2. DIRAC FIELD

Before we can quantize a fermionic field, we need to build up some properties of the Lorentz-group.

2.1. LORENTZ - GROUP

Let us define an arbitrary L-transform as:

$$x^\mu \rightarrow x^{\mu'} = \Lambda^\mu{}_\nu x^\nu \qquad (2.1)$$

Same linear transform applies to any vector field. Now, matrices $\Lambda^\mu{}_\nu$ are restricted by the defining property of L-transform:

$$A \cdot B = A' \cdot B' = \text{invariant} \qquad (2.2)$$

This implies

$$g_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu = g_{\mu\nu} \qquad (2.3)$$

Then it follows $(\det \Lambda)^2 = 1 \Rightarrow \det \Lambda = \pm 1$. Taking $\mu, \nu = 0$ in (2.3) one finds $|\Lambda^0{}_0| \geq 1$. Transformations with $\Lambda^0{}_0 \geq +1$ are called orthocronous. If also $\det \Lambda = +1$, is the transformation called restricted. (This is a group.)

Orthocronous transformations map the forward light-cone into itself.

We can show that $\mathcal{L}_{K\text{-}G}$ is a Lorentz-scalar. First note that since $\phi_\Lambda(x_\Lambda) = \phi(x)$:

$$\phi(x) \xrightarrow{\Lambda} \phi_\Lambda(x) = \phi(\Lambda^{-1} x) \qquad (2.4)$$

So clearly

$$\mathcal{L}_{KG}(x) = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 \longrightarrow \mathcal{L}_{KG}(\Lambda^{-1}x). \qquad (2.6)$$

Since $\mathcal{L}_{K-G}$ is a $\mathcal{L}$-scalar, the action $\int d^4x \mathcal{L}_{K-G}$ is an invariant. (show that $d^4x \xrightarrow{\Lambda} |\det \Lambda| d^4x = d^4x$). It then follows that also the Euler-Lagrange E.o.m are invariant:

$$(\partial^2 - m^2)\phi(x) \xrightarrow{\Lambda} (\partial^2 - m^2)\phi(\Lambda^{-1}x) = 0 \qquad (2.7)$$

The same applies to all relativistic theories. When $\mathcal{L}$-function is constructed to be a Lorentz-scalar, the action and the E.o.m are Lorentz-invariants.

The independent fields in the Lagrange-function are not necessarily scalars. For example the A_μ-field transforms as

$$A_\mu(x) \xrightarrow{\Lambda} \Lambda_\mu{}^\alpha A_\alpha(\Lambda^{-1}x) \qquad (2.8)$$

We say that ϕ belongs to the scalar- (spin 0) and A_μ to the vector (spin-1) representation of the Lorentz group. Also other representations do exist. Let Φ_a be an n-component multiplet of fields, that transform as follows under Lorentz transformations

$$\phi_a(x) \xrightarrow{\Lambda} M_{ab}(\Lambda)\, \Phi_b(\Lambda^{-1}x) \qquad (2.9)$$

The group property then places constraints on $M(\Lambda)$. Since $\Lambda'' = \Lambda'\Lambda$ is also a Lorentz-transform, we must have

$$M(\Lambda')\,M(\Lambda)\,\Phi = M(\Lambda'')\,\Phi \qquad (2.10)$$

Matrices M are said to span an n-dimensional representation of the Lorentz group. We have already met the scalar and vector representations ($n = 1, 4$). For the Dirac eqn. the most important are the spinor representations. Let us now find these:

LIE ALGEBRA

The group-structure of Lorentz-transformations is $O(3,1)$. This is a continuous, non-compact group, and we can specify the generators and algebra to it. To do this we begin first from a more familiar case of the SU(2)-group.

SU(2) Lie-Algebra (the invariant: $\vec{x}^2$) → $\binom{a}{b}$

SO(3) really

doublet

Generators $\hat{J}_i$ satisfy the algebra ($\triangleq SO(3)$)

$$[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k \qquad (2.11)$$

$\hat{J}_i$ is known to have for example the 2-dimensional (spin-$\frac{1}{2}$)-representation

$$\hat{J}_i \triangleq \frac{1}{2}\sigma_i \qquad (2.12)$$

Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The algebra (2.11) can be expressed in a slightly different form by use of the representation $\vec{J} = \vec{x} \times \vec{p} = -i \vec{x} \times \vec{\nabla}$; i.e.

$$\hat{J}^k = -i \epsilon^{ijk} x^i \nabla^j$$

$$= \epsilon^{ijk} \times \left(\frac{-i}{2}\right) \times \left(x^i \nabla^j - x^j \nabla^i\right) \equiv \frac{1}{2} \epsilon^{ijk} \hat{J}^{ij} \quad (2.13)$$

Written in terms of the antisymmetric matrices J^{ij} the algebra (2.11) becomes (ex.)

$$[\hat{J}^{ij}, \hat{J}^{kl}] = -i \left(\delta^{ik} \hat{J}^{jl} - \delta^{jk} \hat{J}^{il} + \delta^{il} \hat{J}^{jk} - \delta^{jl} \hat{J}^{ik}\right). \quad (2.14)$$

The antisymmetric tensor representation generalizes directly to the Lorentz-group. Defining

$$\hat{J}^{\mu\nu} = -i \left(x^\mu \partial^\nu - x^\nu \partial^\mu\right) \quad (2.15a)$$

we find

$$[\hat{J}^{\mu\nu}, \hat{J}^{\rho\sigma}] = i \left(g^{\nu\rho} \hat{J}^{\mu\sigma} - g^{\mu\rho} \hat{J}^{\nu\sigma} + g^{\nu\sigma} \hat{J}^{\mu\rho} - g^{\mu\sigma} \hat{J}^{\nu\rho}\right) \quad (2.15)$$

We use (2.15) as the definition of the Lorentz-group algebra. This structure must be met by all possible, different dimensional representations of the operators $\hat{J}^{\mu\nu}$. For example the 4-dim. vector representation is

$$(J^{\mu\nu})_{\alpha\beta} = i \left(\delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\mu{}_\beta \delta^\nu{}_\alpha\right). \quad (2.16)$$

To find the spinor representations, we trace our treatment with SU(2)-group, but now backwards. So, we have six antisymmetric generators ($\hat{J}^{\mu\nu} = -\hat{J}^{\nu\mu}$). Let us choose them as

$$
\begin{aligned}
\hat{J}^i &\equiv \tfrac{1}{2}\epsilon^{ijk}\hat{J}^{ij} \\
\hat{K}^i &\equiv \hat{J}^{0i}
\end{aligned}
\qquad (2.17)
$$

$\hat{J}^i$:s then clearly represent the rotations, and for example by use of (2.16) one realizes that $\hat{K}^i$:s are the boost generators.

As we know, rotations form an SU(2)-sub-group of the symmetry O(3,1). Boosts on the other hand don't; indeed: two boosts is not always a mere boost, but involves a rotation as well. This physics is manifested in the algebra

$$
\begin{aligned}
[\hat{J}^i, \hat{J}^j] &= i\epsilon^{ijk}\hat{J}^k \\
[\hat{K}^i, \hat{K}^j] &= -i\epsilon^{ijk}\hat{J}^k \\
[\hat{J}^i, \hat{K}^j] &= i\epsilon^{ijk}\hat{K}^k
\end{aligned}
\qquad (2.18)
$$

We can find mutually commuting operators however. Defining

$$\hat{L}^i_{\pm} \equiv \tfrac{1}{2}(\hat{J}^i \pm i\hat{K}^i) \qquad (2.19)$$

we find that

$$[\hat{L}^i_{\pm}, \hat{L}^j_{\pm}] = i\epsilon^{ijk}\hat{L}^k_{\pm} \quad \text{and} \quad [\hat{L}^i_{+}, \hat{L}^j_{-}] = 0 \qquad (2.20)$$

With regards to representations, the Lorentz group is then equivalent with a product group $SU(2)_L \otimes SU(2)_R$. We will place the Dirac fermions to representations $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$, following this decomposition. The vector representation turns out to be $(\frac{1}{2},\frac{1}{2})$ and the scalar one $(0,0)$.

Physically, the duality found above is tied to parity. Indeed, under the parity transformation

$$\hat{P} x^\mu = x_\mu, \qquad (2.21)$$

we have $\hat{P} J^i = J^i$ and $\hat{P} K^i = -K^i$, so that

$$\hat{P} L^i_\pm = L^i_\mp . \qquad (2.22)$$

Thus the representations $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$ are related by the parity transformation. We can thus call them for example left- and right handed representations.

2.2 Dirac equation

To find the explicit spinor representations, we use the Dirac trick: If a set of ($n \times n$-) matrices γ^μ satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad (2.23)$$

then their commutators

$$S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu] \qquad (2.24)$$

satisfy the Lorentz algebra (2.15). It is easy to check that this works in 3d-Euclidian space, where $\gamma^\mu \to \sigma^i$. (There $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$ and $S^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma^k$.)

In Minkowski space (4d) the lowest dimension for γ^μ-matrices is 4. Explicit form in the Weyl representation is:

$$\boxed{\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \quad ; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}} \qquad (2.25)$$

In this representation the boost- and rotation generators read.

$$S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} = -\frac{i}{2}\rho^3 \otimes \sigma^i \qquad (2.26)$$

and

$$S^{ij} = \frac{1}{2}\epsilon^{ijk}\underbrace{\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}}_{\equiv \Sigma^k} = \frac{1}{2}\epsilon^{ijk}\, 1_2 \otimes \sigma^k \qquad (2.27)$$

As usual, a finite Lorentz-transformation can be expressed in terms of the generators as:

$$\Lambda_{\frac{1}{2}} = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}} \qquad (2.28)$$

refers to the spin

Using (2.28) one can show that γ^μ transforms as a vector:

$$\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu \qquad (2.29)$$

This implies that the operator $\not{\partial} \equiv \gamma^\mu\partial_\mu$ is Lorenz invariant. These γ^μ-matrices are of course just the quantities we have seen in the Dirac equation (Particle physics I, QM II):

4-comp. spinor

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \qquad (2.30)$$

Based on above, this is a Lorentz-covariant equation, i.e. it retains it's form when

$$\psi(x) \to \psi_\Lambda(x) = \Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x). \qquad (2.31)$$

Dirac eqn. (2.30) is the Euler-Lagrange equation of the $\mathcal{L}$-function

$$\mathcal{L}_{Dirac} = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi \qquad (2.32)$$

where the conjugated field

$$\boxed{\bar{\psi} \equiv \psi^\dagger\gamma^0.} \qquad (2.33)$$

The extra γ^0 in the definition is needed because $\Lambda_{\frac{1}{2}}^\dagger \neq \Lambda_{\frac{1}{2}}^{-1}$, and instead

$$\Lambda_{\frac{1}{2}}^{-1} = \gamma^0\Lambda_{\frac{1}{2}}^\dagger\gamma^0 \qquad (2.34)$$

- Any solution to Dirac eqn is also a solution to KG-eqn:

$$(\partial^2 - m^2)\psi = -(i\not{\partial}+m)(i\not{\partial}-m)\psi = 0$$

2.3 WEYL SPINORS

The advantage of the Weyl representation (2,25) is obvious from the expressions (2,26) and (2,27): representation is manifestly reducible. That is, if we write 4-comp. spinor ψ as

'left' chirality

$$\boxed{\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},} \qquad (2,35)$$

'right'

we see that the components ψ_L and ψ_R do not mix under L-transformations, because $\Lambda_{\frac{1}{2}}$ is block diagonal in the Weyl basis. Infinitesimally:

$$\psi \to \Lambda_{\frac{1}{2}}\psi \Rightarrow \begin{aligned} \psi_L &\xrightarrow{\Lambda_{\frac{1}{2}}} \left(1 - \tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma} - \tfrac{1}{2}\vec{\beta}\cdot\vec{\sigma}\right)\psi_L \\ \psi_R &\xrightarrow{\Lambda_{\frac{1}{2}}} \left(1 - \tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma} + \tfrac{1}{2}\vec{\beta}\cdot\vec{\sigma}\right)\psi_R, \end{aligned} \qquad (2,36)$$

where $\beta_i \equiv \omega_{0i}$ and $\theta_k \equiv \frac{1}{2}\epsilon_{ijk}\omega_{ij}$. This division of course corresponds to the one we found earlier, i.e. the representations $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$ are representations with different chiralities.

(Hence: $SU(2)_L \otimes SU(2)_R$)

Now:

$$\gamma^0 \hat{P}\left[(i\not{\partial} - m)\psi(x)\right] = \gamma^0(i\gamma^0\partial_t - i\gamma^i\partial_i - m)\psi_p(t,-\vec{x})$$

$\not{\partial} = \gamma^0\partial_0 + \gamma^i\partial_i$

Parity Operator

$$= (i\not{\partial} - m)\underbrace{\gamma^0\psi_p(t,-\vec{x})}$$

$$= \eta_p\psi(t,\vec{x})$$

arbitrary phase

$$\Rightarrow \quad \psi_p(t,-\vec{x}) = \eta_p\gamma^0\psi(t,\vec{x}) \qquad (2,37)$$

the parity conjugated state of the spinor ψ.

In component form (2,37) implies that parity transf. changes chirality:

$$\begin{aligned}(\psi_L)_p &= \eta_p \psi_R \\ (\psi_R)_p &= \eta_p \psi_L \,.\end{aligned} \qquad (2.38)$$

Written in the Weyl basis the Dirac equation becomes

$$\gamma^0 (i\partial\!\!\!/ - m)\psi = \begin{pmatrix} i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}) & -m \\ -m & i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}) \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0 \qquad (2.39)$$

We now see that mass terms mix states of definite chirality.

Notation. Sometimes, for example with SUSY one uses the shorthand notation:

$$\begin{aligned}\sigma^\mu &\equiv (1, \vec{\sigma}) \\ \bar{\sigma}^\mu &\equiv (1, -\vec{\sigma})\end{aligned} \qquad (2.40)$$

$$\Rightarrow \begin{pmatrix} i\bar{\sigma}\cdot\partial & -m \\ -m & i\sigma\cdot\partial \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0 \qquad (2.41)$$

In the limit $m \to 0$ L- and R-chiral sectors decouple and one gets the independent Weyl equations

$$i\bar{\sigma}\cdot\partial\,\psi_L = 0 \,, \qquad i\sigma\cdot\partial\,\psi_R = 0 \,. \qquad (2.42)$$

2,4 FREE PARTICLE SOLUTIONS

Because Dirac eqn. solutions also satisfy the KG-equation, we can write the free solutions as plane waves:

$$\psi(x) = u(p)\, e^{-ip\cdot x}, \qquad (\text{K-G: } p^2 - m^2 = 0) \qquad (2,43)$$

where the 4-spinor $u(p)$ satisfies the algebraic equation

$$\gamma^0(\not{p} - m)\, u(p) = \begin{pmatrix} \bar{\sigma}\cdot p & -m \\ -m & \sigma\cdot p \end{pmatrix} \begin{pmatrix} u_L \\ u_R \end{pmatrix} = 0 \qquad (2,44)$$

The different chirality states, u_L and u_R are still 2-spinors, which can be freely rotated in the SU(2)-spin subspace. We will here choose to use helicity basis in this space: i.e. we require that

2-spinor

$$u_{L,R}(h) = a^h_{L,R}\, \xi_h \qquad (2,45)$$

c-number

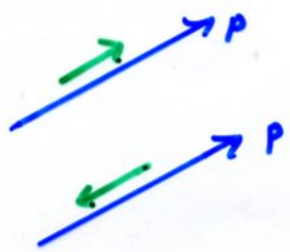
where

note that helicity spinor depends on $\hat{p}$.

$$\boxed{\vec{\sigma}\cdot\hat{p}\, \xi_{h,\hat{p}} \equiv h\, \xi_{h,\hat{p}} \quad ; \quad h = \pm 1} \qquad (2,46)$$

Physically helicity corresponds to the spin of a particle along the direction of its momentum. If we set $\hat{p} = \hat{e}_z$, then $\sigma_z \xi_h = h\xi_h$, ie. $h \triangleq$ spin along the z-axis, and $\xi_+ = \binom{1}{0}$ and $\xi_- = \binom{0}{1}$.

$h=1$;



positive helicity

$h=-1$;

negative helicity.

Assuming now the form (2.46), we get easily from (2.39) that for positive $p_0 \equiv E > 0$:

$$u(p,h) = \begin{pmatrix} \sqrt{E - h|\vec{p}|} \\ \sqrt{E + h|\vec{p}|} \end{pmatrix} \otimes \xi_{h,\hat{p}} \qquad (2.47)$$

The 4-component version of the helicity-operator (2.46) is

$$\hat{h} \equiv \hat{p}\cdot\vec{S} = \hat{p}\cdot\frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \qquad (2.48)$$

$\hat{h}u_h \equiv h u_h$

Equation (2.47) defines two linearly independent solutions for the Dirac equation. The other two can be found by setting $p_0 = -E < 0$, leading to

$$\tilde{u}(p,h) = \begin{pmatrix} \sqrt{E + h|\vec{p}|} \\ -\sqrt{E - h|\vec{p}|} \end{pmatrix} \otimes \xi_{h,\hat{p}} \qquad (2.49)$$

The complete solution with (2.49) is

$$\psi_{p,h}(x) = \tilde{u}(p,h)\, e^{iEt + i\vec{p}\cdot\vec{x}}$$

$$\equiv v(\tilde{p},h)\, e^{i\tilde{p}\cdot x} \quad ; \quad \tilde{p}^{\mu} \equiv (E, -\vec{p}) = p_{\mu}$$

which can be interpreted as a positive energy antiparticle in the Weyl basis

$$v(p,h) = u(\tilde{p},h) = \begin{pmatrix} \sqrt{E + h|\vec{p}|} \\ -\sqrt{E - h|\vec{p}|} \end{pmatrix} \otimes \xi_{-h,\hat{p}} \qquad (2.50)$$

!

Note that in the limit $m \to 0$ (or $E \gg m$):

$$u(p,-1) \longrightarrow \begin{pmatrix} \sqrt{2E} \\ 0 \end{pmatrix} \xi_{-1} = \begin{pmatrix} u_L \\ 0 \end{pmatrix} \qquad (2.51)$$

and

$$v(p,-1) \longrightarrow \begin{pmatrix} 0 \\ -\sqrt{2E} \end{pmatrix} \xi_{+1} = \begin{pmatrix} 0 \\ v_R \end{pmatrix} \qquad (2,52)$$

That is, in the Ultra-relativistic limit negative helicity particles are left-handed, but negative h-antiparticles are righthanded.

Normalization

We will set $\xi_h^\dagger \xi_{h'} = \delta_{hh'}$ so that •

$$u_h^\dagger u_{h'} = v_h^\dagger v_{h'} = 2E\,\delta_{hh'} \qquad (2,53)$$

and

$$\bar{u}_h u_{h'} = 2m\,\delta_{hh'} \quad ; \quad \bar{v}_h v_{h'} = -2m\,\delta_{hh'} \qquad (2,54)$$

When computing scattering matrix elements we will encounter the following spin-sums

$$\boxed{\begin{aligned} \sum_s u_s(p)\,\bar{u}_s(p) &= \not{p} + m \qquad (2,55) \\ \sum_s v_s(p)\,\bar{v}_s(p) &= \not{p} - m \qquad (2,56) \end{aligned}}$$

• It should be clear that the helicity can be replaced by any basis when computing the norms and spin-sums.

2.5 DIRAC MATRICES AND BI-LINEAR FORMS

The Lagrange function (2.32) contains two bilinear forms $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu\psi$, which transform like a scalar and a vector under Lorentz transformations. We can construct other bilinears as well: $\bar{\psi}\Gamma\psi$. How these transform can be worked out when Γ is given in terms of γ-matrices. In general Γ is any constant 4x4 matrix. We can span this space by the following:

1	scalar	1	
γ^μ	vector	4	
γ^5	pseudo-scalar	1	
$\gamma^\mu\gamma^5$	pseudo-vector	4	
$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu,\gamma^\nu]$	antisymm. tensor.	6	(2.57)
		16 ✓.	

Here we defined a new γ-matrix:

$$\boxed{\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3} = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma \tag{2.58}$$

One can easily show that

$$(\gamma^5)^\dagger = \gamma^5 \quad ; \quad (\gamma^5)^2 = 1 \tag{2.59}$$

$$\{\gamma^\mu,\gamma^5\} = 0 \quad \forall \mu. \tag{2.60}$$

In the Weyl-representation

$$\boxed{\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}} \tag{2.61}$$

γ^5 is very important. From identity (2,60) it follows that

$$[S^{\mu\nu}, \gamma^5] = 0, \qquad (2.62)$$

which implies that Dirac representations must reduce to parts with different eigenvalues w.r.t γ^5. This is what we saw already and indeed:

$$\gamma^5 \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} -\psi_L \\ \psi_R \end{pmatrix}, \qquad (2,63)$$

ie this is again but the division to the left- and right chiral representations. With γ^5 we can define the chiral projectors

$$\begin{aligned} P_L &\equiv \tfrac{1}{2}(1-\gamma^5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad (1 \equiv \mathbb{1}_2) \\ P_R &\equiv \tfrac{1}{2}(1+\gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \qquad (2,64)$$

Clearly: $\underline{P_L\psi} = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \equiv \underline{\psi_L}$ and $\underline{P_R\psi} = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} \equiv \underline{\psi_R}$

weyl

all reps.

Of particular interest are the bilinears corresponding to a vector and pseudovector currents:

$$j^\mu = \bar\psi \gamma^\mu \psi \qquad (2,65)$$

$$j_5^\mu \equiv \bar\psi \gamma^\mu \gamma^5 \psi \qquad (2,66)$$

Of these the vector current is conserved in free theory (2,30), since it is the Noether current under invariance in $\underline{\psi \to e^{i\alpha}\psi}$.

The pseudovector current (2.66) would be invariant if $m=0$, based on the invariance of $\bar{\psi}\not{\partial}\psi$ under the chiral rotation

$$\psi \to e^{i\alpha\gamma^5}\psi \qquad (2.67)$$

However, the mass-term $m\bar{\psi}\psi$ breaks this invariance, and so we find

$$\partial_\mu j_5^\mu = 2im\,\bar{\psi}\gamma^5\psi. \qquad (2.68)$$

↳ pseudoscalar

2.6. QUANTIZATION OF DIRAC FIELD

We shall follow the method of canonical quantization again. From the free field Lagrangian

$$\mathcal{L}_{Dirac} = \bar{\psi}(i\not{\partial}-m)\psi \qquad (2.69)$$

we see that the conjugate field to ψ is

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^+ \qquad (2.70)$$

and so the Hamilton function is

$$H = \int d^3x\, \bar{\psi}(-i\vec{\gamma}\cdot\nabla + m)\psi = \int d^3x\, \psi^+\underbrace{(-i\vec{\alpha}\cdot\nabla + \beta m)}\psi \qquad (2.71)$$

where $\vec{\alpha} \equiv \gamma^0\vec{\gamma}$ and $\beta \equiv \gamma^0$.

$= \mathcal{H}_D =$ the Hamilton operator for a 1-particle theory.

When quantizing the K-G theory, we postulated the equal time commutation relations (1.3) between the conjugate fields. From this followed the commutation relation between the creation and annihilation operators, which was consistent with the fact that K-G-particles are bosons. Dirac equation is supposed to describe fermions, which must obey the Pauli principle. Anticipating the statistics issue ● we shall postulate anticommutation relations for ψ and $i\psi^\dagger$:

$$\boxed{\begin{aligned} \{\hat{\psi}_\alpha(x), i\hat{\psi}^\dagger_\beta(y)\} &= i\,\delta^3(\vec{x}-\vec{y})\,\delta_{\alpha\beta} \\ \{\hat{\psi}_\alpha(x), \hat{\psi}_\beta(y)\} &= \{i\psi^\dagger_\alpha(x), i\psi^\dagger_\beta(y)\} = 0 \end{aligned}} \qquad (2.72)$$

These hold at equal times of course. Now construct these field operators in terms of creation and annihilation operators as before:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3 2\omega_p} \sum_s \Big(\underset{\text{destroys a particle}}{a^s_{\vec{p}}}\, u^s(p)\, e^{-ip\cdot x} + \underset{\text{creates an antiparticle.}}{b^{\dagger}_{\vec{p}}}\, v^s(p)\, e^{ip\cdot x} \Big)$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3 2\omega_p} \sum_s \Big(b^s_{\vec{p}}\, \bar{v}^s(p)\, e^{-ip\cdot x} + a^{s\dagger}_{\vec{p}}\, \bar{u}^s(p)\, e^{ip\cdot x} \Big)$$

(2.73)

● Look at chapter 3.5 in Peskin & Schröder to see what goes wrong if you try to impose commutation relations for ψ and $\psi^\dagger$. (positivity of H, causality...)

also below

Similarly to the derivation of (1.13) etc in the bosonic case, one can show that (2.72) imply

$$\boxed{\{a^s_{\vec p}, a^{s'\dagger}_{\vec p'}\} = \{b^s_{\vec p}, b^{s'\dagger}_{\vec p'}\} = 2\omega_p (2\pi)^3 \delta^3(\vec p - \vec p')\delta_{ss'}} \qquad (2.74)$$

while the remaining commutators vanish:

$$\{a,a\} = \{a^\dagger,a^\dagger\} = \{b,b\} = \{b,b^\dagger\} = 0. \qquad (2.75)$$

We can see the necessity for the anticommutation relations by computing the Hamiltonian and requiring its positivity:
From (2.71), using $-i\vec\gamma\cdot\nabla = i\gamma^0\partial_t - (i\not\partial - m)$ and taking the normal ordering:

$$H = :\int d^3x\, \bar\Psi\big(i\gamma^0\partial_t - \underbrace{(i\not\partial - m))\Psi}_{=0}: \;= :\int d^3x\, i\Psi^\dagger\dot\Psi:$$

$$= i\int \widetilde{d^3p}\,\widetilde{d^3p'}\, d^3x \sum_{s,s'} :\big(b^s_{\vec p}\, v^\dagger_s(\vec p)\, e^{-ip\cdot x} + a^{s\dagger}_{\vec p}\, u^\dagger_s(\vec p)\, e^{ip\cdot x}\big)$$

$$\widetilde{d^3p} = \frac{d^3p}{(2\pi)^3 2\omega_p} \qquad \times(-i\omega_{p'})\big(a^{s'}_{\vec p'}\, u_{s'}(\vec p')\, e^{-ip'\cdot x} - b^{s'\dagger}_{\vec p'}\, v_{s'}(\vec p')\, e^{ip'\cdot x}\big):$$

do $\vec x$-integral

$$= \int \widetilde{d^3p} \sum_{ss'} \frac{1}{2} :\big(a^{s\dagger}_{\vec p} a^{s'}_{\vec p}\, \underbrace{u^\dagger_s(\vec p)\, u_{s'}(\vec p)}_{=2\omega_p\delta_{ss'}} - b^s_{\vec p} b^{s'\dagger}_{\vec p}\, \underbrace{v^\dagger_s(\vec p)\, v_{s'}(\vec p)}_{2\omega_p\delta_{ss'}}$$

$$+ b^s_{\vec p} a^{s'}_{-\vec p}\, \underbrace{v^\dagger_s(\vec p)\, u_{s'}(-\vec p)}_{=0} - a^{s\dagger}_{\vec p} b^{s'}_{-\vec p}\, \underbrace{u^\dagger_s(\vec p)\, v_{s'}(-\vec p)}_{=0}\big):$$

$$= \int \frac{d^3p}{(2\pi)^3 2\omega_p} \sum_s \omega_p :\big(a^{s\dagger}_{\vec p} a^s_{\vec p} - b^s_{\vec p} b^{s\dagger}_{\vec p}\big): \qquad (2.76)$$

! order!

Note that until now we have not assumed anything about statistics of the operators, so the normal ordering still needs to be specified. Now, if we had opted for commutation relations, we would have

$$:b b^{+}: \;=\; + b^{+} b, \qquad (2.77)$$

With this choice the Hamiltonian (2.76) would be unbounded from below! Adding more antiparticles would lower the energy indefinitely. Thus, the positivity of H requires that

$$:b b^{+}: \;=\; - b^{+} b. \qquad (2.78)$$

That is, spin-$\frac{1}{2}$ fields must be quantized using anti-commutation rules. (spin-statistics theorem!).

With this choice we get

$$:H: = \int \widetilde{d^3p} \sum_s \omega_p \left(a^{+}_{s,\vec{p}}\, a_{s,\vec{p}} + b^{+}_{s,\vec{p}}\, b_{s,\vec{p}} \right) \qquad (2.79)$$

Anticommutation rules (2.74) also implement the Pauli exclusion rule:

$$a^{s+}_{\vec{p}}\, a^{s+}_{\vec{p}} |0\rangle = - a^{s+}_{\vec{p}}\, a^{s+}_{\vec{p}} |0\rangle = 0 \qquad (2.80)$$

That is ψ-excitations follow Fermi-statistics.

We can also compute the Noether charge for the current $\bar{\psi}\gamma^\mu\psi$:

$$\hat{Q} \equiv q\int d^3x\, \bar{\psi}\gamma^0\psi = q\int d^3x\, \psi^\dagger\psi$$

$$= q\int \widetilde{d^3p} \sum_s \left(a^\dagger_{\vec{p},s} a_{\vec{p},s} - b^\dagger_{\vec{p},s} b_{\vec{p},s} \right)$$

This is consistent: $b^\dagger$-operators create antiparticles with opposite charges $(-q)$ to those of particles (q).

For example in QED the field ψ will describe electrons and positrons:

$a^{s\dagger}_{\vec{p}}|0\rangle$ an electron state

$b^{s\dagger}_{\vec{p}}|0\rangle$ a positron state

Moreover:

$$\underbrace{\langle 0|\, a_{s,\vec{q}}}_{\langle \text{electron}|}\, \hat{Q}\, \hat{\bar{\psi}}(x)|0\rangle = q\,\bar{u}_s(\vec{q})\, e^{iq\cdot x}$$

outgoing electron
$q = -e$

$$\underbrace{\langle 0|\, b_{s,\vec{q}}}_{\langle \text{positron}|}\, \hat{Q}\, \hat{\psi}(x)|0\rangle = -q\, v_s(q)\, e^{iq\cdot x}$$

incoming positron
$-q = +e$

$$\langle 0|\hat{\psi}(x)\, \hat{Q}\, a^\dagger_{s,q}|0\rangle = q\, u_s(q)\, e^{-iq\cdot x}$$

incoming electron

$$\langle 0|\hat{\bar{\psi}}(x)\, \hat{Q}\, b^\dagger_{s,q}|0\rangle = -q\,\bar{v}_s(q)\, e^{-iq\cdot x}$$

outgoing positron

2.7 DIRAC PROPAGATOR

Just as in the case for scalars, we may add a source term to Dirac Lagrangian:

$$\mathcal{L} \to i\bar{\psi}\cancel{\partial}\psi + m\bar{\psi}\psi + \bar{\xi}\psi + \bar{\psi}\xi \qquad (2.85)$$

Whereby

$$(i\cancel{\partial} - m)\psi = \xi(x) \qquad (2.86)$$

The corresponding Greens equation is

Dirac indices

$$(i\cancel{\partial}_x - m)\, S(x-y) = i\delta^4(x-y)\,\delta_{\alpha\beta} \qquad (2.87)$$

and

$$\psi(x) = \psi_0(x) - i\int d^4y\, S(x-y)\,\xi(y). \qquad (2.88)$$

Fourier-transforming (2.87) we get

$$\int \frac{d^4p}{(2\pi)^4} (\cancel{p} - m)\, \tilde{S}(p)\, e^{-ip(x-y)} = i\delta^4(x-y) \qquad (2.89)$$

whereby (using $\cancel{p}\cancel{p} = p^2$)

$$\tilde{S}(p) = i\,\frac{\cancel{p}+m}{p^2-m^2}. \qquad (2.90)$$

And so the yet ambiguous wrt boundary conditions propagator is

$$S_X(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\cancel{p}+m)}{p^2-m^2} e^{-ip\cdot(x-y)}$$

scalar field propagator

$$= (i\cancel{\partial}_x + m)\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2} e^{-ip\cdot(x-y)} = (i\cancel{\partial}_x + m)\,\Delta_X(x-y) \qquad (2.91)$$

Thus we get the four different Fermionic propagators from the bosonic ones (1.44a–1.44d) by simple multiplication by $\not{p}+m$. The Feynman propagator in particular reads

$$\boxed{S_F(p) = \frac{i}{\not{p} - m + i\epsilon}} \qquad (2.92)$$

In the scalar-case we were able to relate the propagator functions to a number of vacuum expectation values. Same goes here. First:

$$\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle =$$

$$= \int \widetilde{d^3p}\,\widetilde{d^3p'} \sum_{ss'} \langle 0| \left(a^s_{\vec{p}} u^s_\alpha(\vec{p}) e^{-ip\cdot x} + b^{s\dagger}_{\vec{p}} v^s_\alpha(\vec{p}) e^{ip\cdot x} \right)$$

$$\times \left(a^{s'\dagger}_{\vec{p}'} \bar{u}^{s'}_\beta(\vec{p}') e^{ip'\cdot y} + b^{s'}_{\vec{p}'} \bar{v}^{s'}_\beta(\vec{p}') e^{-ip'\cdot y} \right) |0\rangle$$

$$= \int \widetilde{d^3p}\,\widetilde{d^3p'} \sum_{ss'} u^s_\alpha(p)\bar{u}^{s'}_\beta(\vec{p}')\, e^{-ip\cdot x + ip'\cdot y} \underbrace{\langle 0| a^s_{\vec{p}} a^{s'\dagger}_{\vec{p}'} |0\rangle}_{= (2\pi)^3 2\omega_p \delta^3(\vec{p}-\vec{p}')\delta_{ss'}}$$

$$= \int \widetilde{d^3p} \sum_s u^s_\alpha(\vec{p}) \bar{u}^s_\beta(\vec{p})\, e^{-ip\cdot(x-y)}$$

$$= \underline{\int \widetilde{d^3p}\, (\not{p}+m)_{\alpha\beta} e^{-ip\cdot(x-y)} = (i\not{\partial}+m)\, D(x-y)} \qquad (2.93)$$

Similarly:

$$\langle 0|\bar{\psi}_\beta(y)\psi_\alpha(x)|0\rangle = \int \widetilde{d^3p} \sum_s \bar{v}^s_\beta(\vec{p})\, v^s_\alpha(p)\, e^{ip\cdot(x-y)}$$

$$= \int d^3\tilde{p}\, (\not{p}-m)_{\alpha\beta} e^{ip\cdot(x-y)} = \underline{-(i\not{\partial}_x+m)\, D(y-x)} \qquad (2.94)$$

From these we can write the propagators in terms of the vacuum expectation values. For example the retarded propagator is

$$\theta(x_0-y_0)\left(D(x-y)-D(y-x)\right)$$

$$S_R(x-y) = (i\not\partial_x+m)\,\Delta_R(x-y)$$

$$= \theta(x_0-y_0)(i\not\partial_x+m)\left(D(x-y)-D(y-x)\right)$$

$$= \theta(x_0-y_0)\langle 0|\{\psi(x),\bar{\psi}(y)\}|0\rangle \qquad (2.95)$$

anti-commutator

Note that the term $\sim\left[\gamma^0\partial_{x_0}\theta(x_0-y_0)\right]\left(D(x-y)-D(y-x)\right) \sim \left[\phi(x_0,\vec{x}),\phi(x_0,\vec{y})\right]\equiv 0$

equal-time commutator.

Similarly the Feynman propagator is

$$S_F(x-y) = \theta(x_0-y_0)\langle 0|\psi_\alpha(x)\,\bar{\psi}_\beta(y)|0\rangle$$

$$- \theta(y_0-x_0)\langle 0|\bar{\psi}_\beta(y)\,\psi_\alpha(x)|0\rangle$$

$$\equiv \langle 0|T[\psi_\alpha(x)\bar{\psi}_\beta(y)]|0\rangle \qquad (2.96)$$

The definition of time-ordering contains a minus sign that can be viewed as arising from change of order of fermionic operators.

2.8 DISCRETE SYMMETRIES

In Dirac theory one encounters the following three discrete symmetries

parity	$(t,\vec{x}) \overset{\hat{P}}{\longleftrightarrow} (t,-\vec{x})$
time reversal	$(t,\vec{x}) \overset{\hat{T}}{\longleftrightarrow} (-t,\vec{x})$
charge conjugation	particle $\leftrightarrow$ antiparticle

The first two are space-time symmetries, which preserve the Minkowski metric invariant $x_\mu x^\mu$, and divide the continuous Lorentz transformations to 4 subclasses:

$$\begin{tikzcd}
L_+^\uparrow \arrow[r, leftrightarrow, "\hat{P}"] \arrow[d, leftrightarrow, "\hat{T}"'] & L_-^\uparrow = \hat{P} L_+^\uparrow \arrow[d, leftrightarrow, "\hat{T}"] \\
L_+^\downarrow = \hat{T} L_+^\uparrow \arrow[r, leftrightarrow, "\hat{P}"'] & L_-^\downarrow = \hat{P}\hat{T} L_+^\uparrow
\end{tikzcd}$$

Orthocronous $(\Lambda^0{}_0 > 1)$

Non-orthocronous $(\Lambda^0{}_0 < -1)$

Proper $(\det \Lambda = 1)$ Improper $(\det \Lambda = -1)$

Any relativistic theory must be invariant under $L_+^\uparrow$, but not necessarily under $\hat{P}$, $\hat{T}$ and $\hat{C}$. Indeed we have for example that:

QED, QCD: P, T, C, CP, $\boxed{CPT}$

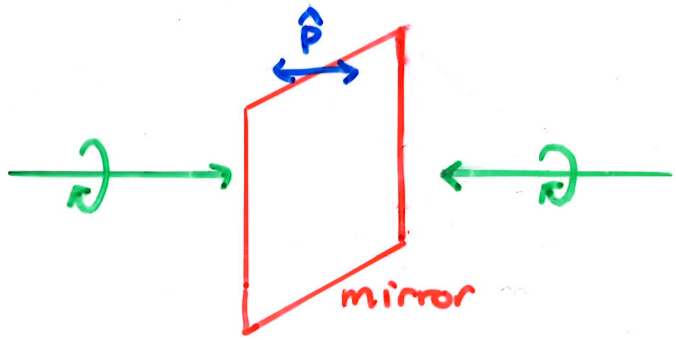
Weak: $\cancel{P}$, $\cancel{T}$, $\cancel{C}$, $\cancel{CP}$, $\boxed{CPT}$

PARITY

$\hat{P}$ changes $\vec{x} \to -\vec{x}$ and $\vec{p} \sim \partial_t \vec{x} \to -\vec{p}$, but leaves spin invariant ($\hat{P}\vec{J} = \hat{P}(\vec{x}\times\vec{p}) = \vec{J}$), so that

$$\hat{P}\, a^s_{\vec{p}}\, \hat{P}^\dagger = \eta_a\, a^s_{-\vec{p}}$$

$$\hat{P}\, b^s_{\vec{p}}\, \hat{P}^\dagger = \eta_b\, b^s_{-\vec{p}} \qquad (2.97)$$



Two applications of parity should return observables to their original values. Since these always built from an even number of these, we can have $\eta_a^2 = \eta_b^2 = 1$. We can now find the transformation law for $\hat{\psi}(x)$:

$$\hat{P}\hat{\psi}(x)\hat{P}^\dagger = \int \widetilde{d^3p} \sum_s \left(\eta_a\, a^s_{-\vec{p}}\, u_s(p)\, e^{-ip\cdot x} + \eta_b^*\, b^{s\,\dagger}_{-\vec{p}}\, v_s(p)\, e^{ip\cdot x} \right) \qquad (2.98)$$

After a change of variables $\vec{p} \to -\vec{p}$ and observing that

$$u_s(p_0, -\vec{p}) = \gamma^0 u_s(p_0, \vec{p})$$

$$v_s(p_0, -\vec{p}) = -\gamma^0 v_s(p_0, \vec{p}) \qquad (2.99)$$

$\tilde{x}^\mu \equiv (t, -\vec{x}) = x_\mu$

we get

$$\equiv -1$$

$$\hat{P}\hat{\psi}(x)\hat{P}^\dagger = \eta_a \gamma^0 \int \widetilde{d^3p} \sum_s \left(a^s_{\vec{p}}\, u_s(p)\, e^{-ip\cdot\tilde{x}} - \frac{\eta_b^*}{\eta_a}\, b^{s\,\dagger}_{\vec{p}}\, v_s(p)\, e^{ip\cdot\tilde{x}} \right)$$

$$= \underline{\eta_a \gamma^0 \hat{\psi}(t, -\vec{x})} \qquad (2.100)$$

see eq. (2.48) on p. 42.

In order to get (2.100) we had to set

$$\eta_b^* = -\eta_a \qquad (2.101)$$

This we can do, because we can choose $\eta_{a,b}$ arbitrarily. Indeed for any bilinear form we have

$$\hat{P}\,\bar{\psi}\,\Gamma\,\psi\,\hat{P}^\dagger \propto |\eta_a|^2 = 1 \qquad (2.102)$$

(So we could have set from the beginning $\eta_{a,b} = \pm 1$)

Given the result (2.102) we can show that

$$\begin{aligned}
\hat{P}\,\bar{\psi}(x)\,\psi(x)\,\hat{P}^\dagger &= \bar{\psi}(\tilde{x})\,\psi(\tilde{x})\\
\hat{P}\,\bar{\psi}(x)\,\gamma^5\,\psi(x)\,\hat{P}^\dagger &= -\,\bar{\psi}(\tilde{x})\,\gamma^5\,\psi(\tilde{x})\\
\hat{P}\,\bar{\psi}(x)\,\gamma^\mu\,\psi(x)\,\hat{P}^\dagger &= \bar{\psi}(\tilde{x})\,\gamma_\mu\,\psi(\tilde{x})\\
\hat{P}\,\bar{\psi}(x)\,\gamma^\mu\gamma^5\,\psi(x)\,\hat{P}^\dagger &= -\,\bar{\psi}(\tilde{x})\,\gamma_\mu\gamma^5\,\psi(\tilde{x})
\end{aligned} \qquad (2.103)$$

"pseudo-" ≙ extra – sign under $\hat{P}$.

where $\tilde{x}^\mu = x_\mu = (t, -\vec{x})$ and $\gamma_\mu = g_{\mu\nu}\gamma^\nu = (\gamma^0, -\vec{\gamma})$.

We call system even- (odd) under parity if its wave-function is symmetric (antisymmetric) under $\hat{P}$.

Indeed, we can choose for example $\eta_a \equiv 1$ whereby $\eta_b^* = \eta_b = -1$. The only sign of physical significance is the relative one between particles and antiparticles:

$$\boxed{\eta_a \eta_b = -1} \qquad (2.101\,b)$$

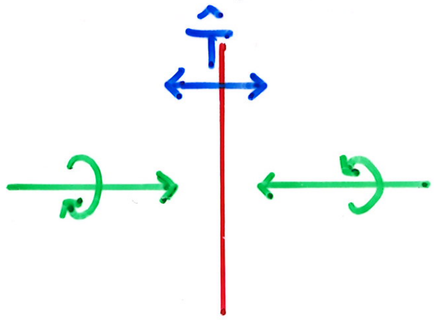
(This is why the parity of the positronium ground state is -1.)

TIME REVERSAL

Under time reversal $\vec{p} \to -\vec{p}$ and $\vec{J} \to -\vec{J}$, i.e. also spin changes. (However helicity, or handedness, does not.)

Thus

$$\hat{T}\, a^h_{\vec{p}}\, \hat{T}^\dagger = \eta^{\hat{T}}_a\, a^h_{-\vec{p}}$$
$$\hat{T}\, b^h_{\vec{p}}\, \hat{T}^\dagger = \eta^{\hat{T}}_b\, b^h_{-\vec{p}} \qquad (2.104)$$



Like parity, the time-reversal obeys $\hat{T}^2 = 1$, so that we can set $(\eta^{\hat{T}}_{a,b})^2 = 1$.

We would like to find an unitary $\hat{T}$ that would do (2.104) and send $\hat{\psi}(\vec{x},t)$ to $A\hat{\psi}(\vec{x},-t)$ where A is some constant matrix. This does not work out like it did for parity however.

Let us again first look at time reversal for the Dirac equation:

$$0 = \hat{T}\,(i\not{\partial} - m)\,\psi(t,\vec{x})$$
$$= (-i\gamma^0\partial_t + i\vec{\gamma}\cdot\vec{\nabla} - m)\,\psi_t(-t,\vec{x}) \quad \Big|^{*}$$
$$= A\,(i\gamma^0\partial_t - i\vec{\gamma}^{*}\cdot\vec{\nabla} - m)\,A^{-1}A\,\psi_t^{*}(-t,\vec{x})$$
$$\equiv (i\not{\partial} - m)\,A\psi_t^{*}(-t,\vec{x}) \qquad (2.105)$$

So we must have $A\gamma^0A^{-1} = \gamma^0$ and $A\vec{\gamma}^{*}A^{-1} = -\vec{\gamma}$

Since $\sigma_2\vec{\sigma}\sigma_2 = -\vec{\sigma}$ and $\sigma_2 1 \sigma_2 = 1$ we see that the appropriate matrix is $A = -i\begin{pmatrix}\sigma_2 & 0\\ 0 & \sigma_2\end{pmatrix} = \gamma^1\gamma^3$; $A^{-1} = -A = \gamma^3\gamma^1$

$$\Rightarrow \quad \underline{\psi_t(t,x) = \gamma^1\gamma^3\,\psi^{*}(-t,\vec{x})} \qquad (2.106)$$

Thus $\hat{T}$ is <u>not</u> linear. Indeed, consider a state

$$\Psi(t,\vec{x}) \equiv \sum_i \alpha_i \psi_i(t,\vec{x}) \quad ; \quad \alpha_i \in \mathbb{C}$$

$$\xrightarrow{\hat{T}} \gamma^1\gamma^3 \Psi^*(-t,\vec{x}) \equiv \Psi_t(t,\vec{x})$$

$$= \sum_i \alpha_i^* (\gamma^1\gamma^3)\psi_i^*(-t,\vec{x}) = \sum_i \alpha_i^* \psi_{i\,t}(t,x) \qquad (2.107)$$

That is, $\hat{T}$ changes complex numbers to their conjugates in linear superpositions. (From this it also follows that $\hat{T}$ cannot be an observable.)

Also if we want to impose $\hat{T}$ as a symmetry of Dirac theory: $[\hat{T}, \hat{H}] \equiv 0$, we encounter problem if we do not conjugate complex numbers: Indeed

$$\hat{\psi}(-t,\vec{x})|0\rangle = e^{-i\hat{H}t}\hat{\psi}(\vec{x})e^{i\hat{H}t}|0\rangle$$

$$= e^{-i\hat{H}t}\hat{\psi}(x)|0\rangle \quad ; \; \hat{H}|0\rangle = 0 \qquad (2.108a)$$

$$\hat{T}\hat{\psi}(t,\vec{x})\hat{T}^\dagger|0\rangle = \hat{T}e^{i\hat{H}t}\psi(\vec{x})e^{-i\hat{H}t}\hat{T}^\dagger|0\rangle$$

$$\equiv e^{+i\hat{H}t}(\hat{T}\psi(x)\hat{T}^\dagger)|0\rangle \qquad (2.108b)$$

assuming no change of c-numbers

Indeed suppose that $\Psi = \sum_i \alpha_i \psi_i$ where ψ_i are normalized $\hat{T}$-eigenstates with $\hat{T}\psi_i = t_i\psi_i$. Then we have

$$\langle\Psi|\Psi\rangle = \sum_i |\alpha_i|^2 \equiv 1$$

and

$$\langle\Psi|\hat{T}|\Psi\rangle = \overbrace{\sum_i (\alpha_i^*)^2 t_i}^{\notin \mathbb{R}} \neq \sum |\alpha_i|^2 t_i$$

this is what it should be

If these manipulations were correct, it would not be possible to relate these states, because the former (2.108a) is a collection of positive and the latter of negative energy states. The way out is to realize that $\hat{T}$ must change e^{+iEt} to e^{-iEt}.

So, with these preliminaries we are ready to write down the operation of $\hat{T}$ on $\hat{\psi}$:

$$\hat{T}\hat{\psi}(\vec{x},t)\hat{T}^{+} = \int \widetilde{d^3p} \sum_h \left(\eta_a^{\hat{T}} a^h_{-\vec{p}} \left(u_h(p)e^{-ip\cdot x}\right)^* + \eta_b^{\hat{T}} b^{h+}_{\vec{p}} \left(v_h(\vec{p})e^{ip\cdot x}\right)^* \right)$$

$$\overset{\vec{p}\to-\vec{p}}{=} \int \widetilde{d^3p} \sum_h \Big(\eta_a^{\hat{T}} a^h_{\vec{p}}\, u_h^*(-p)\, e^{-ip\cdot(-t,\vec{x})}$$

$$+ \eta_b^{\hat{T}} b^{h+}_{\vec{p}}\, v_h^*(-\vec{p})\, e^{ip\cdot(-t,\vec{x})} \Big) \qquad (2.109)$$

Now, from (2.47) we see that

$$u_h^*(-p) = \begin{pmatrix} \sqrt{E-h|\vec{p}|} \\ \sqrt{E+h|\vec{p}|} \end{pmatrix} \otimes \xi^*_{h,-\vec{p}} \, . \qquad (2.110)$$

out since $\vec{\sigma}\cdot\hat{p}\,\xi_{h,\vec{p}} = h\,\xi_{h,\vec{p}}$ and $\sigma_2\vec{\sigma}^*\sigma_2 = -\vec{\sigma}$ we get

$$\overset{\sigma_2\sigma_2}{\checkmark}\ \vec{\sigma}\cdot\hat{p}\left(\sigma_2\,\xi^*_{h,-\vec{p}}\right) = -\sigma_2\,\vec{\sigma}^*\cdot\hat{p}\,\xi^*_{h,-\vec{p}}$$

$$= -\sigma_2\left(\vec{\sigma}\cdot\hat{p}\,\xi_{h,-\vec{p}}\right)^* = +h\,\sigma_2\,\xi^*_{h,-\vec{p}} \quad \Rightarrow \quad \sigma_2\,\xi^*_{h,-\vec{p}} \propto \xi_{h,\vec{p}}$$

We can choose: $\underline{-i\sigma_2\,\xi^*_{h,-\vec{p}} \equiv \xi_{h,\vec{p}}}$. Now define the matrix

$$\underline{A \equiv \gamma^1\gamma^3 = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}} \quad ; \quad A^2 = -1 \qquad (2.111)$$

$$\text{eg. } A^{-1} = -A$$

From the above manipulations it is easy to see that

$$\gamma^1\gamma^3 u_n(-\vec{p})^* = u_n(\vec{p})$$
$$\gamma^1\gamma^3 v_n(-\vec{p})^* = v_n(\vec{p}) \qquad (2.112)$$

So that eventually in Heisenberg picture:

(no complex conjugate)

$$\hat{T}\psi(\vec{x},t)\hat{T}^\dagger = \gamma^1\gamma^3\hat{\psi}(-t,\vec{x}) \qquad (2.113)$$

Using (2,113) one can check various transformation laws for the bilinear forms. For example

$$\hat{T}\,\hat{\bar{\psi}}(1-i\gamma^5)\hat{\psi}\,\hat{T}^\dagger = \hat{\bar{\psi}}(-t,\vec{x})\,(1+i\gamma^5)\,\hat{\psi}(-t,\vec{x})$$
$$\hat{T}\,\hat{\bar{\psi}}\,\gamma^\mu(1-\gamma^5)\,\hat{\psi}\,\hat{T}^\dagger = \hat{\bar{\psi}}(-t,\vec{x})\,\gamma_\mu(1-\gamma^5)\hat{\psi}(-t,\vec{x}) \qquad (2.114)$$

and so on.

CHARGE CONJUGATION

Charge conjugation changes particles to antiparticles, and vice verca.

$$\hat{C}\,\hat{a}^s_{\vec p}\,\hat{C}^\dagger = b^s_{\vec p} \qquad\qquad (2.115)$$

$$\hat{C}\,\hat{b}^s_{\vec p}\,\hat{C}^\dagger = a^s_{\vec p}$$

So that

$$\hat{C}\,\hat{\Psi}(t,\vec x)\,\hat{C}^\dagger = \int \widetilde{d^3p}\,\sum_h \left(b^h_{\vec p}\, u_h(\vec p)\, e^{-ip\cdot x} + a^{h\dagger}_{\vec p}\, v_h(\vec p)\, e^{ip\cdot x} \right) \qquad (2.116)$$

Now, the solutions (2.47) and (2.50) are not all linearly independent. Instead, we can show that

$$\underbrace{C}_{=i\gamma^0\gamma^2\,!}\bar{u}_h^T = -i\gamma^2 u(\vec p)^* = -\begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{E-h|\vec p|} \\ \sqrt{E+h|\vec p|} \end{pmatrix}\otimes \xi^*_{h,\vec p} = \begin{pmatrix} -\sqrt{+} \\ \sqrt{-} \end{pmatrix}\otimes \overbrace{i\sigma_2\,\xi^*_{h,\vec p}}^{=-\xi_{h,-\vec p}}$$

$$= \begin{pmatrix} \sqrt{+} \\ -\sqrt{-} \end{pmatrix}\otimes \xi_{h,-\vec p} = +v_h(\vec p). \qquad (2.117)$$

Similarly one can show that $C\bar{v}_h^T = u_h$. Putting these forms into (2.116) we find

$$\hat{C}\,\hat{\Psi}(t,\vec x)\,\hat{C}^\dagger = \int \widetilde{d^3p}\,\sum_h \Big(\underbrace{-a^{h\dagger}_{\vec p}\, i\gamma^2 \overset{\gamma^0\gamma^0}{\vee} u^*(\vec p)\, e^{ip\cdot x}}_{=\,i\gamma^0\gamma^2\left[\left(a^h_{\vec p}\,u(\vec p)\,e^{-ip\cdot x}\right)^\dagger\gamma^0\right]^T} - b^h_{\vec p}\, i\gamma^2 \overset{\gamma^0\gamma^0}{\vee} v^*(\vec p)\, e^{-ip\cdot x} \Big)$$

$$= C\,\bar{\hat{\Psi}}(t,\vec x)^T \qquad (2.118)$$

One can use this relation to derive transformation laws for bilinears, such as:

$$\hat{C}\,\bar{\psi}(1-\gamma^5)\psi\,\hat{C}^\dagger = \bar{\psi}(1-\gamma^5)\psi$$
$$\hat{C}\,\bar{\psi}\gamma^\mu(1-\gamma^5)\psi\,\hat{C}^\dagger = \bar{\psi}\gamma^\mu(1+\gamma^5)\psi \qquad (2.119)$$

and so on.

Box: Remember that the form (2.118) was expected from Dirac eqn. manipulations, when one introduces coupling to em. field.

$$(i\not\partial + e\not A - m)\psi = 0 \quad \Rightarrow \quad (i\not\partial - e\not A - m)\psi^c \equiv 0$$

↑ charge-conjugated field.

To see what ψ^c is one goes as follows

$$[(i\not\partial + e\not A - m)\psi]^* = \left((-i\partial_\mu + eA_\mu)\gamma^{\mu *} - m\right)\psi^* = 0$$

(γ^0 ; $\gamma^0\gamma^0$)

$$\Leftrightarrow C[(-i\partial_\mu + eA_\mu)\gamma^{\mu T} - m]C^{-1}C\,\underbrace{\gamma^0\psi^*}_{=(\psi^\dagger\gamma^0)^T = \bar{\psi}^T}$$

$$\Leftrightarrow (i\not\partial - e\not A - m)\underline{C\bar{\psi}^T} = 0$$

where one used $\gamma^0\gamma^{\mu\dagger}\gamma^0 = +\gamma^\mu$ and $C\gamma^{\mu T}C^{-1} \equiv -\gamma^\mu$ — def's C up to the sign

$$\underline{C \equiv i\gamma^0\gamma^2} \qquad (2.120)$$

! opposite sign to P&S, consistent with (2.50)

CPT - theorem.

One can easily show that the free Dirac theory $\mathcal{L}_0 = \bar{\psi}(i\not\partial - m)\psi$ is invariant under all discrete transformations $\hat{C}$, $\hat{P}$ and $\hat{T}$. On the other hand a term $\bar{\psi}\gamma^\mu(1-\gamma^5)\psi$ breaks $\hat{P}$ and $\hat{C}$ for example. It still is invariant under the combined $\hat{C}\hat{P}\hat{T}$-transformation. This is an example of a general theorem:

If theory is local, has hermitean $\hat{H}$-operator and it is quantized according to spin-statistics theorem, it necessarily obeys $\underline{\widehat{CPT}\,\mathcal{L} = \mathcal{L}(-x)}$.