Im}(\tilde{T}) \,|\, k_1 k_2\rangle = -2(VT)\,\mathrm{Im}\,T_{k_1k_2 \to k_1k_2}$$

$$= \sum_n \prod_{i=1}^{n} \int \frac{d^3 q_i}{(2\pi)^3 2E_i} \underbrace{\langle k_1 k_2 | \tilde{T}^{+} | \{q_i\}\rangle}_{} \langle \{q_i\} | \tilde{T} | k_1 k_2\rangle \qquad (6.97)$$

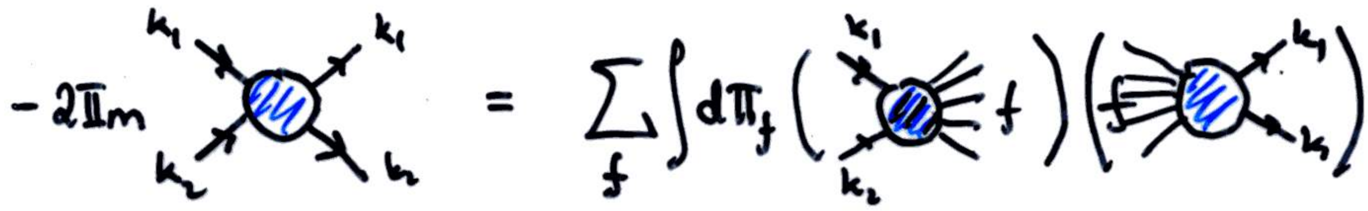
$$= VT \cdot (2\pi)^4\, \delta^4(k_1+k_2 - \textstyle\sum_i q_i)\, |T_{k_1k_2 \to \{q_i\}}|^2$$

i.e.

$$-2\,\mathrm{Im}\,T_{k_1k_2\to k_1k_2} = \sum_n \prod_{i=1}^{n} \int \frac{d^3 q_i}{(2\pi)^3 2E_i} \left| T(k_1k_2 \to \{q_i\}) \right|^2$$

$$= 4E_{cm}\,p_{cm} \sum_n \sigma(k_1k_2 \to n) \cdot S_n \qquad (6.89)$$

where σ is the cross section and the symmetry factor $S_n = n!$ corrects for the inverse S_n-factor in the def. of σ (which avoids multiple counting over identical final states in physical process).



Optical theorem: Imaginary part of the forward scattering amplitude corresponds to sum over all possible physical scattering processes.

Application: $\phi\phi \to \phi\phi$ to order λ^2.

If we compute T to order λ^2, the r.h.s. of eq. (6.98) contains only the tree-level cross-section with $n=2$ ($\phi\phi\to\phi\phi$). Then $n=2$ and $S_n = 2$, and we get

$$\sigma(\phi\phi\to\phi\phi) = -\frac{\mathrm{Im}\,\underbrace{T}_{=i\Gamma}}{2E_{cm}\,p_{cm}}\,\frac{1}{S_n} = -\frac{\mathrm{Re}\,\Gamma^{(4)}_{1PI}}{s\left(1-\frac{4m^2}{s}\right)^{1/2}} \qquad (6.99)$$

where we used eq. (6.92) and (6.93) and the usual kinematics. Now remember our equation (6.73) for the finite part of $\Gamma^{(4)}_{1PI}$ to order λ^2. Clearly $\mathrm{Re}\,\Gamma = \mathrm{Re}\,\tilde{M}(p^2)$, which is nonvanishing (and unambiguous wrt. renormalization) only when $p^2 > 4m^2$, giving (physical region in s-channel)

$$\mathrm{Re}\,\Gamma^{(4)}_{1PI} = -\frac{\lambda^2}{32\pi}\left(1-\frac{4m^2}{s}\right)^{1/2}\Theta(s-4m^2) \qquad (6.100)$$

Putting this back to (6.99) we get the correct result (see eq. 3.108)

$$\underline{\sigma = \frac{\lambda^2}{32\pi s}} \, . \qquad (6.101)$$

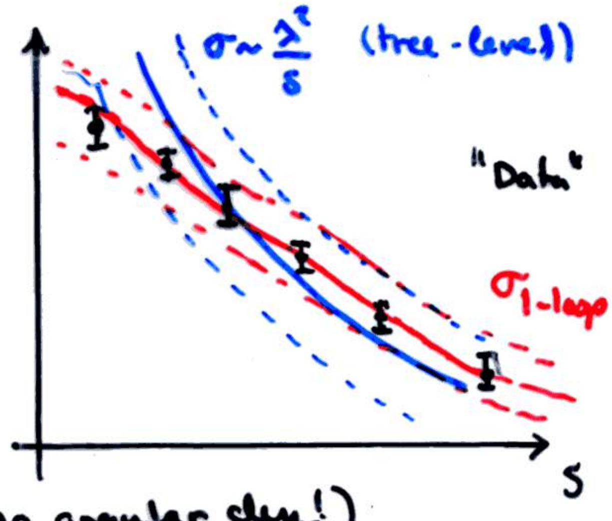
6.5 Scattering $\phi\phi \to \phi\phi$ to order λ^3.

This calculation gives a concrete example of how we get finite, physical correction to observable quantities from apparently divergent PT. We have already shown that

$$\frac{d\sigma}{d\Omega} = \frac{|\Gamma^{(4)}_{1PI}|^2}{64\pi^2 s}\frac{1}{S_2} \qquad (6.102)$$

- Lowest order PT-result: $\Gamma^{(4)}_{1PI} = \lambda$, so

$$\left.\frac{d\sigma}{d\Omega}\right|_{tree} = \frac{\lambda^2}{64\pi^2 s}\cdot\frac{1}{2} \Rightarrow \sigma = \frac{\lambda^2}{32\pi s} \, . \qquad (6.103)$$



We can imagine measuring σ as a function of s. (and also as a function of t (or $\cos\theta$)) Tree level perturbative result predicts $\sigma \sim \lambda^2/s$ with a free parameter λ. (and $\frac{d\sigma}{d\Omega} \sim 1/s$ with no angular dep.!)

We can define λ by fitting at one point. Obviously a good choice would be to adjust λ so as to get best possible fit to the data. However our data might show more complicated s-dependence, as illustrated in the Figure in prev. page (this is not real data or real calculation, but a mere illustration). To improve our prediction we can try to compute $\Gamma^{(4)}_{1PI}$ to second order:

$\tilde{\Gamma} \to 0$ at chosen renormalization point defining λ_R!

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{|\lambda_R + i\tilde{\Gamma}|^2}{64\pi^2 s} \simeq \frac{\lambda_R^2}{128\pi^2 s}\left(1 + \frac{2i\tilde{\Gamma}}{\lambda_R}\right)$$

f vanishes at $p^2=0$

$$= \frac{\lambda_R^2}{128\pi^2 s}\left(1 + \frac{\lambda_R}{16\pi^2}\sum_{s,t,u} f(p^2)\right) \qquad (6.104)$$

where the finite function $f(p^2)$ is defined in eq. (6.73).

- (6.104) predicts more complicated s-dependence
- — " — nontrivial angular dependence
- (6.104) is <u>finite</u>.
- (6.104) is <u>not</u> unique.....

in the sense that <u>the definition of λ_R is not unique.</u> What does this mean? Consider two different possible ways of defining λ_R: In (6.104) we took in fact

$$\lambda_R \equiv \Gamma^{(4)}_{1PI}(0,0,0)$$

Equally well we could have defined

$$\lambda_R' = \Gamma^{(4)}_{1PI}(s_0, t_0, u_0) \qquad (6.105)$$

Clearly

$$\lambda_R' = \lambda_R + \underbrace{\Gamma^{(4)}_{1PI}(s_0,t_0,u_0)}_{\text{infinite (ambiguous)}} - \underbrace{\Gamma^{(4)}_{1PI}(0,0,0)}_{\text{infinite (ambiguous)}} = \underbrace{\lambda_R + \delta\Gamma}_{\text{finite, unambiguous}} \qquad (6.106)$$

$$\delta\tilde{\Gamma} = \delta\Gamma$$

So, rewriting (6.104) using the new parameter λ_R' we would get

$$\frac{d\sigma}{d\Omega} = \frac{\lambda_R'^2}{128\pi^2 s}\left(1 + \frac{\lambda_R'}{16\pi^2}\sum_{s,t,u}\underbrace{\left(f(p^2) - f(p_0^2)\right)}_{\text{vanishes at } p^2=p_0^2}\right) \qquad (6.107)$$

instead of (6.104). This is clearly different from (6.104). However, it

- gives the same parametrical dependence on s and $\cos\theta$ as (6.104).
- it differs numerically from (6.104) only to order $\sim\lambda_R^4$. (which is beyond the order we are calculating to, however!)

SUMMARY.

We've seen that renormalization program removes divergences (to all orders) and their only consequence is the "freedom of choice" in defining the physical parameters. (renormalization point). The numerical effect of these choices is always of higher order, however. $\Rightarrow$ PT well defined.

6.6 Unstable particles

Let us finally use the optical theorem to define a decay width for unstable particles. Early in the course we derived Γ analogously to the derivation of σ. An obvious problem in such an in-out-definition is that we strictly speaking do not have well defined asymptotical states for unstable particles! Here optical theorem gives some improvement.

Consider a propagator


$$= \frac{i}{p^2 - m_0^2 - \Pi(p^2)} \qquad (6.108)$$

We earlier implicitly assumed that $\Pi \in \mathbb{R}$. This is not necessarily the case. In reality $\Pi(p^2)$ picks up an imaginary part for those p^2 for which the state can decay into some other states. If this occurs already for $p^2 = m_R^2$, the physical state is unstable. In this case the pole-mass condition becomes

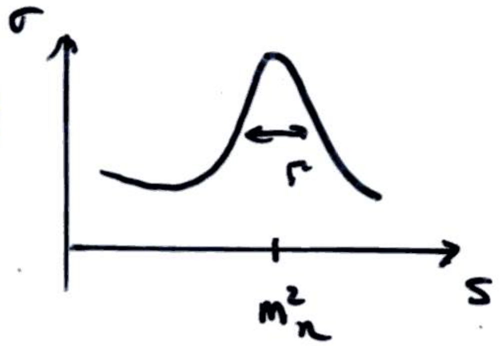
$$m_R^2 - m_0^2 + \mathrm{Re}\,\Pi(m_R^2) = 0 \qquad (6.109)$$

but the pole is not along real axis: $\mathrm{Im}(m_R^2) \neq 0$.


$$\simeq \frac{Z_\phi}{p^2 - m_R^2 - Z_\phi\, i\, \mathrm{Im}\,\Pi(p^2)} \qquad (6.110)$$

close to the pole. When this propagator is used for a cross-section computation in s-channel it predicts the Breit-Wigner shape:

$$\sigma \sim \left| \frac{1}{s - m_R^2 + i m \Gamma} \right|^2 \qquad (6.111)$$



where the decay width

$$\boxed{\Gamma = -\frac{Z_R}{m} \, \mathrm{Im}\, \Pi(m_R^2)} \qquad (6.112)$$

We could compute Γ from the imaginary part of Π-function (and one often does.). However, by optical theorem we can connect (6.112) to our earlier in-out-result. Indeed, by applying (6.96) to 'scattering' $\phi(k) \to \phi(k)$ we get $\left(\Pi(p^2) = i\Gamma^{(2)}_{1PI}\right)$

$$\Gamma = \frac{1}{m} \mathrm{Im}\, \Pi_R(m^2) = \frac{1}{m} \mathrm{Im}\, T(k \to k)$$

$$= \frac{1}{2m} \sum_n \prod_{i=1}^{n} \int \frac{d^3 p_i}{(2\pi)^3 2E_i} \left| T_{k \to \{q_i\}} \right|^2 . \qquad (6.113)$$

This is already the familiar form. The expression strictly speaking holds for $\Gamma \ll m$ (so $\mathrm{Im}\, \Pi \ll m^2$), which allowed to set $\Pi(p^2) \to \Pi(m^2)$ near pole in (6.110). [This is essentially the same condition that allows defining approximate asymptotic states.]

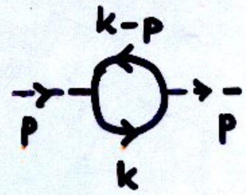
$\phi \to \bar{\Psi}\psi$ in Yukawa theory

Let us consider the ϕ-propagator as an example in the Yukawa theory with

$$m_\phi > 2 m_\psi \qquad (6.114)$$

Lasketaan ϕ-kentän Π-funktio:

$$-i\Pi(p^2) =$$



$$= -(-ig)^2 \cdot \mu^{4-D} \cdot \int \frac{d^D k}{(2\pi)^D} \frac{i^2\, \mathrm{Tr}\left((\not k + m_\psi)((\not k - \not p) + m_\psi)\right)}{(k^2 - m_\psi^2)((k-p)^2 - m_\psi^2)}$$

$$= -\frac{g^2}{8\pi^2} \cdot \left(\frac{\mu^2}{4\pi}\right)^{\frac{D-4}{2}} \cdot \frac{i \cdot 4}{\Gamma(\frac{D}{2})} \int dk_E\, k_E^{D-1} \frac{-k_E \cdot (k-p)_E + m_\psi^2}{(k_E^2 + m_\psi^2)((k-p)_E^2 + m_\psi^2)} \qquad (6.115)$$

Tässä käytimme tuloksia

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad (6.116)$$

$$\mathrm{Tr}[1] \equiv 4. \qquad (6.117)$$

Ainoa asia joka "Dirakologiaan" liittyen riippuu D:stä on tensori $g^{\mu\nu}$. Niinpä

$$g^\mu{}_\mu = D \qquad (6.118)$$

ja sen johdosta mm.

$$\gamma^\mu \gamma^\nu \gamma_\mu = (2-D)\gamma^\nu$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4g^{\nu\rho} - (4-D)\gamma^\nu \gamma^\rho$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu + (4-D)\gamma^\nu \gamma^\rho \gamma^\sigma \qquad (6.119)$$

Edelleen, symmetrisessä integraalissa

$$\int d^D l\, f(l^2)\, \underline{\underline{l_\mu l_\nu}} = \int d^D l\, f(l^2)\, \underline{\underline{\frac{1}{d} g^{\mu\nu} l^2}} \qquad (6.120)$$

Tehdään nyt Feynmanin parametrisaatio täsmälleen kuten kaavan (6.68) yhteydessä:

$$\frac{1}{(k^2+m_\psi^2)((k-p)^2+m_\psi^2)} = \int_0^1 dz \frac{1}{\left((k-zp)^2+m^2+z(1-z)p^2\right)^2}$$

Muuttujanvaihdon $k = k' + zp$ jälkeen (6.115):n osoittajassa

$$k\cdot(k-p) = (k'+zp)\cdot(k'-(1-z)p)$$

$$= k'^2 + (2z-1)\, k' p - z(1-z)p^2$$

pariton termi → 0

ja saamme lopulta:

$$-i\Pi(p^2) = + \frac{ig^2}{4\pi^2}\left(\frac{\mu^2}{4\pi}\right)^{4-D} \frac{1}{\Gamma(\frac{D}{2})} \int_0^1 dz \int dk_E^2 \frac{(k_E^2)^{\frac{D}{2}-1}(k_E^2-a_z^2)}{(k_E^2+a_z^2)^2} \qquad (6.121)$$

missä $a_z^2 \equiv m^2 + z(1-z)p_E^2$. Integraalit on taas helppo suorittaa:

$$\int dk_E^2 \cdots = (a_z^2)^{\frac{D}{2}-1}\Bigg(\underbrace{\frac{\Gamma(\frac{D}{2}+1)\Gamma(1-\frac{D}{2})}{\Gamma(2)} - \frac{\Gamma(\frac{D}{2})\Gamma(2-\frac{D}{2})}{\Gamma(2)}}_{} \Bigg)$$

$$= \Gamma(\tfrac{D}{2})\,\Gamma(1-\tfrac{D}{2})\left(\tfrac{D}{2}-(1-\tfrac{D}{2})\right) = (D-1)\,\Gamma(\tfrac{D}{2})\,\Gamma(1-\tfrac{D}{2})$$

joten

$$-i\Pi(p^2) = -\frac{ig^2}{4\pi^2}\left(\frac{\mu^2}{4\pi}\right)^{\frac{4-D}{2}} (D-1)\,\Gamma\left(1-\frac{D}{2}\right) \int_0^1 dz\, \left(m^2 - z(1-z)p_M^2\right)^{\frac{D}{2}-1} \qquad (6.122)$$

Nyt:

$$\begin{aligned}
\Gamma\left(1-\frac{D}{2}\right) &\simeq -\frac{2}{\epsilon} + \gamma_E - 1 \\
D-1 &\simeq 3-\epsilon = 3\left(1-\frac{2}{3}\cdot\frac{\epsilon}{2}\right) \\
\left(\frac{\mu^2}{4\pi}\right)^{\frac{4-D}{2}} &\simeq 1 - \frac{\epsilon}{2}\ln\left(\frac{\mu^2}{4\pi}\right) \\
(a_z^2)^{\frac{D}{2}-1} &\simeq a_z^2\left(1+\frac{\epsilon}{2}\ln a_z^2\right)
\end{aligned} \qquad (6.123)$$

joten

$$-i\Pi(p^2) \simeq \frac{-3ig^2}{4\pi^2} \times \left(\frac{2}{\epsilon} - \gamma_E + 1\right)\left(1-\frac{2}{3}\frac{\epsilon}{2}\right)\left(1+\frac{\epsilon}{2}\ln 4\pi\right) \times \int_0^1 dz\, a_z^2\left(1+\frac{\epsilon}{2}\ln\frac{a_z^2}{\mu^2}\right)$$

$$= -\frac{3ig^2}{4\pi^2}\left\{\left(\frac{2}{\epsilon} - \gamma_E + \frac{1}{3} + \ln 4\pi\right)\int_0^1 dz\, a_z^2 + \int_0^1 dz\, a_z^2 \ln\frac{a_z^2}{\mu^2}\right\} \qquad (6.124)$$

On helppo osoittaa että

$$\int_0^1 dz\, a_z^2 = -\frac{1}{6}p_M^2 + m_+^2. \qquad (6.130)$$

Log-integraali on työläämpi yleisessä tapauksessa. Kun $p^2=0$ on sekin helppo:

$$\int_0^1 dz\, a_z^2 \ln\frac{a_z^2}{\mu^2} \longrightarrow m_\phi^2 \log\frac{m_\phi^2}{\mu^2}\,. \qquad (6.131)$$

Renormalisoidaan teoria $p^2 = 0$:ssa. Tällöin

$$\Pi(p^2) = \Pi(0) + \Pi'(0)\,p^2 + \tilde{\Pi}(p^2) \qquad (6.132)$$

missä

$$\Pi(0) = \frac{3g^2}{4\pi^2} m_\phi^2 \cdot \left\{ \frac{2}{\epsilon} - \gamma_E + \frac{1}{3} + \ln 4\pi + \ln\frac{m_\phi^2}{\mu^2} \right\} \qquad (6.133)$$

$$\Pi'(0) = -\frac{3g^2}{4\pi^2}\,\frac{1}{6} \left\{ \frac{2}{\epsilon} - \gamma_E + \frac{1}{3} + \ln 4\pi + \ln\frac{m_\phi^2}{\mu^2} + 1 \right\} \qquad (6.134)$$

$$\tilde{\Pi}(p^2) = \left(\int_0^1 dz\, a_z^2 \ln\frac{a_z^2}{m_\phi^2} + \frac{p_M^2}{6} \right) \cdot \frac{3g^2}{4\pi^2} \qquad (6.135)$$

Tarvitaan siis vastatermi-Lagrangen funktio

$$\Delta\mathcal{L} = \frac{\Pi(0)}{2}\phi^2 + \frac{\Pi'(0)}{2}(\partial_\mu\phi)^2, \quad \hat{=} \qquad (6.136)$$

minkä jälkeen

$-i\,\Pi_R(p^2) =$  $+$  $= -i\,\tilde{\Pi}(p^2),$ (6.137)

ja siis

$$G_{\phi R}(p^2) = \frac{i}{p^2 - m_\phi^2 - \tilde{\Pi}(p^2)} \qquad (6.138)$$

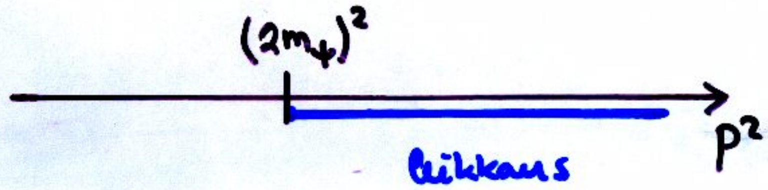
missä siis:

$-i\epsilon$

$$\tilde{\Pi}(p^2) = \frac{3g^2}{4\pi^2}\left(\frac{p^2}{6} + \int_0^1 dz \left(m_\psi^2 - z(1-z)p^2\right)\log\left(1 - z(1-z)\frac{p^2}{m_\psi^2}\right)\right) \qquad (6.139)$$

Tämä on tietysti äärellinen, ja voitaisiin laskea suljetussa muodossa. Tarkastellaan tässä vain $\tilde{\Pi}(p^2)$:in imaginääriosaa. Koska $z(1-z) \leq 1/4$, on $\mathrm{Im}\,\tilde{\Pi}(p^2) \neq 0$ kun $p^2 \geq 4m_\psi^2$. Tällöin $(m_\psi^2 \to m_\psi^2 - i\epsilon)$

$$\log\left(m^2 - z(1-z)p^2 - i\epsilon\right) = \log|\quad| + i\pi\,\theta\left(z(1-z)p^2 - m^2\right) \qquad (6.140)$$



joten:

$$\mathrm{Im}\,\tilde{\Pi}(p^2) = \frac{3g^2}{4\pi}\int_0^1 dz \left(m^2 - z(1-z)p^2\right)\theta\left(z(1-z)p^2 - m^2\right)$$

Kun $p^2 \leq 4m_\psi^2$ on tämä selvästi nolla. Kun $p^2 > 4m_\psi^2$, on integraalin uudet rajat

$$\frac{1}{2}\left(1 - \sqrt{1 - \frac{4m^2}{p^2}}\right) < z < \frac{1}{2}\left(1 + \sqrt{1 - \frac{4m^2}{p^2}}\right)$$

joten pienen laskun jälkeen:

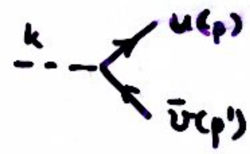
$$\mathrm{Im}\,\tilde{\Pi}(p^2) = -\frac{g^2 p^2}{8\pi}\left(1 - \frac{4m^2}{p^2}\right)^{3/2}\theta\left(1 - \frac{4m^2}{p^2}\right)$$

ja siten

$$\Gamma_{\phi\to\psi\psi} = \frac{g^2 m_\phi}{8\pi}\left(1-\frac{4m_\psi^2}{m_\phi^2}\right)^{3/2}$$

missä lopuksi asetettiin $p^2 = m_\phi^2$. Tämä on tietenkin juuri ϕ:n hajoamisnopeus kahteen fermioniin Yukawa teoriassa.

$$\mathcal{M} = g\,\bar{v}(p')\,u(p)$$



$$\Rightarrow \sum_{\text{spin}} |\mathcal{M}|^2 = g^2 \cdot 4(p\cdot p' - m_\psi^2)$$

$$= 2g^2(s - 4m_\psi^2) \quad ; s = m_\phi^2$$

$$\Rightarrow \Gamma = \frac{|\mathcal{M}|^2}{16\pi m_\phi^3}\,\lambda(m_\phi^2, m_\psi^2, m_\psi^2) \quad = m_\phi\sqrt{m_\phi^2 - 4m_\psi^2}$$

$$= \frac{g^2}{8\pi} m_\phi \left(1 - \frac{4m_\psi^2}{m_\phi^2}\right)^{3/2}$$

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