3. INTERACTING THEORIES

The Lagrange functions of free theories were always quadratic in fields (apart from the linear source terms), and their solutions could be expressed as harmonic oscillator expansions. Interactions will appear as nonlinear terms in Euler-Lagrange equations.

Not all interactions are allowed. Their form is constrained in particular by causality (locality), symmetries and renormalizability.

Causality states that $\mathcal{L} = \mathcal{L}(x)$, so for example a term $\phi^p(x)$ is allowed, but $\phi^n(x)\phi^m(y)$ is not.

EXAMPLE 1. $\lambda\phi^4$-theory

mass, coupling constant = parameters of the theory.

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4} \qquad (3.1)$$

The Lagrangean (3.1) gives rise to the E-L-equation of motion:

$$(\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3 \qquad (3.2)$$

This equation cannot be solved generically by use of the Fourier analysis. However, theory (3.1) can still be quantized by the (free theory) commutation rules (1.3). This is so, because the interaction term does not affect the conjugate momentum: $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$.

EXAMPLE 2. **QED** (An Abelian gauge theory)

$$\mathcal{L}_{QED} = \underbrace{\bar{\Psi}(i\partial\!\!\!/ - m)\Psi}_{\text{Dirac}} - \underbrace{\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{Maxwell}} - \underbrace{e\bar{\Psi}\gamma^\mu\Psi A_\mu}_{\text{Interaction}} \qquad (3.3)$$

$$= \bar{\Psi}(iD\!\!\!/ - m)\Psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (3.4)$$

Here we introduced the covariant derivative

$$D_\mu = \partial_\mu + ieA_\mu \qquad (3.5)$$

The interaction term in fact follows from an invariance of the theory under local $U(1)$ symmetry. Indeed, we have already seen that Dirac theory is invariant under a global $U(1)$-transformation•

$$\Psi \to e^{i\alpha}\Psi \qquad (3.6)$$

where α is some constant. On the other hand, all observables are proportional to Bilinears $\bar{\Psi}\Gamma\Psi$, which are invariant also under local transformations with $\alpha = \alpha(x)$. It would be natural to require that the theory itself satisfies the same invariance. However:

$$\bar{\Psi}\partial\!\!\!/\Psi \xrightarrow{\Psi \to e^{i\alpha(x)}\Psi} \bar{\Psi}\partial\!\!\!/\Psi + \overbrace{i\bar{\Psi}(\partial\!\!\!/\alpha)\Psi}^{\neq 0} \qquad (3.7)$$

• The transformation phases $U_\theta = e^{i\theta}$ are unitary: $U_\theta^+ = U_\theta^{-1}$ and they form a ($U(1)$-) group.

The only way to make the Dirac theory compatible with the local invariance

$$\psi \to e^{i\alpha(x)}\psi \qquad (3.8)$$

is then to extend the idea of the derivative. The form of the eqn suggests that we should add some vector field to ∂_μ, and this leads to form (3.5). Requiring invariance now:

$$\bar\psi \not{D} \psi \xrightarrow{\psi \to e^{i\alpha(x)}\psi} \bar\psi(\not\partial + ie\not{A})\psi + i\bar\psi(\not\partial\alpha)\psi$$

$$\equiv \bar\psi(\not\partial + ie\not{A}')\psi = \bar\psi\not{D}'\psi$$

leads to the transformation law for A_μ

$$A_\mu \longrightarrow A'_\mu = A_\mu - \frac{i}{e}\partial_\mu\alpha(x) \qquad (3.9)$$

This however, one recognizes as the gauge transformation, which is the invariance of the Maxwells theory: $\mathcal{L}_{Maxwell} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

Combining the locally invariant Dirac theory and the Maxwells theory one discovers the Lagrangian (3.3) for the quantum electrodynamics, essentially based on symmetry argument.

The E-L-equations for the QED are easy to derive:

charge — fermionic (vector) current.

$$\partial_\mu F^{\mu\nu} = e\bar\psi\gamma^\nu\psi = j^\nu \qquad (3.10)$$

$$(i\not\partial - m)\psi = e\not{A}\psi \qquad (3.11)$$

(j^μ is of course the conserved Noether current for U(1)-symmetry.)

EXAMPLE 3. (Scalar electrodynamics). The local $U(1)$-invariance can also be imposed on the complex scalar theory (0,21). Again local invariance necessitates introducing the covariant derivative, and one finds

$$\mathcal{L}_\phi = |D_\mu \phi|^2 - m^2 |\phi|^2 - \frac{1}{4}(F_{\mu\nu})^2 \qquad (3.12)$$

This theory contains interactions

$$e\,\phi^*(\partial_\mu \phi) A^\mu \quad \text{and} \quad e^2 |\phi|^2 A^2 \qquad (3.13)$$

EXAMPLE 4. Quantum Chromodynamics (QCD) (Non-Abelian gauge th.)

Let us now assume that the Dirac theory spinor has an internal $SU(3)$-index:

$$\psi \longrightarrow \begin{pmatrix} q_r \\ q_b \\ q_g \end{pmatrix} \xrightarrow{SU(3)} e^{i \frac{\lambda^a}{2} \theta^a} \psi \qquad (3.14)$$

← 8 parameters

Where $\frac{\lambda_a}{2}$ are the (8) generators of the $SU(3)$ Lie-algebra. If θ^a:s are constant, then the free QCD-theory ($i,j = r, b, g$)

$$\mathcal{L}^{QCD}_{free} = \bar{\psi}_i (i \not{\partial} - m_{ij}) \psi_j \qquad (3.15)$$

invariant. However, if $\theta^a = \theta^a(x)$, we must again introduce a covariant Derivative to achieve invariance. It is easy to see that the construction is

$$\partial_\mu \longrightarrow \partial_\mu - i g \frac{\lambda^a}{2} A^a_\mu \equiv D_\mu \qquad (3.16)$$

(one) coupling constant   8 gluon fields.

The transformation law for the gluon field can be worked out:

$$T\cdot A' = U T\cdot A U^{+} - \frac{i}{g} U \partial_\mu U^{+} \qquad (3,16b)$$

where $T^a \equiv \frac{\lambda^a}{2}$ and $U = e^{iT\cdot\theta}$. The generalization of the Maxwell's invariant field strength tensor is

$$F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu] = F^a_{\mu\nu}\cdot T^a \qquad (3,17)$$

By direct evaluation one can show that the Non-abelian term $-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$ contains new types of interactions

$$\shortparallel$$

$$-\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \qquad (\partial_\mu A^\mu) A^2 \quad \text{and} \quad A^4. \qquad (3,18)$$

We shall return to QCD and other non-Abelian Yang-Mills theories later on.

RENORMALIZABILITY

All interactions found above are characterized by the fact that the corresponding coupling constants are dimensionless. Indeed, since the action is dimensionless, we must have

$$[\mathcal{L}] = L^{-4} \qquad (3.19)$$

On the other hand $[\partial] = L^{-1}$, so that $[\phi] = [A] = L^{-1}$ and $[\psi] = L^{-3/2}$. Thus

$$[\bar\psi A\psi] = [\phi^4] = [\phi^2 A^2] = [\phi(\partial_\mu\phi)A^\mu]$$
$$= [A^4] = [A^2\partial_\mu A] = L^{-4} \; ! \qquad (3.20)$$

This is not a coincidence: A theorem states that only interactions whose coupling has zero or negative dimension are renormalizable. What this means is that

1) If $[g] = L^0$, all infinities arising in the PT-calculations can be absorbed to the redefinition of the coupling, (mass) and the fields. $\mathcal{L}$ retains its form and the predictive power.

2) If $[g] = L^{+|\alpha|}$, perturbation theory will create an infinite amount of new interactions. Adjusting these infinities requires ∞ number of new parameters $\Rightarrow$ no predictive power.

3) If $[g] = L^{-|\alpha|}$ the interaction is called super renormalizable. It does not generate any new infinities.

Renormalizability excludes for example all interactions $g_n \phi^n$ with $n > 4$ in the scalar theory. Similarly we cannot have a term $(\bar{\Psi}\Psi)^2$ in a fundamental theory. In fact from our list of possible interactions for the spin $0, \frac{1}{2}$ and 1 fields we are missing only the forms

$$\mu \phi^3 \quad \text{and} \quad y \phi \bar{\Psi} \Psi \qquad (3.21)$$

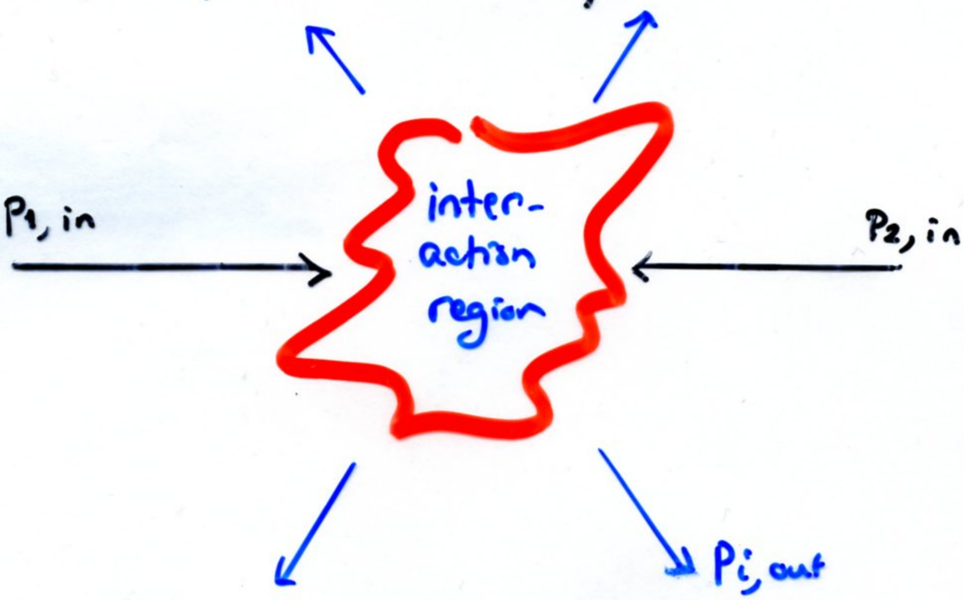
super renorm.
scalar self coupling
(spontaneus symm. breaking, SSB)

Yukawa interaction.
(fermion masses through SSB)

Causality, symmetries and renormalizability are obviously very constraining principles for relativistic QFT. This should be contrasted to the situation in nonrelativistic qm. where potential V is arbitrary.

3.1. S-MATRIX AND CROSS SECTIONS

A very typical application of QFT is to solve the scattering problem by use of the perturbation theory



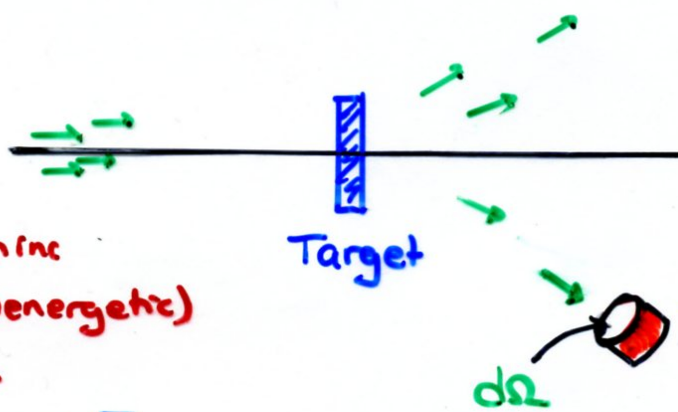
Far away, in the in- and out-states the fields are noninteracting.
(cheating a little here. I will explain better later.)

The formal set up for the problem is as follows: 1 An "in"-state is prepared far from the interaction region (at $t \to -\infty$ in the interaction time scale), typically to momentum eigenstates. 2 One measures the out-going particles at $t \to +\infty$, i.e. constructs the "out"-state.

Experimentally the scattering problem is described by a (differential) scattering cross section. This can be viewed as an effective area of the scatterer as seen by the scattered particle.

Theoretically we wish to compute the cross section. To this end we need to 1 set up an S-matrix formalism that can relate the "in"-states to "out"-states. 2 Express the formal S-matrix in terms of the greens functions of the interacting theory. 3 Develop perturbation theory for evaluation of these interacting theory greens functions.

FLUX AND THE CROSS SECTION



incoming (monoenergetic) beam.

flux $\equiv F$

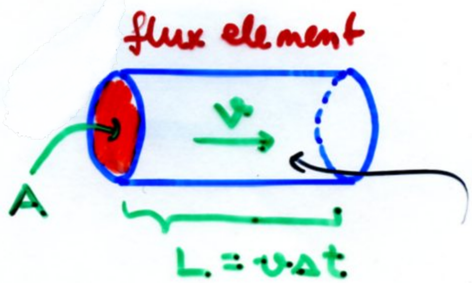
Detector, counts:

$$\frac{d\bar{N}_s}{d\Omega} = \frac{\text{scattered particles}}{\text{unit time}}$$

$$\underline{F} = \frac{\#\ \text{particles}}{(\text{unit time})(\text{unit area})_{\perp}}$$

orthogonal to the flux.

$$= \frac{N_{in}}{\frac{L}{v}\cdot A} = \underline{\rho v} \qquad (3.22)$$



density of states in the flux-element $= \rho$.

Assuming that target has N_k independent scatterers, we get the number of states scattered to the solid angle $d\Omega$ per unit time

$$\frac{d\bar{N}_s}{d\Omega} = F\cdot N_k\left(\frac{d\sigma}{d\Omega}\right) \qquad (3.23)$$

experimental coefficient of proportionality. $\sim L^{-2}$

or

$$\boxed{\frac{d\sigma}{d\Omega} \equiv \frac{1}{F\cdot N_k}\left(\frac{d\bar{N}_s}{d\Omega}\right)} \qquad (3.24)$$

by normalization: property of a single scattering event.

Theoretically we wish to compute this proportionality constant $d\sigma/d\Omega$ in an interacting QFT. First note/assume that

1. The "in" and "out" states can be taken to be eigenstates of the non-interacting field theory.
   $\Rightarrow$ boundary conditions fully understood.

2. The scattering amplitude *

$$ {}_{\text{out}}^{t=+\infty}\langle f \mid i \rangle_{\text{in}}^{t=-\infty} \qquad (3,25) $$

Is still nontrivial because the states are defined at different times and in-state must be developed through the interacting region before it can be related to the out-state.

Indeed, we shall define an $\hat{S}$-operator as a map:

$$ |\alpha\rangle_{\text{in}}^{t=-\infty} = \hat{S}\,|\alpha\rangle_{\text{out}}^{t=+\infty} \qquad (3,26) $$

Thus the scattering amplitude becomes the $\hat{S}$-matrix-element:

$$ \begin{aligned} S_{fi} &= {}_{\text{out}}\langle f|\hat{S}|i\rangle_{\text{out}} \qquad (3,27) \\ &= {}_{\text{in}}\langle f|(\hat{S}^{-1})^{+}|i\rangle_{\text{in}} \\ &= {}_{\text{out}}\langle f|(S^{-1})^{+}|i\rangle_{\text{out}} \end{aligned} $$

---

* Note that amplitudes ${}_{\text{in}}\langle\alpha|\beta\rangle_{\text{in}}$ and ${}_{\text{out}}\langle\alpha|\beta\rangle_{\text{out}}$ are known and trivial based on the first assumption.

The last equivalence followed from the assumed equivalence of the (free) and out-states. This proves that $\hat{S}$-operator is unitary

$$\hat{S}^{-1} = \hat{S}^{+}. \qquad (3.28)$$

The transition probability between states $|i\rangle_{in}$ and $|f\rangle_{out}$ is the square of the amplitude:

$$P_{fi} = \langle f|\hat{S}|i\rangle\langle f|\hat{S}|i\rangle^{*} = S^{+}_{if} S_{fi} \quad \bullet \qquad (3.29)$$

$\hat{S}$-matrix-elements contain also the uninteresting possibility that $|f\rangle = |i\rangle$, i.e. no scattering. To this end one defines the $\hat{T}$-matrix as

$$\boxed{S_{fi} = \underbrace{\delta_{fi}}_{\substack{\text{trivial part}\\ |i\rangle = |f\rangle}} - \overbrace{i(2\pi)^4}^{\text{convention}} \underbrace{\delta^4(P_f - P_i)}_{\substack{\text{4-momentum}\\ \text{conservation}}} \underbrace{T_{fi}}_{\text{only non-trivial parts.}}} \qquad (3.30)$$

Our goal below is to find an expression for T_{fi} from QFT. However, a part of the process of the evaluation of $d\sigma/d\Omega$ does not depend on the precise form of T_{fi} (interactions), but instead involves kinematics of the free in- and out-states and their normalization. Let us first figure this part out.

$\bullet$ Unitarity of $\hat{S}$ can also be seen as a requirement that the total probability of getting from $|i\rangle$ to all possible states is 1, i.e.
$\hat{S}^{+}S = 1 \Rightarrow \sum_f S_{if}^{+} S_{fi} = 1.$

NORMALIZATION AND INTERPRETATION OF P_{fi} FOR CONTINUOUS VARIABLES

Our in- and out-states are collections of free particles described by infinite plane waves. These need careful normalization procedures. Indeed assume that in-state has N and the out-state N' free particles. Then in (3.30):

$$\delta_{fi} = \langle p_1^i, \dots, p_N^i, \alpha_1^i, \dots, \alpha_N^i \mid p_1^f, \dots, p_{N'}^f ; \alpha_1^f, \dots, \alpha_{N'}^f \rangle$$

$$= \delta_{NN'} \prod_{n=1}^{N} \underbrace{(2\pi)^3 \, 2E_n \, \delta^3(p_n^i - p_n^f)}_{\text{our usual normalization for 1-particle states}} \, \delta_{\alpha_n^i, \alpha_n^f} \qquad (3.31)$$

all discrete quantum numbers

The total transition probability can now be computed from the unitarity relation:

$$P_{tot} = \sum_f P_{fi} = \sum_f S^+_{if} S_{fi} = \delta_{ii}$$

$$= \prod_{i=1}^{N} (2\pi)^3 \, 2E_n \, \delta^3(0) \quad = \infty \qquad (3.32)$$

This is actually a distribution. To see what is going on note that we can understand $\delta(0)$ as a volume-factor:

$$(2\pi)^3 \, 2E_p \, \delta(0) = (2\pi)^3 \, 2E_p \left(\lim_{q \to 0} \left(\lim_{L \to \infty} \frac{1}{(2\pi)^3} \iiint_{-L/2}^{L/2} dx\,dy\,dz \; e^{-i\vec{q}\cdot\vec{x}} \right) \right)$$

$$= 2E_p V \quad (V \to \infty) \qquad (3.33)$$

$$= \lim_{p \to p'} \langle \vec{p} \mid \vec{p}' \rangle \qquad \text{the norm of the one-particle state} \qquad (3.34)$$

$\propto N_{TOT}$/phase space element

From (3.33) and (3.34) we conclude that with plane wave normalization the continuum quantities P_{fi} are proportional to particle number / phase space element in an infinite volume $V \to \infty$. (Naturally!). If we normalize "probabilities" to unit volume by dividing with V $(= (2\pi)^3 \delta^3(0))$, the infinities will cancel. Formally the normalization

$$|\vec{p}\rangle \to \frac{1}{\sqrt{V}} |\vec{p}\rangle \quad ; \quad V \to \infty \tag{3.35}$$

would lead to

$$\tilde{P}_{TOT} = \prod_{i=1}^{N} 2E_i \tag{3.36}$$

Again, based on equations (3.33) and (3.34) we find that the quantity

$$\rho(\vec{p}_i) = 2E_i \tag{3.37}$$

can be interpreted as phase-space density of states in a unit volume.

Of course, performing a normalization $|\vec{p}\rangle \to |\vec{p}\rangle/\sqrt{2VE}$ we could remove also the $2E$-factor and find that $\tilde{\tilde{P}}_{TOT} = 1$ with this normalization. This is not necessary however, when we understand that P_{fi}:s are actually not (or necessarily not) probabilities, but phase space densities.

Box: Note that δ_{fi} in (3.31) is a generalization of the Cronecker δ-function in that

$$\sum_{N_f} \sum_{\alpha_1^f, \dots, \alpha_N^f} \int \prod_{i=1}^{N_f} \frac{d^3 p}{(2\pi)^3 2E_p} \delta_{fi} = 1.$$

(although $\delta_{ii} = \infty$)
distribution

For discrete states of course

$$P_{ij} = S^+_{if} S_{fi} = |\langle f | \hat{S} | i \rangle|^2$$

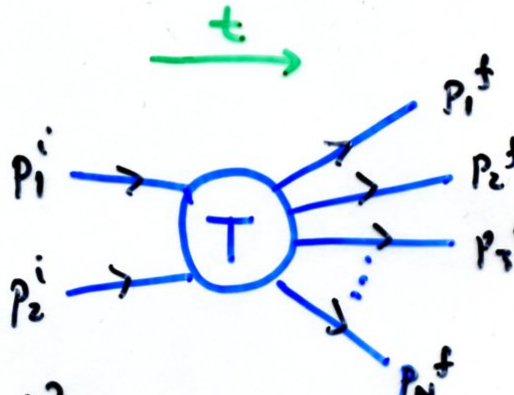
is a C-number, and so

$$P_{TOT} = \sum_f P_{if} = \text{c-number}$$

that can easily be normalized to 1.

To be specific, let us now compute an explicit expression for the cross-section in case of $2 \to N_f$ scattering.

$2 \to N_f$ SCATTERING



$$P_{fi} = S^+_{if} S_{fi}$$

$$\overset{(3.30)}{\underset{f \neq i}{=}} \left[(2\pi)^4 \delta^4 (P_f - P_i) \right]^{2} |T_{fi}|^2$$

!

$$= (2\pi)^4 \delta^4 (P_f - P_i) \cdot \underbrace{(2\pi)^3 \delta^3 (0)}_{\substack{V \\ \text{infinite volume}}} \cdot \underbrace{2\pi \delta(0)}_{\substack{T \\ \text{infinite time}}} |T_{fi}|^2 \qquad (3.38)$$

The problem with the square of the δ-functions was thus seen to be of the same origin as was the infinities in normalization of the states. (Infinite plane waves are scattering off each others everywhere and at all times.) Dividing out the VT-factor we get:

$$\frac{|S_{f\neq i}|^2}{VT} = (2\pi)^4 \delta^4(p_f - p_i)\, |T_{f\neq i}|^2 \qquad (3.38)$$

This expression ought to be related to experimental x-section. To this end note that

$$\frac{|S_{f\neq i}|}{VT} \cdot \underbrace{dN_f}_{= \prod_{k=1}^{N_f} \frac{d^3p_k^f}{(2\pi)^3 2E_k^f}} = \frac{\widetilde{\widetilde{P}}_{f\neq i}}{VT}\, 4E_1^i E_2^i V^2 \underbrace{\prod_{i=1}^{N_f} 2E_k^f \cdot V \; dN_f}_{= \prod_{k=1}^{N_f} \frac{V d^3p_k^f}{(2\pi)^3} \equiv d\widetilde{\widetilde{N}}_f}$$

$$= \frac{N_1^i N_2^i}{VT} \underbrace{\widetilde{\widetilde{P}}_{f\neq i}\, d\widetilde{\widetilde{N}}_f}_{\text{Single scattering property with unit norm.}} = \frac{dN_{scatt}(f)}{VT} \qquad (3.39)$$

Dividing this with the density of the target states (say "1") $\rho_1 = 2E_1 = N_1/V$, and by the flux F of the states 2 (assume Lab-frame with $\vec{p}_1 = 0$):

$$F = \rho_2 v_2^{lab} = \frac{N_a}{T \cdot A} = \left(\frac{N_a}{V} \cdot \frac{L}{T}\right) \qquad (3.40)$$

we get

$$\underbrace{\frac{|S_{f\neq i}|^2}{4E_1^i E_2^i v_2^{lab} VT}}_{\rho,\,F} \cdot dN_f = \frac{dN_{scatt}(f)}{VT\left(\frac{N_1}{V}\frac{N_2}{TA}\right)}$$

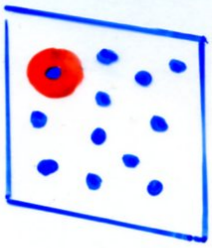
"d" depends of "f".

$$= \left(\frac{A}{N_1}\right) \underbrace{\frac{dN_{scatt}}{N_2}} \equiv \underbrace{d\sigma(f)} \qquad (3,41)$$

The actual area of one particle in the target

Number of events leading to the final state f / incoming particle

~~effective~~ ~~area~~ of the target state leading to outcome "f".



Combining equations (3,41) and (3,38) we find the differential cross section

$$\boxed{d\sigma = \frac{(2\pi)^4 \delta^4(p_f - p_i)\, |T_{f\neq i}|^2}{4\left[(p_1^i \cdot p_2^i)^2 - m_1^2 m_2^2\right]^{1/2}} \prod_{i=1}^{N_f} \frac{d^3 p_i^f}{(2\pi)^3\, 2E_i^f}} \qquad (3,42)$$

where in the final stage one used the invariant form

$$4E_1E_2v_2^{lab} = 4m_2\, p_1^{lab} = 4\left((p_1^i\cdot p_2^i)^2 - m_1^2 m_2^2\right)^{1/2}$$

$$= 2\lambda^{1/2}(s, m_1^2, m_2^2) \quad ; \quad s = (p_1+p_2)^2 \qquad (3,43)$$

Similar expressions can be found for the decay $1 \to N_f$ and any other kinematic process. Let us now turn to the task of evaluating $T_{f\neq i}$ from QFT.

3.2. LSZ - reduction formalism

Above we have derived the connection between the T-matrix and the observable cross-sections by use of the asymptotic properties of the theory. Now we will develop a formalism to express the scattering amplitude and T-matrix in terms of the Greens functions of the interacting theory. In section 3.3 we will then start developing the perturbative methods for evaluating these greens functions. There will be several steps on the way, like Wicks theorem, vacuum normalization, extracting vacuum-to vacuum transitions and irrelevant disconnected graphs. In the end the procedure will finalize into a simple set of Feynman rules for computing arbitrary scattering T-matrices, so do not get scared by the intermediate complications!

- Again we shall introduce the concepts by use of the simple scalar theory. (At this point the form of the interactions is not relevant). Observe that we can express the creation and annihilation operators in terms of the field operators as follows:

$$\begin{aligned} a_{in} &= i\int d^3x\, e^{ip\cdot x} \overleftrightarrow{\partial_0} \phi_{in}(x) \\ a_{in}^{+} &= -i\int d^3x\, e^{-ip\cdot x} \overleftrightarrow{\partial_0} \phi_{in}(x) \end{aligned} \qquad (3.44)$$

where $A\overleftrightarrow{\partial_0}B \equiv A\partial_0 B - (\partial_0 A)B$.

- Now assume that the matrix elements of the interacting and noninteracting fields ($\hat{\phi}$ and $\hat{\phi}_{in}$) can be related as:

multiplicative $\overset{(\infty\to)}{\text{normalization}}$

$$\boxed{\langle f|\phi|i\rangle_{in}^{(out)} \xrightarrow[T\to \mp\infty]{} Z^{1/2}\,{}^{(out)}\langle f|\phi_{in}^{(out)}|i\rangle_{in}} \qquad (3.45)$$

This is a very important relation. Intuitively it is well expected: asymptotically, far outside the interaction region, the complete field operator should approach adiabatically the free field limit. The factor was also to be expected. Indeed $\hat{\varphi}_{in}$ creates only 1-particle states out of the vacuum but $\hat{\phi}$ will create also the all extra pairs. So, for example the matrix element $\langle 1|\hat{\phi}|0\rangle$ does not exhaust the state $\hat{\phi}|0\rangle_g$ and thus one would expect $Z \leq 1$. ●

Let us now consider amplitude for a process $N_i \to N_f$ in the form (3.25)

$$\underline{{}_{out}\langle q_1,\dots,q_{N_f}|p_1,\dots,p_{N_i}\rangle_{in}} = {}_{out}\langle q_1,\dots,q_{N_f}|\,a^{+}_{in}(p_1)\,|p_2,\dots,p_{N_i}\rangle_{in}$$

$$= -i\int d^3x\, e^{-ip_1\cdot x}\overleftrightarrow{\partial_t}\; {}_{out}\langle q_1,\dots,q_{N_f}|\,\hat{\phi}_{in}(x)\,|p_2,\dots,p_{N_i}\rangle_{in}$$

$$= -i\lim_{\substack{t\to-\infty \\ (\text{past})}} Z^{-1/2}\int d^3x\, e^{-ip_1\cdot x}\overleftrightarrow{\partial_t}\; {}_{out}\langle q_1,\dots,q_{N_f}|\,\hat{\phi}(x)\,|p_2,\dots,p_{N_i}\rangle_{in} \tag{3.46}$$

one state is removed

We now want to get rid of the limit $t\to-\infty$ (past) and replace it by action of $\hat{\phi}$ over all space + a future term. We use:

$$\underline{\lim_{t\to\infty} \Big|_{-t}^{t} \int d^3x\, A(\vec{x},t) = \int_{-\infty}^{\infty} dt\, \partial_t \int d^3x\, A(\vec{x},t)} \tag{3.47}$$

$$\Rightarrow {}_{out}\langle\cdot\cdot|\cdot\cdot\rangle_{in} = +i Z^{-1/2}\overbrace{\int d^4x}^{\text{all space}}\, \partial_t\Big(e^{-ip_1\cdot x}\overleftrightarrow{\partial_t}\; {}_{out}\langle q_1,\dots,q_{N_f}|\hat{\phi}(x)|p_2,\dots,p_{N_i}\rangle_{in}$$

$$- i\lim_{\substack{t\to+\infty \\ (\text{future})}} Z^{-1/2}\int d^3x\, e^{-ip_1\cdot x}\overleftrightarrow{\partial_t}\; {}_{out}\langle q_1,\dots,q_{N_f}|\hat{\phi}(x)|p_2,\dots,p_{N_i}\rangle_{in} \tag{3.48}$$

● Note that we cannot assume that $\hat{\phi} = Z^{1/2}\hat{\phi}_{in}$. If we did, we could use equal time commutation relations to immediately prove that $Z=1 \Rightarrow \hat{\phi} = \hat{\phi}_{in}$!

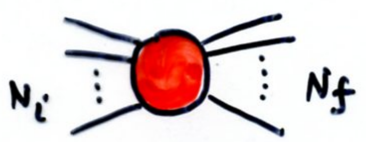
In (3,48) we have succeeded in removing one field from the past in-state and replaced it with the action of the field operator $\hat{\phi}$ through entire space (note that (3.47) could not be used before $\hat{\phi}_{in} \to \hat{\phi}$!), and an additional term involving a limit $t \to +\infty$. This term can be written as (using (3,44–3,45)):

$$-i \lim_{t\to+\infty} (\dots) = {}_{out}\langle q_1, \dots, q_{N_f} | a^{+}_{out}(p_1) | p_2, \dots, p_{N_i} \rangle$$

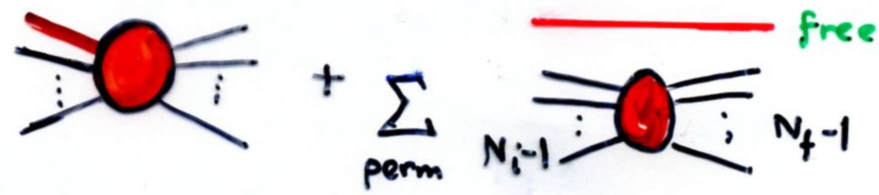
$$= \sum_{i=1}^{N_f} \underbrace{(2\pi)^3 2E_{q_i} \delta^3(q_i - p_1)}_{= {}_{free}\langle q_i | p_i \rangle_{free}} \underbrace{{}_{out}\langle q_1, \dots, \overset{\text{a removed state}}{\hat{q}_i}, \dots, q_{N_f} | p_2, \dots, p_{N_i} \rangle_{in}}_{\text{amplitude with one particle less in both initial \& final states.}} \qquad (3.49)$$

We now realize that the term (3.49) is a sum of all processes where at least one particle does not interact at all. Graphically:

$${}_{out}\langle \prod_{i=1}^{N_f} q_i | \prod_{j=1}^{N_i} p_j \rangle_{in} =$$



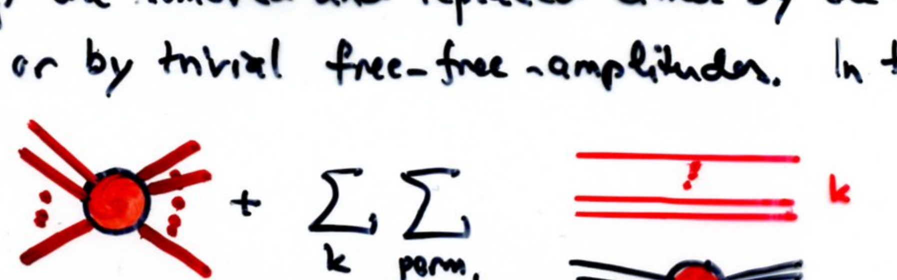
$$=$$



$$\underbrace{}_{\text{1. interaction leg.}} + \underbrace{\sum_{perm} }_{\text{Disconnected processes}}$$

After this realization, we will simply iterate the process until all $a^{+}_{in}(p_i)$ and $a^{+}_{out}(q_j)$ are removed and replaced either by the expectation values involving $\hat{\phi}$ or by trivial free–free amplitudes. In the end

$${}_{out}\langle \prod_{i=1}^{N_f} q_i | \prod_{j=1}^{N_i} p_j \rangle = \underbrace{}_{\text{Connected interaction term.}} + \underbrace{\sum_k \sum_{perm_k} }_{\text{all possible disconnected processes.}}$$



From the definitions of S- and T-matrices: (3.26) and (3.30) it is clear that in complicated processes also DC-terms contribute to the T-matrix. If the initial states are uncorrelated (as they usually are) the disconnected processes can be built from the connected subprocesses. It is therefore sufficient to concentrate only on generic connected processes from now on, discarding DC-graphs.

well, always

Having thus rid ourselves of the DC-term in (3.48) let us now rewrite the formula in a covariant form by use of the identity:

$$\int d^4x\, \partial_0 \left(e^{-ip\cdot x} \overleftrightarrow{\partial_0}\, g\right) = \int d^4x \Big(\underbrace{-(\partial_0^2 e^{-ip\cdot x})}_{= -(\partial_\mu^2 - \nabla^2) e^{-ip\cdot x} = +(m^2 + \nabla^2) e^{-ip\cdot x}} g + e^{-ip\cdot x} \partial_0^2 g \Big)$$

$$= \int d^4x \left((\nabla^2 e^{-ip\cdot x}) g - e^{-ip\cdot x} (\partial_0^2 + m^2) g \right)$$

two partial integrations on 1st term

$$= \int d^4x\, e^{-ip\cdot x} (\partial_\mu^2 + m^2) g \,, \quad \text{since (if) } g \xrightarrow{x\to\infty} 0$$

Using this with $g = \langle \cdots | p_2 \cdots \rangle$, we finally get that after one reduction step:

gain. (reduction)

$$\boxed{\begin{aligned} &{}_{out}\langle q_1, \ldots, q_{N_f} | p_1, \ldots, p_{N_i} \rangle_{in} = \\ &= i Z^{-1/2} \int d^4x\, e^{-ip_1\cdot x} (\partial_x^2 + m^2)\, {}_{out}\langle q_1, \ldots, q_{N_f} | \hat{\phi} | p_2, \ldots, p_{N_i} \rangle_{in} \\ &(+\text{ DC-terms}) \qquad\qquad (3.50) \end{aligned}}$$

"prize"

states that become the free vacuum in the limits $t \to \pm\infty$.

There is one more complication that arises in the second step of reduction. (Let us now remove a particle from the final state)

$$_{out}\langle q_2,\dots,q_{N_f}|\, a_{out}(q_1)\, \hat{\phi}(x)\, |p_2,\dots,p_{N_i}\rangle_{in}$$

$$= i \lim_{y_0\to+\infty} Z^{-1/2} \int d^3y\, e^{iq_1\cdot y} \overleftrightarrow{\partial}_{y_0}\, {}_{out}\langle q_2,\dots,q_{N_f}|\, \hat{\phi}(y)\hat{\phi}(x)\, |p_2,\dots,p_{N_i}\rangle \qquad (3.51)$$

We could use the trick (3.47) again to convert this into a 4-space-integral form + a term with the limit $y_0\to-\infty$. Interpreting the latter as a DC-term is not possible however, because the operators $\hat{\phi}(y)$ and $\hat{\phi}(x)$ would be in wrong order. This forces us to use the time-ordered identity:

$$\lim_{y_0\to\infty} \Big/_{-\infty}^{\infty} T(\phi(y)\phi(x)) = \int_{-\infty}^{\infty} dy_0\, \partial_{y_0} T(\phi(y)\phi(x)). \qquad (3.52)$$

That is, introducing the time-ordering to the 4-space term we get the operator order exchanged in the surface terms! With this it is easy to show that

$$\boxed{\begin{aligned} {}_{out}\langle q_1,\dots,q_{N_f}|p_1,\dots,p_{N_i}\rangle_{in} = \\ i^2(Z^{-\frac{1}{2}})^2 \int d^4x_1 d^4y_1\, e^{iq_1\cdot y_1 - ip_1\cdot x_1} (\partial_{y_1}^2+m^2)(\partial_{x_1}^2+m^2)\,\cdot \\ \times\, {}_{out}\langle q_2,\dots,q_{N_f}|\, T(\phi(y_1)\phi(y_2))\, |p_2,\dots,p_{N_i}\rangle_{in} \\ +\ \text{DC-terms}. \qquad (3.53) \end{aligned}}$$

The issue with operator ordering comes back at each reduction step, and it can always be accounted by introducing time-ordering and one eventually finds:

$$
\boxed{
\begin{aligned}
{}_{out}\langle q_1,\dots,q_{N_f} | p_1,\dots,p_{N_i}\rangle_{in} &= \text{DC-terms} + \\
&+ (iZ^{-\frac{1}{2}})^{N_i+N_f} \int d^4y_1 \dots d^4y_{N_f}\, d^4x_1 \dots d^4x_{N_i} \times \\
&\times e^{i\sum^{N_f} q_i\cdot y_i - i\sum^{N_i} p_i \cdot x_i} \prod_{j=1}^{N_f} \prod_{i=1}^{N_i} (\partial^2_{y_j}+m^2)(\partial^2_{x_i}+m^2) \times \\
&\quad \langle\Omega| T(\hat\phi(y_1)\dots\hat\phi(y_{N_f})\,\hat\phi(x_1)\dots\hat\phi(x_{N_i}))|\Omega\rangle
\end{aligned}
} \qquad (3{,}54)
$$

This is the a Lehmann-Symanzik-Zimmermann reduction formula, which expresses an on-shell transition amplitude ${}_{out}\langle f|i\rangle_{in}$ in terms of the N_i+N_f-point greens function of the interacting field theory:

$$
G(x_1 \dots x_m) \equiv \langle 0|^{(t=+\infty)} T(\hat\phi(x_1)\dots\hat\phi(x_m))|0\rangle^{(t=-\infty)} \;* \qquad (3{,}55)
$$

Our next task is to develop methods for computing $G(x_1,\dots,x_m)$ = perturbation theory.

* (One often leaves out the labels $t=\pm\infty$ over the vacua in expressions 3,54 & 3,45. The idea is that one is implicitly assuming that transitions are between infinite past and infinite future.)

3.3 PERTURBATION THEORY

We still need to work out the vacuum-to vacuum Greens functions (over infinite time!) left out from LSZ-reduction. The idea will be to write everything in terms of the non-interacting theory operators, treating interactions as perturbations.

TIME EVOLUTION OPERATOR. In Heisenberg picture we have:

$$\hat{\phi}(\vec{x},t) = e^{i\hat{H}(t-t_0)}\,\hat{\phi}(\vec{x},t_0)\,e^{-i\hat{H}(t-t_0)} \qquad (3.56)$$

Taking $t_0=-\infty$, this can be used to relate full $\hat{\phi}(\vec{x},t)$ to the asymptotic in-state operators. However most of this evolution is trivial free field evolution. To extract this we divide

$$\hat{H} = \hat{H}_0 + \hat{H}_I \qquad (3.57)$$

where H_0 is the free Hamiltonian and H_I is the interaction:

$$H_I = \int d^3x\, \mathcal{H}_I(x) = -\int d^3x\, \mathcal{L}_I(x)\,. \qquad (3.58)$$

what field?

For example in $\lambda\phi^4$-th theory: $\quad \mathcal{H}_I(x) = \frac{\lambda}{4!}\phi^4(x). \qquad (3.59)$

Separating out the free evolution that takes $\hat{\phi}_{free}(t_0) \to \hat{\phi}_{free}(t)$, we can write

$$\hat{\phi}(\vec{x},t) = U^{-1}(t,t_0)\underbrace{\left(e^{i\hat{H}_0(t-t_0)}\,\hat{\phi}(\vec{x},t_0)\,e^{-i\hat{H}_0(t-t_0)}\right)}_{\equiv\,\phi_I(\vec{x},t)}U(t,t_0) \qquad (3.60)$$

where we have defined $\neq e^{i\hat{H}_I(t-t_0)}$!

$$U(t,t_0) \equiv e^{+i\hat{H}_0(t-t_0)} e^{-i\hat{H}(t-t_0)}. \qquad (3.61)$$

Taking $t_0 \to -\infty$ we can identify $\hat{\phi}(\vec{x},t_0) = \hat{\phi}_{in}$, and so $\hat{\phi}_I(x,t)$ becomes the free in-field at time t: $\hat{\phi}_I = \hat{\phi}_{in}(\vec{x},t)$. Our task is thus reduced to finding a usable form for the time-evolution operator U.(•) We can derive an e.o.m for it from

$$\dot{\phi}(t,\vec{x}) = i\left[H(t), \phi(t,\vec{x})\right] \qquad (3.62a)$$

$$\dot{\phi}_{in}(t,x) = i\left[H_0^{in}, \phi^{in}(t,\vec{x})\right] \qquad (3.62b)$$

On the other hand from (3.60):

$$\begin{aligned}
\dot{\phi}_{in} &= \frac{d}{dt}(U\phi U^{-1}) \\
&= \dot{U}\phi U^{-1} + U\dot{\phi}U^{-1} - U\phi \underbrace{U^{-1}\dot{U}U^{-1}}_{=\dot{U}^{-1}} \\
&= \dot{U}U^{-1}\phi_{in} - \phi_{in}\dot{U}U^{-1} + U\, i[H,\phi]\, U^{-1} \\
&= [\dot{U}U^{-1}, \phi_{in}] + [iUHU^{-1}, U\phi U^{-1}] \quad : \quad UHU^{-1} = H(\phi_{in}, \pi_{in}) \\
&= [\underbrace{\dot{U}U^{-1} + iH(\phi_{in},\pi_{in})}_{\overset{3.62a}{=} iH_0^{in} + \cancel{c}\, 0}, \phi_{in}]
\end{aligned}$$

(•) By definition $U(t_1,t_2)$ satisfies $U(t,t) = 1$ and

$$U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)$$

$$U^{-1}(t_1,t_2) = U(t_2,t_1)$$

From this it follows that:

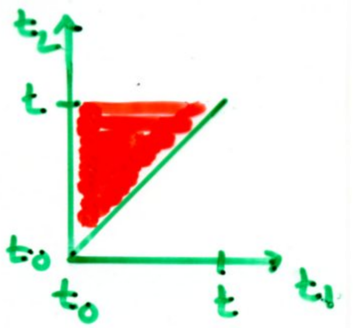
$$i\frac{d}{dt}U(t,t_0) = H_I(t)\,U(t,t_0) \qquad (3.63)$$

where

$$H_I(t) \equiv H(\phi_{in},\pi_{in}) - H_0^{in} = \int d^3x\, \frac{\lambda}{4!}\,\phi_{in}^4 \qquad (3.64)$$

ϕ_{in}-field!

When integrating (3.63) one must be careful to account for the non-commutativity at different times: $[\hat{H}_I(t_1), \hat{H}_I(t_2)] \neq 0$. We get by iteration:



$$U(t,t_0) = U(t_0,t_0) - i\int_{t_0}^{t} dt'\, H_I(t')\,U(t',t_0)$$

$$= 1 - i\int_{t_0}^{t} dt'\, H_I(t')\left(1 - i\int_{t_0}^{t'} dt''\, H_I(t'')\,U(t'',t)\right)$$

$$\ldots = 1 - i\int_{t_0}^{t} dt_1\, H_I(t_1) + (-i)^2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1)\,H_I(t_2) + \ldots$$

$$= 1 - i\int_{t_0}^{t} dt\, H_I(t_1) + \frac{(-i)^2}{2!}\int_{t_0}^{t}\int_{t_0}^{t} dt_1\, dt_2\, T\big(H_I(t_1)\,H_I(t_2)\big) + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{(-i)^n}{n!}\int_{t_0}^{t} dt_1 \ldots dt_n\, T\big(H_I(t_1) \ldots H_I(t_n)\big)$$

$$\equiv T\exp\left(-i\int_{t_0}^{t} dt\, \hat{H}_I(t)\right) \qquad (3.65)$$

Thus $U(t,t_0)$ is a time-ordered exponent of the interaction Hamiltonian.

PT-EXPANSION FOR THE N-POINT GREEN FUNCTION

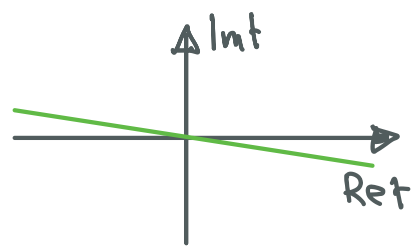
We have established that a scattering matrix $S_{fi}^{(n)}$ involving n particles in the initial or final states is related to the n-point Green function:

$$S_{fi}^{(n)} = (iR^{-1})^n \int \prod_{i=1}^{n} \left[d^4x_i\, e^{-ip_i\cdot x_i}(\Box_i + m_i^2)\right] \langle\Omega|T(\hat{\phi}(x_1)\dots\hat{\phi}(x_n))|\Omega\rangle \qquad (3.66)$$

where all momenta are pointing into the graph. (For out-states set $p_i \to -p_i$ then).
We now want to compute $\langle\Omega|T(\hat{\phi}(x_1)\dots\hat{\phi}(x_n))|\Omega\rangle$ perturbatively, using free theory states and free theory vacuum. The main distinction is that for full theory $H|\Omega\rangle = 0$ whereas in free theory $H_0|0\rangle \equiv 0$.
Now assume that $|0\rangle$ can be expanded in full theory Fock space:
$|0\rangle = |\Omega\rangle\langle\Omega|0\rangle + \sum_{n=1}^{\infty} |n\rangle\langle n|0\rangle$. This then implies that

$$e^{-iHT}|0\rangle = e^{-iE_\Omega t}\langle\Omega|0\rangle\,|\Omega\rangle + \sum_{n=1}^{\infty} e^{-iE_n t}\langle n|0\rangle\,|n\rangle$$

Now set $T \to (1-i\epsilon)\infty$, which implies that only the vacuum state will remain in the sum. (This is our preparation of the system into collection of free particles in both asymptotics.) Using also $e^{iH_0 t}|0\rangle = |0\rangle$, we get



$$|\Omega\rangle = \lim_{T\to(1-i\epsilon)\infty} \frac{e^{-iH(T+t)}\,e^{iH_0(T+t)}|0\rangle}{e^{-iE_\Omega(T+t)}\langle\Omega|0\rangle} = \lim_{T\to(1-i\epsilon)\infty} \frac{U(t,-T)|0\rangle}{e^{-iE_\Omega(T+t)}\langle\Omega|0\rangle} \qquad (3.67)$$

and

$$\langle\Omega| = \lim_{T\to(1-i\epsilon)\infty} \frac{e^{-iH(T-t)}\,e^{iH_0(T-t)}|0\rangle}{e^{-iE_\Omega(T-t)}\langle\Omega|0\rangle} = \lim_{T\to(1-i\epsilon)\infty} \frac{\langle 0|U(T,t)}{e^{-iE_\Omega(T+t)}\langle\Omega|0\rangle} \qquad (3.68)$$

(Note that $\langle 0|e^{iHT} = (e^{-iHT}|0\rangle)^\dagger \to (e^{-iE_\Omega T - E\epsilon T}\langle\Omega|0\rangle|\Omega\rangle)^\dagger = e^{-E\epsilon T}e^{-iE_\Omega T}\langle 0|\Omega\rangle\langle\Omega|$. converges as too.) Then using normalization $\langle\Omega|\Omega\rangle = 1$, we moreover get

$$1 = \lim_{T\to(1-i\epsilon)\infty} \frac{\langle 0|U(T,t)U(t,-T)|0\rangle}{e^{-2iE_\Omega T}|\langle\Omega|0\rangle|^2} \Rightarrow \langle 0|U(T,-T)|0\rangle = e^{-2iE_\Omega T}|\langle\Omega|0\rangle|^2 \quad (3.69)$$

We can now use (3.68-3.70) to write: (let us take $t > x_{1_0} > x_{2_0} > \ldots > x_{n_0} > -t$)

$$\langle\Omega|T(\hat\phi(x_1)\ldots\hat\phi(x_n))|\Omega\rangle = \frac{\langle 0|U(T,t)\,\hat\phi(x_1)\ldots\hat\phi(x_n)\,U(t,-T)\,|0\rangle}{\langle 0|U(T,-T)|0\rangle} \quad (3.70)$$

Next write

$$\begin{aligned}\hat\phi(\bar x_i, x_{0i}) &= e^{i\hat H(x_{0i}-t)}\,\hat\phi(\vec x,t)\,e^{-i\hat H(x_{0i}-t)} \\ &= U(t,x_{0i})\,e^{i\hat H_0(x_{0i}-t)}\,\hat\phi(\bar x,t)\,e^{-i\hat H_0(x_{0i}-t)}\,U(x_{0i},t) \\ &= U(t,x_{0i})\,\hat\phi_I(\bar x,x_{0i})\,U(x_{0i},t) \end{aligned} \quad (3.71)$$

a state evolved from t to x_{0i} by free field evolution operator.

$\hat{=}$ 1-particle state of full theory at t evolved to x_{0i} as 1-particle state with no interactions.

We can then write the nominator in the r.h.s of equation (6) as

$$= \langle 0|\,\underbrace{U(T,t)U(t,x_{01})}_{=U(T,x_{01})}\,\hat\phi_I(x_1)\,\underbrace{U(x_{01},t)U(t,x_{02})}_{U(x_{01},x_{02})}\,\hat\phi_I(x_2)\cdots\hat\phi_I(x_n)\,\underbrace{U(x_{0n},t)U(t,-T)}_{U(x_{0n},-T)}\,|0\rangle$$

This applies for any ordering of x_{0i}, which means it applies also to the T-ordered product.

$$= \langle 0|\,T(\,U(T,x_{01})\,\hat\phi_I(x_1)\,U(x_{01},x_{02})\,\hat\phi_I(x_2)\cdots\hat\phi_I(x_{0n})\,U(x_{0n},-T)\,)\,|0\rangle$$

$$= \langle 0|T(\hat{\phi}_I(x_1)\cdots\hat{\phi}_I(x_n)\,U(T,-T))|0\rangle$$

Here we used first $-T \ll x_{0i} \ll T$, used the fact that $U(t_1,t_2)$ is T-ordered operator and finally relied on time-ordering to split $U(T,-T)$ correctly around and between the field operators. Finally using (3.65), we get

$$\boxed{\langle\Omega|T(\hat{\phi}(x_1)\cdots\phi(x_n))|\Omega\rangle = \frac{\langle 0|T(\hat{\phi}_I(x_1)\cdots\hat{\phi}_I(x_n)\exp(i\int_{-T}^{T}d^4x\,\mathcal{L}_I(x)))|0\rangle}{\langle 0|T(\exp(i\int_{-T}^{T}d^4x\,\mathcal{L}_I(x)))|0\rangle}} \qquad (3.72)$$

The r.h.s. of eqn. (3.72) can be expanded as a series of time-ordered vacuum expectation values. The series has hope of converging if $\mathcal{L}_I$ is in some sense small. This series expansion = perturbation theory.

at least in the sense of Borel-summability.

Putting (3.72) back to (3.54) we have a calculable (approximation) scheme for computing the T-matrix from the QFT!

3.4 Wick's THEOREM.

Perturbation theory is quite cumbersome tool, and one must be good at bookkeeping when using it. The following Wick's theorem is an invaluable tool in reducing complicated free-theory vacuum expectation values:

$$\boxed{\langle 0|T(\hat{\phi}(x_1)\cdots\hat{\phi}(x_n))|0\rangle = \sum_{\text{combinations}} D_F(x_2-x_1)\cdots D_F(x_n-x_{n-1})} \qquad (3.73)$$

Feynman propagator

More precisely, (3,73) follows from Wick's theorem, that actually states a connection between time-ordered and normal-ordered operator products. To appreciate this, consider first a product of two fields.

Define:

$$\hat{\phi}^+(x) = \int \widetilde{d^3p}\; a_{\vec{p}}\, e^{ip\cdot x}$$
$$\hat{\phi}^-(x) = \int \widetilde{d^3p}\; a^+_{\vec{p}}\, e^{-ip\cdot x} \qquad (3,74)$$

Then obviously

annihilation operator to left

$$\hat{\phi}^+|0\rangle = \langle 0|\hat{\phi}^- = 0 \qquad (3,75)$$

annihilation op. to right.

We can now express a normal-ordered product:

$$:\hat{\phi}(x)\,\hat{\phi}(y): \;=\; \underbrace{\hat{\phi}^+(x)\hat{\phi}^+(y)}_{=\hat{\phi}^+(y)\hat{\phi}^+(x)} + \hat{\phi}^-(x)\hat{\phi}^+(y) + \hat{\phi}^-(y)\hat{\phi}^+(x) + \underbrace{\hat{\phi}^-(x)\hat{\phi}^-(y)}_{=\hat{\phi}^-(y)\hat{\phi}^-(x)} \qquad (3,76)$$

Now, if $x_0 > y_0$ we have*

$$T(\hat{\phi}(x)\hat{\phi}(y)) = \hat{\phi}(x)\,\hat{\phi}(y) = :\hat{\phi}(x)\hat{\phi}(y): + [\hat{\phi}^+(x), \hat{\phi}^-(y)] \qquad (3,77a)$$

and if $x_0 < y_0$

$$T(\hat{\phi}(x)\hat{\phi}(y)) = \hat{\phi}(y)\,\hat{\phi}(x) = :\hat{\phi}(x)\hat{\phi}(y): + [\phi^+(y), \phi^-(x)] \qquad (3,77b)$$

---

* Rem.: $T(\phi(x)\phi(y)) \equiv \Theta(x_0-y_0)\,\phi(x)\phi(y) + \Theta(y_0-x_0)\,\phi(y)\phi(x)$

Defining a contraction

$$\boxed{\overline{\hat{\phi}(x_1)\hat{\phi}(x_2)} \equiv \theta(x_0-y_0)\left[\hat{\phi}^+(x), \hat{\phi}^-(y)\right] + \theta(y_0-x_0)\left[\hat{\phi}^+(y), \hat{\phi}^-(x)\right]} \qquad (3.78)$$

we can write the Wick's theorem for two fields:

$$T(\hat{\phi}_1(x)\hat{\phi}_2(x)) = :\hat{\phi}(x_1)\hat{\phi}(x_2): + \overline{\hat{\phi}(x_1)\hat{\phi}(x_2)} \qquad (3.79)$$

It is easy to see that the contraction is just the Feynman propagator (without taking vac. expectation values!): (see (1.45) and (1.49))

$$\overline{\hat{\phi}(x_1)\hat{\phi}(x_2)} = D_F(x_2-x_1) \qquad (3.80)$$

Since $\langle 0| :\hat{\phi}(x_1)\hat{\phi}(x_2): |0\rangle = 0$, the result (3.73) for two fields follows from (3.79).

The most general form of the Wick theorem states that

$$\boxed{T(\hat{\phi}_1, \dots, \hat{\phi}_n) = :\hat{\phi}_1, \cdots \hat{\phi}_n + \text{all possible contractions}:} \qquad (3.81)$$

Examples:

$$T(\hat{\phi}_1\hat{\phi}_2) = :\hat{\phi}_1\hat{\phi}_2 + \overline{\hat{\phi}_1\hat{\phi}_2}: \quad = :\hat{\phi}_1\hat{\phi}_2: + \overline{\hat{\phi}_1\hat{\phi}_2}$$

$$T(\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3) = :\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3 + \hat{\phi}_1\overline{\hat{\phi}_2\hat{\phi}_3} + \overline{\hat{\phi}_1\hat{\phi}_2}\hat{\phi}_3 + \overline{\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3}:$$

and so on.

Theorem (3.81) then holds for $n=2$. Let us sketch its proof for an arbitrary G_n by induction. (Again take $x_1^0 \geq x_2^0 \geq \dots \geq x_n^0$.)

If not this order, just relabel.

$$T(\hat\phi_1,\dots,\hat\phi_n) = \hat\phi_1,\dots,\hat\phi_n$$

induction step

$$= \hat\phi_1 :\hat\phi_2,\dots\hat\phi_n + \text{contractions}:$$

$$= (\hat\phi_1^+ + \hat\phi_1^-) :\hat\phi_2,\dots,\hat\phi_n + \text{contractons}: \qquad (3.81a)$$

It is sufficient to prove theorem for a generic $(\phi_1^+ + \phi_1^-):\underbrace{\hat\phi_2\cdots\hat\phi_m}:$

some n-ordered sub-sequence.

Write $:\hat\phi_2\cdots\hat\phi_m:$

sub-order does not matter.

$$= \sum_{k=0}^{m-1}\binom{m-1}{k}\phi_{i_1}^-\cdots\phi_{i_k}^-\,\phi_{i_{k+1}}^+\cdots\phi_{i_{m-1}}^+ \quad ; \quad \sum_{k=0}^{m-1}\binom{m-1}{k} = 2^{m-1}\text{ terms.}$$

$$\binom{m-1}{k} = \frac{(m-1)!}{k!(k-m+1)!}$$

Then

$$(\phi_1^+ + \phi_1^-):\hat\phi_2\cdots\hat\phi_m: = \underbrace{\phi_1^- :\hat\phi_2\cdots\hat\phi_m: + :\hat\phi_2\cdots\hat\phi_m:\phi_1^+}_{2\cdot 2^{m-1} = \sum_{k=0}^{m-1}\binom{m-1}{k}\text{ terms} = :\hat\phi_1\hat\phi_2\cdots\hat\phi_m:} + [\hat\phi_2^+, :\hat\phi_2\cdots\hat\phi_m:]$$

Commutator:

\# of equivalent combinations

$$[\hat\phi_2^+ : \cdots :] = \sum_{k=0}^{m-1}\binom{m-1}{k}\underbrace{[\phi_1^+, \phi_{i_1}^-\cdots\phi_{i_k}^-]}\cdot\phi_{i_{k+1}}^+\cdots\phi_{i_n}^+ \qquad [\phi_i^+,\phi_j^+] = 0$$

$$= \underbrace{[\phi_1^+,\phi_{i_1}^-]}_{=\overline{\phi_1\phi_{i_1}}}\phi_{i_2}^-\cdots\phi_{i_k}^- + \phi_{i_1}^-\overbrace{[\phi_1^+,\phi_{i_2}^-]}^{\overline{\phi_1\phi_{i_2}}}\phi_{i_3}^-\cdots\phi_{i_k}^-$$

$$+ \cdots \phi_{i_1}^-\cdots\phi_{i_{k-1}}^-\underbrace{[\phi_{i_1}^+,\phi_{i_k}^-]}_{=\overline{\phi_{i_1}\phi_{i_k}}} = k\text{ terms}$$

this adds all contractions with $\hat\phi_1$ to all substrings

$\Downarrow$

Thus $\hat\phi_1 :\hat\phi_2\cdots\hat\phi_m: = :\hat\phi_1\hat\phi_2\cdots\hat\phi_m: +$ all possible contractions with ϕ_1.

So, going through all terms in the series we get all possible new normal orderings and all possible contractions with all fields, which proves the theorem.

Wicks theorems most important consequence is (3.73). It follows trivially from the fact that $\langle 0 | : \text{anything} : | 0 \rangle = 0$, i.e. only the fully contracted term in the Wicks expansion survives.

3.5 FEYNMANIN DIAGRAMMAT

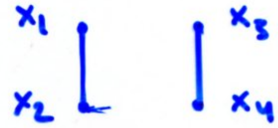
are nothing but a nice way to represent graphically the different terms contributing to function (3.73).

Example 1.

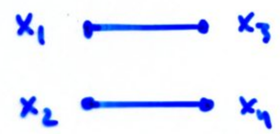
etc

$$\langle 0 | T(\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)) | 0 \rangle$$

$$= D_F(x_2 - x_1)\, D_F(x_4 - x_3)$$



$$+ D_F(x_3 - x_1)\, D_F(x_4 - x_2)$$



$$+ D_F(x_4 - x_1)\, D_F(x_3 - x_2)$$

This function, and the corresponding graphs follow from (3.72) in the lowest order of the perturbation theory. These are not interesting scatterings, and we shall see that they actually do not contribute to the T-matrix. (Note that despite the appearance, these are unlike the DC-processes in the LSZ-step. Terms (3.82) are part of a fully connected process, but not interesting due to PT-expansion.)

The higher order terms can be found by expanding the operator $e^{i\int \mathcal{L}_I}$ as a Taylor series. Indeed:

$$\boxed{\begin{aligned}&\langle 0|T(\hat\phi_1,\dots,\hat\phi_n e^{i\int d^4y\,\mathcal{L}_I})|0\rangle \\ &\quad = \sum_{n=0}^{\infty}\frac{1}{m!}\int_{-\infty}^{\infty} d^4y_1\dots d^4y_m \langle 0|T\Big(\hat\phi(x_1)\dots\hat\phi(x_n)\cdot \\ &\qquad\qquad \times i\mathcal{L}_I(y_1)\dots i\mathcal{L}_I(y_m)\Big)|0\rangle\end{aligned}} \qquad (3.82)$$

Example 2

$$\langle 0|T(\hat\phi_1\hat\phi_2\; i\int d^4y\, \overbrace{\mathcal{L}_I(y)}^{=-\frac{\lambda}{4!}\phi^4(y)})|0\rangle$$

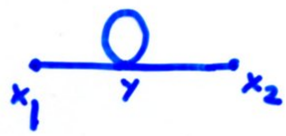
(3 ways)

$$= -\frac{\lambda}{4!}\int d^4y\,\langle 0|T(\hat\phi(x_1)\hat\phi(x_2)\,\hat\phi(y)\hat\phi(y)\hat\phi(y)\hat\phi(y))|0\rangle$$

4·3 ways

$$= -\frac{\lambda}{4!}\int d^4y\,\Big(\underline{3}\, D_F(x_1-x_2)\, D_F^2(0)\,\cdot$$

$$+\underline{4\cdot 3}\, D_F(y-x_1)\,D_F(y-x_2)\,D_F(0)\Big)$$



(3.83)

First of these is again a disconnected graph. In the end it is also cancelled in the expression (3.72). (It contains a vacuum graph 8 and all such terms go away when one accounts for the denominator in the PT-expansion.)

The combinatoric factors 3 and $4\cdot 3 = 12$ appearing in (3.83)

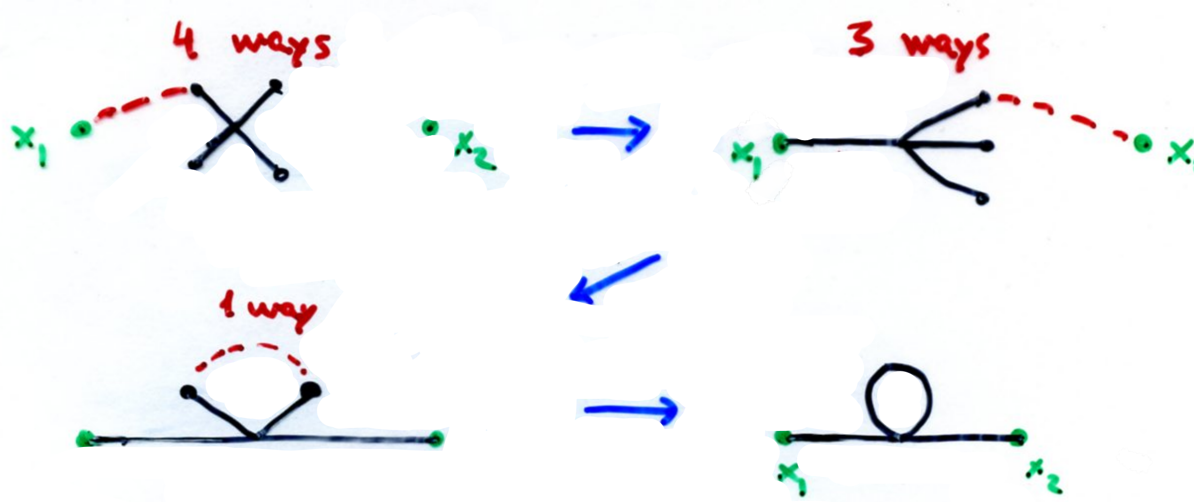
are coefficients expressing the number of equivalent contractions.
For example the vacuum diagram 8 has three of these:

$$8 = \langle 0| \cdots \phi_y \phi_y \phi_y \phi_y + \cdots \phi_y \phi_y \phi_y \phi_y + \cdots \phi_y \phi_y \phi_y \phi_y |0\rangle$$

Combinatoric factors are very important, and at first sight very cumbersome. Fortunately $\lambda\phi^4$-theory provides the worst case scenario in combinatorics!
CF's can be defined graphically. A contraction means just connecting two points in a graph. Each field operator creates a dot to which a line can be connected. Thus the interaction term $\sim \phi_y^4$ is a "4-dot" to which 4 lines can be connected, let us denote it as follows:

each "end" (operator) can be connected anywhere

Example The diagram can be constructed as follows.



We are thus getting 12 equivalent terms that will have the same numerical value. Kombinatorics factor defines the symmetry factor S for the graph. Above

$$S(\text{—○—}) \equiv \frac{4!}{12} = 2$$

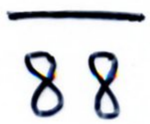
More generally it holds that the symmetry factor of an arbitrary graph of order λ^n (n:th order graph) is:

$$S = \frac{(4!)^n \, n!}{\text{comb. factor}} \qquad (3.85)$$

In higher orders things get more baroque of course.

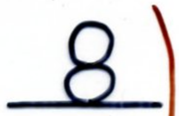
Example 3

$$\frac{1}{2!}\left(\frac{\lambda}{4!}\right)^2 \langle 0|\, \phi_1 \phi_2 \;\; \phi_y\phi_y\phi_y\phi_y \;\; \phi_z\phi_z\phi_z\phi_z \,|0\rangle$$

$= \dfrac{\lambda^2}{2!(4!)^2} \Big\{$ $\overset{(128)}{3\cdot 3}$  $+ \overset{(96)}{4\cdot 3}$  $+ \overset{(48)}{4!}$  (DC + vac.)

$+ \overset{(16)}{8\cdot 3\cdot 3}$  (vacuum)

$+ \overset{(S=4)}{8\cdot 4\cdot 3\cdot 3}$  (resummable)

$+ \Big(\overset{(S=6)}{8\cdot 4\cdot 3\cdot 2}$  $+ \overset{(S=4)}{8\cdot 4\cdot 3\cdot 3}$  $\Big) \Big\}$

( ) = Important! 1PI (one particle irreducible graphs).
(later)

$$= \lambda^2 \sum_i \frac{1}{S_i} (\text{Diagram}). \qquad (3.86)$$

coupling constant — symm. factor — Diagram (from Feynman rules)

In practice one does not develop the PT-expansion from the vacuum expectation values and their operator product expansions. Rather, one draws the appropriate (1PI-) diagrams up to the desired order in λ and determines the relevant symmetry factor. You can already guess what they are:

Feynman rules for evaluating Greens functions (direct space)

In each separate graph mark each

1) propagator  $\rightarrow$ $D_F(x-y)$

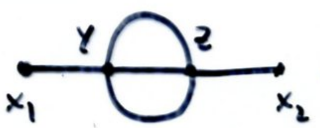
2) vertex  $\rightarrow$ $-i\lambda \int d^4z$

and

3) divide by the graphs-symmetry factor (3.87)

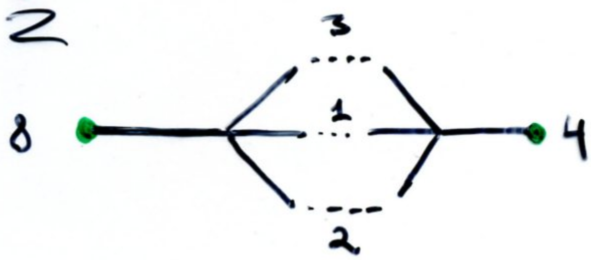
Example

$\frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^2 \times$(combinatorics factor)



$$= \frac{(-i\lambda)^2}{S} \int d^4y\, d^4z\, D_F(y-x_1)\left(D_F(z-y)\right)^3 D_F(x_2-z)$$

where $S = \frac{(4!)^2 2!}{8\cdot 4\cdot 3\cdot 2} = \frac{(4!)^2 2!}{\frac{1}{3}(4!)^2} = 6$ ~~2~~



comb. fac = $8\cdot 4\cdot 3\cdot 2$

3.6 CLASSIFYING GRAPHS

As has become clear from the examples above, PT creates a large number of different graphs, so at high orders computations become involved. Fortunately by far the most of the graphs we have drawn turn out to be uninteresting. There is two main reasons for this:

1. Disconnected graphs are killed in (3.54).
2. Vacuum graphs get cancelled in (3.72)

REMOVING THE DC-GRAPHS

While performing the LSZ-reduction we systematically dropped all DC-graphs. Yet, when computing the Greens function in (3.54), we immediately get more DC-processes. This apparent contradiction is removed when we realize that those new DC-graphs do not contribute to the amplitude ${}_{out}\langle f | i \rangle_{in}$. Indeed, each "new" DC-graph in the PT-expansion involves at least one contraction between the external operators (not containing any interaction operators):

$$DC \sim \langle 0 | T(\hat{\phi}_1 \cdots \hat{\phi}_i \cdots \hat{\phi}_j \cdots \hat{\phi}_n \, e^{i\int \mathcal{L}_I} | 0 \rangle$$

$$\propto \; D_F(x_j - x_i) \times \ldots$$

Such a term is always accompanied in (3.54) by the integrals:

$$DC \sim \int d^4x_i\, d^4x_j\, e^{\pm i p_i \cdot x_j \pm i p_j \cdot x_j} (\partial_{x_i}^2 + m^2)(\partial_{x_j}^2 + m^2)\, D_F(x_j - x_i) \cdots$$

But since $D_F(x_j - x_i)$ is the Greens function of the free field theory, we have

$$(\partial_x^2 + m^2)\, D_F(x-y) = -i\,\delta^4(x-y).$$

Using (3.90) we can evaluate the integrals in (3.89) with the result

$$DC \sim \cdots (2\pi)^4\, \delta^4(p_j - p_i)\, \underbrace{(-p_i^2 + m^2)}_{=0}$$

$= 0$, because this is an on shell state.

Thus despite the fact that PT-expansion for interacting Greens functions creates DC-graphs, these are not part of the amplitude.

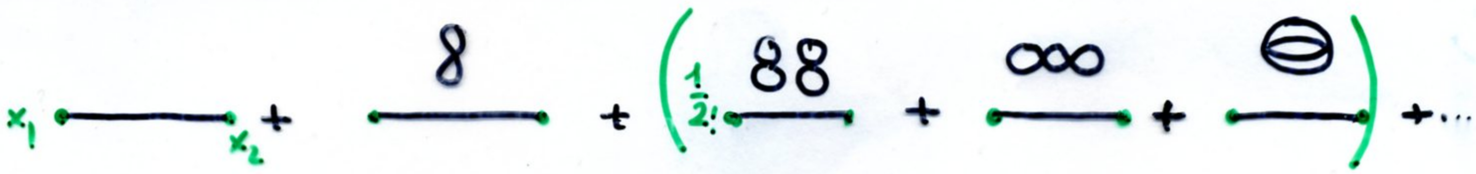
$\Rightarrow$ When employing rules (3.87) never draw any DC-graphs.

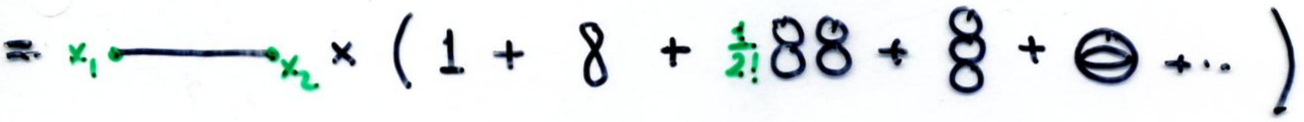
etc

For example for the 2^{nd} order corrections in example 3 on p 99 this implies that we can throw the 1^{st} line to trash right away.

VACUUM GRAPH CANCELLATION

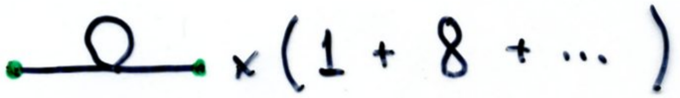
Reorganizing the perturbation expansion, we can show that each connected graph is accompanied by an identical infinite series of vacuum diagrams. Moreover, this series will be identified with the perturbative expansion of the denominator in (3.72). Indeed for example:





(3.41)

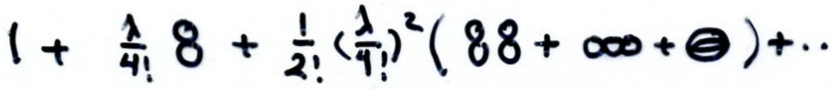
Similarly one can see that with the graph  we get



(3.42)

The diagrams appearing these multiplicative expansions are not connected to any of the external points. Such graphs are thus vacuum-to-vacuum transitions, or vacuum diagrams.

In these cases it is fairly easy to show that the series is an exponent:



$$= \langle 0|0\rangle + \langle 0|T(i\int d^4y\,\mathcal{L}_I(y))|0\rangle + \frac{1}{2!}\langle 0|T(i\int d^4y\,\mathcal{L}_I(y)\, i\int d^4x\,\mathcal{L}_I(x))|0\rangle + \dots$$

$$= \langle 0|e^{i\int d^4y\,\mathcal{L}_I(y)}|0\rangle \qquad (3.43)$$

Formal proof for an arbitrary connected diagram, or contraction in straightforward:

a) Let Γ_i be some connected contraction, which first appears in the n:th order of the PT's

$$\Gamma^m_{n,i} = \langle 0| T(\underbrace{\hat{\phi}_1, \dots, \hat{\phi}_m \left[\frac{1}{n!} \prod_{j=1}^{n} i \int d^4y_j \, \mathcal{L}_I(y_j) \right]}_{\Gamma^m_{n,i}} |0\rangle$$

b) In all higher orders we can make exactly the same contractions, after we have chosen n interaction vertices for the contraction. In order $n+k$ this selection can be done in

$$\binom{n+k}{n} = \frac{(n+k)!}{n!\,k!}$$

different ways. Thus

self contracted (vacuum)

$$\langle 0| T(\hat{\phi}_1, \dots, \hat{\phi}_m \left[\frac{1}{(n+k)!} \prod_{j=1}^{n+k} i \int d^4y_j \, \mathcal{L}_I(y_j) \right]) |0\rangle$$

$$= \langle 0| T(\underbrace{\hat{\phi}_1, \dots, \hat{\phi}_m \left[\frac{1}{n!} \prod_{j=1}^{n} i \int d^4y_j \, \mathcal{L}_I(y_j) \right]}_{\Gamma^m_{n,i}} \left[\frac{1}{k!} \prod_{i=1}^{k} i \int d^4y_i \, \mathcal{L}_I(y_i) \right]) |0\rangle$$

$$+ \dots\dots$$

$$= \underline{\Gamma^m_{n,i}} \cdot \underline{\langle 0| \frac{1}{k!} T(\prod_{i=1}^{k} i \int d^4y_i \, \mathcal{L}_I(y_i)) |0\rangle}$$

c) Sum over all orders $n+k$, $k=0,1,\dots,\infty$. One obviously gets

$$= \Gamma^{m}_{n,i} \cdot \sum_{k=0}^{\infty} \frac{1}{k!} \langle 0| T\Big(\prod_{i=1}^{k} i\int d^4y_i\, \mathcal{L}_I(y_i)\Big)|0\rangle$$

$$= \Gamma^{m}_{n,i} \cdot \langle 0| T\big(e^{i\int d^4y\, \mathcal{L}_I(y)}\big)|0\rangle$$

d) Because our derivation a)–c) applies to an arbitrary connected diagram, we can write

$$\boxed{\begin{aligned}&\langle\Omega|T(\hat{\phi}_1,\dots,\hat{\phi}_m\, e^{i\int d^4y\, \mathcal{L}_I(y)})|\Omega\rangle \\ &\quad = \sum(\text{connected graphs}) \times \langle 0|T e^{i\int d^4y\, \mathcal{L}_I(y)}|0\rangle\end{aligned}} \tag{3.94}$$

$\square$

This is a remarkable result, because the full Greens function in (3.72) has the vacuum-factor in the denominator! Thus

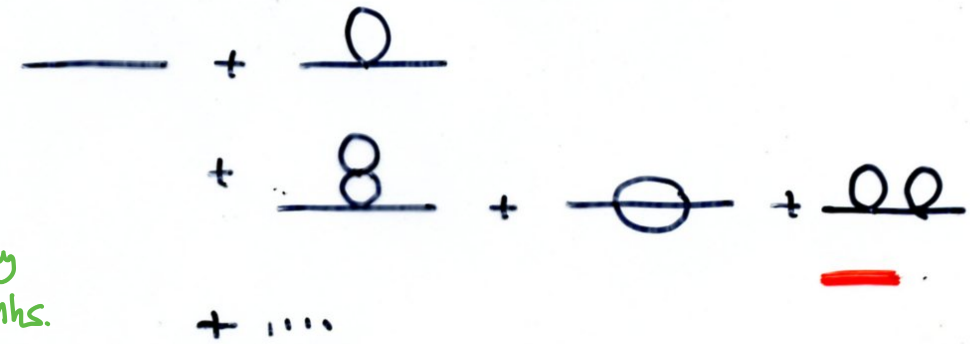
full field op.

$$\boxed{\begin{aligned}\langle\Omega|T(\hat{\phi}_1,\dots,\hat{\phi}_m)|\Omega\rangle &= \frac{\langle 0|T(\hat{\phi}^{in}_1,\dots,\hat{\phi}^{in}_m e^{i\int d^4y\,\mathcal{L}_I})|0\rangle}{\langle 0|T e^{i\int d^4y\,\mathcal{L}_I}|0\rangle} \\ &= \sum(\text{connected graphs}) \\ &\equiv \langle 0|T(\hat{\phi}^{in}_1,\dots,\hat{\phi}^{in}_m e^{i\int d^4y\,\mathcal{L}_I})|0\rangle_{\underline{c}}\end{aligned}} \tag{3.95}$$

connected

Example. For two point function we get

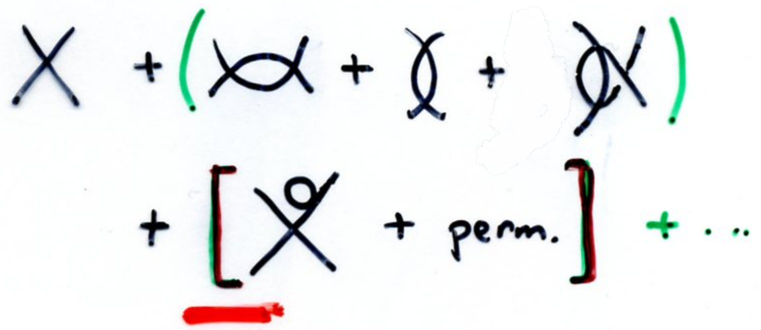
$\langle\Omega|T(\hat{\phi}_1\hat{\phi}_2)|\Omega\rangle \rightarrow$



perturbatively, accounting only connected graphs.

And for the 4-point function:

$\langle\Omega|T(\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4)|\Omega\rangle \rightarrow$



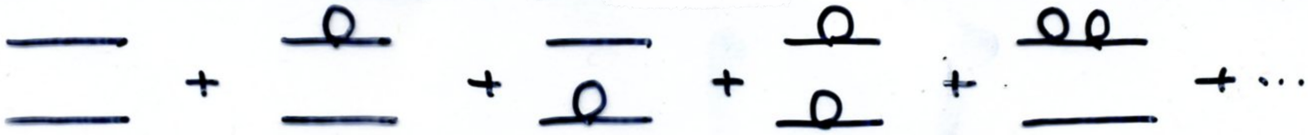
1PI - diagrams

It turns out there are even further simplifications. Namely all one-particle reducible diagrams can be accounted for by resummation. You will learn to appreciate this best when we learn about renormalization, but we can give a 'correct' heuristic argument here. First, a graph is 1-particle reducible if it breaks to two by cutting a single internal line.

1P- reducible

1P-Irreducible : 1PI

To see how this works consider for example the expansion



(3,96)

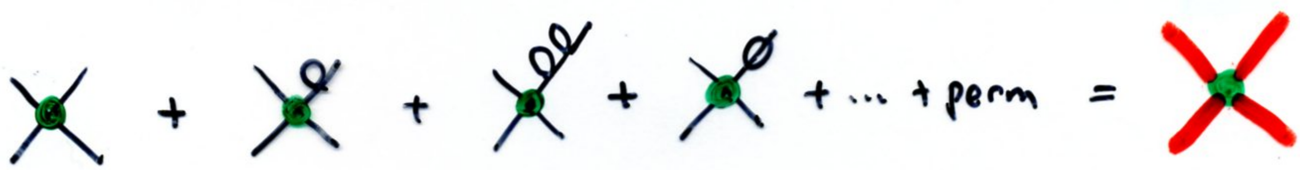
If we define a free propagator of the interacting theory as a sum:

(3,97)

we see that (3,96) is describing an uninteresting DC-process:

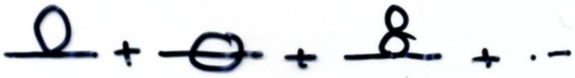
which does not contribute to the T-matrix.

Similarly for example:



here  can contain arbitrary connected subgraphs.
That is, the 1-particle reducible processes merely describe how the noninteracting state $|k\rangle$ evolves to the propagating mode of the interacting theory. They have nothing to do with scatterings.

As a result we only need to consider connected 1PI-graphs!



PHYSICAL SIGNIFICANCE OF VACUUM GRAPHS

Due to corrections on p. 90-92 this page contains some repetition

Remember the time-evolution operator (3.61) that connects the operators of free and interacting theory.

$$U(t,T) = e^{iH_0(t-T)} e^{-iH(t-T)} \qquad (3.98)$$

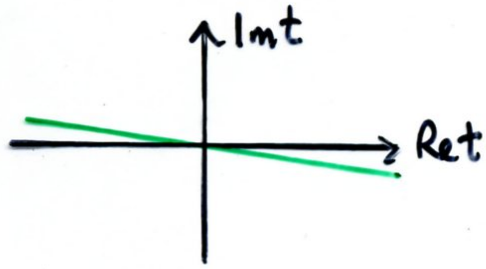
First note that the free vacuum under the full interacting theory on

vacuum of the interacting th at time t.

$$e^{-iHT}|0\rangle = e^{-iE_\Omega T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n=1}^{\infty} e^{iE_n T}|n\rangle\langle n|0\rangle \qquad (3.99)$$

We can keep other states from entering the vacuum state if we turn the interactions on adiabatically. $T \to (1-i\epsilon)\infty$.

Given this complex time contour the extra terms with $n \neq 0$ die quickly in (3.99) in comparison to $|\Omega\rangle$. Normalizing this adiabatic vacuum we get



$$|\Omega\rangle \equiv \lim_{T\to(1-i\epsilon)\infty} \frac{e^{-iH(T+t)} e^{iH_0(T+t)}|0\rangle}{e^{-iE_\Omega(T+t)}\langle\Omega|0\rangle}$$

$$= \lim_{T\to(1-i\epsilon)\infty} \frac{U(t,-T)|0\rangle_{in}}{e^{-iE_\Omega(T+t)}\langle\Omega|0\rangle} \qquad (3.100)$$

Similarly:

$$\langle\Omega| = \lim_{t\to(1-i\epsilon)\infty} \frac{{}_{in}\langle 0|U(T,t)}{e^{-iE_\Omega(T-t)}\langle 0|\Omega\rangle} \qquad (3.101)$$

From this we see that

$$1 = \langle \Omega | \Omega \rangle = \lim_{T \to \infty(1-i\epsilon)} \frac{\langle 0 | \overbrace{U(T,t)\,U(t,-T)}^{= U(T,-T)} | 0 \rangle}{e^{-2iE_\Omega T} |\langle 0 | \Omega \rangle|^2}$$

$$\Rightarrow \quad \langle 0 | T e^{i\int \mathcal{L}_I} | 0 \rangle \propto e^{-2iE_\Omega T} \qquad (3.102)$$

Vacuum graph expansion is thus proportional to the vacuum energy of the interacting theory. This gets even clearer when you note that: (Ex.)

$$\langle 0 | T e^{i\int d^4y\, \mathcal{L}_I} | 0 \rangle = \sum_{\{n_i\}} \prod_i \frac{1}{n_i!} (V_i)^{n_i}$$

$$= \prod_i \sum_{n_i} \frac{1}{n_i!} (V_i)^{n_i} = \prod_i e^{V_i} = e^{\sum V_i} \qquad (3.103)$$

That is

$$\sum_i V_i \propto 2VT \cdot \left(\frac{E_\Omega}{V}\right) \qquad (3.104)$$

vacuum energy density of the interacting th.

ok. it turns out that $\forall i: V_i = 2VT \cdot \tilde{V}_i$ — finite part

Graphically:

$\langle 0 | T e^{i\int \mathcal{L}_I} | 0 \rangle = \exp\{$  +  +  + ... $\}$ (3.105)

3.7 FEYNMAN RULES

In momentum space, for computing $\hat{T}$-matrix directly.

We are now ready to collect what we have learned to a set of rules to compute the LSZ-reduced transition amplitudes. Again it is best go through a couple of examples:

Example 1

$$ {}_{out}\langle q_1 | p_1 \rangle_{in} = \overbrace{(2\pi)^3 2E_{p_1} \delta^3(\vec{p}-\vec{q})}^{\text{DC-term}} $$

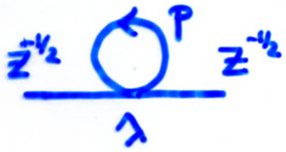
$$ + i^2 Z \int d^4y\, d^4x\, e^{-ip\cdot x + iq\cdot y} (\partial_x^2+m^2)(\partial_y^2+m^2) $$

δ-functions

$$ \cdot \left(-\frac{i\lambda}{2}\right) \int d^4z\, D_F(z-x)\, D_F(z-y)\, D_F(0) $$

$$ = \text{DC-term} + Z \cdot \frac{\lambda}{2} \underbrace{\int d^4z\, e^{-i(p-q)z}}_{(2\pi)^4 \delta^4(p-q)} \, iD_F(0) $$

$$ = \text{DC-term} + \underbrace{(2\pi)^4 \delta^4(p-q)}_{\text{4-momentum conservation.}} \underbrace{\left(\frac{\lambda}{2}\right) \int \frac{d^4p}{(2\pi)^4} \frac{i Z^{-1}}{p^2-m^2+i\epsilon}}_{\text{T-matrix}} \qquad (3.106) $$



Example 2. Consider 2–2 scattering at lowest order:

We have

4! identical contractions

$$ = \langle 0 | T\left(\hat{\phi}_1 \hat{\phi}_2 \hat{\phi}_3 \hat{\phi}_4 \int d^4z\, \frac{i\lambda}{4!} \hat{\phi}_z \hat{\phi}_z \hat{\phi}_z \hat{\phi}_z \right) | 0 \rangle_C $$

$$= i\lambda \int d^4z \; D_F(x_1-z) D_F(x_2-z) D_F(z-x_3) D_F(z-x_4)$$

Using this result for the amplitude in (3,54) we get

$${}_{\text{out}}\langle q_1 q_2 | p_1 p_2 \rangle_{\text{in}} = \text{DC-terms}$$

$$+ i \int d^4y_1 \, d^4y_2 \, d^4x_1 \, d^4x_2 \; e^{iq_1\cdot y_1 + iq_2\cdot y_2 - ip_1\cdot x_1 - ip_2\cdot x_2}$$

$$\times i\lambda \int d^4z \left(\prod_{i=1}^{2} \underbrace{(\partial_{x_i}^2 + m^2) D_F(x_i - z)}_{= i\delta^4(x_i - z)} \right) \left(\prod_{j=1}^{2} \underbrace{(\partial_{y_j}^2 + m^2) D_F(z - y_j)}_{-i\delta^4(y_j - z)} \right)$$

$$= \text{DC-terms}$$

$$+ i\lambda \int d^4z \; e^{i(q_1+q_2-p_1-p_2)\cdot z}$$

$$= \text{DC-terms} + \underline{(2\pi)^4 \delta^4(p_1+p_2-q_1-q_2)} \cdot \underbrace{i\lambda}_{\text{T-matrix}}$$

SIMPLE!

We can now immediately write down the lowest order prediction of this theory for the 2–2 scattering cross section:

$$\boxed{\frac{d\sigma}{d\Omega_{cm}} = \frac{\lambda^2}{64\pi^2 s}}$$

(Here I used $m_1 = m_2 = m_3 = m_4 = m$, so the 1-functions cancel.)

Note that I left out the Z-factors. They reduce to 1 in the lowest order in PT.

These examples are sufficient for us to define the general Feynman rules to compute the T-matrix: directly:

1. Draw all connected 1PI-Feynman graphs relevant for the process.

2. To every internal propagator put $\hat{=} \quad \frac{i}{p^2-m^2+i\epsilon}$

3. To every vertex put  $\hat{=} \quad -i\lambda$

4. To every external leg  $\quad 1$

5. To every closed loop insert $\quad \int \frac{d^4p}{(2\pi)^4}$

6. Divide by the symmetry factor. $\quad (3.109)$

These rules immediately give a T-matrix relevant for the scattering event under investigation.

Box 8

- You should quickly note that the simplest loop-diagram in eqn. (3.106) actually diverges:

$$\int d^4p \frac{1}{p^2-m^2} \sim \lim_{\Lambda\to\infty} \int^{\Lambda} dp \frac{p^3}{p^2} \sim \lim_{\Lambda\to\infty} \Lambda^2$$

This is an example of a singularity, whose elimination requires the renormalization procedure. We shall return to this issue in chapter 5.

- Note that the F-rules (3.109) would lead to a horrible result for disconnected graphs

q 

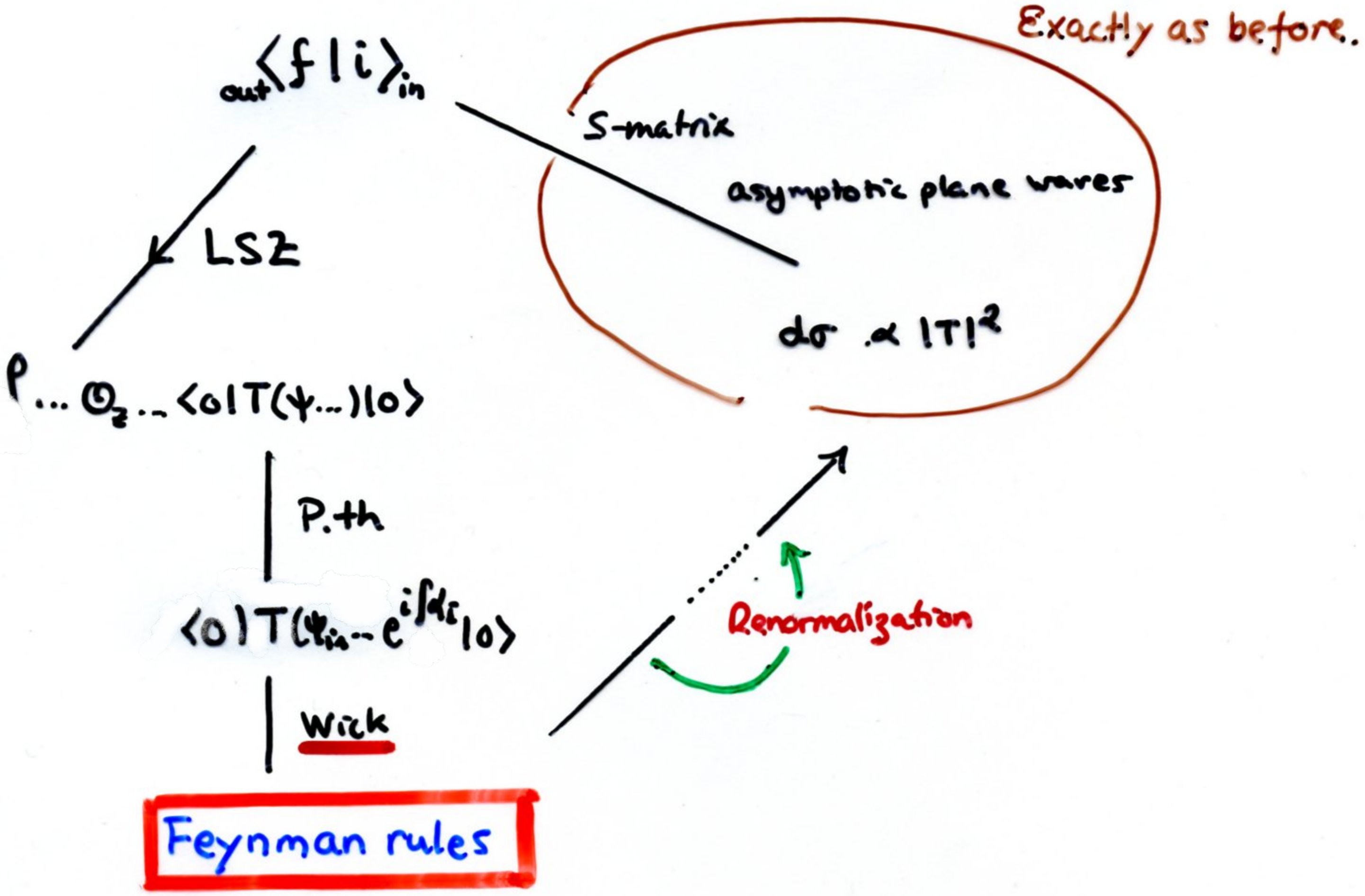
$$\sim \lambda^2 \left(\int \frac{d^4p}{(2\pi)^4} \frac{iZ}{p^2-m^2}\right) \underbrace{\frac{1}{q^2-m^2}}_{=\frac{1}{0}=\infty\,!} \left(\int \frac{d^4p}{(2\pi)^4} \frac{iZ}{p^2-m^2}\right)$$

So, the resummation is not a convenience, but a necessity!

- The Z-factor in the LSZ-reduction step is related to both of the issues above. It's role as a wave function renormalization factor becomes clear later. For time being (before touching renormalization) we can set $Z=1$ however.

3.8 Feynman rules for fermions

We shall follow exactly the same recipe as for the scalar fields. This can be pictured as follows:



We only need to concentrate on a couple of additional small features while doing LSZ-reduction, and deriving the Wick theorem.

LSZ-reduction for fermions

We now have four operators (see. 2.73)

$$a_{\vec{p}}^{s\,\dagger}, \quad b_{\vec{p}}^{s\,\dagger}, \quad a_{\vec{p}}^{s}, \quad b_{\vec{p}}^{s}$$

$a_{\vec{p}}^{s\,\dagger}$: creates a particle

$b_{\vec{p}}^{s\,\dagger}$: creates an antiparticle

Their field-operator representations can be found by using the orthogonality & normalization relations for spinors:

$$u^{\dagger}_{\vec{p},s} u_{\vec{p},s'} = v^{\dagger}_{\vec{p},s} v_{\vec{p},s'} = 2E_{\vec{p}}\,\delta_{ss'} \quad \text{and} \quad u^{\dagger}_{\vec{p}} v_{p} = v^{\dagger}_{\vec{p}} u_{\vec{p}} = 0$$

Namely:

$$\begin{aligned}
a^{s}_{\vec{p}} &= \int d^3x\, \bar{u}(p,s)\, e^{ip\cdot x}\, \gamma^0 \hat{\Psi}(x) \\
b^{s\dagger}_{\vec{p}} &= \int d^3x\, \bar{v}(p,s)\, e^{-ip\cdot x}\, \gamma^0 \hat{\Psi}(x) \\
a^{s\dagger}_{\vec{p}} &= \int d^3x\, \hat{\bar{\Psi}}(x)\, \gamma^0\, u(p,s)\, e^{-ip\cdot x} \\
b^{s}_{\vec{p}} &= \int d^3x\, \hat{\bar{\Psi}}(x)\, \gamma^0\, v(p,s)\, e^{ip\cdot x}
\end{aligned} \qquad (3.111)$$

Furthermore, we are setting the asymptotic condition as before

$$\langle f|\hat{\Psi}|i\rangle \xrightarrow{x^0 \to \infty} Z_\Psi^{1/2} \langle f|\hat{\Psi}_{in}|i\rangle \qquad (3.112)$$

Based on these we get for example

$$_{out}\langle f|(k,s)\,;i\rangle_{in} = {}_{out}\langle f|a^{s\dagger}_{\vec{k}\,in}|i\rangle_{in}$$

$$= \lim_{t\to-\infty} Z_\Psi^{-1/2} \int d^3x\, {}_{out}\langle f|\hat{\bar{\Psi}}(x)|i\rangle_{in}\, \gamma^0 u(k,s)\, e^{-ik\cdot x}$$

By use of $\quad \int_{-\infty}^{\infty} \partial_{x_0} A(x) = \Big|_{-\infty}^{\infty} A(x) \quad$ this again becomes

$$= {}_{out}\langle f|a^{s\dagger}_{\vec{k},out}|i\rangle$$

$$- Z_\Psi^{-1/2} \int d^4x\, \partial_{x_0} \left[{}_{out}\langle f|\hat{\bar{\Psi}}(x)|i\rangle_{in}\, \gamma^0 u(k,s)\, e^{-ik\cdot x} \right]$$

Now

$$\gamma^0 \partial_0 \underbrace{u(k,s)\, e^{-ik\cdot x}}_{\equiv\, \varphi_{\vec{k},s}(x)} = (\not{\partial} - \vec{\gamma}\cdot\nabla)\, \varphi_{\vec{k},s}(x) = (-im - \underbrace{\vec{\gamma}\cdot\nabla}_{\to\, +\vec{\gamma}\cdot\overleftarrow{\nabla} \text{ after partial integration.}})\, \varphi_{\vec{k},s}(x)$$

so

$$_{out}\langle f|\, a^{s\,+}_{\vec{p},in}\, |i\rangle_{in} = \text{DC-term}$$

$$+\, \underline{i Z_\psi^{-1/2}} \int d^4x\ _{out}\langle f|\, \underbrace{\hat{\bar{\Psi}}(x)}_{\text{field-op}}\, |i\rangle_{in}\ \underbrace{(i\overleftarrow{\not{\partial}} + m)\, u(k,s)\, e^{-ik\cdot x}}_{\text{multiplying factor}} \qquad (3.113)$$

Similarly one finds: (dropping DC-terms)

$$_{out}\langle f|\, b^{s\,+}_{\vec{k},in}\, |i\rangle = \underline{i Z_\psi^{-1/2}} \int d^4x\ \underline{\bar{v}(k,s)\, e^{-ik\cdot x}\, (i\not{\partial} - m)}\ _{out}\langle f|\, \hat{\Psi}(x)\, |i\rangle_{in}$$

$$_{out}\langle f|\, a^{s}_{\vec{k},out}\, |i\rangle = \underline{-i Z_\psi^{-1/2}} \int d^4x\ \underline{\bar{u}(k,s)\, e^{ik\cdot x}\, (i\not{\partial} - m)}\ _{out}\langle f|\, \hat{\Psi}(x)\, |i\rangle_{in}$$

$$_{out}\langle f|\, b^{s}_{\vec{k},in}\, |i\rangle = \underline{-i Z_\psi^{-1/2}} \int d^4x\ _{out}\langle f|\, \hat{\bar{\Psi}}(x)\, |i\rangle_{in}\ \underline{(i\overleftarrow{\not{\partial}} + m)\, v(k,s)\, e^{ik\cdot x}} \qquad (3.114)$$

By use of (3.113 – 3.114) the whole fermionic matrix element can be reduced to a vacuum expectation value. Just as with bosons, the next steps give rise to time-ordering, where T-ordering follows the fermion statistics. See (2.102). Other than that the proof is similar to that for bosons.

If we define

$$_{out}\langle f|i\rangle_{in} = {}_{out}\langle \underbrace{q_1, \ldots, q_{n_0}}_{\text{fermions}}\, \underbrace{q_1', \ldots, q_{n_0'}'}_{\text{antiferm.}} \,|\, \underbrace{p_1, \ldots, p_{n_i}}_{\text{ferm.}}\ \underbrace{p_1', \ldots, p_{n_i'}'}_{\text{antif.}} \rangle_{in}$$

We eventually get:

$$
\begin{aligned}
{}_{\text{out}}\langle f | i \rangle_{\text{in}} = & \left(i Z_\psi^{-1/2}\right)^{n_i + n_i'} \left(-i Z_\psi^{-1/2}\right)^{n_o + n_o'} \int d^4y_1 \ldots d^4y_1' \ldots d^4x_1 \ldots d^4x_1' \ldots \\
& \times \underbrace{\prod_{i=1}^{n_o} \bar{u}(q_i, s_i^o)\, e^{i q_i \cdot y_i} (i \partial\!\!\!/_{y_i} - m)} \times \underbrace{\prod_{i=1}^{n_i'} \bar{v}(p_i', \bar{s}_i')\, e^{-i p_i \cdot x_i'} (i \partial\!\!\!/_{x_i'} - m)} \\
& \times \langle \Omega | T\big(\underbrace{\hat{\bar{\psi}}(y_1) \ldots \hat{\bar{\psi}}(y_{n_o})}\, \underbrace{\hat{\bar{\psi}}(x_1') \ldots \hat{\bar{\psi}}(x_{n_i'}')} \cdot \\
& \qquad \underbrace{\hat{\psi}(x_1) \ldots \hat{\psi}(x_{n_i})}\, \underbrace{\hat{\psi}(y_1') \ldots \hat{\psi}(y_{n_o'}')} \big) | \Omega \rangle \\
& \times \underbrace{\prod_{i=1}^{n_i} u(p_i, s_i) (i \overleftarrow{\partial\!\!\!/}_{x_i} + m)\, e^{-i p_i \cdot x_i}} \cdot \underbrace{\prod_{i=1}^{n_o'} v(q_i, \bar{s}_i) (i \overleftarrow{\partial\!\!\!/}_{y_i'} + m)\, e^{i q_i \cdot y_i}}
\end{aligned}
\qquad (3.115)
$$

This is a simple formula which looks complicated due to lengthy notation. Note however, that for fermions and antifermions the "in" and "out" function is reversed.

Wicks theorem for fermions

The −signs arising from fermion field reorderings match in the time-ordered and normal ordered products. For example

$$T(\psi_1 \psi_2 \psi_3 \psi_4) = (-1)^3 \psi_3 \psi_1 \psi_4 \psi_2 \qquad \text{if} \quad x_3^0 > x_1^0 > x_4^0 > x_2^0$$

over ψ_1

This is matched by

to make same order

$$:\psi_1^- \psi_2^- \psi_3^+ \psi_4^-: \; = (-1)^2 \psi_3^+ \psi_1^- \psi_2^- \psi_4^- \; = (-1)^3 \psi_3^+ \psi_1^- \psi_4^- \psi_2^-$$

must

So, for example

$$T(\hat{\psi}(x)\hat{\bar{\psi}}(y)) = :\psi(x)\bar{\psi}(y): + \overbracket{\psi(x)\bar{\psi}(y)} \qquad (3.116)$$

where

$$\overbracket{\psi(x)\bar{\psi}(y)} \equiv \begin{cases} \{\psi^+(x), \bar{\psi}^-(y)\} & , \; x^0 > y^0 \\ -\{\bar{\psi}^+(y), \psi^-(x)\} & , \; x^0 < y^0 \end{cases} = S_F(x-y) \qquad (3.117)$$

When one notes also that

$$:\overbracket{\psi_1 \psi_2 \bar{\psi}_3}\, \bar{\psi}_4: = -\overbracket{\psi_1 \bar{\psi}_3}\, :\psi_2 \bar{\psi}_4:$$

ie, making a contraction one pulls the fields next to each other by anticommuting sufficient number of times and counts the $-$ signs.

Eventually one gets

$$\boxed{T(\psi_1 \bar{\psi}_2 \psi_3 \ldots) = :\psi_1 \bar{\psi}_2 \psi_3 \ldots + \text{all contractions}:} \qquad (3.118)$$

Anticommutation rules introduce a number of signs to contractions, as indicated above, but that is essentially the only difference to bosons. These signs will have consequences for the Feynman rules however.

We have not defined a specific interaction yet, but we can write formally:

other fields coupling to ψ.

$$\langle\Omega| T(\psi_1 \ldots \bar{\psi}_N)|\Omega\rangle =$$

$$= \langle 0| T(\psi^1_{in} \ldots \bar{\psi}^N_{in}\, e^{i\int \mathcal{L}_I(\psi_{in}, \bar{\psi}_{in}, \phi_{in} \ldots)})|0\rangle_c \qquad (3.119)$$

Computation of the matrix element in (3,119) proceeds analogously to the bosonic case, through use of the Wick-theorem, which reduces it to a products of contractions. These are then easily integrated in (3,119) by using the fact that $(i\not\partial - m) S_F(x-y) = i\delta^4(x-y)$

Example:

$$\ldots \int d^4x_i \, \bar{u}(q_i, s_i) \, e^{iq_i \cdot x_i} (i\not\partial - m) \ldots \langle 0| \ldots \psi(y_i) \ldots \bar{\psi}(x_i) \ldots |0\rangle$$

$$\overset{\ldots}{=} (-1)^{n_{cont}} \int d^4x_i \, \bar{u}(q_i, s_i) \, e^{iq_i \cdot x_i} \underbrace{(i\not\partial - m) S_F(x_i - y)}_{i\delta^4(x_i - y)} \ldots$$

$$= \ldots (-1)^{n_{cont}} \, e^{iq_i \cdot y} \, i \, \bar{u}(q_i, s_i) \qquad (3,120)$$

→ wave function to be put to the external leg.

Contributes to the relative sign of the graph in the overall amplitude

goes to build up the 4-momentum conservation law.

LSZ - theorem generalizes along the same lines to the case where in and out states contain both bosons and fermions. It is completely straightforward, but notationally cumbersome, so we won't write it down explicitly.

3.9 Yukawa-theory

This is the simplest theory involving interacting fermions and bosons. The Lagrange density is:

$$\boxed{\mathcal{L}_{\text{Yukawa}} = \underbrace{\frac{1}{2}(\partial\phi)^2 - \frac{m_\phi^2}{2}\phi^2}_{\substack{\text{free scalar theory}\\ \text{(Klein-Gordon)}}} + \underbrace{i\bar{\psi}\not{\partial}\psi + m\bar{\psi}\psi}_{\substack{\text{Free Dirac}\\ \text{theory}}} - \underbrace{g\bar{\psi}\psi\phi}_{\substack{\text{Yukawa interaction}\\ \text{term } \mathcal{L}_I}}} \tag{3.121}$$

Example 1. $\psi\psi \to \psi\psi$ -scattering (particle-particle)

All external legs are fermions, so from (3.115) we get

$$\begin{aligned}
{}_{\text{out}}\langle q_1 s_1^o; q_2 s_2^o | p_1 s_1^i; p_2 s_2^i \rangle &= \int d^4x_1\, d^4x_2\, d^4y_1\, d^4y_2\, e^{-ip_1\cdot x_1 - ip_2\cdot x_2 + iq_1\cdot y_1 + iq_2\cdot y_2} \\
&\times \left[-i\bar{u}(q_1, s_1^o)(i\not{\partial}_{y_1} - m)\right]_\alpha \left[-i\bar{u}(q_2, s_2^o)(i\not{\partial}_{y_2} - m)\right]_\beta \\
&\times \langle\ | T(\psi(y_1)_\alpha \psi(y_2)_\beta \bar{\psi}(x_1)_\gamma \bar{\psi}(x_2)_\delta) |\ \rangle \\
&\times \left[iu(p_1, s_1^i)(i\overleftarrow{\not{\partial}}_{x_1} + m)\right]_\gamma \left[iu(p_2, s_2^i)(i\overleftarrow{\not{\partial}}_{x_2} + m)\right]_\delta
\end{aligned} \tag{3.122}$$

Where

$$\begin{aligned}
\langle\Omega| T(\cdots) |\Omega\rangle &= \langle 0| T(\psi_{in}(y_1)\psi_{in}(y_2)\bar{\psi}_{in}(x_1)\bar{\psi}_{in}(x_2)\, e^{i\int \mathcal{L}_I\, d^4y}) |0\rangle_c \\
&= \frac{1}{2!}\langle 0| T(\underbrace{\psi_{y_1}\psi_{y_2}\bar{\psi}_{x_1}\bar{\psi}_{x_2}\left[-ig\int d^4w\, \bar{\psi}_w\psi_w\phi_w\right]\left[-ig\int d^4z\, \bar{\psi}_z\psi_z\phi_z\right]}_{\text{(all in-fields)}}) |0\rangle_c \\
&+ \mathcal{O}(g^4)
\end{aligned} \tag{3.123}$$

Using the properties $[\psi_x, \bar\psi_y \psi_y] = [\bar\psi_x, \bar\psi_y \psi_y] = 0$ we can write this expectation value as

← one sign change →

$$= \frac{(-ig)^2}{2!}\int d^4w\, d^4z \langle 0|T(\psi_{y_1\alpha}\psi_{y_2\beta}\bar\psi_{w\epsilon}\bar\psi_{z\eta}\,\phi_w\phi_z\,\psi_{z\eta}\psi_{w\epsilon}\bar\psi_{x_1\gamma}\bar\psi_{x_2\delta})|0\rangle \qquad (3{,}124)$$

We are getting four different contractions, (pay attention to Dirac indices)

$$= \frac{(-ig)^2}{2}\int d^4z\, d^4w \begin{matrix} (-1)\, S_F(y_1-w)_{\alpha\epsilon} S_F(y_2-z)_{\beta\eta} \\ (-1)^0\, S_F(y_1-z)_{\alpha\eta} S_F(y_2-w)_{\beta\epsilon} \end{matrix}\; D_\phi(w-z)\; \begin{matrix} (-1)\, S_F(z-x_1)_{\epsilon\gamma} S_F(w-x_2)_{\eta\delta} \\ (-1)^0\, S_F(z-x_2)_{\epsilon\delta} S_F(w-x_1)_{\eta\gamma} \end{matrix}$$

(3,125)

This notation assumes that you form all possible products of terms in the two columns. However, since we can change $z \leftrightarrow w$ in the integrand we see that there is only 2 different terms

$$= -g^2 \int d^4z\, d^4w\, S_F(y_1-w)_{\alpha\epsilon} S_F(y_2-z)_{\beta\eta} D_\phi(w-z)\cdot\left(S_F(z-x_1)_{\epsilon\delta} S_F(w-x_2)_{\eta\delta} - S_F(z-x_2)_{\epsilon\delta} S_F(w-x_2)_{\eta\gamma}\right)$$

(3.125a)

putting this back to (3.122) and using $\left(S_F^\dagger(x-y) = \gamma^0 S_F(y-x)\gamma^0 \right)$

$$\boxed{\begin{aligned} &\int d^4y\, [-i\bar u(q,s)\underbrace{(i\overleftarrow{\not\partial}_y - m)]_\beta\, S_F(y-w)_{\beta\epsilon}}_{i\delta^4(y-w)} = e^{iq\cdot w}\,\bar u(q,s)_\epsilon \\ &\int d^4x\, e^{-ip\cdot x} S_F(z-x)_{\eta\varphi}\, [iu(p,s)(i\overleftarrow{\not\partial}_x + m)]_\varphi = e^{-ip\cdot z}\, u(p,s)_\eta \end{aligned}} \qquad (3{,}126)$$

Dirac index always follows the position index.

we easily get:

$${}_{out}\langle f|i\rangle_{in} = g^2 \int d^4z\, d^4w\, e^{iq_1\cdot w + iq_2\cdot z}\, \bar u(q_1,s_1')_\epsilon\, \bar u(q_2,s_2')_\eta \times D_\phi(z-w)$$

$$\times \left\{ e^{-ip_1\cdot w - ip_2\cdot z}\, u(p_1,s_1^i)_\epsilon\, u(p_2,s_2^i)_\eta \right.$$
$$\left. - \, e^{-ip_1\cdot z - ip_2\cdot w}\, u(p_1,s_1^i)_\eta\, u(p_2,s_2^i)_\epsilon \right\}$$

Observing that $z - w \equiv z'$

$$\int d^4z\, d^4w\, e^{-iP_1\cdot w - iP_2\cdot z} D_\phi(z-w) = (2\pi)^4 \delta^4(P_1+P_2)\, D_\phi(P_1) \qquad (3.127)$$

where of course $D_\phi(p) = i/(p^2 - m_\phi^2 + i\epsilon)$.

Now in the first term in the brackets $\{\}$ we have

$$P_1 = p_1 - q_1 \quad ; \quad P_2 \equiv p_2 - q_2$$

and in the second term

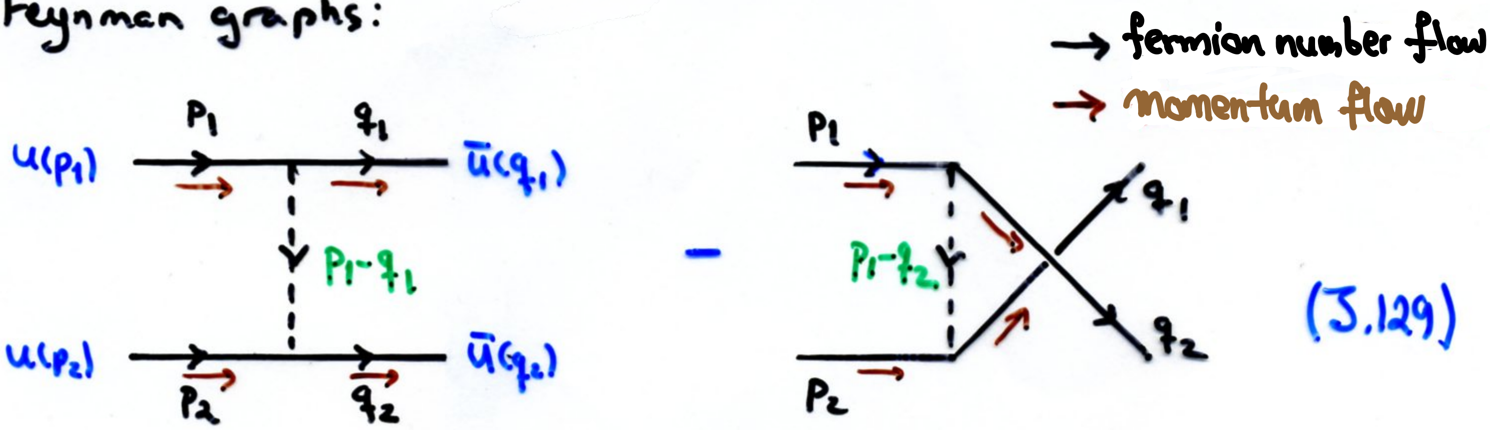
$$P_1 \equiv p_1 - q_2 \quad ; \quad P_2 = p_2 - q_1$$

So we get

$$\boxed{\begin{aligned} {}_{out}\langle \dots | \dots \rangle_{in} = {} & (2\pi)^4 \delta^4(q_1+q_2-p_1-p_2)\, \times \\ & \times \Big\{ g^2\, \bar{u}(q_1)\, u(p_1)\, \frac{i}{(p_1-q_1)^2 - m_\phi^2 + i\epsilon}\, \bar{u}(q_2)\, u(p_2) \\ & - g^2\, \bar{u}(q_1)\, u(p_2)\, \frac{i}{(p_1-q_2)^2 - m_\phi^2 + i\epsilon}\, \bar{u}(q_2)\, u(p_1) \Big\} \end{aligned}} \qquad (3.128)$$

$$= \underline{(2\pi)^4 \delta^4(P_f - P_i)\; T}$$

These terms in T-matrix correspond to the following Feynman graphs:



"t-channel" "u-channel"

$$(3.129)$$

The names t- and u-channel follow from the Lorentz-invariant Mandelstam variables

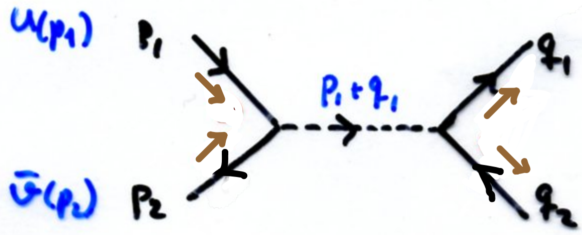
$$s \equiv (p_1+p_2)^2 \,;\quad t = (p_1-q_1)^2 \quad ; \quad u = (p_1-q_2)^2 \qquad (3.130)$$

Example 2 $\psi_{p_1} \psi^c_{p_2} \to \psi_{q_1} \psi^c_{q_2}$ (particle-antiparticle scattering)

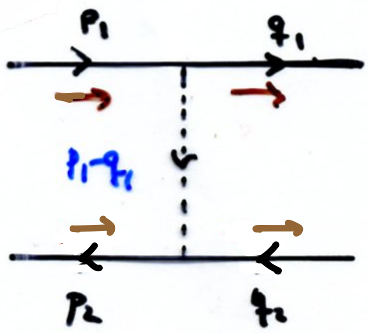
It is easy enough to do the same calculation for particle-antiparticle scattering. The Greens function and the contractions remain the same, so (3.125a) is unchanged. However, in (3.122) particle operators corresp. to p_2 and q_2 become antiparticle operators, which also skip the sides wrt. Green function. Eventually one has (EX):

$$
\begin{aligned}
{}_{out}\langle q_1 \bar{q}_2 | p_1 \bar{p}_2 \rangle_{in} = \; & (2\pi)^4 \delta^4(p_1 + \bar{q}_2 - q_1 - \bar{p}_2) \\
& \times \Big\{ g^2 \bar{v}(p_2) u(p_1) \frac{i}{(p_1+p_2)^2 - m_y^2 + i\epsilon} \bar{u}(q_1) v(q_2) \\
& - g^2 \bar{u}(q_1) u(p_1) \frac{i}{(p_1-q_1)^2 - m_y^2 + i\epsilon} \bar{v}(p_2) v(q_2) \Big\} \qquad (3.131)
\end{aligned}
$$

These terms correspond to diagrams:



s-channel



t-channel

→ fermion number flow

→ momentum flow

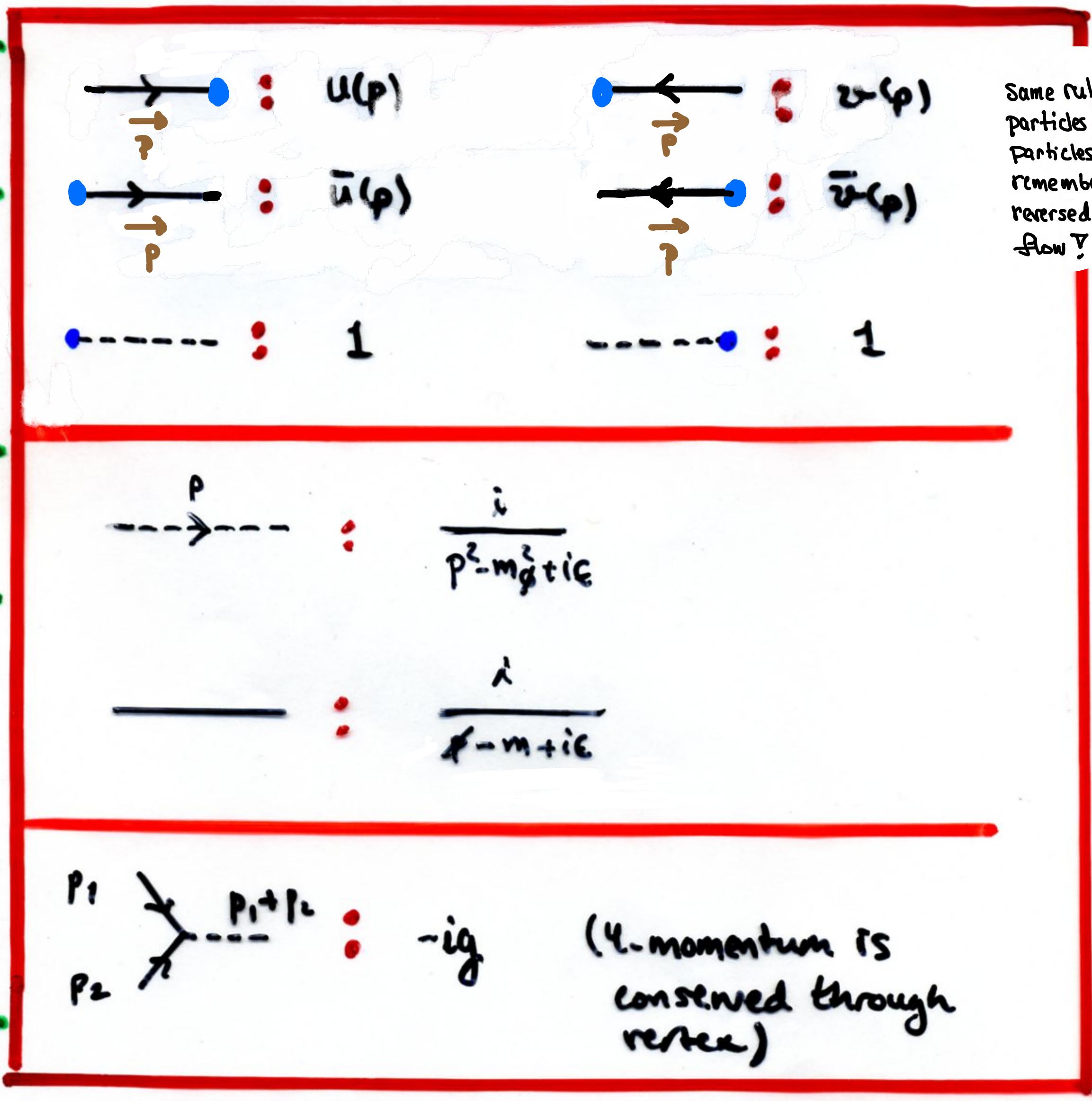
(f-flow reversed for antip. w.r.t momentum flow)

With help of these examples it is already easy to see what are the Feynman rules for the Yukawa theory.*

① External legs

② Propagators

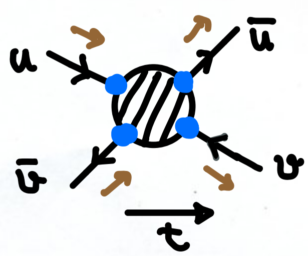
③ Inter-action vertex



Same rule for both particles & anti-particles. But remember the reversed f-number flow!

(3.133)

* Dot represents the outer point in the 1PI part of the graph. eg



reversed f-number-flow for anti-particles.

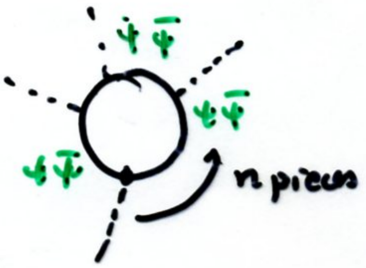
In addition to these there are the following rules:

④ Integrate over the momenta in closed loops: $\int \frac{d^4p}{(2\pi)^4}$

⑤ Figure out the relative signs of the diagrams by working out the complete contraction signs. (3.133b)

Some other rules that we will learn later will be needed. Let us mention one of those here:

⑥ Each fermion loop introduces a Trace over Dirac indices and induces a − sign.



$$\propto \bar{\psi}_\alpha \psi_\beta \bar{\psi}_\beta \psi_\gamma \cdots \bar{\psi}_\delta \psi_\alpha$$

$$= (-1)^{2n-1}\,\mathrm{Tr}\left[\psi\bar{\psi} \cdots \psi\bar{\psi}\right]$$

$$= -\mathrm{Tr}\left[S_F \cdots S_F\right] \qquad (3.134)$$

To compute the cross section we will need the square of the T-matrix. This is easily done for our examples using the Diracology we have learned so far. We leave this to an exercise however. Explicit calculations for σ will be done later on in connection with QCD.

Symmetry factor in this theory is always 1.