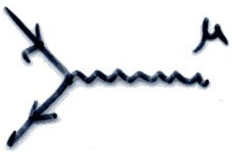
3.10 QED

Based on what we have seen, we can read off the Feynman rules for fermions in QED:

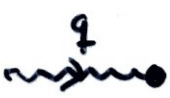
$$\mathcal{L}_{QED} = \bar{\psi}(i\not{\partial}-m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e\bar{\psi}\not{A}\psi \qquad (3.135)$$

① External Fermion legs. Just as in Yukawa th. on p. 124.

② Vertex:  : $-ie\gamma^\mu$ (3.136)

In addition we will need rules for external photons and for the photon propagator. They are

③  : $\epsilon_\mu(q)$

 : $\epsilon^*_\mu(q)$

④ Propagator:  $-\dfrac{ig_{\mu\nu}}{q^2+i\epsilon}$ (3.137)

(Feynman- or Lorentz gauge.)

We will not derive the rules (3.137) exactly in the canonical quantization, where the elimination of the gauge d.o.f's is a little delicate. We shall argue for them heuristically and leave a better proof for Path Integral quantization, where removing gauge d.o.f.'s is very easy. Choose to work in the Feynman gauge (or Lorentz gauge)

$$\partial_\mu A^\mu \equiv 0 \qquad (3.138)$$

It then follows that free A_μ-fields satisfy (for all μ)

$$\partial^2 A_\mu = 0 \qquad (3.139)$$

Eqs. (3.138) and (3.139) have a solution

$$A_\mu = \epsilon_\mu(q) e^{-iq\cdot x} \qquad (3.140)$$

where

$$\underbrace{q^2}_{\text{on-shell}} = \underbrace{\epsilon\cdot q = 0}_{\text{gauge condition}} \qquad (3.141)$$

We can now write the photon field operator by combining our experience from the KG- and Dirac fields:

$$A_\mu(x) = \int \frac{d^3q}{(2\pi)^3 2\omega_q} \underbrace{\sum_{\lambda=0}^{3}}_{\text{polarization sum } \lambda=0,1,2,3\,!} \left(a_q^\lambda \, \epsilon_\mu^\lambda(q) \, e^{-iq\cdot x} + a_q^{\lambda\,\dagger} \, \epsilon_\mu^{\lambda\,*}(q) \, e^{iq\cdot x} \right) \qquad (3.142)$$

- Feynman rules for external legs now follow just as before with fermions and KG-fields.

- Note that the polarization sum has 4 d.o.f's. In quantization one must make sure that the spurious extra gauge dof's are not part of the physical spectrum.* As mentioned, this is easiest to implement (through (3.138)) in PI-formalism.

- Propagator can (in a given gauge) be derived in the same way we did before. Introducing a current j^μ $(= e\bar{\psi}\gamma^\mu\psi)$ we get (Feynman gauge)

$$\partial^2 A^\mu(x) = j^\mu(x) \tag{3.143}$$

The corresponding Greens function is:

$$\partial^2 D_{\mu\nu}(x-y) = i g_{\mu\nu}\,\delta^4(x-y) \tag{3.144}$$

Fourier transforming and choosing the T-ordered boundary conditions one immediately arrives to the expression (3.137).

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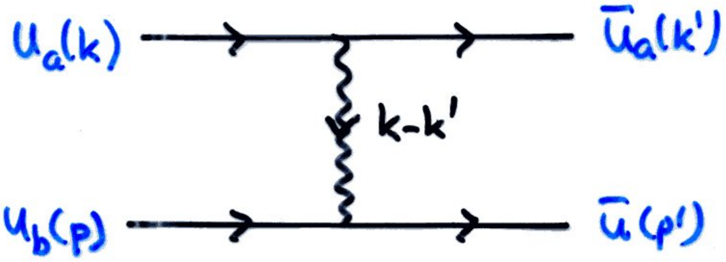
* For example, the operator $\hat{A}_0$ has a negative norm. See P&S p.128.

Coulomb potential

As first example, we consider Coulomb scattering. We have two (different) fermions a and b, so to lowest order $\psi_a \psi_b \to \psi_a \psi_b$ has one graph:



(3.145)

Note that this diagram comes from a contraction similar to (3.124). The only difference is that now $a \neq b$, so that the cross-contraction is not possible:

$$2 \cdot \frac{1}{2!} \langle 0 | T(\psi^a_{y_1} \psi^b_{y_2} \quad \bar{\psi}^a_w A_w \psi^a_w \quad \bar{\psi}^b_z A_z \psi^b_z \quad \bar{\psi}^a_{x_1} \bar{\psi}^b_{x_2}) | 0 \rangle \tag{3.146}$$

This contraction has a relative $(-1)^3$ sign factor and the symmetry factor is 1. The extra 2 in (3.146) arises from the fact that

$$\mathcal{L}_I = \mathcal{L}_I^a + \mathcal{L}_I^b \tag{3.147}$$

so that to second order

$$\frac{1}{2!} (\mathcal{L}_I)^2 = \frac{1}{2} (\mathcal{L}_I^a + \mathcal{L}_I^b)^2 = \frac{1}{2!} 2 \mathcal{L}_I^a \mathcal{L}_I^b + \dots \tag{3.148}$$

Feynman rules now give the T-matrix elements:

$$T_{ab\to ab} = -(-ie)^2 Q_a Q_b\, \bar{u}_a(k_1')\gamma^\mu u_a(k_1) \frac{-i g_{\mu\nu}}{(k-k')^2+i\epsilon} \bar{u}_b(p')\gamma^\nu u_b(p) \qquad (3.149)$$

Let us now consider the case when

$$m_b \gg |p_b| \qquad m_b \gg m_a \qquad (3.150)$$

It is now easy to show that

$$\bar{u}_b(p')\gamma^\mu u_b(p) \approx \delta^{\mu,0}\, u(p)^\dagger u(p) \approx 2m\,\delta^{\mu,0} \qquad (3.151)$$

and furthermore, since total momentum is conserved:

$$-E_k + E_{k'} = E_p - E_{p'} \approx 0 \qquad (3.152)$$

Then

$$T_{ab\to ab} \approx Q_a Q_b \frac{-ie^2\cdot 2m_b}{|\vec{k}-\vec{k}'|^2 - i\epsilon}\, \bar{u}_a(k')\gamma^0 u(k) \qquad (3.153)$$

Because particle b is now stationary, we can reformulate our $2\to 2$ scattering as a scattering off a potential. Indeed:

infinite time

$$\int \underbrace{\frac{d^3q'}{(2\pi)^3 2E_{q'}}}_{\text{integral over final states}} \frac{|S_{fi}|^2_{2\to 2}}{\underbrace{2E_q}_{\#\text{ of target particles}} V} = \int \frac{d^3q'}{(2\pi)^3 2E_{q'} 2E_q}\left(T(2\pi)^4\delta^4(p+q-p'-q')\,|T_{fi}|^2_{2\to 2}\right)$$

$$= T\,2\pi\,\delta(E_k-E_k')\left|\frac{T_{fi}}{2m_b}\right|^2 \equiv \left|2\pi\,\delta(E_k-E_k')\,T_{fi}^{1-1}\right|^2$$

phase space density of states b.

$$T\,2\pi\,\delta(E_k-E_k') = (2\pi\,\delta(E_k-E_k'))^2$$

When you further observe that

$$\frac{1}{4\pi}\int d^3x \, \frac{1}{|\vec{x}|} e^{i\vec{q}\cdot\vec{x}} = \frac{1}{|\vec{q}|^2 - i\epsilon} \qquad (3.155)$$

and

$$2\pi\,\delta(E_k - E_{k'}) = \int dx_0 \, e^{-i(k_0 - k_0')x_0}, \qquad (3.156)$$

we can rewrite (3.153) as

$$S^{1-1}_{f\neq i} = -i\int d^4x \, e^{i(k-k')\cdot x} \frac{Q_a Q_b e^2}{4\pi|\vec{x}|} \bar{u}(k)\gamma^0 u(k')$$

See exercise!

$$= -i\int d^4x \, \bar{\psi}_k(x) \left[A_0(x)\gamma^0\right] \psi_{k'}(x) \qquad (3.157)$$

That is, scattering off a heavy target reduces to scattering off a classical Coulomb potential (to 1st order in PT):

$$A_0 = \frac{Q_a Q_b}{4\pi|\vec{x}|}. \qquad (3.158)$$

Potential is repulsive for $Q_a Q_b > 0$ and attractive for $Q_a Q_b < 0$.
(Signs, the relative $(-1)^3$ in particular were important!)

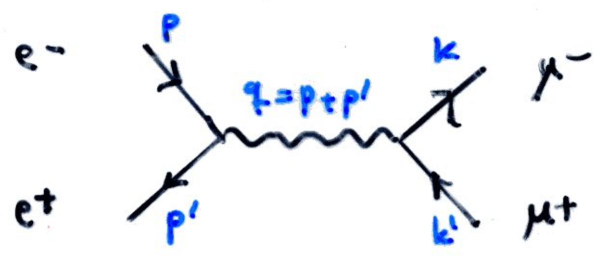
4 BASIC QED PROCESSES

The computationally simplest QED process is the annihilation process, where initial and final state particles are different. These processes include $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to q\bar{q}$.

4.1. $e^+e^- \to \mu^+\mu^-$

We will first compute the unpolarized cross section for the process $e^+e^- \to \mu^+\mu^-$. This is often the desired quantity, given that the spins of the incoming particles are arbitrary (beams are unpolarized), and one does not measure the spins of the final state. (Inclusion of initial and final state polarization can be done of course; calculations only get a little messier.

- The only Feynman diagram, to lowest nontrivial order is



$$\left(\mathcal{L}_I = +eQ_\ell \bar{\psi}_\ell \gamma^\mu \psi A_\mu \right)$$

- The matrix element can be read off using the rules (3.136–3.137)

$$T = (-ie)^2 \underbrace{\bar{v}_e(p',s')\gamma^\mu u_e(p,s)}_{\text{Lorentz-vectors}} \frac{-ig_{\mu\nu}}{q^2+i\epsilon} \underbrace{\bar{u}_\mu(k,r)\gamma^\nu v_\mu(k',r')}_{\text{Lorentz-vectors}} \qquad (4.1)$$

$$\equiv -\frac{ie^2}{q^2+i\epsilon}\, a^\mu b_\mu \qquad (4.1a)$$

The cross section $\sigma \sim |T|^2 = TT^+$, so we have to compute

$$|T|^2 = TT^+ = \frac{e^4}{q^4}\left[a^\mu a^{+\nu}\right]\left[b_\mu b^+_\nu\right] \qquad (4.2)$$

Now

$$[\bar{v}\gamma^\mu u]^+ = u^+ \gamma^0 \gamma^0 \underbrace{\gamma^{\mu +}\gamma^{0+}}_{=\gamma^\mu} v = \bar{u}\gamma^\mu v \qquad (4.3)$$

Unpolarized cross section is found by summing over the final state - and averaging over the initial state spins:

$$\langle |T|^2 \rangle \equiv \frac{1}{4}\sum_{\substack{s,s'\\ r,r'}} |T|^2$$

So we must compute e.g.

$$\sum_{s,s'} a^\mu a^{+\nu} = \sum_{s,s'} \bar{v}(p',s')_a \gamma^\mu_{ab}\, u(p,s)_b\, \bar{u}(p,s)_c\, \gamma^\nu_{cd}\, v(p',s')_d$$

$$= (\not{p}' - m_e)_{da}\gamma^\mu_{ab}(\not{p} + m_e)_{bc}\gamma^\nu_{cd}$$

$$= \mathrm{Tr}\left[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right] \equiv \underline{L^{\mu\nu}_{(e)}(p',p)} \qquad (4.4)$$

An identical calculation gives

$$\sum_{r,r'} b_\mu b^+_\nu = L^{(\mu)}_{\mu\nu}(k',k) = L^{(\mu)}_{\mu\nu}(k,k') \qquad (4.5)$$

To compute these Leptonic Tr – tensors, we need a few Trace-theorems derived in the excercizes:

$$\mathrm{Tr}[1] = 4$$

$$\mathrm{Tr}[\gamma_{\alpha_1} \cdots \gamma_{\alpha_{2n+1}}] = 0$$

$$\mathrm{Tr}[\gamma_\mu \gamma_\nu] = 4 g_{\mu\nu}$$

$$\mathrm{Tr}[\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma] = 4(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho})$$

$$\mathrm{Tr}[\gamma^5] = 0$$

$$\mathrm{Tr}[\Gamma \gamma^5] = 0 \quad ; \quad \Gamma = \gamma, \gamma\gamma, \gamma\gamma\gamma$$

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5] = -4i\epsilon^{\mu\nu\rho\sigma} \qquad (4.6)$$

With help of these we find

$$L^{\mu\nu}_{(e)}(p,p') = \mathrm{Tr}[\not{p}' \gamma^\mu \not{p} \gamma^\nu] - m^2 \mathrm{Tr}[\gamma^\mu \gamma^\nu]$$

$$= 4\Big(p'^\mu p^\nu + p'^\nu p^\mu - \underbrace{(m^2 + p \cdot p')}_{= s/2\ ?} g^{\mu\nu} \Big) \qquad (4.7)$$

and

$$L^{(\mu)}_{\mu\nu}(k,k') = 4\left(k'_\mu k_\nu + k'_\nu k_\mu - \frac{s}{2} g_{\mu\nu} \right) \qquad (4.7a)$$

and so:

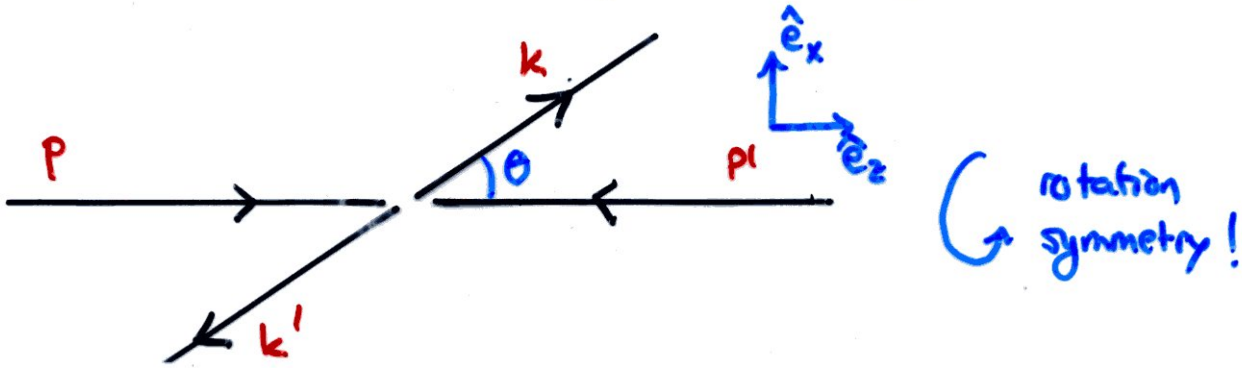
$$L^{\mu\nu}_{(e)} L^{(\mu)}_{\mu\nu} = 16\Big(2\, p\cdot k\, p'\cdot k' + 2\, p\cdot k'\, p'\cdot k - s\,(k\cdot k' + p\cdot p') + s^2 \Big) \qquad (4.8)$$

where we used $m_e^2 + p\cdot p' = \frac{1}{2}(p+p')^2 = \frac{s}{2} = \frac{1}{2}(k+k')^2 = m_\mu^2 + k\cdot k'$.

All together then:

$$\langle |T|^2 \rangle = \frac{1}{4} \cdot \frac{32 e^4}{q^4} \left(p\cdot k\, p'\cdot k' + p\cdot k'\, p'\cdot k + \frac{1}{2}(m_\mu^2 + m_e^2)\, s \right) \quad (4.9)$$

At this point we must fix the coordinate system and express the dot-products in terms of the angles and energies.



$$p \equiv (E;\, |\vec{p}|\hat{e}_z) \qquad p' \equiv (E, -|\vec{p}|\hat{e}_z)$$
$$k = (E;\, \vec{k}) \qquad k' = (E, -\vec{k}) \quad (4.10)$$

where I used: $E_{CM}^{e(e')} = E_{CM}^{\mu(\mu')} = E$ and defined

$$\vec{k} \equiv |\vec{k}| \sin\theta\, \hat{e}_x + |\vec{k}| \cos\theta\, \hat{e}_z \quad (4.11)$$

Thus

$$p\cdot k = p'\cdot k' = E^2 - |\vec{p}||\vec{k}|\cos\theta$$
$$p'\cdot k = p\cdot k' = E^2 + |\vec{p}||\vec{k}|\cos\theta \quad (4.12)$$

- Alternatively one could express them using the Mandelstam variables s, t and u.

and of course $q^2 = (p+p')^2 = s$. So

$$\int d\Omega \frac{\langle |T|^2\rangle}{\left(\frac{1}{4}\frac{32e^4}{q^4}\right)} = 2\pi \int_{-1}^{1} d\cos\theta \left((E^2 - pk\cos\theta)^2 + (E^2 + pk\cos\theta)^2 + \frac{s}{2}(m_e^2+m_\mu^2) \right)$$

$$= 8\pi \left(E^4 + \frac{s}{4}(m_e^2+m_\mu^2) + \frac{1}{3}p^2k^2 \right) \qquad (4.13)$$

Remembering the expression for the cross-section in the CM-frame:

$$\frac{d\sigma}{d\Omega} = \frac{\langle |T|^2\rangle}{64\pi^2 s}\frac{p_f}{p_i} \qquad (4.14)$$

and noting that

$$E = \frac{\sqrt{s}}{2}$$

$$p_i = p = \frac{\sqrt{s}}{2}\sqrt{1-\frac{4m_e^2}{s}} \qquad (4.15)$$

$$p_f = k = \frac{\sqrt{s}}{2}\sqrt{1-\frac{4m_\mu^2}{s}}$$

We eventually get (also: $e^2 = 4\pi\alpha$)

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{\int d\Omega \langle |T|^2\rangle}{64\pi^2 s}\frac{k}{p}$$

$$= \frac{1}{\cancel{64\pi^2} s}\frac{(4\pi\alpha)^2}{q^4}\cdot\cancel{\frac{1}{4}}\cdot 32\cdot\cancel{8\pi}\,\frac{1}{16}s^2\left(1 + \frac{2(m_e^2+m_\mu^2)}{s} + \frac{1}{3}\left(1-\frac{4m_e^2}{s}\right)\left(1-\frac{4m_\mu^2}{s}\right)\right)\cdot\frac{k}{p}$$

$$= \frac{16\pi^2\alpha^2}{\cancel{\pi}\, s^2}\,\frac{1}{\cancel{16}}\cancel{s}\cdot\frac{4}{3}\underbrace{\left(1 + \frac{2(m_\mu^2+m_e^2)}{s} + \frac{4m_\mu^2 m_e^2}{s^2}\right)}\frac{\sqrt{1-\frac{4m_\mu^2}{s}}}{\sqrt{1-\frac{4m_e^2}{s}}}$$

$$= \underbrace{\frac{4\pi\alpha^2}{3s}}_{\text{order of magnitude}}\cdot\underbrace{\left(\frac{\sqrt{1-\frac{4m_\mu^2}{s}}}{\sqrt{1-\frac{4m_e^2}{s}}}\right)}_{\text{phase space!}}\cdot\underbrace{f(m_\mu^2, m_e^2, s)}_{\mathcal{O}(1)\text{ factor.}} \xrightarrow[m_i \ll s]{} \boxed{\frac{4\pi\alpha^2}{3s}} \qquad (4.16)$$

Reaction $e^+e^- \to \mu^+\mu^-$ sets the scale for many high energy e^+e^--reactions. Let us define

$$R \equiv \frac{\pi\alpha^2}{3}\left(\frac{\text{GeV}}{E_{cm}}\right)^2\left[\frac{1}{\text{GeV fm}}\right]^2 \text{fm}^2$$

$$= \underline{20\ \text{mb}\left(\frac{\text{MeV}}{E_{cm}}\right)^2} \qquad (4,17)$$

millibarn $= 10^{-3}$ barn.

Quark-antiquark pair production

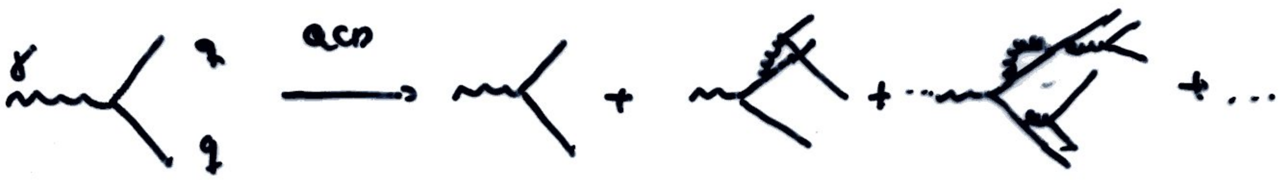
From the previous results we get directly the cross section for $\bar{q}q$-final state; we only need to replace

$$e \to Q_q|e| \qquad (4,18)$$

and multiply by N_c to account for $N_c = 3$ different (and indistinguishable) colours for the quarks. So we have at high $\sqrt{s} \gg m_q$:

$$\underline{\sigma_{e^+e^-\to q\bar{q}} = 3Q_q^2 R} \qquad (4.19)$$

Of course, we cannot observe free quarks, but instead the final state quarks get "dressed" by the QCD-interactions into showers of hadrons:



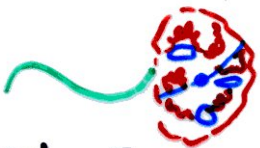
and so on.

In experiment the $q\bar{q}$ final state is observed as two back to back jets

It is highly remarkable that the cross section for hadron production at high energies can be computed from expression (4,19), such that

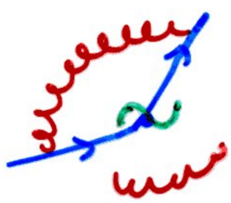
$$\sigma_{e^+e^- \to \text{Hadrons}} \overset{E_{cm}\to\infty}{\simeq} \left(3 \sum_{i=1}^{N_f} q_i^2\right) \cdot R \qquad (4,20)$$

Experimental proof of this simple prediction (done!) verifies the important property of the QCD: asymptotic freedom. At high enough energies, the photons "see" through the cloud of QCD-virtual states surrounding each quark.



$\lambda^{-1} \sim E$

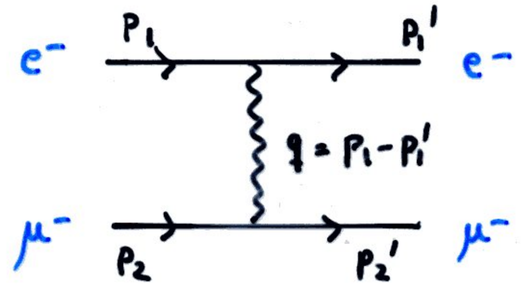
$\ll \Lambda_{QCD} \sim 100$ MeV



$\lambda \ll \Lambda_{QCD}^{-1}$

4.2 $e^- \mu^- \to e^- \mu^-$; Crossing symmetry ; s,t,u -variables

Consider the reaction $e^-\mu^- \to e^-\mu^-$. The lowest order diagram is



and the corresponding matrix element

$$T = (-ie)^2 \left[\bar{u}_e(p_1') \gamma^\mu u_e(p_1) \right] \frac{-i g_{\mu\nu}}{q^2 + i\epsilon} \left[\bar{u}_\mu(p_2') \gamma^\nu u_\mu(p_2) \right] \qquad (4.21)$$

The spin-averaged matrix element is

$$\frac{1}{4} \sum_{\text{spins}} |T|^2 = \frac{e^4}{4q^2} Tr\left[(\not{p}_1' + m_e) \gamma^\mu (\not{p}_1 + m_e) \gamma^\nu \right] \cdot Tr\left[(\not{p}_2' + m_\mu) \gamma_\mu (\not{p}_2 + m_\mu) \gamma_\nu \right]$$

$$= \underline{\frac{e^4}{4q^4} \left(- L_{(e)}^{\mu\nu}(-p_1', p_1) \right) \left(- L_{\mu\nu}^{(\mu)}(-p_2', p_2) \right)}, \qquad (4.22)$$

Where $L^{\mu\nu}$:s are the leptonic trace-tensors defined in (4.4). Noting further that here $q = (p_1 - p_1')^2 = (p_1 + (-p_1'))^2$, we see that (4.22) can be obtained directly from (4.2) by replacements:

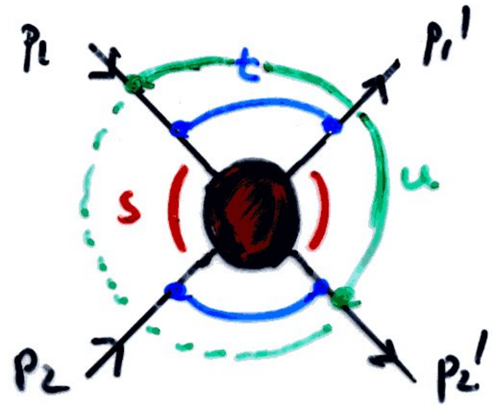
$$\boxed{\begin{array}{ll} p' \to -p_1' & p \to p_1 \\ k' \to -p_2 & k \to p_2' \end{array}} \qquad (4.23)$$

With help of these substitutions we can read $\frac{1}{4}\sum|T|^2$ directly from eqn. (4.9):

$$\frac{1}{4}\sum|T|^2_{e\mu\to e\mu} = \frac{8e^4}{q^4}\left(p_1\cdot p_2'\; p_2\cdot p_1' + p_1\cdot p_2\; p_1'\cdot p_2' + \frac{1}{2}(m_\mu^2+m_e^2)t\right) \quad (4.24)$$

Let us now compute this using Mandelstam invariants rather than a specific coordinate system. We have

$$\begin{aligned} s &= (p_1+p_2)^2 = (p_1'+p_2')^2 \\ t &= (p_1-p_1')^2 = (p_2-p_2')^2 \\ u &= (p_1-p_2') = (p_2-p_1')^2 \end{aligned} \quad (4.25)$$



Because $p_i^2 = m_i^2$, we have $p_1^2 = p_1'^2 = m_e^2$ and $p_2^2 = p_2'^2 = m_\mu^2$ i.e.:

- $p_1\cdot p_2' = -\frac{1}{2}\left((p_1-p_2')^2 - p_1^2 - p_2'^2\right) = \frac{1}{2}(m_e^2+m_\mu^2-u)$
  $= p_2\cdot p_1' \quad (4.26)$
- $p_1\cdot p_2 = p_1'\cdot p_2' = \frac{1}{2}(s-m_e^2-m_\mu^2) \quad (4.27)$

Using these results we get (4.24) into the following form

$$\frac{1}{4}\sum|T|^2_{e\mu\to e\mu} = \frac{2e^4}{t^2}\left((u-M)^2 + (s-M)^2 + \frac{1}{2}M^2 t\right) \quad (4.28)$$

where $M^2 \equiv m_e^2 + m_\mu^2$.

Written in Mandelstam variables, the matrix element for $e^+e^- \to \mu^+\mu^-$ is:

$$\frac{1}{4}\sum |T|^2 = \frac{2e^4}{s^2}\left((t-M^2)^2 + (u-M^2)^2 + \frac{1}{2}M^2 s\right). \qquad (4.29)$$

Thus, as is clearly visible from (4.25) the crossing relations (4.23), in terms of Mandelstam variables, read:

$$\boxed{s \to t \quad ; \quad t \to u \quad ; \quad u \to s.} \qquad (4.30)$$

Let us now return to eq. (4.28). We could compute σ using (Ex.)

$$\frac{d\sigma}{dt} = \frac{|T|^2}{16\pi\lambda(m_1^2, m_2^2, s)} \cdot S \qquad (4.31)$$

↑ final state symmetry factor (here = 1)

and the identity

$$s+t+u = \sum_{i=1}^{4} m_i^2 \,. \qquad (4.32)$$

(Here $s+t+u = 2(m_e^2+m_\mu^2) = 2M^2$.) The problem is however, that in $\mu^- \to e\mu^-$ the upper limit for t is $t_{max}=0$. Indeed

$$t = (p_1-p_1')^2 = -2(E_1E_1' + p_1p_1'\cos\theta + m_e^2) \overset{\text{CM-frame}}{=} -2p^2(1-\cos\theta) \qquad (4.33)$$

That is, for small angles $\frac{d\sigma}{d\Omega} \sim \frac{1}{\theta^4}$. This is the same infrared singularity already observed in connection with Rutherford scattering (excercises). It results from exchange of nearly on-shell photons with $q^2 \approx 0$. Partly the singularity is due to approximation of "collision over infinite time" of plane waves: Soft photon exchange takes time $t \sim \frac{1}{\sqrt{q^2}} \to \infty$ as $q^2 \to 0$. So finite interaction time cuts it.

Nevertheless, we can write $d\sigma/d\Omega$ in terms of invariants too:

$$\frac{d\sigma}{d\Omega_{CM}} = \frac{\alpha^2}{2st^2}\left((u-M)^2 + (s-M)^2 + \frac{1}{2} t M^2 \right), \qquad (4.34)$$

where $M^2 = m_e^2 + m_\mu^2$. This is often the most useful form. We can express it in terms of s and θ_{CM} by use of (4.31), (4.32) and the result

$$P_{CM} = \frac{1}{2\sqrt{s}} \lambda^{1/2}(s, m_e^2, m_\mu^2) \qquad (4.35)$$

where $\lambda(x,y,z) \equiv (x-y-z)^2 - 4yz$. In ultrarelativistic limit: $s \gg M^2$ we have $p \approx \sqrt{s}/2$ and

$$\left.\begin{aligned} t &\approx -\frac{s}{2}(1-\cos\theta) \\ u &\approx -\frac{s}{2}(1+\cos\theta) \end{aligned}\right. \qquad (4.36)$$

"complicated" angle dependence

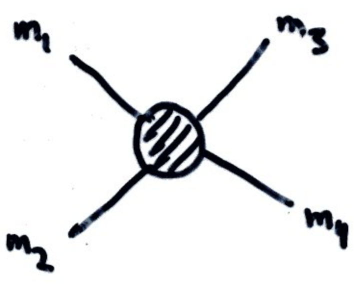
So that

$$\frac{d\sigma^{e^-\mu^-\to e^-\mu^-}}{d\Omega_{CM}} \approx \frac{\alpha^2}{2s}\left[\frac{1+\frac{1}{4}(1+\cos\theta)^2}{(1-\cos\theta)^2}\right] \qquad (4.37)$$

essentially the same scale as with $e^+e^- \to \mu^+\mu^-$

CROSSING SYMMETRY AND PHYSICAL REGION IN THE s,t - PLANE

- Expression (4.35) is an example of a way to present kinematical variables in terms of invariants. More generally in scattering $1+2 \to 3+4$:



we get the results (CM-frame):

$$p_i = \frac{1}{2\sqrt{s}} \lambda^{1/2}(s, m_1^2, m_2^2) \quad ; \qquad p_f = \frac{1}{2\sqrt{s}} \lambda^{1/2}(s, m_3^2, m_4^2) \qquad (4.38)$$

$$E_1 = \frac{1}{2\sqrt{s}} (s + m_1^2 - m_2^2)$$

$$E_2 = \frac{1}{2\sqrt{s}} (s - m_1^2 + m_2^2) \qquad (4.39)$$

$E_{3,4}$ can be found from (4.39) by substitution $(1,2) \to (3,4)$. Similar expressions can be found also for Lab-frame quantities.

From (4.38-4.39) one finds that for $e\mu^- \to e\mu^-$ reaction:

$$t = -\frac{\lambda(s, m_e^2, m_\mu^2)}{2s} (1 - \cos\theta_{CM}) \qquad (4.40)$$

$$u = \frac{(m_e^2 - m_\mu^2)^2}{s} - \frac{\lambda(s, m_e^2, m_\mu^2)}{2s} (1 + \cos\theta_{CM}) \qquad (4.41)$$

So in particular

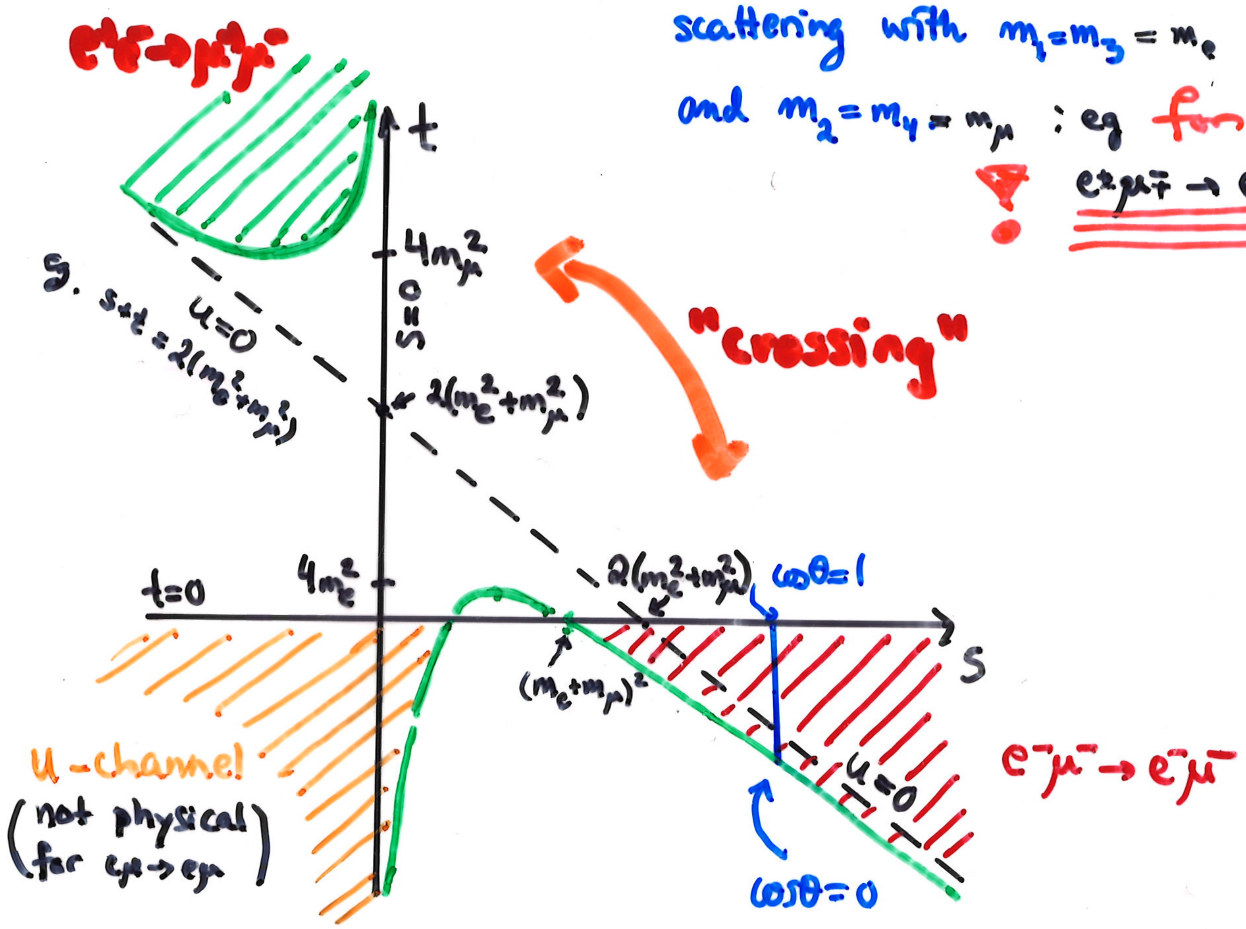
eqn. for a hyperbola.

$$t_{min} = -\frac{1}{s} \lambda(s, m_e^2, m_\mu^2)$$

$$t_{max} = 0. \qquad (4.42)$$

The area falling between limits (4.42) in the st-plane is scetched in Fig. 1 next page. In the physical region (red area) we must have $s > 0$. Eqs. (4.42) define curves also for $s < 0$ however and also for $t > 0$. These curves give rise to the physical regions of the other processes, through crossing relations.

Daliz-plot for 2→2 scattering with $m_1 = m_3 = m_e$ and $m_2 = m_4 = m_\mu$: eg for $e^\pm\mu^\mp \to e^\pm\mu^\mp$!



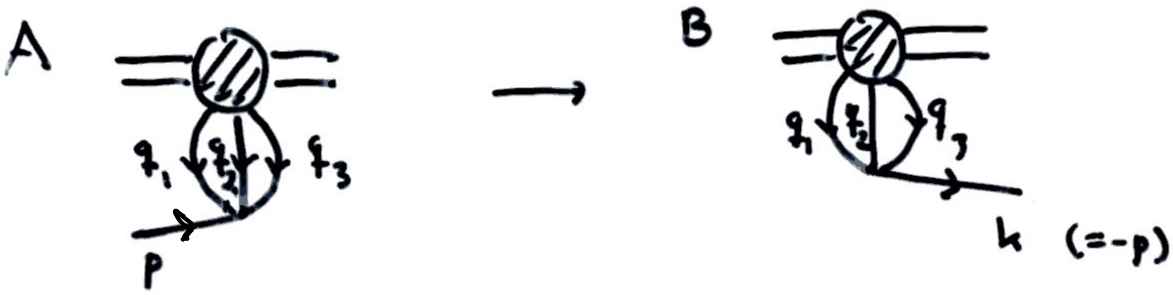
Because the physical $e^+e^- \to \mu^+\mu^-$ process must have $s>0$, we see that going from one physical region to another needs analytic continuation. It is easy to see that

$$s_{e\mu\to e\mu} = (p_1+p_2)^2 = (p_1-(-p_2))^2 \xrightarrow{p_2 \to -p_3} (p_1-p_3)^2 = t_{ee\to\mu\mu} < 0 \quad ✓.$$

$$u_{e\mu\to e\mu} = (p_1-p_3)^2 = (p_1+(-p_3))^2 \xrightarrow{p_3 \to -p_2} (p_1+p_2)^2 = s_{ee\to\mu\mu} > 4m_\mu^2 \quad ✓.$$

Similar procedure holds for the u-channel processes (orange area) if such exists. In eμ-scattering we do not have it, but other examples exist where it is there.

- These crossing relations were examples of a more general theorem: If processes A and B differ only by one initial state particle in A being moved to final state in B, the amplitude for B can be obtained from the amplitude for A by analytic continuation $p \to -k$.



According to Feynman rules, the only difference between A and B has to come from external legs, because the rest of the diagrams are identical. For scalar external leg is $\phi = 1 \Rightarrow \square$.
For fermions A will contain

$$\sum_s u(p)\,\bar{u}(p) = \not{p} + m$$

whereas B will have

$$\sum v(k)\,\bar{v}(k) = \not{k} - m = -(\not{p} + m)$$

overall sign is irrelevant

4.3 Helicity amplitudes

Remember the definition of helicity and the helicity projector (2.46) and (2.48):

$$\underline{\hat{P}_h = \frac{1}{2}\left(\mathbb{1}_4 \pm h\,\vec{\Sigma}\cdot\hat{p}\right)} \quad ; \quad \vec{\Sigma} = -\gamma^5\vec{\gamma}\gamma^0. \qquad (4.43)$$

$\pm$ = sgn(p_0)

where +(-) refers to particles (antiparticles). Clearly with this definition $\hat{P}_{h'}u(\vec{p},h) = \frac{1}{2}(1+hh')u(\vec{p},h) = \delta_{hh'}u(\vec{p},h)$, and $\hat{P}_{h'}v(\vec{p},h) = \delta_{hh'}v(\vec{p},h)$. This is not a covariant form however. A covariant generalization was derived in ex. 4/6:

$$\underline{P_h = \frac{1}{2}\left(1+\gamma^5\not{s}_h\right)} \qquad (4.44)$$

where

$$\underline{s_h^\mu = \frac{h}{m}\left(|\vec{p}|,\, E\hat{p}\right)}. \qquad (4.45)$$

It is easy to check that in rest frame $\gamma^0\begin{cases}u\\v\end{cases} = \text{sgn}(p_0)\begin{cases}u\\v\end{cases}$ so that (4.44) reduces to (4.43) there. The spin-vector s_h satisfies

$$\underline{s\cdot p = 0 \;;\quad s^2 = -1 \quad \text{and} \quad \vec{s}_h \parallel \hat{p}}$$

(by construction). It is easy to show that indeed

$$P_{h'}u(\vec{p},h) = \delta_{hh'}\,u(\vec{p},h) \qquad P_{h'}\,v(\vec{p},h) = \delta_{hh'}\,v(\vec{p},h)$$

Covariant projection operators allow to compute polarized cross sections using the Dirac matrix Trace theorems.

It suffices to insert an explicit projector into the T-matrix and continue summing over the spins. For example the polarized version of (4.1) is

$$T_{h_e \bar{h}_e \to h_\mu \bar{h}_\mu} = (-ie)^2 \, \bar{v}_e(p',s') \, P_{\bar{h}_e} \gamma^\mu P_{h_e} u(p,s) \times \frac{-i g_{\mu\nu}}{q^2} \times$$

$$\times \; \bar{u}_\mu(k,r) P_{h_\mu} \gamma^\nu P_{\bar{h}_\mu} v(k',r') \qquad (4.46)$$

So, when we replace $\gamma^\mu \to P_+ \gamma^\mu P_-$ combination(s), we can go through the computations as before. Obviously things get much more complicated, though!

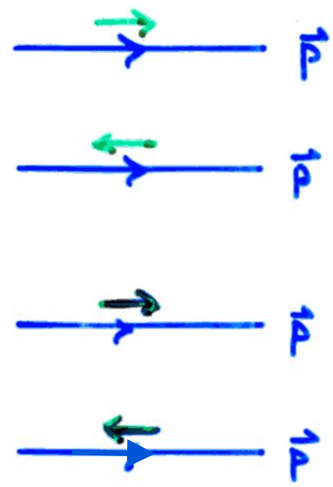
The situation simplifies tremendously in the UR-limit. There one finds (see (2.47) and (2.50))

$$u_+(p) \longrightarrow \sqrt{2E} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xi_{+,\hat{p}}$$

$$u_-(p) \longrightarrow \sqrt{2E} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xi_{-,\hat{p}}$$

$$v_+(p) \longrightarrow \sqrt{2E} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xi_{-,\hat{p}}$$

$$v_-(p) \longrightarrow -\sqrt{2E} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xi_{+,\hat{p}} \qquad (4.47)$$



↑helicity

On the other hand these states are the eigenstates of the chiral projectors

$$u_{\underset{(R)}{L}}(p,h) \equiv \frac{1}{2}(1 \underset{(+)}{-} \gamma_5)\, u(p,h) \xrightarrow{E\to\infty} u_{\mp}(p,h)$$

$$v_{\underset{(R)}{L}}(p,h) \equiv \frac{1}{2}(1 \mp \gamma_5)\, v(p,h) \longrightarrow v_{\pm}(p,h) \qquad (4.48)$$

In other words, in the UR-limit, spin projectors have operative limits

$$P_{+(-)}\, u \longrightarrow P_{R(L)}\, u$$

$$P_{+(-)}\, v \longrightarrow P_{L(R)}\, v \qquad (4.49)$$

where we defined the chiral projectors

$$\boxed{P_{\substack{L \\ (R)}} \equiv \frac{1}{2}\left(1 \underset{(+)}{-} \gamma_5\right)} \qquad (4.50)$$

rem. $\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ in Weyl basis

Now trivially

$$\underline{P_R \gamma^\mu P_L} = \gamma^\mu P_L$$

$$\underline{P_L \gamma^\mu P_R} = \gamma^\mu P_R$$

$$\underline{P_L \gamma^\mu P_L} = P_R \gamma^\mu P_R = 0 \qquad (4.51)$$

As a result only nonvanishing processes at UR-limit are

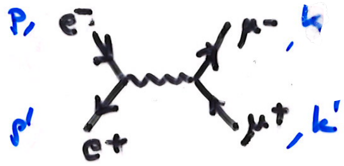
$$\begin{aligned}
&\underset{(L)}{e^-_{-1}}\ \underset{(L)}{e^+_{+1}} \longrightarrow \underset{(L)}{\mu^-_{-1}}\ \underset{(L)}{\mu^+_{+1}} \\
&\qquad\qquad \longrightarrow \underset{(R)}{\mu^-_{+1}}\ \underset{(R)}{\mu^+_{-1}} \\
&\underset{(R)}{e^-_{+1}}\ \underset{(R)}{e^+_{-1}} \longrightarrow \underset{(R)}{\mu^-_{+1}}\ \underset{(R)}{\mu^+_{-1}} \\
&\qquad\qquad \longrightarrow \underset{(L)}{\mu^-_{-1}}\ \underset{(L)}{\mu^+_{+1}}
\end{aligned} \qquad (4.52)$$

<u>Physical spin configuration</u> for $e^-_{-1} e^+_{+1}$ state is.

$(\sim u_L)$ $h = -1$ $\quad e^-_{-1} \longrightarrow \quad$ $\vec{s}$ $\qquad\qquad$ $\longleftarrow e^+_{+1}$ $\quad$ $\vec{s}$ $\quad h = +1$ $(\sim \bar{v}_L)$

Using results (4.48) we can rewrite for example $e^-_{-1} e^+_{+1} \to \mu^-_{-1}\mu^+_{+1}$ as $e^-_L e^+_L \to \mu^-_L \mu^+_L$ ie:

$$T_{e^-_{-1}e^+_{+1} \to \mu^-_{-1}\mu^+_{+1}} \overset{E\gg m}{\simeq} \frac{ie^2}{q^2}\, a^\mu_L\, b_{L\mu} \qquad (4,53)$$



$$a^\mu_L = \bar{v}_L(p')\gamma^\mu u_L(p) = \bar{v}(p')\gamma^\mu P_L u(p)$$

$$b^\mu_L = \bar{u}_L(k)\gamma^\mu v_L(k') = \bar{u}(k)\gamma^\mu P_L v(k') \qquad (4,54)$$

Firstly.

$$a^{\mu\dagger}_L = \bar{u}\gamma^\mu P_L v$$

$$b^{\mu\dagger}_L = \bar{v}\gamma^\mu P_L u \qquad (4,55)$$

So that

$$\sum_{\text{spins}} a^\mu_L a^{\nu\dagger}_L = \mathrm{Tr}\left((\not{p}'-m_e)\gamma^\mu P_L(\not{p}+m_e)\gamma^\nu P_L\right)$$

$$= \tfrac{1}{2}\mathrm{Tr}\left(\not{p}'\gamma^\mu\not{p}\gamma^\nu(1-\gamma^5)\right)$$

$$= 2\left(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu} p'\cdot p + i\epsilon^{\alpha\mu\beta\nu} p'_\alpha p_\beta\right) \qquad (4,56)$$

and correspondingly:

$$\sum_{\text{spins}} b_{L\mu} b^\dagger_{L\nu} = \mathrm{Tr}\left((\not{k}+m_\mu)\gamma_\mu P_L(\not{k}'-m_\mu)\gamma_\nu P_L\right)$$

$$= 2\left(k'_\mu k_\nu + k'_\nu k_\mu - g_{\mu\nu}\, k\cdot k' + i\,\epsilon_{\alpha\mu\beta\nu} k'^\alpha k^\beta\right) \qquad (4.57)$$

$$(\epsilon_{\alpha\mu\beta\nu} = -\epsilon^{\alpha\mu\beta\nu})$$

The square of the matrix elements is easily computed noting:

$$\underline{T^{symm}_{\mu\nu}\, T^{\mu\nu}_{antisymm} = 0} \qquad (5,48)$$

and

$$\underline{\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\rho\sigma} = -2(g^{\alpha}{}_{\rho}g^{\beta}{}_{\sigma} - g^{\alpha}{}_{\sigma}g^{\beta}{}_{\rho})}\,. \qquad (5,49)$$

We gets

$$\sum_{spins} |T|^2 = \frac{e^4}{q^4} a_L^{\mu} a_L^{\nu +} b_{L\mu} b_{L\nu}^{+}$$

$$= \frac{16 e^4}{q^4}(p.k')(p'.k) = \underline{e^4(1+\cos\theta)^2} \qquad (4,60)$$

Thus

$$\frac{d\sigma}{d\Omega_{cm}}(\underset{(L)}{e^-_{-1}}\underset{(L)}{e^+_{+1}} \to \underset{(L)}{\mu^-_{-1}}\underset{(L)}{\mu^+_{+1}}) = \underline{\frac{\alpha^2}{4s}(1+\cos\theta)^2} \qquad (4,61)$$

Similarly one finds (others $\to 0$ at $E \gg m$)

$$\frac{d\sigma}{d\Omega_{cm}}(\underset{(L)}{e^-_{-1}}\underset{(L)}{e^+_{+1}} \to \underset{(R)}{\mu^-_{+1}}\underset{(R)}{\mu^+_{-1}}) = \frac{\alpha^2}{4s}(1-\cos\theta)^2$$

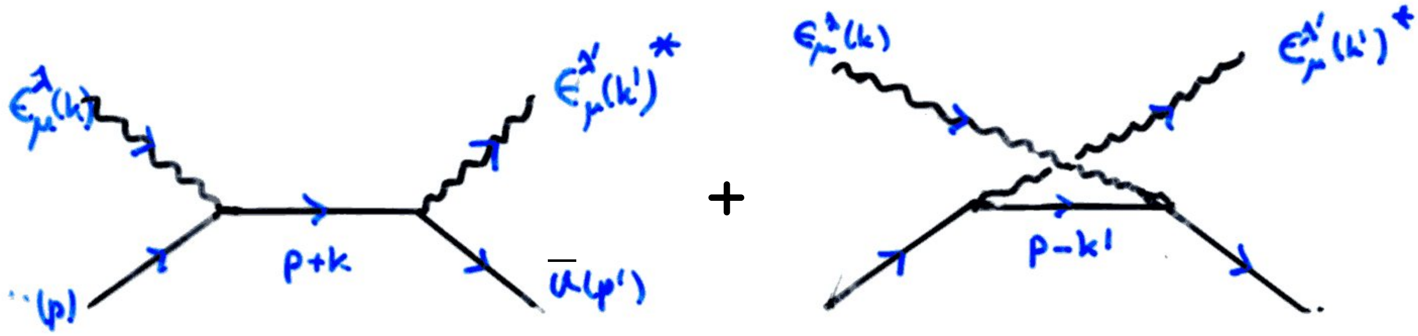
$$\frac{d\sigma}{d\Omega_{cm}}(\underset{(R)}{e^-_{+1}}\underset{(R)}{e^+_{-1}} \to \underset{(L)}{\mu^-_{-1}}\underset{(L)}{\mu^+_{+1}}) = \frac{\alpha^2}{4s}(1-\cos\theta)^2$$

$$\frac{d\sigma}{d\Omega_{cm}}(\underset{(R)}{e^-_{+1}}\underset{(R)}{e^+_{-1}} \to \underset{(R)}{\mu^-_{+1}}\underset{(R)}{\mu^+_{-1}}) = \frac{\alpha^2}{4s}(1+\cos\theta)^2 \qquad (4,62)$$

Note that $\underline{\frac{1}{4}\sum_{helicities}\int d\Omega \frac{d\sigma}{d\Omega} = \frac{4\pi\alpha^2}{3s}}$.

4.4 Compton scattering

Next consider photon scattering off an electron. A new thing to be learned here will be external photon polarization sums. Process $e^-\gamma \to e^-\gamma$ has following two diagrams to lowest nontrivial order:



Symmetry factors are just 1 and we immediately get

$$T^{\lambda\lambda'} = (-ie)^2\, \bar{u}(p') \Big\{ \not{\epsilon}^{\lambda'}(k')^* \frac{i}{\not{p}+\not{k}-m_e} \not{\epsilon}^{\lambda}(k) + \not{\epsilon}^{\lambda}(k) \frac{i}{\not{p}-\not{k}'-m_e} \not{\epsilon}^{\lambda'}(k')^* \Big\} u(p) \qquad (4.63)$$

Working the square of this matrix-element is already much harder than the μ-processes above. First it is good idea to simplify T as much as possible using e.o.m.:

$$(\not{p}+\not{k}+m_e)\not{\epsilon}_k^{\lambda}\, u(p) = \left(\not{k}\not{\epsilon}_k^{\lambda} + 2p\cdot\epsilon_k^{\lambda}\right) u(p)$$

$$(\not{p}-\not{k}'+m_e)\not{\epsilon}_{k'}^{\lambda'*}\, u(p) = -\left(\not{k}'\not{\epsilon}_{k'}^{\lambda'*} - 2p\cdot\epsilon_{k'}^{\lambda'*}\right) u(p) \qquad (4.64)$$

where we used $\not{p}\not{\epsilon} = -\not{\epsilon}\not{p} + 2p\cdot\epsilon$ and $(\not{p}-m)u = 0$.

Thus

$$= \epsilon_\mu^{\lambda' *} \epsilon_\nu^{\lambda} \gamma^\mu (\not k \gamma^\nu + 2p^\nu)$$

$$T^{\lambda\lambda'} = e^2 \bar u(p') \left\{ \frac{\not\epsilon_{k'}^{\lambda' *}(\not k \not\epsilon_k^\lambda + 2p\cdot\epsilon_k^\lambda)}{(p+k)^2 - m_e^2} - \frac{\not\epsilon_k^\lambda (\not k' \not\epsilon_{k'}^{\lambda' *} - 2p\cdot\epsilon_{k'}^{\lambda' *})}{(p-k')^2 - m_e^2} \right\} u(p) \tag{4.65}$$

The square of the unpolarized matrix element is:

$$\langle |T|^2 \rangle = \frac{e^4}{4} \sum_{\lambda,\lambda'} \epsilon_\mu^{\lambda'}(k')^* \epsilon_\nu^\lambda(k) \; \epsilon_\rho^{\lambda'}(k') \; \epsilon_\sigma^\lambda(k)^*$$
$$\cdot \mathrm{Tr}\left((\not p' + m_e) \left\{ \frac{\gamma^\mu(\not k \gamma^\nu + 2p^\nu)}{s - m_e^2} - \frac{\gamma^\nu(\not k' \gamma^\mu - 2p^\mu)}{u - m_e^2} \right\} \right.$$
$$\left. \cdot (\not p + m_e) \left\{ \frac{(\gamma^\sigma \not k + 2p^\sigma)\gamma^\rho}{s - m_e^2} - \frac{(\gamma^\rho \not k' - 2p^\rho)\gamma^\sigma}{u - m_e^2} \right\} \right) \tag{4.66}$$

where one used eg $\gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu$ and $\gamma^0 (\gamma^\alpha \gamma^\beta \gamma^\delta)^\dagger \gamma^0 = \gamma^\delta \gamma^\beta \gamma^\alpha$.

Photon polarization sums

In calculations of scattering cross sections involving photons (or other gauge bosons) we often encounter polarization sums over the transverse (physical) states: $\sum_\lambda \epsilon_{T\mu}^\lambda \epsilon_{T\nu}^{*\lambda}$, where $\epsilon_T \cdot k = 0$. There are precisely 2 vectors that satisfy this condition, which can be chosen spacelike, using the residual gauge freedom that allows shifting $\epsilon_{T\mu} \to \epsilon_{T\mu} + \alpha k_\mu$. Then given $k^\mu = k(1, \hat k)$, $\epsilon_T \cdot k = 0 \Rightarrow \vec\epsilon_T \cdot \hat k = 0$. In practice we can replace the polarization sums by the completeness relation

$$\sum_{\lambda=1}^{2} \epsilon_{T\mu}^\lambda \epsilon_{T\nu}^{*\lambda} \longrightarrow -g_{\mu\nu} \tag{4.67}$$

even though $\epsilon_{T\mu}^{1,2}$ do not span the whole Minkowski space. We can amend the situation by adding two unphysical polarization vectors to the l.h.s. of (4.67). These are conveniently chosen as the following two light-like vectors:

$$\epsilon_\mu^\pm(k) = \frac{1}{\sqrt2}(1, \pm\hat k)$$

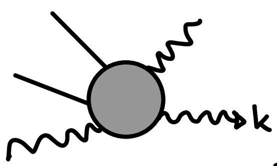
These satisfy $\epsilon^\pm(k)\cdot\epsilon_T^i = 0$ and $\epsilon^+ \cdot \epsilon^{-*} = 1$, as well as $\epsilon^\pm \cdot \epsilon^{\pm *} = 0$, whereas

$\epsilon_T^i \cdot \epsilon_T^j = -\delta_{ij}$. We can now write the full completeness relation as $g_{\mu\nu} = \epsilon^-_\mu \epsilon^{+*}_\nu + \epsilon^+_\mu \epsilon^{-*}_\nu - \sum_{\lambda=1}^{2} \epsilon^\lambda_{T\mu} \epsilon^{*\lambda}_{T\nu}$. Or, we can define the transverse polarization sum as the idempotent projection operator onto the physical subspace:

$$\sum_{\lambda=1}^{2} \epsilon^\lambda_{T\mu} \epsilon^{*\lambda}_{T\nu} = -P_{\mu\nu} = -g_{\mu\nu} + \underline{\epsilon^-_\mu \epsilon^{+*}_\nu + \epsilon^+_\mu \epsilon^{-*}_\nu} \qquad (4.68)$$

Indeed, it is easy to show that $P_{\mu\alpha} P^\alpha{}_\nu = P_{\mu\nu}$ as well as $\epsilon^{\pm\mu} P_{\mu\nu} = \epsilon^{\pm\nu} P_{\mu\nu} = 0$, but $\epsilon^{\lambda\mu}_T P_{\mu\nu} = \epsilon^\lambda_{T\mu}$.

Now, in all photon scattering events, the matrix element of the form



$$T = \mathcal{M}^\mu(k) \epsilon^\lambda_{T\mu}(k)$$

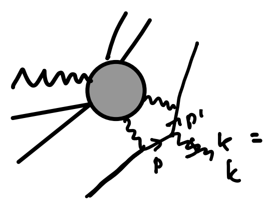
$$\Rightarrow \sum_\lambda |T|^2 = \sum_\lambda \epsilon^\lambda_{T\mu} \epsilon^{\lambda *}_{T\nu} \mathcal{M}^\mu(k) \mathcal{M}^{\nu *}(k)$$

$$= P_{\mu\nu} \mathcal{M}^\mu(k) \mathcal{M}^{\nu *}(k).$$

$$= -\mathcal{M}^\mu(k) \mathcal{M}^*_\mu(k) + \underline{\epsilon^- \cdot \mathcal{M} (\epsilon^+ \cdot \mathcal{M})^* + \epsilon^+ \cdot \mathcal{M} (\epsilon^- \cdot \mathcal{M})^*}$$

In all scattering diagrams we will encounter, the matrix element $\mathcal{M}^\mu(k)$ is of the form

$$\mathcal{M}^\mu(k) = \ldots \bar{u}(p') \gamma^\mu u(p) \ldots \quad \text{where } p - p' = k$$



$= \mathcal{M}^\mu \epsilon_{T\mu}$

so that $\epsilon^+_\mu \mathcal{M}^\mu = \frac{1}{\sqrt{2}|k|} k_\mu \mathcal{M}^\mu \propto \bar{u}(p') \not{k} u(p) = \bar{u}(p')(\not{p} - \not{p}') u(p) = 0$!

That is, the $\underline{\epsilon^-_\mu \epsilon^{+*}_\nu + \epsilon^+_\mu \epsilon^{-*}_\nu}$ - part of the projection operator $P_{\mu\nu}$ never contributes to the scattering process!

This fact can be expressed such that if we replace $\epsilon_{T\mu}$ by k_μ in an arbitrary diagram the matrix element vanishes.

$$\boxed{k_\mu \mathcal{M}^\mu = 0} \qquad (4.69)$$

This identity in fact follows from the U(1)-symmetry behind QED, which gives rise to current conservation $\partial_\mu j^\mu = 0$, which in momentum space reads just $k_\mu j^\mu = 0$! The result (4.69) is an example of important gauge-symmetry imposed Ward identities.

So, because of the Ward-identity (4.69) we can use

$$\sum_\lambda \epsilon_\mu^\lambda \epsilon_\nu^{*\lambda} \equiv -g_{\mu\nu} \qquad (4.70)$$

Using (4.70) we can rewrite (4.66) as:

$$\langle|T|^2\rangle = \frac{e^4}{4}\mathrm{Tr}\Big[(\not p' + m_e)\Big\{\frac{\gamma^\mu(\not k\gamma^\nu+2p^\nu)}{s-m_e^2} - \frac{\gamma^\nu(\not k'\gamma^\mu - 2p^\mu)}{u-m_e^2}\Big\}$$
$$\times(\not p+m_e)\Big\{\frac{(\gamma_\nu\not k+2p_\nu)\gamma_\mu}{s-m_e^2} - \frac{(\gamma_\mu\not k'-2p_\mu)\gamma_\nu}{u-m_e^2}\Big\}\Big]$$

$$\equiv \frac{e^4}{4}\left(A_{ss}+A_{su}+A_{us}+A_{uu}\right) \qquad (4.71)$$

It is easy to see that

- $A_{uu} = A_{ss}(k \to -k')$
- $A_{us} = A_{su}$ $\quad(\mathrm{Tr}(\gamma_1\gamma_2\cdots\gamma_n) = \mathrm{Tr}(\gamma_n\cdots\gamma_2\gamma_1))$

So we only need to compute two Tr-terms.

$$\not k\gamma^\nu(\not p+m_e)\gamma_\nu\not k = -2\not k\not p\not k$$

$$2\not k\not p(\not p+m_e)$$

$$(s-m_e^2)^2A_{ss} = \mathrm{Tr}\Big(\underbrace{\gamma_\mu(\not p'+m_e)\gamma^\mu}_{=-2\not p'+4m_e}(\not k\gamma^\nu+2p^\nu)(\not p+m_e)(\gamma_\nu\not k+2p_\nu)\Big)$$

$$2(\not p+m_e)\not p\not k \qquad 4p^2(\not p+m_e)$$

$$= \mathrm{Tr}\Big((-2\not p'+4m_e)\big(-2\not k\not p\not k + 4m_e^2(\not p+m_e) + \underbrace{2\not k\overbrace{\not p(\not p}^{m^2}+m_e)}_{2m_e\not k(\not p+m_e)} + \underbrace{2\overbrace{(\not p+m_e)\not p}^{m^2}\not k}_{2m_e(\not p+m_e)\not k}\big)\Big)$$

$$= 4m_e^2\not k + 2m_e(\overbrace{\not k\not p+\not p\not k}^{=2k\cdot p})$$

$$= \mathrm{Tr}\Big((-2\not{p}' + 4m_e)\big(-2\not{k}\not{p}\not{k} + 4m_e^2(\underbrace{\not{k}+\not{p}}_{=\not{k}'+\not{p}'}) + 4m_e(m_e^2 + k\cdot p)\big)\Big)$$

$$= 4\,\mathrm{Tr}\Big(\not{p}'\not{k}\not{p}\not{k} - 2m_e^2(\not{p}'\not{k}' + \underbrace{\not{p}'\not{p}'}_{m_e^2}) + 4m_e^2(m_e^2 + k\cdot p)\Big)$$

$$= 16\big(2m_e^4 + \underbrace{4m_e^2\, k\cdot p - 2m_e^2\, k'\cdot p'}_{=2m_e^2 k\cdot p} + 2p\cdot k\; p'\cdot k\big)$$

In terms of invariants

$$s = (p+k)^2 = 2p\cdot k + m_e^2 = 2p'\cdot k' + m_e^2$$

$$t = (p-p')^2 = -2p\cdot p' + 2m_e^2 = -2k\cdot k'$$

$$u = (k-p')^2 = -2k\cdot p' + m_e^2$$

$$\Rightarrow (s-m_e^2)^2 A_{ss} = 16\big(2m_e^4 + m_e^2(s-m_e^2) - \tfrac{1}{2}(s-m_e^2)(u-m_e^2)\big)$$

By a direct substitution (same internal propagator here)

$$(u-m_e^2)^2 A_{uu} = 16\big(2m_e^4 + m_e^2(u-m_e^2) - \tfrac{1}{2}(u-m_e^2)(s-m_e^2)\big)$$

A similar computation gives for the interference:

$$(u-m_e^2)(s-m_e^2)(A_{su}+A_{us}) = -16\big(4m_e^4 + m_e^2(s-m_e^2) + m_e^2(u-m_e^2)\big)$$

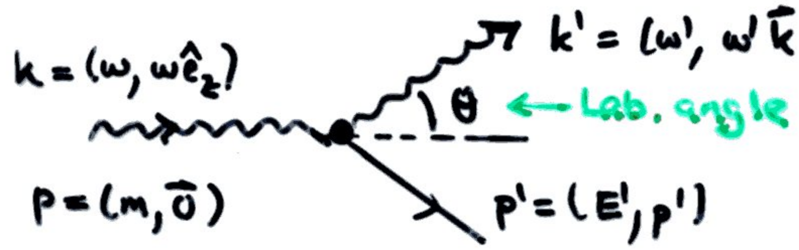
So that alltogether

$$\langle |T|^2\rangle = 4e^2\Bigg(\underbrace{\frac{2m_e^4}{(s-m_e^2)^2} + \frac{m_e^2}{(s-m_e^2)} - \frac{1}{2}\,\frac{u-m_e^2}{s-m_e^2}}_{A_{ss}} + \underbrace{\frac{2m_e^4}{(u-m_e^2)^2} + \frac{m_e^2}{u-m_e^2} - \frac{1}{2}\,\frac{s-m_e^2}{u-m_e^2}}_{A_{uu}} +$$

$$+ \frac{4m_e^4}{(s-m_e^2)(u-m_e^2)} + \frac{m_e^2}{u-m_e^2} + \frac{m_e^2}{s-m_e^2} \Bigg)$$

$$= 2e^4 \left(-\left(\frac{s-m_e^2}{u-m_e^2} + \frac{u-m_e^2}{s-m_e^2}\right) + \left(\frac{2m_e^2}{s-m_e^2} + \frac{2m_e^2}{u-m_e^2} + 1\right)^2 - 1 \right) \qquad (4{,}76)$$

This is an invariant form that can be used in any coordinate system. Most often we study Compton scattering in Lab-frame:



The initial state photon energy follows from s:

$$s - m_e^2 = 2k\cdot p = 2\omega m_e \Rightarrow \omega = \frac{s-m_e^2}{2m_e} \qquad (4.77)$$

Similarly we can compute

$$t = -2kk' = -2\omega\omega'(1-\cos\theta) \qquad (4{,}78)$$

kinematical variables of the final state phase space.

ω' and $\cos\theta$ are not independent. Indeed:

$$\cancel{m_e^2} = p'^2 = (k+p-k')^2 = p^2 + 2p\cdot(k-k') + \overbrace{(k-k')^2}^{=t}$$

$$= \cancel{m_e^2} + 2m_e(\omega-\omega') - 2\omega\omega'(1-\cos\theta) \qquad (4{,}79)$$

From this we get useful equalities:

$$t = -2m_e(\omega - \omega') \qquad (4.80)$$

and equivalently

$$\underline{\omega' = \frac{\omega}{1+\frac{\omega}{m}(1-\cos\theta)}} \quad \text{or} \quad t = -\frac{2\omega^2(1-\cos\theta)}{1+\frac{\omega}{m}(1-\cos\theta)} \qquad (4.81)$$

Moreover then

$$\underline{u - m_e^2} = \underline{-2\omega' m_e} \qquad (4.82)$$

and

$$\boxed{dt = 2\omega'^2 \, d\cos\theta} \qquad (4.83)$$

Using results (4.77, 4.78, 4.80) and (4.82) we can rewrite (4.76) as

$$= 2e^4\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + \Big(\underbrace{\frac{m_e}{\omega} - \frac{m_e}{\omega'} + 1}_{=\cos\theta}\Big)^2 - 1\right)$$

$$\underline{= 2e^4\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right).} \qquad (4.84)$$

Using now

$$\frac{d\sigma}{dt} = \frac{|T|^2}{16\pi \underbrace{\lambda(s, m_e^2, 0)}_{\lambda = (s-m_e^2)^2 = 4\omega^2 m_e^2}} \qquad (4.85)$$

$$\frac{d\sigma}{d\cos\theta} = \frac{\alpha^2\pi}{m_e^2}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right) \tag{4.86}$$

This is deceivingly simple expression. However, if one wants to compute σ one must remember that $\omega' = \omega'(\theta)$, ie eq. (4.81). Eq. (4.86) is called Klein-Nishina formula for Compton scattering. It was first derived in 1929!

low energy limit $\omega \ll m_e$, we get $\omega'/\omega \to 1$ and

$$\frac{d\sigma}{d\cos\theta} \approx \frac{\alpha^2\pi}{m_e^2}(1+\cos^2\theta) \Rightarrow \sigma_{TOT} \approx \frac{8\pi\alpha^2}{3m^2} \tag{4.87}$$

This is just the well known Thomson scattering cross section for a low energy photon scattering off electrons.

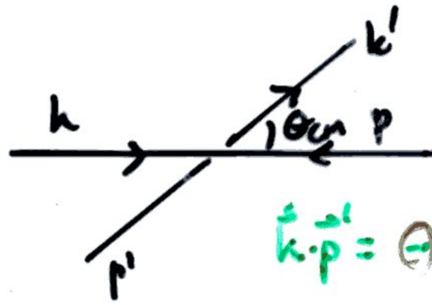
High energy limit : $\omega \gg m_e$

This one in easier to understand using the CM-frame. From (4.38) and (4.39) we see that in CM-frame:

$$p = p' = k = k' = \omega = \omega' = \frac{1}{2\sqrt{s}}(s-m_e^2) \approx \frac{\sqrt{s}}{2}$$

$$\omega_p = \omega_{p'} = \frac{1}{2\sqrt{s}}(s+m_e^2) \simeq \frac{\sqrt{s}}{2}$$

$$\Rightarrow \left| \begin{aligned} t &\simeq -\frac{s}{2}(1-\cos\theta_{cm}) \\ u &\simeq -\frac{s}{2}(1+\cos\theta_{cm}) \end{aligned} \right. \tag{4.88}$$



$\vec{k}\cdot\vec{p}' = -kp'\cos\theta$

Invariant matrix element (4.76) now becomes

$$\langle |T|^2 \rangle_{UR}^{CM\text{-frame}} \simeq 2e^4 \left(-\frac{s}{u-m_e^2} - \frac{u}{s} \right)$$

$$\approx 4e^2 \left(\frac{1}{1+c\Theta_{cm}+\frac{2m_e^2}{s}} + \frac{1}{4}(1+c\Theta_{cm}) \right)$$

↑ kept so that $\frac{d\sigma}{dc\Theta_{cm}}$ remains integrable

$$\Rightarrow \sigma_{TOT} = \frac{8\pi e^4}{64\pi^2 s} \int_{-1}^{1} dz \left(\underbrace{\frac{1}{1+\delta+z}}_{= \log\frac{2+\delta}{\delta} \simeq \log\frac{2}{\delta} + \mathcal{O}(\delta)} + \underbrace{\frac{1}{4}(1+z)}_{\to \frac{2}{3}} \right)$$

$$\approx \underline{\frac{2\pi\alpha^2}{s} \left(\log\frac{s}{m_e^2} + \frac{2}{3} \right)} \qquad (4.90)$$

The logarithmic enhancement of the cross section at large s results from photons tendency to want to scatter backwards: $z \simeq -1$, or $\Theta = \pi$ at large energies. Such photons also tend to be polarized, which opens a possibility to generate polarized photon and electron beams. (See P&S pages 165-167.)

The log-term seen above is actually typical for all t- and u-channel processes. They break partial wave unitarity at scale

$$\log\frac{s}{m_e^2} \simeq \underset{\substack{\uparrow \\ \text{here} \approx 3}}{\mathcal{O}(1)} \frac{1}{\alpha^2}$$

This limit is called the Landau Pole. At energies higher than this perturbation theory breaks down and weak interactions become strong. This topic is however beyond the scope of this course.

Other important QED-processes

Most important electron-photon interactions are

done above — $\boxed{e^-\gamma \to e^-\gamma}$ — Compton scattering

crossing — $\boxed{e^+e^- \to \gamma\gamma}$ — Pair annilation

like $e\mu$-scattering, but a little more demanding (2 graphs) — $\boxed{e^-e^- \to e^-e^-}$ — Møller scattering

$\boxed{e^-e^+ \to e^-e^+}$ — Bhabha scattering (detector calibration)

3-body phase space unless [illegible]. — $\boxed{e^-N \to e^-N\gamma}$ — Brehmsstrahlung (photon interactions in matter. Mean free path)

$\boxed{\gamma N \to e^+e^-N}$ — Pair production

infrared divergences.

Processes enclosed in boxes are in crossing relation to each others.

In case of brehmsstrahlung and pair production we would encounter additional infrared singularities. Taking these properly into account would require computing also radiative corrections, so we leave them to future.