5. PATH INTEGRALS

PI provides an alternative, in many ways superior way to quantize field theories. (Feynman -48) It's advantages over canonical quantization are

1) Simple, covariant way to quantize complicated systems (gauge theories).

2) Intuitive, compared to canonical quantization

3) Also suitable for nonperturbative considerations

4) Based on c-numbers instead of operators (easier to manipulate)

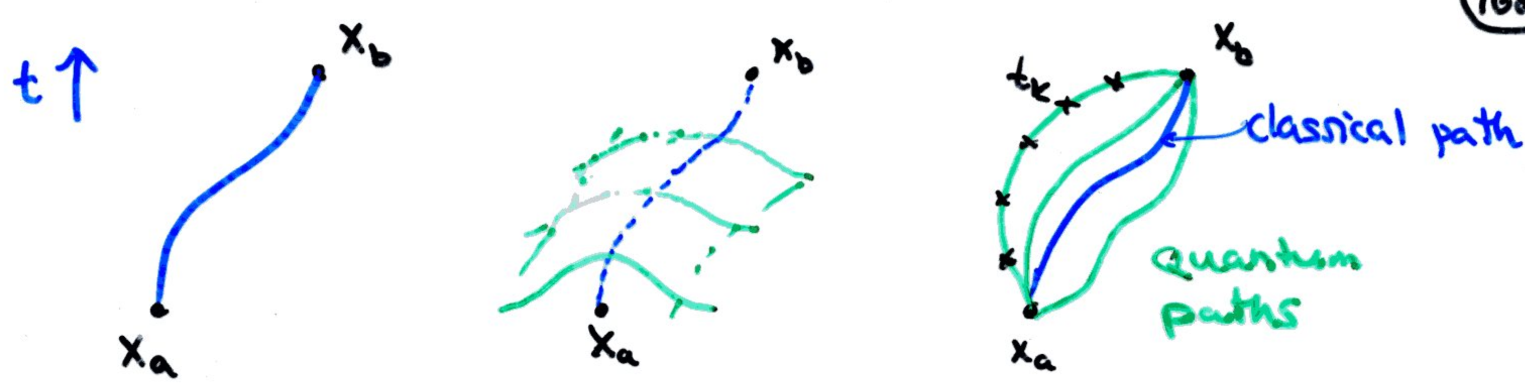
Many details of renormalization theory are easier to understand from path integral point of view (for example anomalies and Ward identities).

We will introduce path integrals through simple examples and using nonrelativistic quantum mechanics as example.

Moreover, I will introduce PI thru. postulates and derive its connection to operator formalism later (KAKU).

5.1 Non-relativistic quantum mechanics and P.I.

Fundamental concept in both classical and quantum physics is a path of motion of a particle from point x_a to point x_b.



Classical path | Schrödinger | Path integral

$m\ddot{x}_a = -\nabla V$ | $\psi(\vec{x},t)$

The deterministic path of classical physics is replaced by the (deterministic) evolution of prob. wave function. In class. physics the path is deterministic even when only the end positions were measured. In qm. the path is not deterministic. Performing a series of gedanken experiments (or real measurements for a large number of identically prepared systems) at times $t_1, \dots, t_N \in [t_a, t_b]$, we can visualize how a qm. particle can follow many different paths between points x_a and x_b. This idea is exactly what lies behind path integral formalism. Let us now make these ideas into specific mathematical postulates:

- Transition probability from point x_a to x_b

$$P(a,b) = |K(a,b)|^2 \qquad (5.1)$$

where transition function $K(a,b)$ is a linear superposition

← classical action

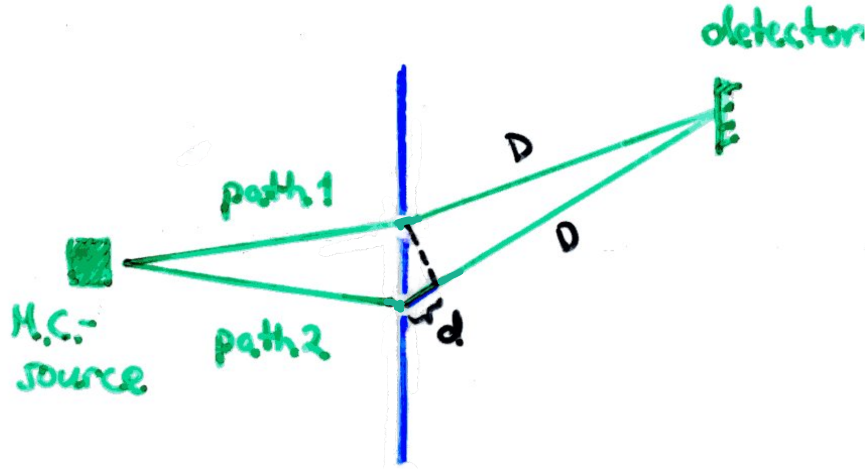
$$K(a,b) = \sum_{\forall \text{ paths}} k\, e^{iS/\hbar} \qquad (5.2)$$

The constant k is set by a natural condition

$$K(c,a) = \sum_{\text{paths}} K(c,b)\, K(b,a) \qquad (5.3)$$

where the sum is over all intermediate points at time $t_b \in [t_a, t_c]$ and all paths connecting those points to x_a and x_c.

- Two-slit experiment



Now $S = \int L\, dt$

$$= \int \frac{1}{2} m v^2 dt$$

$d \ll D$

$$\Rightarrow \quad \text{Path 1.} \quad S_1 = \frac{1}{2} m v_1^2 t = \frac{1}{2} \frac{m D^2}{\hbar t}$$

$$\text{Path 2.} \quad S_2 = \frac{1}{2} m v_2^2 t = \frac{1}{2} \frac{m (D+d)^2}{\hbar t}$$

$$\Rightarrow \Delta S \approx \frac{m D d}{\hbar t} \approx \frac{p}{\hbar} d = 2\pi \frac{d}{\lambda}$$

deBroglie wave interpret.

$\Rightarrow$ Interference maxima when $d = 0, \lambda, 2\lambda, \ldots$

— " — minima — " — $d = \frac{\lambda}{2}, 3\lambda/2, \ldots$

- Correspondence

functional derivative (formally equiv. with ordinary $\frac{d}{dx}$)

The action S reaches its extremal on classical path, i.e.

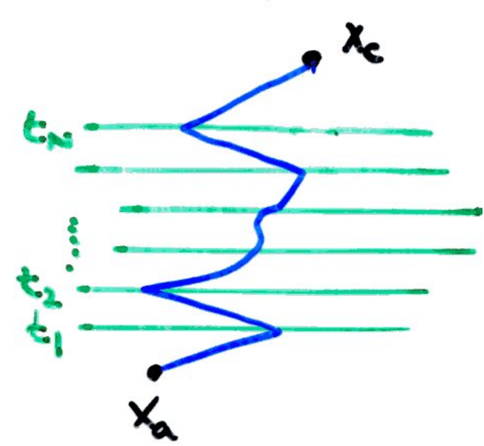
$$\frac{\delta}{\delta x(t)} S[x(t)]_{cl} = 0 \qquad (5.4)$$

In the limit $\hbar \to 0$ even the smallest deviation from classical path causes a large change in the phase $\delta S/\hbar \gg 1$. Such contributions tend to sum to zero in the expression for $K(b,a)$ in (5,2) (rapidly oscillating integral ≈ 0). That is

$$\text{P.I} + \hbar \longrightarrow 0 \quad \hat{=} \quad \text{Classical physics (Lagrange formalism)}$$

Path integral $\hat{=}$ Hamilton e.o.m

The path integral measure in (5,2) can be found by discretizing the paths.



$$\sum_{\text{paths}} = \lim_{N\to\infty} k \prod_{i=1}^{N-1} \int dx_i \, k_i \equiv \int \mathcal{D}x \qquad (5{,}5)$$

$\delta t_i \equiv \epsilon$

Let us now consider:

$$S = \int dt \, L = \int dt \left(\tfrac{1}{2} m \dot{x}^2 - V(x) \right)$$

$$\longrightarrow \sum_k \left(\frac{m}{2} \frac{(x_{k+1}-x_k)^2}{\epsilon} - \epsilon V\left(\frac{x_{k+1}+x_k}{2}\right) \right) \qquad (5{,}6)$$

Let us now apply eqn. (5,3) to the case where $x_b = x_{N-1} \equiv x'$ and the last time step has a length ϵ. Clearly

$$K(x_c, x_a; T) \equiv \int_{-\infty}^{\infty} dx' \, k \, e^{\frac{i}{\hbar}\left(\frac{m}{2}\frac{(x_c - x')^2}{\epsilon} - \epsilon V\left(\frac{x_c + x'}{2}\right)\right)} \underbrace{K(x', x_a; T-\epsilon)}_{\text{expand in series around } x_c} \qquad (5{,}7)$$

$x_c - x' \approx \sqrt{\epsilon}$

$$\simeq \int_{-\infty}^{\infty} dx' \, k_\epsilon \, e^{\frac{im}{2\hbar\epsilon}(x_c - x')^2} \left[1 - \frac{i\epsilon}{\hbar} V(x_c) + \dots \right]$$

$$\times \left(1 + \underbrace{\cancel{(x' - x_c)\frac{\partial}{\partial x_c}}}_{0 \text{ (odd)}} + \frac{1}{2!}\underbrace{(x'-x_c)^2}_{\approx O(\epsilon)} \frac{\partial^2}{\partial x_c^2} + \dots \right) K(x_c, x_a; T-\epsilon) \qquad (5.8)$$

Convergence of the Gaussian integrals here requires changing the definition of K such that T gets a small complex part (see. eq. (3.99) on p. 108).

$$\underline{K(b,a;T) \longrightarrow K(b,a;T(1-i\delta))} \qquad (5.9)$$

This changes $\epsilon \to \epsilon - i\delta'$ ($\delta' = \delta/\omega$). Now the integrals in (5.8) converge as $x \to \pm\infty$, and we can use the results

$$\int_{-\infty}^{\infty} dy \, y^{2n} e^{-by^2} = (-1)^n \partial_b^n \sqrt{\frac{\pi}{b}}$$

$$\int_{-\infty}^{\infty} dy \, y^{2n+1} e^{-by^2} = 0 \qquad (5.10)$$

With these results one easily gets:

$$K(x_c, x_a; T) = k_\epsilon \sqrt{\frac{i2\pi\hbar\epsilon}{m}} \left(1 - \frac{i\epsilon}{\hbar} V(x_c) + \frac{i\epsilon\hbar}{2m} \frac{\partial^2}{\partial x_c^2} \overbrace{}^{+O(\epsilon^2)} \right) K(x_c, x_a; T-\epsilon) \qquad (5.11)$$

(lowest order)

- when $\epsilon \to 0$: LH = RH $\Rightarrow$ $\underline{k_\epsilon = \sqrt{\frac{m}{2\pi\hbar\epsilon i}}}$ (5.12)

Putting this back to (5.11) we get

$$\boxed{i\hbar \frac{\partial}{\partial T} K(c,a) = \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_c^2} + V(x_c) \right) K(c,a)} \qquad (5.13)$$

furthermore

$$\lim_{x_a \to x_b} K(b,a) = \lim_{\epsilon \to 0} k_\epsilon \, e^{\frac{i}{\hbar}\frac{m(x_b-x_a)^2}{2\epsilon}} = \delta(x_b - x_a) \qquad (5,14)$$

Derivation of Schrödinger equation from (5.1–5.3).

In path integral formalism the transition amplitude $K(b,a)$ tells also the evolution of the wave function, or the state vector. According to Huygens principle

$$\psi(x_j, t_j) = \int_{-\infty}^{\infty} dx_i \; K(x_j, t_j \, ; x_i, t_i) \, \psi(x_i, t_i) \qquad (5,15)$$

(The amplitude at x_j (at t_j) is the sum (integral) of amplitudes transported from point x_i at t_i). Choosing again $t_i = t$ and $t_j = t + \epsilon$, and using the infinitesimal form for K we get:

$$\psi(x, t+\epsilon) \approx \int_{-\infty}^{\infty} dy \, k_\epsilon \, e^{-\frac{im}{2\hbar\epsilon}(x-y)^2} \left(1 - i\frac{\epsilon}{\hbar} V(x)\right)$$

$$\times \left(1 + \frac{1}{2}(x-y)^2 \frac{\partial^2}{\partial x^2}\right) \psi(x,t)$$

$$\overset{(5,12)}{\Rightarrow} \quad \approx \psi(x,t) - i\frac{\epsilon}{\hbar}\left(V(x) - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi(x,t) + \mathcal{O}(\epsilon^2)$$

$$\Rightarrow \quad \underline{i\hbar\,\partial_t \psi = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi} \qquad (5,16)$$

- This indeed is a derived eqn. So P.I. allows deriving S.E. from general principles of qm: (5.1–5.3).

Lagrange - Hamilton formalism

A simple c-number relation

$$\sqrt{\frac{ia}{\pi}} \int dp \; e^{iap^2 + ibp} = e^{-ib^2/4a} \qquad (5,17)$$

makes it possible to express the $\frac{1}{2}m\dot{x}^2$-term using variables $\dot{x}$ and p. Insert the identity (5,17) to each discrete integral in the discretized form of $K(b,a)$, using $a = 1/2m\hbar$ and $b = \dot{x}_i$. We get

$$K(b,a) = \int_{x_a}^{x_b} \mathcal{D}x \; e^{\frac{i}{\hbar}\int_{t_a}^{t_b} dt \underbrace{\left(\frac{1}{2}m\dot{x}^2 - V(x)\right)}_{L(x,\dot{x})}}$$

$$= \int_{x_a}^{x_b} \mathcal{D}x \mathcal{D}p \; e^{\frac{i}{\hbar}\int dt \underbrace{\left(p\dot{x} - \frac{p^2}{2m} - V(x)\right)}_{= L(x,p)}} \qquad (5,18)$$

$(L = p\dot{x} - H)$

Moving from Lagrange formalism to the Hamilton formalism with twice the number of d.o.f, goes through a simple identity.

Operator picture

In the operator formalism (Heisenberg picture), the transition amplitude is represented by the time evolution operator

$$U(x_b, x_a; T) \equiv \langle x_b | e^{-i\hat{H}T} | x_a \rangle \quad ; \; T = t_b - t_a$$

$$= \langle x_b ; t_b | x_a ; t_a \rangle \qquad (5,19)$$

where $|x\rangle$ is the eigenstate of the position operator: $\hat{x}|x\rangle = x|x\rangle$.

The equivalence of $K(b,a)$ and $U(b,a)$ is actually clear already based (5,13) and (5,14); they both obey the same e.o.m with same boundary condition. We can see the connection also using (5,18). Indeed:

$$K(b,a) = \int \mathcal{D}x\, \mathcal{D}p\, e^{\frac{i}{\hbar}\int dt\,(p\dot{x} - H(x,p))}$$

$$= \lim_{N\to\infty} \prod_{i=1}^{N} \int \frac{dx_i\, dp_i}{2\pi\hbar}\, e^{-\frac{i}{\hbar}\left(H(x,p)_i\,\epsilon\, -\, p_i\,(x_{i+1}-x_i)\right)} \qquad (5,20)$$

$x_0 \equiv x_a$

$x_{N+1} \equiv x_b$

$$= e^{-\frac{i}{\hbar}H(x,p)_i\epsilon}\, e^{\frac{i}{\hbar}p_i\overbrace{(x_{i+1}-x_i)}^{=r}}$$

$$= e^{-\frac{i}{\hbar}\hat{H}(\bar{x}_i,\, -i\partial_{r_i})}\, e^{+\frac{i}{\hbar}p_i r_i}$$

$\equiv \frac{x_{i+1}+x_i}{2}$

operator

$-i\partial_r e^{irp} = p e^{irp}$

Non-relativistic norms:

$$\langle x|y\rangle = \int \frac{dp}{2\pi\hbar} \underbrace{\langle x|p\rangle}_{=e^{ip\cdot x}} \langle p|y\rangle \equiv \delta(x-y) \qquad (5,21)$$

no index

$$\Rightarrow \int \frac{dp_i}{2\pi\hbar}\, e^{-\frac{i}{\hbar}\hat{H}(\bar{x}_i,\, -i\partial_{r_i})\epsilon}\, e^{\frac{i}{\hbar}p_i(x_{i+1}-x_i)} = \langle x_{i+1}| e^{-\frac{i}{\hbar}\hat{H}\epsilon} |x_i\rangle \qquad (5,22)$$

Note that in the Hamiltonian form the k_ϵ-factor and the $\sqrt{\frac{ia_0}{\pi}} = \sqrt{\frac{i\epsilon}{2m\hbar\pi}}$ -factor from $\mathcal{D}p$ combine to

$$k_\epsilon\sqrt{\frac{ia_0}{\pi}} = \sqrt{\frac{m}{2\pi\hbar\epsilon i}} \cdot \sqrt{\frac{i\epsilon}{2m\hbar\pi}} = \frac{1}{2\pi\hbar}\ ;$$

When (5,22) is inserted into (5,20) we get

$$K(b,a) = \lim_{N\to\infty} \prod_{i=1}^{N} \int dx_i \, \langle x_{i+1} | e^{-\frac{i}{\hbar}\hat{H}\epsilon} | x_i \rangle \quad ; \; x_{N+1} \equiv x_b, \; x_0 \equiv x_a$$

$$= \lim_{N\to\infty} \langle x_b | e^{-\frac{i}{\hbar}\hat{H}\epsilon} \underbrace{\int dx_N |x_N\rangle\langle x_N|}_{=1} e^{-\frac{i}{\hbar}\hat{H}\epsilon} \dots |x_a\rangle$$

$$= \underline{\langle x_b | e^{-\frac{i}{\hbar}\hat{H}T} | x_a \rangle} \quad \square \qquad (5,23)$$

In arriving to this form, we actually (had to) make the following identifications

obvious

not so..

$$\hat{p} \longrightarrow -i\overleftrightarrow{\partial}_x$$

$$V(\hat{x}) \equiv \lim_{x\to y} V\left(\frac{x+y}{2}\right) \qquad (5,24)$$

Where the derivative has a − sign when operating to the left. This is natural since $\langle x| i\overleftarrow{\partial}_x = (-i\partial_x |x\rangle)^*$. More deeply, these choices have to do with the fundamental ambiguity with operator ordering in qm. Indeed

$$q^2p^2 \longrightarrow \hat{q}^2\hat{p}^2 , \; \hat{q}\hat{p}^2\hat{q} , \; \hat{p}^2\hat{q}^2 \; ? \qquad (5,25)$$

classical quantum mechanics

Because

$$[x, \partial_r] = \left[\frac{x_{i+1}+x_i}{2}, \partial_{x_{i+1}-x_i}\right] = 0 \qquad (5,26)$$

The prescription

$$H \equiv H(x, -i\partial_r) \qquad (5,27)$$

Always implies

$$\langle x_{i+1} | \hat{H}(\hat{x}, -i\partial_r) | x_i \rangle \longrightarrow H\left(\frac{x_{i+1}+x_i}{2}, p_i\right) \qquad (5,28)$$

However we wish to express our results in terms of conjugate variables (x and ∂_r are not) $\hat{x}$ and $\hat{p}$. One can show that the result

↑normal, not averaged.

$$\langle x_{i+1} | \hat{H}(\hat{x}, \hat{p}) | x_i \rangle \longrightarrow H\left(\frac{x_{i+1}+x_i}{2}, p_i\right) \qquad (5,29)$$

is true only if the operators $\hat{x}$ and $\hat{p}$ are Weyl ordered in $\hat{H}(\hat{x},\hat{p})$. This means that operator $\hat{x}$ is maximally symmetrized wrt $\hat{p}$: For example

$$\begin{aligned} \mathrm{Weyl}(\hat{x}\hat{p}^n) &= \tfrac{1}{2}(\hat{x}\hat{p}^n + \hat{p}^n\hat{x}) \\ \mathrm{Weyl}(\hat{x}^2\hat{p}^n) &= \tfrac{1}{4}(\hat{x}^2\hat{p}^n + 2\hat{x}\hat{p}^n\hat{x} + \hat{p}^n\hat{x}^2) \end{aligned} \qquad (5,30)$$

Operator time ordering

Another operator ordering relation between the operator- and PI-formalisms concerns the time ordering. Consider the following matrix element

$$\langle x_n, t_n | \hat{x}(t_j)\hat{x}(t_k) | x_1, t_1 \rangle \qquad (5,31)$$

where $t_n > t_j > t_k > t_1$. We can derive a PI expression for this matrix element going backwards the steps from (5.23) to (5.20) while keeping the internal operators $\hat{x}(t_j)$ and $\hat{x}(t_k)$.

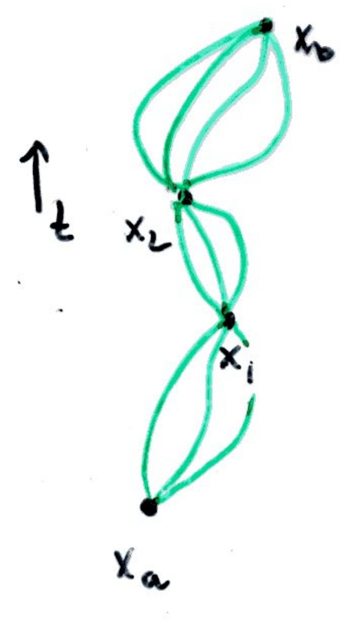
One only needs to take $\hat{x}(t_j)$ and $\hat{x}(t_k)$ in the appropriate time slice, and as the slicing comes continuous, they just appear as c-numbers in the PI. The result is obvious:

$$\langle x_N, t_N | \underbrace{\hat{x}(t_j)\hat{x}(t_k)}_{\text{operators}} | x_1; t_1 \rangle \underset{t_j > t_k}{=} \int_{\substack{x(t_1)=x_1 \\ x(t_N)=x_N}} \mathcal{D}x \; \underbrace{x(t_j)x(t_k)}_{\text{c-numbers}} \, e^{\frac{i}{\hbar}S} \qquad (5.32)$$

Now, what if $t_k > t_j$ (but still both between t_N and t_1). Because $x(t_j)$ and $x(t_k)$ are just c-numbers, one can just swap them to show that $\langle x_N, t_N | \hat{x}(t_k)\hat{x}(t_j) | x_1; t_1 \rangle_{t_k > t_j}$ has exactly the same PI-expression as (5.32). That is, the PI-expression automatically corresponds to the time-ordered operator product expectation value. It is easy to see that this result is independent of the number of operators, and so

$$\int_{\substack{x(t_a)=x_a \\ x(t_b)=x_b}} \mathcal{D}x \; x(t_1) \cdots x(t_N) \, e^{\frac{i}{\hbar}S} = \langle x_b, t_b | T(\hat{x}_1(t_1) \cdots \hat{x}_N(t_N)) | x_a, t_a \rangle \qquad (5.33)$$

$\hat{=}$



sum over all possible paths such that $x(t_a) = x_a$, $x(t_b) = x_b$ and $x(t_1) = x_1$ and $x(t_2) = x_2$.

2^{nd} Quantization

Above we treated the transition amplitude in 1. quantization. Generalization to 2. quantization (point particle → field) is straightforward. The recipe is to extend our earlier PI for conjugate variables:

$$(q, p) \longrightarrow \{q_i, p_i\}_{\substack{i=1,\dots,N \\ N\to\infty}} \qquad (5,34)$$

Obviously, for such infinite set of conjugate variables we can define

$$K(b,a) = \prod_{i=1}^{\substack{N \\ N\to\infty}} \int \mathcal{D}q_i \, \mathcal{D}p_i \, e^{i\int_0^T dt \left(\sum_i p^i \dot{q}^i - H(q,p)\right)} \qquad (5,35)$$

If H is quadratic in p, the p_i - integrals can be done to obtain the Lagrange formulation.

It is clear that these expressions are valid for <u>any</u> conjugate variable pairs; they need not be positions and momenta. For example we can take

$$q_i = \psi(x_i) \quad ; \quad \pi_i = \psi^*(x_i) \qquad (5,36)$$

I.e. probability field amplitude ψ and its conjugate in a space which we can for a moment consider discretized (continuum $N\to\infty$, limit, loosely speaking). In each space-time location ψ_i and ψ_i^* get arbitrary values, so that

Then

$$K(\psi_b,\psi_a) = \lim_{N\to\infty} \int \prod_{i=1}^{N} \mathcal{D}\psi_i \mathcal{D}\pi_i \; e^{i\int dt \left(\sum_i \pi^i \dot{\psi}^i - H(\psi_i,\pi_i)\right)}$$

$$= \int \prod_{\vec{x}} \mathcal{D}\psi(x) \mathcal{D}\psi^*(x) \; e^{i\int dt dx \, \underbrace{\psi^*(i\partial_t - H)\psi}_{\mathcal{L}_{\text{Schrödinger}}}} \qquad (5.37)$$

$\psi(x,t_a) = \psi_a(x)$
$\psi(x,t_b) = \psi_b(x)$

What we have here, is a path integral over field configuration paths in an infinite dimensional Hilbert space. The initial and final points have been replaced by initial and final field configurations

$$x_{a,b} \to \psi(x_{a,b}) \qquad (5.38)$$

Operator connection is of course

$$K(\psi_b,\psi_a) = \langle \psi_b(x) | e^{i\hat{H}T} | \psi_a(x) \rangle \qquad (5.39)$$

One can already see what there is to come: PI provides a nice way to compute transition amplitudes.• If we take $t_0 \to -\infty$ and $t_0 \to +\infty$ we can consider asymptotic vacuum to vacuum amplitudes. Introducing intermediate field-values to PI in form $\psi(t_i,\vec{x})$, we can then generate PI-expressions for the N-point Greens functions needed in the LSZ-formulae for the cross sections.

• This is but one application of P.I.

5.2 Scalar field quantization

The Hamiltonian for scalar field is

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \right], \qquad (5,37)$$

The P.I-representation for a transition amplitude in the configuration space then is

$$\langle \phi_b(t_b) | \phi_a(t_a) \rangle = \int \mathcal{D}\phi\, \mathcal{D}\pi\, e^{i\int_{t_a}^{t_b} d^4x \left[\pi\dot{\phi} - H\right]} \Bigg|_{\substack{\phi(t_a,\vec{x}) = \phi_a(\vec{x}) \\ \phi(t_b,\vec{x}) = \phi_b(\vec{x})}}$$

$$= \underbrace{\int \mathcal{D}\phi}_{\text{continuous product } \prod_{\vec{x},t} d\phi(\vec{x},t)}\, e^{i\int_{t_a}^{t_b} d^4x\, \mathcal{L}(\phi,\partial\phi)} \Bigg|_{\substack{\phi(t_a) = \phi_a \\ \phi(t_b) = \phi_b}} \qquad (5,38)$$

We shall be using the second form, defined by the Lagrange function $\mathcal{L}$ from now on. In P.I-form it is particularly clear that $\mathcal{L}$ defines the theory completely. Covariance is explicit and internal symmetries are obvious from $\mathcal{L}$.

Generating functions

Our task is now to derive our former results for Greens functions (correlation functions), perturbation theory and Feynman rules based on (5,38).

From now on I shall not mark explicitly the initial and final configurations in the P.I. The idea is that eventually $t_a \to -\infty$ and $t_b \to +\infty$, and $|\phi_{a,b}\rangle \to |0\rangle$, so that our P.I. will describe vacuum-to-vacuum transitions over infinite time.

Generalize (5.38) to a generating function $Z[J]$:

$$Z[J] = \frac{1}{N} \int \mathcal{D}\phi \, e^{i \int d^4x \, (\mathcal{L} + J_x \phi_x)} \qquad (5.39)$$

where the normalization factor N is just the amplitude (5.38) so that $Z[0] = 1$. The results from point-particle PI time ordering carry over here directly. The correlators

$$\frac{1}{N} \int \mathcal{D}\phi \; \phi(x_1) \cdots \phi(x_n) \, e^{i \int d^4x \, \mathcal{L}} \equiv G_n(x_1, \dots, x_n) \qquad (5.40)$$

are just the time ordered Greens functions appearing in the LSZ-formulae (when we take $|t_{a,b}| \to \infty$ and $|\phi_{a,b}\rangle \to |0\rangle_{\text{free}}$). The normalization factor N is necessary so that $|\langle 0|0\rangle| = 1$. Now using (5.39) it is easy to see that

$$\boxed{G_n(x_1, \dots, x_n) = \frac{(-i)^n \delta^n}{\delta J(x_1) \cdots \delta J(x_n)} Z[J] \Big|_{J=0}} \qquad (5.41)$$

Here we encounter the functional derivative

$$\frac{\delta}{\delta f(x)} E[f(y)] = \lim_{\epsilon \to 0} \frac{E[f(x) + \epsilon \delta^n(x-y)] - E[f(y)]}{\epsilon} \qquad (5.42)$$

$$\Rightarrow \frac{\delta f(x)}{\delta f(y)} = \delta^n(x-y) \; ; \; \frac{\delta}{\delta f(x)} \int d^n y \, K(x',y) f(y) = K(x',x) \quad \text{etc.} \qquad (5.43)$$

Computing $Z[J]$ and therefore $G_n(x_1,\dots,x_n)$ in closed form is impossible when $\mathcal{L}_\phi$ describes a generic interacting theory. Let us therefore consider the free Klein-Gordon model. Now

surface term

$$Z[J]_{KG} = \frac{1}{N_{KG}}\int \mathcal{D}\phi\, e^{i\int d^4x\left(\frac{1}{2}\underbrace{(\partial_\mu\phi)^2}_{=\cancel{\partial_\mu(\phi\partial^\mu\phi)}-\phi\Box\phi}-\frac{m^2}{2}\phi^2+J_x\phi_x\right)} \quad ; N_{KG}=Z_{KG}[0]$$

$$= \frac{1}{N_{KG}}\int \mathcal{D}\phi\, e^{i\int_{x,y}\phi_x\, i\hat{D}^{-1}_{y,x}\,\phi_y + i\int_x J_x\phi_x} \qquad (5.44)$$

where $\int_x \equiv \int d^4x$ and we defined the differential operator

$$\int_y i\hat{D}^{-1}_{y,x}\,\phi_y \equiv -(\partial_x^2+m^2)\phi_x \qquad (5.45)$$

i.e.

$$iD^{-1}_{y,x} = -(\partial_y^2+m^2)\,\delta^4(y-x)$$

$$\doteq -(\partial_y^2+m^2)\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)} \quad \bullet\ ; (y-x)_0 \to T(1-i\epsilon)$$

$$= \int\frac{d^4p}{(2\pi)^4}(p^2-m^2+i\epsilon)\,e^{-ip\cdot(y-x)} \qquad (5.46)$$

Thus

$$D_{y,x} = \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip\cdot(y-x)} = \Delta_F(y-x) \qquad (5.47)$$

Feynman propagator

$$\left(\int d^4y\, D_{x,y}D^{-1}_{y,z} = \delta^4(x-z)\right)$$

• This was necessary for consistence of $P_\pm$. See eq. (5.9)

Note that the integrals appearing in (5.44) are Gaussian. Indeed:

$$\frac{i}{2}\int_{y,x} \phi_y (iD^{-1}_{y,x}) \phi_x + \int_x J_x \phi_x$$

$$= \frac{i}{2}\int_{y,x} \left(\phi_y - \int_z J_z\, iD_{zy}\right)\left(iD^{-1}_{y,x}\right)\overbrace{\left(\phi_x - \int_{z'} iD_{x,z'} J_{z'}\right)}^{\equiv \phi'_x}$$

$$+ \frac{i}{2}\int_{y,x} J_y\, iD_{y,x}\, J_x$$

$$= \frac{i}{2}\int_{y,x} \phi'_y (iD^{-1}_{y,x}) \phi'_x + \frac{i}{2}\int_{y,x} J_y\, iD_{y,x}\, J_x \qquad (5.48)$$

Now the integral over ϕ' can be done at every position $(t,\vec{x})$, and it merely returns the normalization factor N_{KG}, so that

$$\boxed{Z_{KG}[J] = e^{\frac{i}{2}\int d^4x\, d^4y\, J(x)\, i\Delta_F(x-y)\, J(y)}} \qquad (5.49)$$

Now that we have $Z_{KG}[J]$, we can compute different Greens functions from (5.41):

- $$G_2(x_1,x_2) = \frac{(-i)^2\, \delta^2}{\delta J(x_1)\,\delta J(x_2)} Z_{KG}[J]\Big|_{J=0}$$

$$= (-1)\cdot i \cdot i\Delta_F(x_1-x_2) = \underline{\Delta_F(x_1-x_2)} \quad ! \qquad (5.50)$$

Thus

$$\boxed{\frac{1}{N_{KG}}\int \mathcal{D}\phi\, \phi(x_1)\,\phi(x_2)\, e^{i\int \mathcal{L}_{KG}} = \Delta_F(x_1,x_2) = \langle 0 | T(\phi(x_1)\phi(x_2)) | 0\rangle} \qquad (5.51)$$

Wick's theorem

It is easy to see how (5,50) generalizes to n-point functions in KG-theory. Firstly:

$$G^{KG}_{2n+1}(x_1,\dots,x_{2n+1}) = 0 \qquad (5,52)$$

because an odd number of $J(x_i)$-derivatives in (5,41) leaves at least one J in each term outside the exponent. Moreover

$$G^{KG}_{2n}(x_1,\dots,x_{2n}) = \frac{(-i)^{2n}\,\delta^{2n}}{\delta J(x_1)\cdots\delta J(x_{2n})}\left(\frac{i}{2}\right)^n \frac{1}{n!}\prod_{k=1}^{n}\int_{x_k,x_k'} J_{x_k}\, i\Delta_F(x_k-x_k')\, J_{x_k'}$$

$$= (-1)^n \frac{\delta^{2n-1}}{\delta J(x_1)\cdots\delta J(x_{2n-1})}\frac{1}{2^{n-1}}\frac{1}{(n-1)!}\int_x i\Delta_F(x_{2n}-x)\, \underbrace{J_x \prod_{k=1}^{n-1}\int_{x_k x_k'} J_{x_k}\, i\Delta_F\, J_{x_k'}}_{(n-1)\times} \qquad (5,53)$$

$i\Delta_F(x_{2n}-x_{2n-1})$

Quite obviously, doing all derivatives creates all Wick-contractions

$$G^{KG}_{2n}(x_1,\dots,x_{2n}) = \prod_{i<j} i\Delta_F(x_i-x_j) \qquad (5,54)$$

$$= \langle 0|\,T\big(\hat\phi(x_1)\dots\hat\phi(x_{2n})\big)|0\rangle$$

For example

$$G^{KG}_4(x_1,\dots,x_4) = i\Delta^F_{12}\, i\Delta^F_{34} + i\Delta^F_{13}\,\Delta^F_{24} + i\Delta^F_{14}\,\Delta^F_{23} \qquad (5,54)$$

and so on ...

Perturbation theory. $\lambda\phi^4$.

We just showed explicitly that in free theory:

$$G_n(x_1,\dots,x_n) = \frac{1}{N_{KG}}\int \mathcal{D}\phi\, \phi(x_1)\cdots\phi(x_n)\, e^{i\int \mathcal{L}_{KG}}$$

$$= \langle 0|T(\hat{\phi}(x_1)\cdots\hat{\phi}(x_n))|0\rangle. \qquad (5,55)$$

Let us now see how this generalizes to interacting theories.

Problem is that we do not know how to integrate the expression (5,39) to compute $Z[J]$. However, if we know that the coupling λ is small, it makes sense to expand in $\mathcal{L}_I$ and compute the series term by term. Indeed

$$Z[J] = \frac{1}{N_{int}}\int \mathcal{D}\phi\, e^{i\int d^4x(\mathcal{L}+J\phi)}$$

$$= \frac{1}{N_{int}}\int \mathcal{D}\phi\, \underbrace{e^{i\int d^4x\,\mathcal{L}_I}}_{=\sum_{n=0}^{\infty}\frac{1}{n!}\left(i\int d^4x\,\mathcal{L}_I\right)^n} \cdot e^{i\int d^4x(\mathcal{L}_{KG}+J\phi)} \qquad (5.56)$$

In particular

$$\underline{N_{int}} = \int \mathcal{D}\phi \sum_{n=0}^{\infty}\frac{1}{n!}\left(i\int d^4x\,\mathcal{L}_I\right)^n e^{i\int d^4x\,\mathcal{L}_{KG}}$$

$$\overset{(5.55)}{=} N_{KG}\langle 0|T\left(\sum_{n=0}^{\infty}\frac{1}{n!}\left(i\int d^4x\,\hat{\mathcal{L}}_I\right)^n\right)|0\rangle$$

$$= \underline{N_{KG}\langle 0|T\left(e^{i\int d^4x\,\mathcal{L}_I}\right)|0\rangle} \qquad (5,57)$$

← all vacuum-to-vacuum diagrams

Moreover, we can write

$$Z[J] = \frac{1}{N_{int}} \int \mathcal{D}\phi \, e^{i\int d^4x \mathcal{L}_I(\phi)} \, e^{i\int d^4x (\mathcal{L}_{KG} + J\phi)}$$

$$= \frac{1}{\cancel{N_{KG}} \langle 0|T(e^{i\int \mathcal{L}_I})|0\rangle} e^{i\int d^4x \mathcal{L}(-i\frac{\delta}{\delta J})} \underbrace{\int \mathcal{D}\phi \, e^{i\int d^4x (\mathcal{L}_{KG} + J\phi)}}_{= \cancel{N_{KG}} \cdot Z_{KG}[J]}$$

where we pulled $\mathcal{L}_I$ out from the integral by identity

$$f(\phi) \, e^{i\int J\phi} = f(-i\frac{\delta}{\delta J}) \, e^{i\int J\phi}$$

That is:

$$\boxed{Z[J] = \frac{1}{\langle 0|T(e^{i\int \mathcal{L}_I})|0\rangle} e^{i\int d^4x \mathcal{L}(-i\frac{\delta}{\delta J})} Z_{KG}[J]} \qquad (5.58)$$

with

$$\langle 0|T(e^{i\int \mathcal{L}_I})|0\rangle = e^{i\int d^4x \mathcal{L}(-i\frac{\delta}{\delta J})} Z_{KG}[J]\Big|_{J=0} . \qquad (5.59)$$

where $Z_{KG}[J] = \exp(-i\int_{x,y} J_x i\Delta_F(x-y) J_y)$. This creates the entire perturbation series. For example for the vacuum-amplitudes using (5.59):

- 0^{th} order: $\langle |T()|0\rangle^{(0)} = 1$
- 1^{st} order: $$\langle \; \rangle^{(1)} = i\frac{\lambda}{4!}\int d^4x \, (-i)^4 \frac{\delta^4}{\delta J(x)^4} \Big[\underbrace{\frac{1}{2!}\left(\frac{i}{2}\right)^2}_{\text{from } e^{J\Delta J} \text{ with 4 } J\text{'s only}} \prod_{k=1}^{2} \int_{x_k, x_k'} J_{x_k} i\Delta_F(x_k - x_k') J_{x_k'}\Big]_{J=0}$$

$$= -\frac{i\lambda}{8}\int d^4x \, [i\Delta_F(0)]^2 = 8 . \qquad (5.60)$$

And so on. Similarly, from (5.58) we get for the two-point function:

$$G_2(x_1,x_2) = (-i)^2 \frac{\delta^2}{\delta J(x_1)\,\delta J(x_2)} Z[J] \Big|_{J=0}$$

$$= \frac{-1}{\langle 0|T(e^{i\int \mathcal{L}_I})|0\rangle} e^{i\int \mathcal{L}_I(-i\frac{\delta}{\delta J})} \frac{\delta^2}{\delta J_1 \delta J_2} Z_{KG}[J] \Big|_{J=0}$$

$$= \frac{-1}{\langle 0|T(\;)|0\rangle} e^{i\int \mathcal{L}_I(-i\frac{\delta}{\delta J})} \left(-\Delta^F_{12} + \left(\int \Delta J\right)_1 \left(\int \Delta J\right)_2\right) e^{-\frac{i}{2}\iint J\Delta J} \Big|_{J=0} \qquad (5.61)$$

- 0^{th} order:

$$G_2(x_1,x_2) = \Delta_F(x_1-x_2)$$

- 1^{st} order:

$$G_2(x_1,x_2) = -\frac{\lambda}{4!}(-i)^4 \int d^4x \frac{\delta^4}{\delta J(x)^4} \times \left(-\frac{i}{2}\right) \int (\Delta J)_1 (\Delta J)_2 \iint J\Delta J$$

$$= \frac{\lambda}{2} \int d^4x\, \Delta_F(x_1-x)\, \Delta_F(x_2-x)\, \Delta_F(0) \qquad (5.61a)$$

There actually would have been also a vacuum graph  from expanding the exponent to second order in connection with the leading $-\Delta^F_{12}$-term in (5.61). These are cancelled by the denominator however, as we expect. In fact, this cancellation is obvious to all orders for the lowest order propagator, since Δ^F_{12} is multiplied by just $\frac{1}{\langle\;\rangle}\cdot\langle\;\rangle = 1$. (see 5.59).

This result generalizes to all orders so that eventually

$$\underline{G_n(x_1,\dots,x_n)} = \frac{(-i)^n\,\delta^n}{\delta J_1\cdots\delta J_n} Z[J]\Big|_{J=0} = \underline{\text{all non-vacuum graphs}}$$

$$= \underline{\frac{\langle 0|T(\hat\phi_1\cdots\hat\phi_n e^{i\int \mathcal{L}_I})|0\rangle}{\langle 0|T(e^{i\int\mathcal{L}_I})|0\rangle}} \,. \qquad (5.62)$$

Proof. Some combinatorics. (Ex).

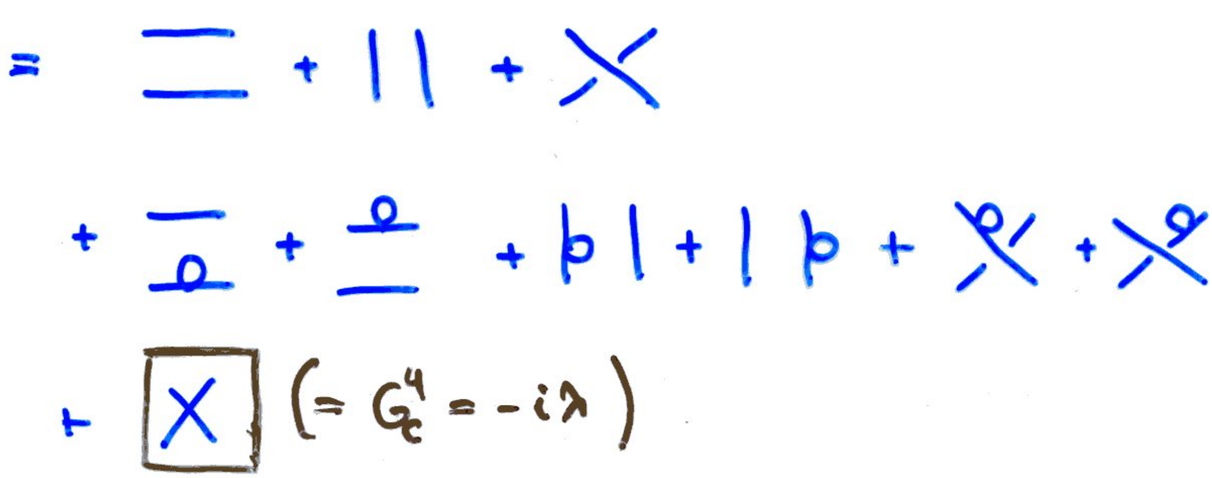
However, the green function (5.62) does contain all non-connected diagrams and also all non 1PI-diagrams. For example

$$G_4(x_1,\dots,x_4) = \underset{\equiv\,\langle 0|T(e^{i\int\mathcal{L}_I})|0\rangle}{\frac{1}{N_{vac}}}\, e^{i\int \mathcal{L}_I(-i\frac{\delta}{\delta J})} \underbrace{\frac{(-i)^4\,\delta^4}{\delta J_1\cdots\delta J_4}\, e^{-\frac{1}{2}\iint J\Delta J}}\Big|_{J=0}$$

$$= \frac{\delta^2}{\delta J_1 \delta J_2}\left(-\Delta_{34} + (\Delta J)_3(\Delta J)_4\right)e^{-\frac{1}{2}J\Delta J}$$

$$= \Big(\left(\Delta_{12} - (\Delta J)_1(\Delta J)_2\right)\left(\Delta_{34} - (\Delta J)_3(\Delta J)_4\right) + \text{perm}$$

$$- 2(\Delta J)_1(\Delta J)_2(\Delta J)_3(\Delta J)_4\Big) \times e^{-\frac{1}{2}J\Delta J}$$



$(= G_c^4 = -i\lambda)$

$+ \cdots$

Generating function for connected graphs

Path integral methods show their strength when we start to consider restricted sets of diagrams. For example, we know that only connected diagrams contained in $G_n(x_1,\dots,x_n)$ contribute to the S-matrix. It turns out that these graphs are generated by the functional

$$W[J] \equiv -i \ln Z[J] \qquad (5.63)$$

That is

$$G_c(x_1,\dots,x_n) = \frac{(-i)^n\,\delta^n}{\delta J_1 \cdots \delta J_n}\, iW[J]\Big|_{J=0} \qquad (5.64)$$

Let us consider a few examples.

- Free theory. From (5.49) we get

$$W_{KG}[J] = \frac{1}{2}\iint d^4x\,d^4y\; J(x)\, i\Delta_F(x-y)\, J(y) \qquad (5.65)$$

This clearly generates only one nonvanishing function:

$$\frac{(-i)^2\delta^2}{\delta J_1\,\delta J_2} W[J]_{KG} = \Delta_F(x-y) \qquad (5.66)$$

But this is as it should, because Δ_F is the only connected diagram in the KG-theory.

- **2-point function** in the interacting theory. Now

$$i \frac{\delta^2 W}{\delta J_1 \delta J_2}\bigg|_{J=0} = \left(-\frac{1}{Z^2} \frac{\delta Z}{\delta J_1} \frac{\delta Z}{\delta J_2} + \frac{1}{Z} \frac{\delta^2 Z}{\delta J_1 \delta J_2} \right)_{J=0}$$

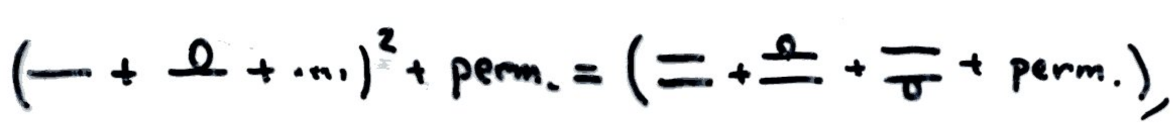
$$= \frac{\delta^2 Z}{\delta J_1 \delta J_2}\bigg|_{J=0} \qquad (5.67)$$

if only $\delta Z/\delta J|_J = 0$ (and we used $Z[0]=1$). This is the case when $\delta W[J]/\delta J = \delta Z/\delta J = \langle \phi \rangle_J = 0$ when $J=0$, i.e. whenever the expectation value of the field vanishes with the external source.

- **4-point function** Using again $\delta^{2n+1} Z/\delta J^{2n+1} = 0$ (true in $\lambda\phi^4$-theory) we get

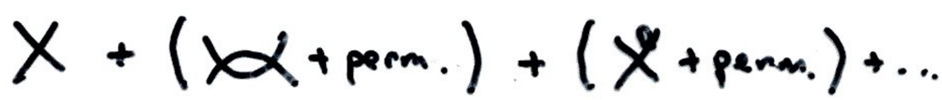
$$i \frac{\delta^4 W}{\delta J_1 \cdots \delta J_4} = \frac{\delta^4 Z}{\delta J_1 \cdots \delta J_2} \underbrace{-}_{\text{from differentiation of } \frac{1}{Z}.} \left(\frac{\delta^2 Z}{\delta J_1 \delta J_2} \frac{\delta^2 Z}{\delta J_3 \delta J_4} + \overset{(2\text{ more})}{\text{permutations}} \right) \qquad (5.68)$$

The terms in brackets clearly generate, with a - sign, all graphs that arise from products of 2-point expansions:



ie all disconnected graphs! Therefore, what is left, are only the connected ones:

$$i \frac{\delta^4 W}{\delta J_1 \cdots \delta J_4} =$$



$$= -i\lambda + \dots \qquad (5.69)$$

Generating Function for 1PI-graphs

In chapter 3 (p.106) we mentioned the special role of the 1PI-diagrams. This issue will become even clearer now. In functional integral formalism 1PI-graphs are generated by function

$$\Gamma_{1PI}[\phi_c] = W[J] - \int d^4x\, J(x)\,\phi_c(x) \qquad (5.70)$$

where

$$\frac{\delta W[J]}{\delta J} \equiv \phi_c \quad \Rightarrow \quad \frac{\delta \Gamma_{1PI}}{\delta \phi_c} = -J \qquad (5.71)$$

(5.70) is an example of a Legendre transformation, familiar from thermodynamics. It is important to understand that it is just a change of variables: $\phi_c = \phi_c[J]$. From (5.71) we get immediately:

$$\left.\frac{\delta^2 W}{\delta J_x \delta J_y}\right|_{J=0} = \left.\frac{\delta \phi_c(x)}{\delta J(y)}\right|_{J=0} \quad ; \quad \left.\frac{\delta^2 \Gamma_{1PI}}{\delta \phi_x \delta \phi_y}\right|_{\phi_c} = -\left.\frac{\delta J(x)}{\delta \phi(y)}\right|_{\phi_c} \qquad (5.72)$$

so that

$$\int d^4y\, \frac{\delta^2 W}{\delta J_x \delta J_y}\, \frac{\delta^2 \Gamma_{1PI}}{\delta \phi_y \delta \phi_{x'}} = -\int d^4y\, \frac{\delta \phi(x)}{\delta J(y)}\, \frac{\delta J(y)}{\delta \phi(x')}$$

$$= -\frac{\delta \phi(x)}{\delta \phi(x')} = -\delta^4(x-x') \qquad (5.73)$$

- For example consider internal energy $U = U(S,V)$ vs. Free energy $F \equiv U - ST$:

$dU = TdS - pdV$: $\Rightarrow dF = -SdT - pdV$; ie $F = F(T,V)$

That is, $\frac{\delta^2 \Gamma_{1PI}}{\delta\phi_x \delta\phi_y}$ is the inverse of the two-point function $-\frac{\delta^2 W}{\delta J_x \delta J_y}$. (The Gibbs condition.) Consider some examples.

- **free theory** We can compute explicitly:

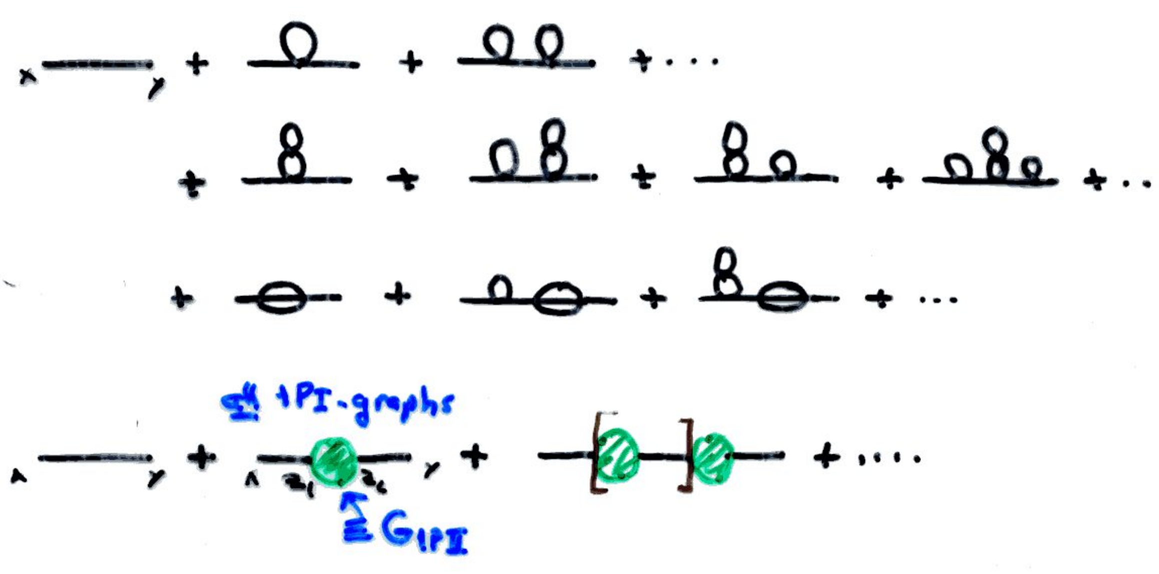
$$\frac{\delta W^{KG}}{\delta J_x} = i\int_y i\Delta_{xy} J_y \equiv \phi_c(x) \Rightarrow J_y = i\int_z i\Delta^{-1}_{y,z}\phi_c(z)$$

$$\Rightarrow \Gamma^{KG}_{1PI} = \frac{i}{2}\iint d^4x\, d^4y\, \phi_c(x)\, i\Delta_F^{-1}(x-y)\, \phi_c(y) \qquad (5.74)$$

- **$\lambda\phi^4$- theory. Propagator.**

Suitably reorganizing the perturbation expansion generated by $\frac{\delta^2 W}{\delta J_x \delta J_y}$ we can prove that $\delta^2\Gamma/\delta\phi_c\delta\phi_c$ is 1PI:

$$-\frac{\delta^2 W}{\delta J_1 \delta J_2}\Big|_{J=0} =$$



$$= \Delta_{xy} + \Delta_{xz_1} G^{1PI}_{z_1 z_2} \Delta_{z_2 y} + \Delta_{xz_1} (G^{1PI}\Delta)^2_{z_2 y} + \dots$$

$$= \Delta_{xz}\left(\frac{1}{1 - G^{1PI}\Delta}\right)_{zy}$$

free th. only 1PI-graphs

$$\Rightarrow \frac{\delta^2\Gamma}{\delta\phi_x\delta\phi_y} = +(1 - G^{1PI}\Delta)_{xz}\Delta^{-1}_{zy} = \Delta^{-1}_{xy} - G^{1PI}_{xy} \qquad (5.75)$$

$\square$

• 4-point function in $\lambda\phi^4$-theory

Now differentiate the l.h.s of (5.73) twice w.r.t. J directly and by using the chain rule:

$$\frac{\delta}{\delta J_x} = \int_y \frac{\delta \phi_y}{\delta J_x}\frac{\delta}{\delta \phi_y} = -\int_y \frac{\delta^2 W}{\delta J_y \delta J_x}\frac{\delta}{\delta\phi_y} \qquad (5.76)$$

We get a new identity:

$$\int d^4y \, \frac{\delta^4 W}{\delta J_{x_1}\delta J_{x_2}\delta J_{x_3}\delta J_y}\frac{\delta \Gamma_{1PI}}{\delta\phi_y\,\delta\phi_{x'}}$$

$\left(W^3 = \Gamma^3_{1PI} = 0 \text{ in } \lambda\phi^4\text{-th.}\right)$

$$= \int d^4y\, d^4y_1\, d^4y_2 \, \frac{\delta^2 W}{\delta J_{x_1}\delta J_{y_1}}\frac{\delta^2 W}{\delta J_{y_2}\delta J_{x_2}}\frac{\delta^2 W}{\delta J_{x_3}\delta J_y}\cdot\frac{\delta^4\Gamma_{1PI}}{\delta\phi_{y_1}\delta\phi_{y_2}\delta\phi_{y_3}\delta\phi_{x'}} + 0$$

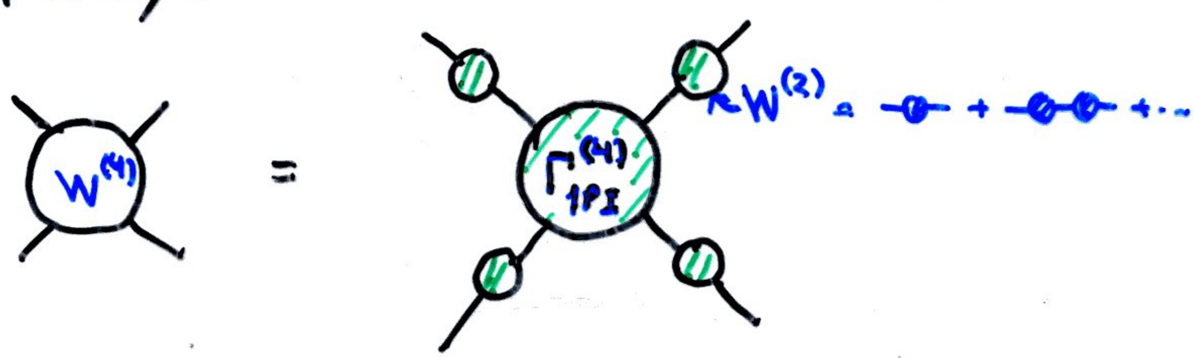
Multiplying both sides by $-\int d^4x' \frac{\delta^2 W}{\delta J_{x'}\delta J_{x_4}}$, and using (5.73) we get

$$\underline{\frac{\delta^4 W}{\delta J_{x_1}\cdots\delta J_{x_4}}} = \underline{\int \prod_{i=1}^4 \left(d^4y_i \frac{\delta^2 W}{\delta J_{x_i}\delta J_{y_i}}\right)\cdot\frac{\delta^4\Gamma}{\delta\phi_{y_1}\cdots\delta\phi_{y_4}}}$$

$$= \underbrace{\int \prod_{i=1}^4 \left(d^4y_i\, W^{(2)}(x_i, y_i)\right)\, \Gamma_{1PI}(y_1,\dots,y_4)}_{} = W^{(4)}$$

full inverse propagator.

Graphically:



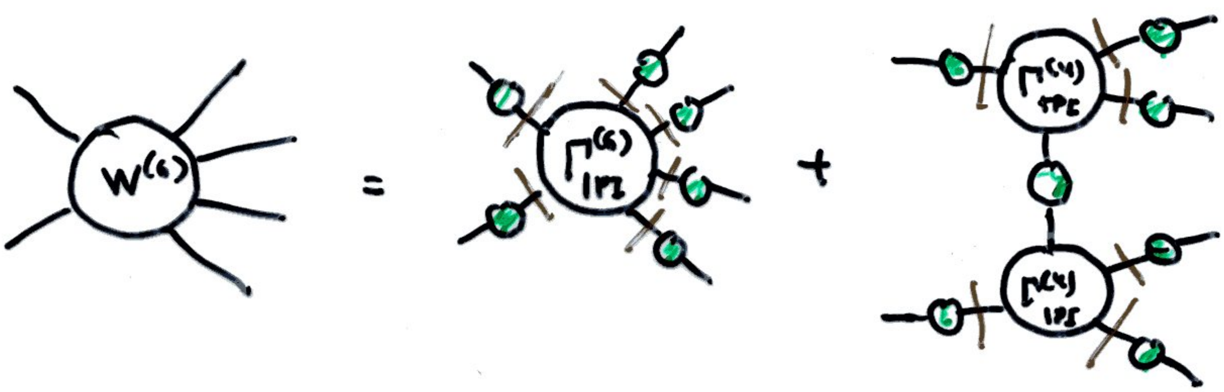
This remarkable identity explicitly shows that we can compute the $2 \to 2$ scattering process in $\lambda\phi^4$-theory by computing only the 1PI 2- and 4-point functions:

$G^{-1}_{1PI} =$  $\propto$ propagator of interacting th.

$\Gamma^{4}_{1PI} =$  $\propto$ interacting th. vertex.

This division to propagator and vertex functions will be extremely useful in renormalization theory.

We can follow this construction further. For example the 6-point function decomposes as follows:



and so on. There is a formal way to derive such identities using Taylor expansions of W and Γ. See for example Kaku p. 282–283.

– Amputation for $2 \to 4$ scattering

- Moreover, the amputation process in LSZ-reduction formulae works with physical on-shell state of the interacting theory:

(return to this later)

$$\int d^4y\, e^{-ip\cdot y} \underbrace{(\partial^2 - m^2_{phys})}_{\text{d.o.f.}} W^{(2)}(y,z) \equiv e^{-ipz}$$

↑ external leg ↔ on-shell inverse prop.

5.3 Connection to statistical physics

Above we introduced several generating functionals, $Z[J]$, $W[J]$ and $\Gamma_{1PI}[\phi_c]$ purely from technical grounds. Let us now study the analogy between QFT and statistical ferromagnetic system, to bring some physical intuition to these concepts.

Ferromagnetic (spin) system

Let H be an external magnetic field. The partition function of a ferromagnetic system is

$$Z[H] = \int \mathcal{D}s \, e^{-\beta \int d^3x \left(\mathcal{H}[s] - H s(x) \right)}$$

energy density of a spin configuration (internal energy)

$$= e^{-\beta F(H)} \qquad (5.78)$$

Helmholtz free energy

where $\beta \equiv 1/T$. We can compute the magnetization of the system as:

$$M \equiv -\left(\frac{\partial F}{\partial H}\right)_T = -\frac{1}{\beta}\frac{\partial}{\partial H} \log Z[H]$$

$$= \frac{1}{Z_1} \int d^3x \int \mathcal{D}s \; s(\vec{x}) \, e^{-\beta \int d^3x \left(\mathcal{H}[s] - H s(x) \right)}$$

$$= \int d^3x \, \langle s(x) \rangle_H = \langle s \rangle_H \qquad (5.79)$$

We can also define a Gibbs free energy

$$G = F + MH \qquad (5.80)$$

Which satisfies

$$\left(\frac{\partial G}{\partial M}\right)_T = \overbrace{\cancel{\left(\frac{\partial F}{\partial M}\right)_T}}^{\left(\frac{\partial F}{\partial H}\right)_T \left(\frac{\partial H}{\partial M}\right)_T} + M\cancel{\left(\frac{\partial H}{\partial M}\right)_T} + H$$

$$= H \qquad (5.81)$$

Comparing these results with the definitions for our generating functions we get the following formal analogies:

Quantum theory	Magnetic system	
x^μ	$\vec{x}$	(no time) but has T
$\phi(x)$	$S(\vec{x})$	
$J(x)$	H	
$\mathcal{L}[\phi]$	$\mathcal{H}[s]$	
$Z[J]$	$Z[H]$	
$W[J]$	$F(H)$	
$\phi_c(x)$	M	
$\Gamma_{1PI}[\phi_c]$	$G(M)$	

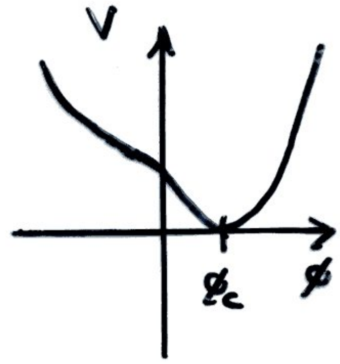
(5.82)

• One is usually interested in global response to a constant H. There is no problem to extend results to local $H(\vec{x})$, $M(\vec{x})$ and functional $F[H]$ and $G[M]$ however.

Different generating functionals of the QFT are thus analogous to the statistical physics thermodynamical potentials. Analogy also uncovers the physical meaning of $\Gamma_{1PI}[\phi_c]$. Solution to equation

$$(J=) \qquad \left.\frac{\delta\Gamma_{1PI}[\phi_c]}{\delta\phi_c}\right|_{J=0} = 0 \qquad (5,83)$$

The solution to this equation is the lowest energy configuration of the system (when the source function J vanishes). Note that for a constant ϕ at tree level $\Gamma_{1PI} = H = V(\phi)$, so (5.83) reduces to $dV/d\phi = 0$. The solution to (5,83) then gives the expectation value of ϕ. Unlike with tree level potential, solution to (5.83) gives the extremum to arbitrary order in the perturbation theory, however.



Magnetization analogy is but one example. More generally we can write the statistical physics partition function as

$$Z = \sum_n e^{-E_n/T} = Tr\{e^{-\hat{H}/T}\}$$

$$\xrightarrow{\text{"}\hat{Q}\text{-bath"}} Tr\{e^{-(\hat{H}-\mu\hat{Q})/T}\} \equiv e^{-F(\mu)/T} \qquad (5,84)$$

$$\Rightarrow \langle Q\rangle = T\frac{\partial}{\partial\mu_a}\ln Z = \frac{T}{Z}\frac{\partial Z}{\partial\mu} = \frac{\partial F}{\partial\mu} \qquad (5,85)$$

i.e.

$$Z[J]\ ,\ W[J] \quad \longleftrightarrow \quad Z[\mu,T]\ ;\ F[\mu,T]$$

$$\phi_c(x) \quad \longleftrightarrow \quad \langle Q \rangle \equiv N_Q$$

$$\Gamma_{1PI}[\phi_c] \quad \longleftrightarrow \quad G(N_Q) \qquad (5.86)$$

Connection between formulations goes deeper than these analogies. It turns out that in many ways QFT- systems are very closely analogous to statistical systems near critical point. Indeed, the renormaizability of QFT, i.e. form invariance of the Lagrangian under inclusion of radiative correction stems from scale invariance of the theory. Similarly the universal characteristics of critical phenomena follow from a similar scale invariance. (eg. Kadanoff scaling ...). Connection is most fully exploited through Wilson renormalization theory (spring semester...)

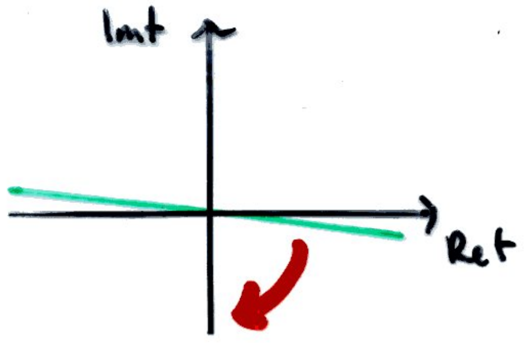
Complex time path, Wick rotation, thermal field theory

Our complex time contour allows performing a Wick rotation

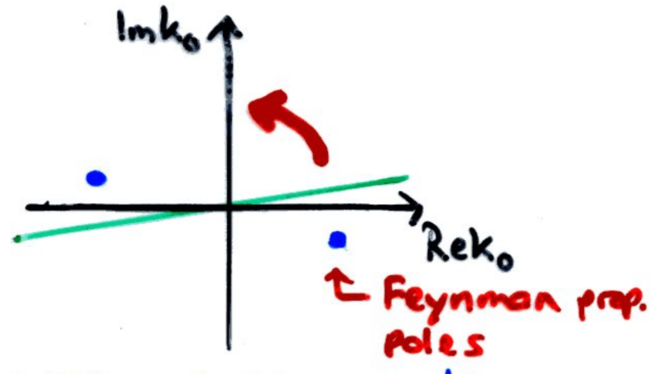
$$t \to -i x_E^0 \qquad (5.87)$$

4D-Euclidian length of a vector

$$\Rightarrow \quad x^2 = t^2 - \vec{x}^2 = -x_E^2 \qquad (5.88)$$



Wick rotation in time

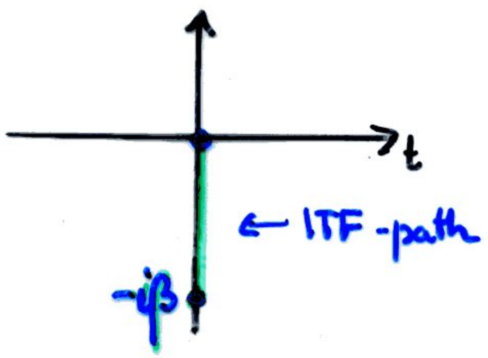


Wick rotation in k_0

Wick rotation allows analytic continuation of Minkowski space correlators to Euclidian space and backwards.

$$\langle 0|T(\phi_1,\dots,\phi_n)|0\rangle \overset{\text{Wick}}{\longleftrightarrow} \text{4-D Euclidian space Greens function} \qquad (5.89)$$

One can bring the idea of a complex time path further. Remember that originally we wanted to compute with $t_{in} \to -\infty$ and $t_{end} \to +\infty$, but consistency of PI (or correct adiabatic vacuum limit, or Feynman boundary conditions) requested adding $T \to T(1-i\delta)$. However, we might want to compute, not "in-out" transitions, but say thermal (quantum) correlations in a plasma. Such "in-in" systems can also be considered in QFT. We just need different path-choices. Eg.



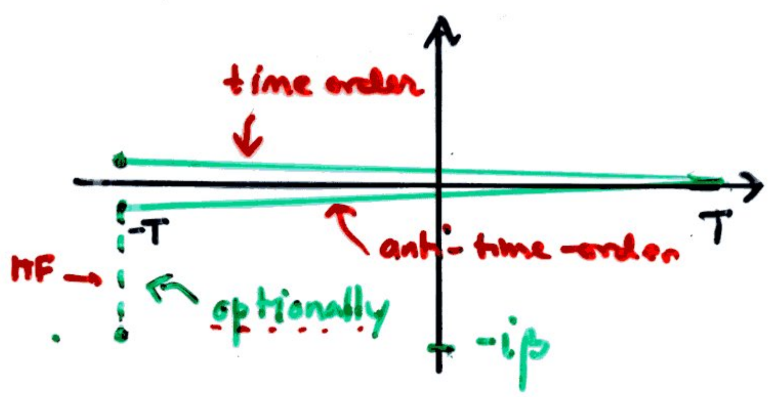
⇒ Imaginary-time formalism

- No time at all (quantum statistical syst
- Require that correlators are periodi in complex time

$$\Rightarrow \quad \beta = \frac{1}{T} \qquad (5.90)$$

↖ Temperature!

This is not the only possibility. Time-dependent phenomena in out-of-eq. thermal system can be accommodated using e.g. the Keldysh-paths



⇒
- Real time thermal QFT
- Fully nonthermal QFT (Schwinger-Keldysh.)

5.5 Fermionic path integral; Grassmann variables

For fermionic fields the PI formalism needs extension from c-numbers to a new class of anticommuting variables. Such variables are called Grassmann-variables. So, if θ_i and θ_j are G-numbers, they obey

$$\theta_i \theta_j = -\theta_j \theta_i \qquad (5.91)$$

From this it of course follows that $\theta_i^2 = 0$ $\Rightarrow$ strange phenomena! For example, it is immediately clear that the most complicated function of a Grassmann variable is

$$\phi(\theta) = a + b\theta \quad ! \qquad (5.92)$$

Indeed eg

$$e^{a\theta} = 1 + a\theta = \frac{1}{1-a\theta}, \text{ etc.}$$

Integration over G-numbers is defined such that translational invariance of ordinary c-numbers:

$$\int_{-\infty}^{\infty} dx\, \phi(x) = \int_{-\infty}^{\infty} dx\, \phi(x+c) \qquad (5.93)$$

carries over:

$$\int d\theta\, \phi(\theta) \equiv \int d\theta\, \phi(\theta+\xi) \qquad (5.94)$$

Using (5.93) then:

only 2 indep. integrals

$$a\left[\int d\theta\right] + b\left[\int d\theta\, \theta\right] = \underline{\left[\int d\theta\right]}(a+b\xi) + b\underline{\int d\theta\, \theta}$$

since this must hold for all ϕ ($\forall a, b$) $\Rightarrow$ $\int d\theta = 0$. Furthermore defines $\int d\theta\, \theta \equiv 1$, such that

$$\int d\theta\,(A + B\theta) = B \qquad (5.96)$$

From this it follows that G-integral is the same as G-derivative!

$$\int d\theta\, \phi(\theta) = \frac{\partial}{\partial\theta}\phi(\theta) \qquad (5.97)$$

Moreover:

$$\int d\theta \int d\eta\, \eta\theta = +1$$

$$\theta \equiv \frac{\theta_1 + i\theta_2}{\sqrt{2}} \Rightarrow \theta^* = \frac{\theta_1 - i\theta_2}{\sqrt{2}},$$

$$\int d\theta\, d\theta^*\, \theta^*\theta = 1 \Rightarrow \int d\theta\, d\theta^*\, e^{\theta^* b\theta} = \int d\theta\, d\theta^* (1 + \theta^* b\theta)$$

$$= +b \int d\theta^* d\theta\, \theta^*\theta = +b.$$

etc...

Multidimensional Gaussian G-integrals

Above formulae are easily extended to ∞ number of variables ($\lim N \to \infty$). Consider integral

$$I_F(A) = \int \prod_{i=1}^{N} d\theta_i d\bar\theta_i\; e^{\sum_{ij} \bar\theta_i A_{ij} \theta_j} \qquad \zeta_j \equiv \sum_{k_j=1}^{N} A_{jk_j}\theta_{k_j}$$

$$= \int \prod_{i=1}^{N} d\theta_i d\bar\theta_i\; \frac{1}{N!}\Big(\sum_{jk} \bar\theta_j A_{jk}\theta_k\Big)^N = \int \prod_{i=1}^{N} d\theta_i d\bar\theta_i\; \frac{1}{N!}\Big(\sum_{j=1}^{N} \bar\theta_j \zeta_j\Big)^N$$

$$= \int \prod_{i=1}^{N} d\theta_i d\bar\theta_i\; (\bar\theta_1 \zeta_1 \cdots \bar\theta_N \zeta_N) = \int \prod_{i=1}^{N} d\theta_i d\bar\theta_i\; \prod_{j=1}^{N} \bar\theta_j \Big(\sum_{k_j=1}^{N} A_{jk_j}\theta_{k_j}\Big)$$

$$= \int \prod_{i=1}^{N} d\theta_i d\bar\theta_i \sum_{\text{perm.}} A_{1k_1} A_{2k_2} \cdots A_{Nk_N}\, \bar\theta_1 \theta_{k_1} \underbrace{\bar\theta_2 \theta_{k_2}}_{\text{antisymm}} \cdots \bar\theta_N \theta_{k_N}$$

$$= \epsilon_{k_1 \dots k_N} A_{1k_1} \cdots A_{Nk_N} \overbrace{\int \prod_{i=1}^{N} d\theta_i \, d\bar{\theta}_i \; \bar{\theta}_i \theta_i}^{=1} = \det A \qquad (5.98)$$

For comparision, bosonic (c-number) Gaussian integral gives:

$$I_B(A) \equiv \int \prod_{i=1}^{N} dx_i \; e^{-x_i A_{ij} x_j} = (2\pi)^{N/2} \frac{1}{\sqrt{\det A}} . \qquad (5.99)$$

Using (5.98) we can express a functional determinant as a functional integral over Gaussian grassmann distribution. This will become handy shortly, when we will be quantizing gauge fields using Faddeev-Popov trick.

Generating function for fermions

Just going to the continuum limit by $\theta_i \to \psi(x)$, where ψ is a Grassmann valued field, we can write:

$$Z_D[\eta,\bar{\eta}] = \frac{1}{N_D} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \; e^{i \int d^4x \left(\bar{\psi} \overbrace{(i\partial\!\!\!/ - m)\psi}^{\equiv \int d^4y \, S_F^{-1}(x-y)\psi(y)} + \bar{\eta}\psi + \bar{\psi}\eta \right)} \qquad (5.100)$$

This is the generating functional for the free Dirac theory. Since (5.100) is quadratic (at most) we can compute it through the rules we developed above for G-integrals. Shifting:

$$\psi_x \longrightarrow \psi_x - \int_y i S_F(x-y)\, \eta(y) = \psi'_x \qquad (5.101)$$

and performing the Grassmann integral over ψ'_x (which gives just N_D) we get

$$Z_D[\eta,\bar{\eta}] = e^{i \iint d^4x \, d^4y \; \bar{\eta}(x) \, i S_F(x-y)\, \eta(y)} \qquad (5.102)$$

where

$$iS_F^{-1}(x-y) \equiv (i\not{\partial}-m)\,\delta^4(x-y) \tag{5.103}$$

and thus

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\epsilon}\,e^{-ip\cdot(x-y)} \tag{5.104}$$

Yukawa theory

We are now ready to generalize the PI-perturbation theory to more generic theories with scalar and fermion fields. The Yukawa model of section (3.9) is represented by

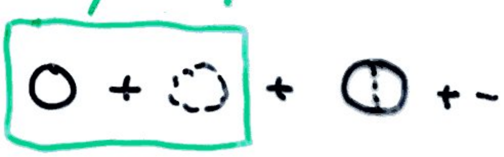
$$Z_{\text{Yukawa}}[J,\eta,\bar\eta] = \int \mathcal{D}\phi\,\mathcal{D}\bar\psi\,\mathcal{D}\psi\; e^{i\int(\mathcal{L}_{\text{Yukawa}} + J\phi + \bar\eta\psi + \bar\psi\eta)d^4x}$$

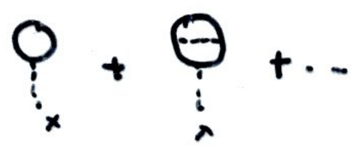
$$= e^{ig\int d^4x\,\frac{\delta}{\delta J_x}\frac{\delta}{\delta\eta_x}\frac{\delta}{\delta\bar\eta_x}}\; Z_{KG}[J]\,Z_{\text{Dirac}}[\eta,\bar\eta] \tag{5.105}$$

where $Z_{KG}[J]$ and $Z_{\text{Dirac}}[\eta,\bar\eta]$ are given above in eqs. (5.49) and (5.100). (Normalization as before $Z[0,0,0]\equiv 1$).

Examples.

actually not here with my norm. There are det(··) factors from N_{KG} and N_{Dirac}.

$Z \Rightarrow$ 

$\frac{\delta Z}{\delta J_x} =$ 

$\frac{\delta^2 Z}{\delta J_x\,\delta J_y} =$ 

5.5 Quantization of gauge fields, QED

Earlier we took the photon propagator to be, heuristically,

$$= -\frac{i g_{\mu\nu}}{k^2 + i\epsilon} . \qquad (5.106)$$

However, there were problems, for example with the norm of the field A_0. Polarization sum also contained terms $\sim k_\mu k_\nu / k^2$ (remember, however also the exercise .). These were found to be irrelevant in external lines due to Ward identity. Thus (5.106) did give correct results in the practical calculations. Let us now derive (5.106) – and other forms – for the photon propagator from P.I-methods.

Free photon field

Let us consider the path integral

$$\int \mathcal{D}A \, e^{iS[A]} \qquad (5.107)$$

where $\mathcal{D}A = \mathcal{D}A_0 \mathcal{D}A_1 \mathcal{D}A_2 \mathcal{D}A_3$ and

$$\begin{aligned} S[A] &= \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] \\ &= \int d^4x \, \frac{1}{2} A_\mu \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu \right) A_\nu \\ &= \int \frac{d^4k}{(2\pi)^4} \, \frac{1}{2} A_\mu(k) \overbrace{\left(-k^2 g^{\mu\nu} + k^\mu k^\nu \right)}^{= i D^{-1}_{\mu\nu}(k)} A_\nu(k) . \qquad (5.108) \end{aligned}$$

Can we interpret this as a transition amplitude? The action (5.108) vanishes when $A_\mu(k) = k^\mu \alpha$, i.e. when $A_\mu(x)$ is a pure gauge: $\partial_\mu \alpha$. Because this set is infinite (arbitrary $\alpha(x)$), the PI. (5.107) is badly divergent● ($e^{i\cdot 0}=1$). Similarly, each field $A_\mu \neq \partial_\mu \alpha$ can be associated with ∞ set of equivalent configurations

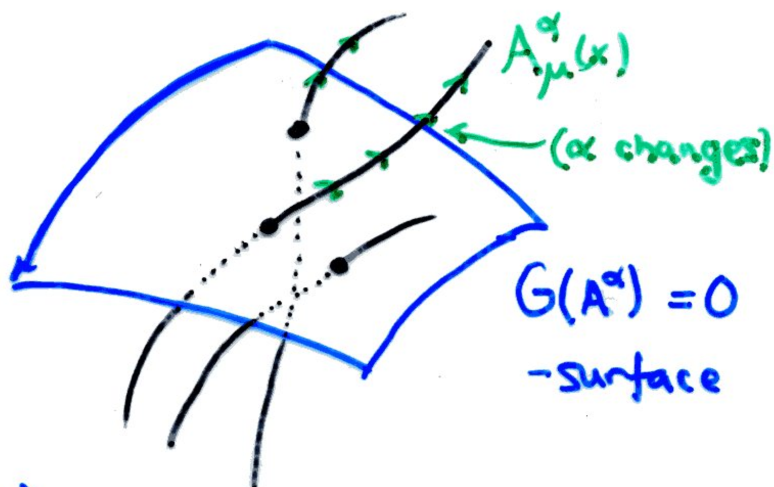
$$A_\mu^\alpha = A_\mu + \frac{1}{e}\partial_\mu \alpha \qquad (5.109)$$

PI (5.107) is thus badly defined, and unnormalizable. This obviously results from it containing integral over ∞ many physically equivalent configurations. Indeed, if we tried to include a source to (5.107) and perform the Gaussian integral, we would fail because the function (inverse propagator)

$$-iD^{-1}_{\mu\nu}(k) = -k^2 g_{\mu\nu} + k_\mu k_\nu$$

is singular. That is, it does not have an inverse, i.e. we cannot define a propagator.

In PI.-formalism this problem formally trivial to solve: we only need to constrain the PI with a suitable gauge constraint



$$G(A^\alpha) = 0 \qquad (5.111)$$

such that out of each α-path only one member contributes. So, we "only" need to set a functional δ-function $\delta(G(A^\alpha))$ into integral (5.107) to constrain it to surface $G(A^\alpha)=0$. This, however,

● Clearly $\int \mathcal{D}A\, e^{iS} \propto \int \mathcal{D}\alpha \ldots = \infty \times \ldots$

has to be done without changing the PI-measure•. To define a consistent procedure we introduce the following unity into (5.107):

$$\underline{1 \equiv \Delta_{FP}(A_\mu) \int \mathcal{D}\alpha\, \delta(G(A_\mu^\alpha))}, \qquad (5.112)$$

where the Faddeev-Popov determinant Δ_{FP} guarantees that the measure of the PI is conserved. FP-determinant defined by (5.112) is gauge-invariant

$$\begin{aligned}\Delta_{FP}^{-1}(A_\mu^{\alpha'}) &= \int \mathcal{D}\alpha\, \delta(G(A^{\alpha'\alpha}))\\ &= \int \mathcal{D}(\alpha'\alpha)\, \delta(G(A^{\alpha'\alpha}))\\ &= \int \mathcal{D}\alpha''\, \delta(G(A^{\alpha''})) = \Delta_{FP}^{-1}(A_\mu)\end{aligned} \qquad (5.113)$$

Let us now integrate (5.112) into integral (5.107). We get

$$\int \mathcal{D}A \left(\Delta_{FP}(A_\mu) \int \mathcal{D}\alpha\, \delta(G(A_\mu^\alpha))\right) e^{iS[A_\mu]}. \qquad (5.114)$$

Now make a gauge transformation $A_\mu \to A_\mu^{\alpha^{-1}}$ and note that $\mathcal{D}A$, $S[A]$ and $\Delta_{FP}(A_\mu)$ are all gauge invariant. We get

$$= \underline{\underbrace{\left(\int \mathcal{D}\alpha\right)}_{\text{The unphysical gauge d.o.f. infinity is now extracted}} \cdot \int \mathcal{D}A_\mu\, \underbrace{\Delta_{FP}(A_\mu)}_{\text{The price to pay. Additional FP-determinant.}}\, \underbrace{\delta(G(A_\mu))}_{\text{constraint to the surface } G(A)=0}\, e^{iS[A_\mu]}} \qquad (5.115)$$

• If for example $f(x)$ has a zero at $x=a$, we can use either $\delta(x-a)$ or $\delta(f(x)) = \frac{\delta(x-a)}{|f'(a)|}$ to constrain to $x=a$, but the latter also changes the measure.

In general Δ_{FP} can depend on A_μ, and thus change the PI essentially on surface $G(A_\mu)$. We will see this behaviour in connection with non-Abelian gauge theories in the next section. For QED however there is, for example, a class of Lorenz-covariant gauges for which $\Delta_{FP} = \text{const.}$ Let us now choose:

w.r.t A_μ.

$$\underline{G_\omega(A_\mu) = \partial_\mu A^\mu - \omega(x) = 0} \qquad (5.116)$$

Then

$$\Delta_{FP} = \mathrm{Det}\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right) = \det\left(\frac{1}{e}\partial^2\right) \qquad (5.117)$$

Now that Δ_{FP} does not depend on A_μ, it can be taken as a front factor in (5.115). After that we do one more trick. Since Δ_{FP} depends not on $\omega(x)$ either, we can integrate (5.115) over $\omega(x)$ with a Gaussian weight function:

$$= N(\xi)\int \mathcal{D}\omega\, e^{-i\int d^4x \frac{\omega^2}{2\xi}} \left(\int \mathcal{D}\alpha\right) \mathrm{Det}\left(\tfrac{1}{e}\partial^2\right) \int \mathcal{D}A\, \delta(\partial_\mu A^\mu - \omega)\, e^{iS[A]}$$

$$= \underbrace{N(\xi)\left(\int \mathcal{D}\alpha\right) \mathrm{Det}\left(\tfrac{1}{e}\partial^2\right)}_{\equiv \tilde{N}\ :\ A\text{-independent constant}} \int \mathcal{D}A\, e^{iS[A] - i\int d^4x \frac{1}{2\xi}(\partial_\mu A^\mu)^2}$$

additional piece to the action.

$$\underline{= \tilde{N} \int \mathcal{D}A\, e^{i\int d^4x \left(\mathcal{L}(A) - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right)}} \qquad (5.118)$$

gauge-infinity is now here.

the result of all slavery. An extra term to Lagrangian!

The new additional term looks modest but is all-important. First note that for a pure gauge

$$\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \longrightarrow \frac{k^4}{2\xi}\alpha^2$$

If $\xi \neq 0$ this integral is now strongly damped. Moreover:

$$iS[A] - \frac{i}{2\xi}\int d^4x\,(\partial_\mu A^\mu)^2 = \frac{i}{2}\int d^4x\, A_\mu(x)\left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu + \frac{1}{\xi}\partial^\mu\partial^\nu\right)A_\nu$$

$$= \frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{A}^\mu(k)\underbrace{\left(-k^2 g_{\mu\nu} + (1-\frac{1}{\xi})k_\mu k_\nu\right)}_{iD^{-1}_{\mu\nu}(k)}\tilde{A}^\nu(k) \qquad (5.120)$$

This inverse propagator function is no longer singular, and its inverse, the photon propagator is easily found:

$$D_{\mu\nu}(k) = \frac{-i}{k^2+i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) \qquad (5.121)$$

As special cases of this ξ-dependent propagator we get

$$\xi = 1: \quad D_{\mu\nu} = \frac{-i g_{\mu\nu}}{k^2+i\epsilon} \qquad \text{Feynman gauge}$$

$$\xi = 0: \quad D_{\mu\nu} = -\frac{i}{k^2+i\epsilon}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \qquad \text{Landau gauge} \qquad (5.122)$$

To conclude. Instead of the fundamental pi (5.107), we must base our theory on explicitly gauge-parameter dependent, but cured from gauge-infinities PI (5.118).

This can indeed be done. Consider a gauge-invariant operator $\mathcal{O}(A)$. Then obviously

$$\langle 0|\,T[\mathcal{O}(A)]\,|0\rangle = \frac{\int \mathcal{D}A\, \mathcal{O}(A)\, e^{i\int d^4x\,(\mathcal{L} - \frac{1}{2\xi}(\partial A)^2)}}{\int \mathcal{D}A\, e^{i\int d^4x\,(\mathcal{L} - \frac{1}{2\xi}(\partial A)^2)}} \qquad (5.123)$$

To derive this one just starts from the fundamental PI with $\mathcal{O}(A)$ inserted and extracts the gauge-infinity by FP-trick. Eventually all infinities cancel in the ratio. Note that gauge-invariance of $\mathcal{O}(A)$ is essential requirement for obtaining (5.123).

It can be shown that FP-procedure produces a gauge-invariant and unitary S-matrix for asymtotic states. (See PS p. 208).

We are now ready to write QED in terms of path integrals:

$$\mathcal{L}^{\xi}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \bar{\psi}(i\not{\partial} - m)\psi \qquad (5.124)$$

For example (5.118), that is. Actually this is but one choice. Other gauges lead to different $\mathcal{L}_{eff}$, but all gauges give eventually the same physical results.

and

$$Z^{\xi}_{QED}[J^\mu, \eta, \bar\eta] = \frac{1}{N^{\xi}_{vac}}\, e^{-ie\int d^4x \frac{\delta}{\delta\eta}\gamma^\mu \frac{\delta}{\delta J^\mu}\frac{\delta}{\delta\bar\eta}}\, Z^{\xi}_{FP}[J^\mu]\, Z_D[\eta,\bar\eta] \qquad (5.125)$$

where the free theory generating function $Z_D[\eta,\bar\eta]$ for fermions is defined in (5.100). Now that our propagator-functions can be inverted, we can compute straightforwardly for the free gauge field:

$$Z^{\xi}_{FP}[J^\mu] = \int \mathcal{D}A\, e^{\frac{i}{2}\int_{x,y} A^\mu(x)\, iD^{-1,\xi}_{\mu\nu}(x-y)\, A^\nu(y) + i\int_x J_\mu(x) A^\mu(x)}$$

$$= e^{\frac{i}{2}\int_{x,y} J^\mu(x)\, iD^{\xi}_{\mu\nu}(x-y)\, J^\nu(y)} \qquad (5.126)$$

where the Fourier-space form for $iD^{\xi}_{\mu\nu}$ is already given in (5.121).

The normalization factor N^{ξ}_{vac} is just the vacuum-vacuum-transition amplitude:

$$N^{\xi}_{vac} \equiv e^{-ie\int d^4x \frac{\delta}{\delta\eta}\gamma^\mu\frac{\delta}{\delta J^\mu}\frac{\delta}{\delta\bar\eta}}\, Z^{\xi}_{FP}[J^\mu]\, Z_D[\eta,\bar\eta]\Big|_{J^\mu=\eta=\bar\eta=0} \qquad (5.127)$$

Generating function Z^{ξ}_{QED} generates the QED perturbation theory to arbitrary order. Functions for connected- and 1PI-graphs can be also defined

$$W^{\xi}_{QED}[J^\mu;\eta,\bar\eta] = -i\ln Z^{\xi}_{QED}[J^\mu;\eta,\bar\eta]$$

$$\Gamma^{\xi}_{1PI}[A^\mu_c,\psi_c,\bar\psi_c] = W[J^\mu,\eta,\bar\eta] - \int d^4x\,(J_\mu A^\mu_c + \bar\psi_c\eta + \bar\eta\psi_c) \qquad (5.128)$$

and they generate expansions:

$$\frac{\delta^2 W}{\delta\bar\eta\,\delta\eta} \sim$$

$\frac{\delta^2 W}{\delta J_\mu \delta J_\nu} \sim$ 

$\frac{\delta^3 W}{\delta J_\mu \delta\bar{\eta} \delta\eta} \sim$ 

etc.

Box. FP-trick in case of an ordinary function.

Consider integral

$$W \equiv \int dx\,dy\; e^{iS(x,y)} = \int d\vec{r}\; e^{iS(\vec{r})} \qquad \text{(FP1)}$$

Now assume that $S(\vec{r}) = S(|\vec{r}|)$, ie the (action) $S(\vec{r})$ is invariant:

$$S(\vec{r}) = S(\vec{r}_\phi) \qquad \text{(FP2)}$$

in rotations (gauge-transforms)

$$\vec{r} = (r, \theta) \longrightarrow \vec{r}_\phi = (r, \theta - \phi). \qquad \text{(FP3)}$$

Now the volume factor arising from the symmetry is clearly $\int d\phi = 2\pi$. Let us extract this symmetry factor in FP-fashion, however. Introducing unity

$$1 = \int d\phi\; \delta(\theta - \phi) \qquad \text{(FP4)}$$

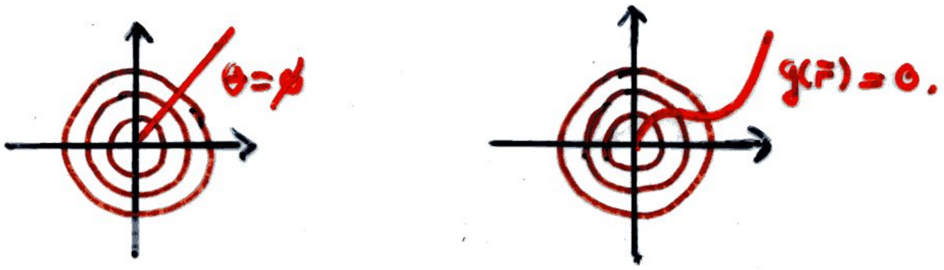
into (FP1) we get

$\Gamma = \int dr\; e^{iS(r)} \equiv w$

$$W = \int d\phi \left[\int d\vec{r}\; e^{iS(\vec{r})}\, \delta(\theta - \phi) \right] = \int d\phi\, [W_\phi] = w \int d\phi = 2\pi w$$

where we used the invariance $S(\vec{r}) = S(r)$ to get $W_\phi = W_{\phi'} = W$. The gauge condition we used here: $\theta = \phi$ was so simple that no FP-determinant arose. More generally we can set instead

$$g(\vec{r}) = 0 \qquad \text{(FP5)}$$



Then our unity in (FP4) becomes

$$1 = \Delta_g(\vec{r}) \int d\phi \, \underbrace{\delta[g(r_\phi)]}_{= \frac{1}{\left|\frac{\partial g}{\partial \theta}\right|_{g(r_\phi)=0}} \delta(r - r_\phi)}$$

where

$$\Delta_g(\vec{r}) \equiv \left.\frac{\partial g(\vec{r})}{\partial \theta}\right|_{g(\vec{r})=0} .$$

This is invariant in rotations: $\vec{r}_{\phi+\phi'} = (r, \phi+\phi')$ because $d(\phi+\phi') = d\phi$

$$\Delta_g^{-1}(\vec{r}_{\phi'}) = \int d\phi \, \delta\left[g(\vec{r}_{\phi+\phi'})\right] = \int d\phi'' \delta\left[g(\vec{r}_{\phi''})\right] = \Delta_g^{-1}(\vec{r}) .$$

We can now repeat our calculation above with arbitrary $g(\vec{r})$:

$$W = \int d\phi \underbrace{\int d\vec{r} \, e^{iS(\vec{r})} \Delta_g(\vec{r}) \, \delta(g(\vec{r}_\phi))}_{W_\phi = W.} = W \int d\phi = 2\pi W$$

determinant is left in.

because indeed

$$W_{\phi'} = \int dr \, e^{iS(\vec{r})} \Delta_g(\vec{r}) \, \delta(g(\vec{r}_{\phi'})) = \int d\vec{r}' \, e^{iS(\vec{r}')} \Delta_g(\vec{r}') \, \delta(g(r'_\phi)) = W_\phi = W$$

5.6. Non-Abelian gauge theories

QED is an Abelian gauge theory. With this one refers to the fact that all group-elements U_θ commute: $U_\theta U_{\theta'} = U_{\theta'} U_\theta$. Theories that are based on higher order unitary groups $SU(N)$, $N>1$, are called Non-Abelian, because then

$$[U_\theta, U_{\theta'}] \neq 0.$$

In QED $U_\theta = e^{i\theta}$ was a simple phase. In $SU(N)$, whose generators τ^a obey die algebra

$$[\tau^a, \tau^b] = i f^{abc} \tau^c, \qquad (5.129)$$

↑ structure functions: $SU(2): \begin{cases} f^{abc} \to \epsilon^{ijk} \\ \tau^a \to \frac{1}{2}\sigma^i \end{cases}$

the transform becomes

$$\psi_i \to U_{\theta_{ij}} \psi_i = \left(e^{-i\theta^a \tau^a}\right)_{ij} \psi_i \qquad (5.130)$$

(Normally (but not necessarily) fermions are put to the fundamental representation of the group. This reps. has dimension N for $SU(N)$)

The transform (5.130) can be local, when $\theta^a = \theta^a(x)$. Now, if we wish to impose a local $SU(N)$-invariance on our theory with the kinetic term $\bar{\psi}_i \not{\partial} \psi_i$. (sum implied), we need to introduce the covariant derivative:

$$D_\mu^{ij} = \delta^{ij} \partial_\mu - i g \tau^{ij} \cdot A_\mu \qquad (5.131)$$

↑ representation indices for fermions. Often suppressed.

It is easy to show that the invariance

$$\bar{\Psi}\not{D}\Psi \equiv \bar{\Psi}_\theta \not{D}_\theta \Psi_\theta \qquad (5.132)$$

is obeyed if

$$\tau \cdot A^\theta_\mu = U_\theta \, \tau \cdot A_\mu \, U^\dagger_\theta - \frac{i}{g}(\partial_\mu U_\theta) U^\dagger_\theta \qquad (5.133)$$

If we insert (5.130) into this expression and keep θ^a infinitesimal, we find

$$\delta A^a_\mu = -\frac{1}{g}\partial_\mu \theta^a + f^{abc}\theta^b A^c_\mu \qquad (5.134)$$

$$\delta \Psi = i\theta^a \tau^a \Psi \qquad (5.135)$$

A gauge invariant and covariant field strength tensor can be defined as

$$F_{\mu\nu} \equiv \frac{i}{g}[D_\mu, D_\nu]$$

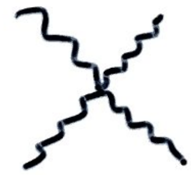
$$= (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu)\tau^a \equiv F^a_{\mu\nu} \cdot \tau^a \qquad (5.136)$$

And with this we define the Yang-Mills action:

$$iS_{YM} \equiv -\frac{i}{2}\int d^4x \, \mathrm{Tr}(F^2_{\mu\nu}) = i\int d^4x \overbrace{\left(-\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu}\right)}^{\mathcal{L}_{YM}} \qquad (5.137)$$

This action differs radically from QED in that now the gauge-fields have self-interactions. For example:

$$F^a_{\mu\nu}F^{a,\mu\nu} \supset g^2 f^{abc} f^{ade} A^a_\mu A^b_\nu A^{d,\mu} A^{e,\nu} \quad \sim \tag{5.138}$$



The corresponding Feynman rules are fairly complicated. Before computing them we still have to cure the action (5.137) from the gauge infinities associated with (5.137), just like we did with the QED. So we start from

$$\underline{\int \mathcal{D}A^a_\mu \, e^{i\int \mathcal{L}_{YM} d^4x}} \tag{5.139}$$

and insert the Faddeev-Popov unity, based on some specific gauge. We can choose e.g. the Lorentz-gauge:

$$\underline{G_\omega(A^a_\mu) \equiv \partial_\mu A^{a,\mu} - \omega^a = 0.} \tag{5.140}$$

Going through steps exactly analogous to derivation in QED we can rewrite (5.139) as

$$= \left(\int \mathcal{D}\theta\right) \cdot \int \mathcal{D}A^a_\mu \, \Delta^{YM}_{FP}(A^a_\mu)\, \delta\big(G(A^a_\mu)\big)\, e^{i\int d^4x\, \mathcal{L}_{YM}}$$

$$\underline{= N_\xi \left(\int \mathcal{D}\theta\right) \cdot \int \mathcal{D}A^a_\mu \, \Delta^{YM}_{FP}(A^a_\mu)\, e^{i\int d^4x \left(\mathcal{L}_{YM} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2\right)}} \tag{5.141}$$

where on the last line we again performed a Gaussian weighted integral over all possible gauge-configurations. The essential difference between YM-theories and QED is that here Δ_{FP} is not a constant. It can therefore not be removed from the integral. We can actually compute it explicitly, given (5.140)

$$\Delta_{FP}^{-1}(A_\mu^a) = \int \mathcal{D}\theta\, \delta(G(A_\mu^{\theta,a}))$$

$$= \mathrm{Det}^{-1}\left(\frac{\delta G}{\delta \theta}\right)_{\theta=0}$$

(note that only the infinitesimal form is needed.)

$$\overset{(5,140)\,\&\,(5,139)}{=} \mathrm{Det}^{-1}\left(\frac{\partial^\mu \delta A_\mu^{\theta,a}(x)}{\delta\theta^b(y)}\right)$$

Thus

$$\Delta_{FP}^{YM}(A_\mu^a) = \mathrm{Det}\left(\frac{\partial^\mu\left(-\frac{1}{g}\partial_\mu\theta^a(x) + f^{adc}\,\theta^d(x)\,A_\mu^c(x)\right)}{\delta\theta^b(y)}\right)$$

$$\doteq \mathrm{Det}\left(\partial^\mu\left[-\frac{1}{g}\delta^{ab}\partial_\mu + f^{abc}A_\mu^c(x)\right]\delta^4(x-y)\right)$$

$$\equiv \mathrm{Det}(M_{ab}(x-y)). \qquad (5,142)$$

Such determinant can be rewritten as a Gaussian functional integral over some fictitious fermionic fields $\bar{c}_a(x)$ and $c_b(x)$:

$$\mathrm{Det}(M_{ab}) = \int \mathcal{D}\bar{c}\,\mathcal{D}c\; e^{i\int_{x,y} \bar{c}^a(x)\, M_{ab}(x-y)\, c^b(y)}$$

$$= \int \mathcal{D}\bar{c}\,\mathcal{D}c\; e^{i\int d^4x\, \bar{c}^a\, \partial^\mu[\delta^{ab}\partial_\mu - g f^{abc} A_\mu^c]\, c^b} \qquad (5,143)$$

Kinetic term of a massless bosonic field (Yet c:s are fermionic variables)

coupling to gauge fields

These new (non-physical) degrees of freedom are called Faddeev-Popov ghosts.

The entire effective action for Y-M-theory then is:

$$\mathcal{L}^{\xi}_{Y\text{-}M} = -\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar{c}^a\partial^\mu[\delta^{ab}\partial_\mu - gf^{abc}A^c_\mu]c^b$$

$$= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar{c}^a\partial^2 c^a + \mathcal{L}_I \qquad (5,144)$$

where the interaction term contains the ghost-gauge - and gauge-self interactions:

Did one partial integration here.

$$\boxed{\begin{aligned}\mathcal{L}_I = &+ g f^{abc}(\partial_\mu \bar{c}^a)A_\mu c^b \\ &- \frac{1}{2}g f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)A^{b,\mu}A^{c,\nu} \\ &+ \frac{1}{4}g^2 f^{abc}f^{ade}A^b_\mu A^c_\nu A^{d,\mu}A^{e,\mu}\end{aligned}} \qquad (5,145)$$

The propagators of the theory are easily read from (5,144):

μ,a ~~~A~~~ ν,b : $$-i\delta_{ab}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]\frac{1}{k^2+i\epsilon} \qquad (5,146)$$

a - - - c - - - b : $$-i\delta_{ab}\frac{1}{k^2+i\epsilon} \equiv iG^{ab}(k) \qquad (5,147)$$

For perturbation theory we will also need the generating functions and the vertex Feynman rules. Generating function Z is:

$$Z[J^a_\mu, \bar{\eta}^b_c, \eta^c_c] = e^{i\int d^4x\, \mathcal{L}_I\left(-\frac{i\delta}{\delta J^a_\mu}, -\frac{i\delta}{\delta \eta^b_c}, -\frac{i\delta}{\delta \bar{\eta}^c_c}\right)} Z^{\xi}_{YM}[J^a_\mu]\, Z[\eta^b_c, \bar{\eta}^c_c] \qquad (5,148)$$

where $Z_{YM}[J^a_\mu]$ is just like the QED-function in (5.126), but with a sum over a, and the ghost-function is

$$\underline{Z_i[\bar{\eta},\eta] = e^{-i\int d^4y\, d^4x\, \bar{\eta}^a(x)\, i\, G^{ab}(x-y)\,\eta^b(y)}} \qquad (5.149)$$

Vertex functions can be deduced by use of symmetry from (5.145). (See eg. P&S) Let us derive here eg. the AAA-vertex through a little heavier machinery involving 1PI-functions. First start from generic 3-point function:

$$\frac{\delta^3 W}{\delta J^a_\mu(x)\,\delta J^b_\nu(y)\,\delta J^c_\rho(z)} \overset{\text{lowest order}}{=} -\frac{i}{2} g f^{def} \int d^4w \left(\partial_\alpha \frac{\delta}{\delta J^d_{\beta,w}} - \partial_\beta \frac{\delta}{\delta J^d_{\alpha,w}}\right) \frac{\delta^2}{\delta J^{e,\alpha}_w\,\delta J^{f,\beta}_w}$$

$$\times \underbrace{\frac{\delta^3}{\delta J^a_{\mu,x}\,\delta J^b_{\nu,y}\,\delta J^c_{\rho,z}}\, e^{-\frac{i}{2}\int_{z_1 z_2} J_{z_1}\, i\Delta_{z_1 z_2} J_{z_2}}}$$

$$= (i\Delta J)^a_{\mu,x}\,(i\Delta J)^b_{\nu,y}\,(i\Delta J)^c_{\rho,z}\, e^{\int J\Delta J} + (\to 0)$$

(5.150)

$$= -\frac{i}{2} g f^{def} \int d^4w \Big\{ \left[\partial_\alpha\, i\Delta^{ad}_{\mu\beta}(x-w) - \partial_\beta\, i\Delta^{ad}_{\nu\alpha}(x-w)\right] \times$$

$$\times \Big(i\Delta^{be}(y-w)^\alpha{}_\nu\; i\Delta^{cf}(z-w)^\beta{}_\rho$$

$$+\, i\Delta^{bf}(y-w)^\beta{}_\nu\; i\Delta^{ce}(z-w)^\alpha{}_\rho \Big)$$

$$+ \text{ 2 more terms } (\downarrow \text{ and } \downarrow) \Big\} \qquad (5.151)$$

We are interested in the 1PI-function $\Gamma^{(3)}_{1PI}$ however. This can be found by inverting the expression

$$W^{(3)} = \int \prod_{i=1}^{3} d^4y_i\; W^{(2)}(x_i,y_i)\,\Gamma^{(3)}_{1PI}(y_1,y_2,y_3) \qquad (5.152)$$

to get $\Gamma^{(3)}_{1PI}$ in terms of $W^{(3)}$:

$$\Gamma_{1PI}(y_1,y_2,y_3) = \int \prod_{i=1}^{3} d^4x_i \, \underbrace{W^{(2)^{-1}}(y_i,x_i)}_{=\,\Gamma^{(2)}_{1PI}(y_i,x_i)} \, W^{(3)}(x_1,x_2,x_3) \qquad (5.153)$$

Apply this formula to (5.151) to lowest order, and use the identity

$$\sum_{a'} \int d^4x \, (i\Delta^{-1})^{aa'}_{}(y-x)_{\mu}{}^{\mu'} \, \partial_\alpha \, i\Delta^{a'd}_{\beta\mu'}(x-w)$$

$$= \int \frac{d^4q}{(2\pi)^4} e^{iq\cdot(w-y)} iq_\alpha g_{\beta\mu} \delta^{ab} \qquad (5.154)$$

we get

$$\Gamma^{(3)}(y_1,y_2,y_3)^{abc}_{\mu\nu\rho} = \frac{1}{2} g f^{def} \int d^4w \, \int \prod_{i=1}^{3} \frac{d^4q_i}{(2\pi)^4} \overbrace{e^{i\sum_i q_i w}}^{\to (2\pi)^4\delta^4(\Sigma_i q_i)} e^{-i\sum_i q_i\cdot y_i}$$

$$\times \Big\{ (q_{1\alpha} g_{\beta\mu} - q_{1\beta} g_{\alpha\mu}) \delta^{ad} (\delta^{be} g^\alpha{}_\nu \delta^{cf} g^\beta{}_\rho + \delta^{bf} g^\beta{}_\nu \delta^{ce} g^\alpha{}_\rho)$$

$$+ 2 \text{ terms} \Big\}$$

$$\equiv \int \prod_{i=1}^{3} \frac{d^4q_i}{(2\pi)^4} e^{-i\sum q_i\cdot y_i} \, \tilde{\Gamma}^{(3)}(q_1,q_2,q_3)^{abc}_{\mu\nu\rho} \qquad (5.155)$$

Where eventually

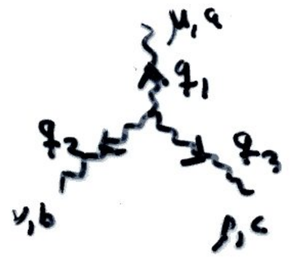
$$i\Gamma^{(3)\,abc}_{\mu\nu\rho}(q_1,q_2,q_3) = \underbrace{(2\pi)^4\delta^4(\Sigma q_i)}_{\text{4-momentum conservation}} \times$$

$$\times \underbrace{igf^{abc}}_{\text{vertex Feynman rule.}} (q_{1\rho} g_{\mu\nu} - q_{1\nu} g_{\rho\mu} + 4 \text{ other terms}) \qquad (5.156)$$

We could continue working out laborously the four remaining terms. However, when one term is found the rest can be found by symmetrizing the result. Indeed since A-fields are bosonic the vertex function must be symmetric under exchange of any two A's. However f^{abc} is antisymmetric, so the remaining part must also be antisymmetric under exchange of $\mu_i \leftrightarrow \mu_j$ or $q_i \leftrightarrow q_j$. So we must have:

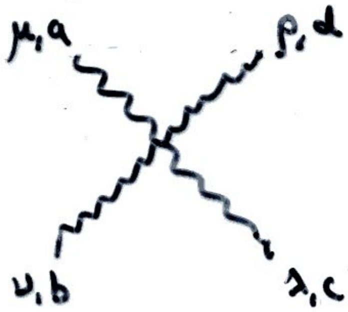
$$i\Gamma^{(3)}(q_1,q_2,q_3)^{abc}_{\mu\nu\rho} = igf^{abc}\Big((q_1-q_2)_\rho g_{\mu\nu} + (q_2-q_3)_\mu g_{\nu\rho} + (q_3-q_1)_\nu g_{\mu\rho}$$

(5,157)



$$q_1+q_2+q_3=0$$

Similarly

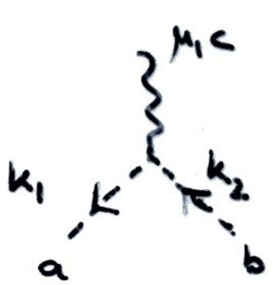


$$\begin{aligned} i\Gamma^{abcd}_{\mu\nu\lambda\rho} = ig^2\Big(& f^{abe}f^{cde}(g_{\mu\lambda}g_{\nu\rho} - g_{\nu\lambda}g_{\mu\rho}) \\ & + f^{ace}f^{bde}(g_{\mu\nu}g_{\lambda\rho} - g_{\lambda\nu}g_{\mu\rho}) \\ & + f^{ade}f^{cbe}(g_{\mu\lambda}g_{\rho\nu} - g_{\rho\lambda}g_{\mu\nu})\Big) \end{aligned}$$

$$q_1+q_2+q_3+q_4=0$$

(5,158)

And finally



$$i\Gamma^{abc}_{\mu} = gf^{abc}k_{1\mu}$$

(5,159)